mathematics

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Pulse Processes in Networks and Evolution Algebras

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Received: 18 January 2020; Accepted: 18 February 2020; Published: 10 March 2020

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Abstract: In this paper, we merge two theories: that of pulse processes on weighted digraphs and that of evolution algebras. We enrich both of them. In fact, we obtain new results in the theory of pulse processes thanks to the new algebraic tool that we introduce in its framework, also extending the theory of evolution algebras, as well as its applications.

Keywords: pulse process; stability; weighted digraph; evolution algebra

1. Introduction

A complex system can be understood as a system determined by many components which may interact with each other (see [1] for a deeper discussion about the term). Such systems are usually described by a weighted digraph (that is, a network) where the nodes represent the components of the system, and the arcs their interactions. The study of complex networks is a modern, active, and interdisciplinary area of research addressed to the empirical study of complex systems, such as computer networks, technological networks, brain networks, and social networks.

To understand a complex network, a mathematical framework is needed in order to determine the properties of the underlying weighted digraph, for instance, to make predictions on the evolution of the system.

A pulse process is a structural model to analyze a complex network. Its mathematical foundation is established in [2] (see also [3]) and is summarized in [4,5]. Such a process is a simple dynamic model to study the propagation of changes, through the vertices of a weighted digraph, after introducing an "initial pulse" in the system at a particular vertex. It is based on a spectral analysis of the corresponding weighted digraph to face large scale decision making problems.

Pulse processes have been applied to topics, such as food production, energy, air pollution, transportation systems, coastal resources, health care delivery, manpower, water policy, inland waterway traffic, ecosystems, and the analysis of historical events, to make decisions (see for instance [2,6–17]). Particularly, the pulse process analysis has been used in many reports of the National Science Foundation, especially in those of the study entitled "Evaluation Measures to Conserve Energy" achieved by the Rand Corporation (see National Science Foundation (NSF) reports R-756-NSF, R-926-NSF, R-927/1-NSF, R-927/2-NSF, and R-1578-NSF referenced in [3,7,18–20]). In the last report [20], pulse processes were used in the context of energy demand, air pollution, and related environmental problems in order to analyze the transportation system of a hypothetical metropolitan area similar to San Diego, California. As it is stated in R-756-NSF, the result of using graph theory to model such problems, "while it does not necessarily provide a complete solution to the problem, it often brings a better understanding of what the possible solutions are or an insight into the qualitative interrelationships that underlie the problem, or an identification of significant or vulnerable points of attack".

Evolution algebras are non-associative algebras with a dynamic nature. They were introduced in 2008 by J. P. Tian [21] for the study of Non-Mendelian Genetics. As it is shown in this pioneering monograph, they have strong connections with group theory, Markov processes, theory of knots, adynamic systems and graph theory. Because of this, a vast literature around them has grown since 2008, with direct applications in Biology, Physics and Mathematics itself (see for instance [22–27] and references therein).

In Section 2 of this paper, we summarize the mathematical substrate of the pulse processes analysis and, similarly, in Section 3, we briefly review the notion of evolution algebra with emphasis in the associated weighted digraph relative to a natural basis.

In Section 4, we merge the theory of evolution algebras with the theory of pulse processes, enlightening an original way to introduce algebraic techniques into the study of pulse processes that simultaneously enriches the theory of evolution algebras. We illustrate this approach with new results that help to understand in a deeper way many aspects of the aforementioned NSF's reports.

In Section 5, we explore the role of the ideals of the evolution algebra associated to a pulse process. We apply this to get a better knowledge of what, in report [20], are called "interesting strong connected components" by showing, in algebraic terms, that the behavior of such components is not always the same. More precisely, we describe when the stability of one or more of these components determines the stability of the given pulse process (see Examples 8,10,11). To do this, we apply some results such as Theorem 9, Theorem 11 and Corollary 5. Moreover, we use these results in Section 6 to describe what we name the "reduction process". This is a method to obtain, from a given pulse process, another very simplified one, called the reduced process, which is such that its stability (in pulse and/or value) is equivalent to that of the original pulse process. With the examples quoted above, we also show in an explicit way how the reduction process simplifies and enriches the analysis achieved in [20].

2. Pulse Processes on Weighted Digraphs: A Brief Review

In this section, we review the main results about the stability of the pulse processes, following [2,4,5].

Let D be a weighted digraph with vertices $x_1, x_2, ..., x_n$. We suppose that each vertex x_i attains a value $v_i(t)$ at each discrete time $t = 0, 1, 2, ...$ Then, the successive value $v_i(t+1)$ is determined from the last time period t according to the following model:

$$v_i(t+1) = v_i(t) + p_i^0(t+1) + \sum_j w(x_j, x_i) p_j(t), \tag{1}$$

where

- $v_i(t)$ is the value of vertex x_i at time t,
- $p_i^0(t+1)$ is the value of the external pulse introduced at vertex x_i at time $t+1$ (therefore, the possibility of externally influencing the variables of the system at each time is considered in this model),
- $w(x_j, x_i)$ is the weight of the arc $x_j x_i$ (that is the value which measures the strength of the effect that vertex x_j has over x_i), and
- $p_j(t)$ is the *pulse* at vertex x_j at time t, defined by:

$$p_j(t) = \begin{cases} v_j(t) - v_j(t-1) & \text{if } t > 0 \\ p_j^0(0) & \text{if } t = 0. \end{cases} \tag{2}$$

Hence, the value $v_i(t+1)$ of vertex x_i at time $t+1$ is obtained by the value $v_i(t)$ that x_i had at the last time period, the external pulse $p_i^0(t+1)$ introduced in x_i at time $t+1$, and the weighted pulse that the vertices x_j, adjacent to x_i, transmit to vertex x_i from t to $t+1$.

Consequently, the **pulse process** on a weighted digraph D, with vertices $x_1, x_2, ..., x_n$, is defined by Equation (1), along with an initial vector of values

$$V(0) = (v_1(0), v_2(0), ..., v_n(0)),$$

and the vector providing the value of the external pulse introduced at each vertex at each time period, denoted by

$$P^0(t) = (p_1^0(t), p_2^0(t), ..., p_n^0(t)).$$

Finally, the vector $P(t) = (p_1(t), ..., p_n(t))$, defined by Equation (2), is called the **pulse vector** and shows the evolution of the system.

It is clear that Equation (1) describes a discrete-time system with parameters $w(x_j, x_i)$, which can be rewritten as:

$$p_i(t+1) = p_i^0(t+1) + \sum_j w(x_j, x_i) p_j(t). \tag{3}$$

The weights of the weighted digraph associated to the pulse process have a specific interpretation. In fact, $w(x_j, x_i)$ means that an increase of k units in vertex x_j at any time t leads to an increase of $k \times w(x_j, x_i)$ units in vertex x_i at time $t+1$ (or a decrease, whenever the value $w(x_j, x_i)$ is negative).

Definition 1. *A pulse process with vertices $x_1, ..., x_n$, is called an **autonomous pulse process** when*

$$P^0(t) = (p_1^0(t), p_2^0(t), ..., p_n^0(t)) = (0, ..., 0), \quad \text{for } t > 0.$$

Therefore, these are pulse processes with no external pulses introduced in the system (that is at any vertex) for $t > 0$. An autonomous pulse process for which

$$P^0(0) = (0, ..., \overset{(i)}{1}, ..., 0),$$

*for some $i \in \{1, ..., n\}$, is called a **simple pulse process** starting at vertex x_i.*

Example 1. *Consider the following weighted digraph Figure 1.*

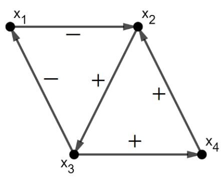

Figure 1. Signed digraph.

(here, the values of the weights are ± 1, and because of this it is said that it is signed digraph). The simple pulse process starting at x_3 is described by the following values and pulses:

If $t = 0$, then $V(0) = (0, 0, 1, 0) = P(0)$.

If $t = 1$, then the value of the vertex x_1 decreases in one unit, while x_4 increases one unit and x_2 does not change. Therefore, $V(1) = (-1, 0, 1, 1)$ with a pulse value $P(1) = (-1, 0, 1, 1) - (0, 0, 1, 0) = (-1, 0, 0, 1)$.

If $t = 2$ then, $V(2) = (-1, 2, 1, 1)$ because

$$\begin{aligned}
v_1(2) &= v_1(1) + p_1^0(2) - p_3(1) = -1 + 0 - 0 = -1 \\
v_2(2) &= v_2(1) + p_2^0(2) - p_1(1) + p_4(1) = 0 + 0 + 1 + 1 = 2 \\
v_3(2) &= v_3(1) + p_3^0(2) + p_2(1) = 1 + 0 + 0 = 1 \\
v_4(2) &= v_4(1) + p_4^0(2) + p_3(1) = 1 + 0 + 0 = 1,
\end{aligned}$$

with pulse value $P(2) = V(2) - V(1) = (-1, 2, 1, 1) - (-1, 0, 1, 1) = (0, 2, 0, 0)$, etc.

In this paper, we will consider only autonomous pulse processes. From the above definition, we obtain that an autonomous pulse process with vertices $x_1, ..., x_n$ is given by

$$p_i(t+1) = \sum_j w(x_j, x_i) p_j(t), \tag{4}$$

or, equivalently, by the weighted digraph D with vertices $x_1, x_2, ..., x_n$ and weights $w(x_i, x_j)$, along with the initial pulse vector $P(0)$. In this case, the adjacency matrix of the graph D is given by

$$A = \begin{pmatrix} w(x_1, x_1) & \cdots & w(x_1, x_n) \\ & \ddots & \\ w(x_n, x_1) & \cdots & w(x_n, x_n) \end{pmatrix},$$

and in [2], Theorem 3, it was established the following fact, easy to check.

Theorem 1. *In an autonomous pulse process on a weighted digraph with adjacency matrix A, the pulse vector $P(t)$ is given by*

$$P(t) = P(0)A^t, \text{ for } t \geq 0. \tag{5}$$

Example 2. *In the simple pulse process starting at vertex x_3 of the signed digraph of Figure 1, we have*

$$P(0)A = (0,0,1,0) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (-1,0,0,1) = P(1),$$

$$P(1)A = (-1,0,0,1) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (0,2,0,0) = P(2), \text{ etc.}$$

Therefore, for $t = 0$, we have $(0,0,1,0) = V(0) = P(0) = P(0)A^0$.

For $t = 1$, we obtain:

$$V(1) = V(0) + P(1) = P(0)A^0 + P(0)A = (0,0,1,0) + (-1,0,0,1) = (-1,0,1,1).$$

For $t = 2$,

$$V(2) = V(0) + P(2) = P(0)A^0 + P(0)A + P(0)A^2 = (-1,2,1,1).$$

Similarly, $V(3) = (-1,2,3,1)$ and $P(3) = (0,0,2,0)$, etc.

The main qualitative property studied on a complex system is the stability. Within the framework of the pulse processes associated to a weighted digraph, two notions of stability are considered in [2]. These are the following:

Definition 2. *Let D be a weighted digraph. We say that a vertex x_j of D is **pulse stable under a pulse process** if the sequence*

$$\{|p_j(t)| : t = 0,1,2,...\}$$

*is bounded. Similarly, x_j is **value stable under a pulse process** if the sequence*

$$\{|v_j(t)| : t = 0, 1, 2, ...\}$$

is bounded. (Pulse and/or value) ***unstable*** *processes are those that are not stable. A weighted digraph D is* ***pulse*** *(resp.* ***value****)* ***stable under the pulse process*** *if each vertex of D is pulse (resp. value) stable.*

Let D be a weighted digraph. Under any pulse process, the value stability at a vertex x_j of D implies the pulse stability at x_j. This is due to the fact that,

$$|p_j(t)| = |v_j(t) - v_j(t-1)| \leq |v_j(t)| + |v_j(t-1)|,$$

for $t > 0$. The next result (see [2], Theorem 4) provides a sufficient condition for pulse and value instability.

Theorem 2. *Let D be a weighted digraph, with adjacency matrix A. If D has an eigenvalue λ with $|\lambda| > 1$, then D is pulse unstable under some simple pulse process.*

The next result is nothing but [2], Theorem 5, and describes the pulse stability.

Theorem 3. *Let D be a weighted digraph, with adjacency matrix A. Then, the following assertions are equivalent:*

(i) *D is pulse stable under all autonomous pulse processes.*
(ii) *D is pulse stable under all simple pulse processes.*
(iii) *If λ is an eigenvalue of A then $|\lambda| \leq 1$, and if the algebraic multiplicity of λ differs from its geometric multiplicity then $|\lambda| < 1$.*

The next result (see [2], Theorem 6) characterizes the value stability.

Theorem 4. *Let D be a weighted digraph, with adjacency matrix A. Then, the following assertions are equivalent:*

(i) *D is value stable under all autonomous pulse processes.*
(ii) *D is value stable under all simple pulse processes.*
(iii) *D is pulse stable under all simple pulse processes and $\lambda = 1$ is not an eigenvalue of A.*

3. Evolution Algebras and Weighted Digraphs: A Brief Review

An **algebra** is a vector space A over a field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ provided with a bilinear map $A \times A \to A$, called the product of A, given by $(a, b) \to ab$, for $a, b \in A$. An algebra is said to be associative if $(ab)c = a(bc)$ for every $a, b, c \in A$, and commutative if $ab = ba$ for every $a, b \in A$. Through this paper, all the algebras that we consider are finite-dimensional.

Definition 3 ([21])**.** *A finite-dimensional* ***evolution algebra*** *is an algebra A over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) provided with a* ***natural basis****. This is a basis $B = \{e_1, ..., e_n\}$ such that $e_ie_j = 0$ if $i \neq j$, with $i, j \in \{1, ..., n\}$. This is nothing but a basis B such that the multiplication table of A relative to B is diagonal. If such a table is*

	e_1	$\cdots$	e_n
e_1	$\sum_{k=1}^{n} w_{k1}e_k$	0	0
$\vdots$	0	$\sum_{k=1}^{n} w_{ki}e_k$	0
e_n	0	0	$\sum_{k=1}^{n} w_{kn}e_k$

*then, the coefficients $w_{ij} \in \mathbb{K}$ determine a matrix $M_B(A)$, named the **structure matrix** of A relative to B, that encodes the product of A as well as the dynamic nature of A. More precisely, this matrix is obtained by writing the coefficients of e_i^2 in columns as follows:*

$$M_B(A) = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ & \ddots & \\ w_{n1} & \cdots & w_{nn} \end{pmatrix}.$$

It is easy to check that if $a = \sum_{i=1}^n \alpha_i e_i$ and $b = \sum_{i=1}^n \beta_i e_i$ then, $ab = \sum_{i=1}^n \gamma_i e_i$ where

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ & \ddots & \\ w_{n1} & \cdots & w_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix}. \tag{6}$$

*The dynamic nature of A is described by the **evolution operator** of A associated to B, which is defined as the unique linear operator $L_B : A \to A$ such that $L_B(e_i) = e_i^2$. It is easy to check that $L_B(a) = ea$, where $e = e_1 + ... + e_n$. Therefore, fixed the natural basis B, we have that L_B is the linear operator determined by the structure matrix $M_B(A)$.*

Evolution algebras are non-associative algebras (in fact, it is easy to see that they are not even power-associative), and they are commutative. On the other hand, except in very special cases, they do not have a unit as we show next.

Definition 4. *An evolution algebra A with a natural basis $B = \{e_1, ..., e_n\}$ is said to be a **non-zero trivial evolution algebra** if there exist constants $w_{ii} \neq 0$ such that $e_i^2 = w_{ii}e_i$, for every $i = 1, ..., n$.*

The evolution algebras that have a unit were characterized in [28] as follows.

Theorem 5. *The only evolution algebras that have a unit are the trivial finite-dimensional evolution algebras.*

In [29], the following notion of spectrum of an element in a non-necessarily associative algebra was introduced, and therefore many results of the spectral theory of Banach algebras were extended to the non-associative framework.

Definition 5. *The **multiplicative spectrum** (or **m-spectrum**) of an element a in a complex algebra A with a unit e is defined as the set:*

$$\sigma_m^A(a) := \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not m-invertible}\}. \tag{7}$$

*An element $b \in A$ is said to be **m-invertible** if the left (L_b) and right (R_b) multiplication operators by the element b are bijective. If A does not have a unit then $\sigma_m^A(a) := \sigma_m^{A_1}(a)$, where A_1 denotes the unitization of A. Similarly, if A is real then $\sigma_m^A(a) := \sigma_m^{A_\mathbb{C}}(a)$, where $A_\mathbb{C}$ denotes the complexification of A.*

Recall that, as usual, $A_\mathbb{C} := \{a + ib : a, b \in A\}$ and $A_1 = A \oplus \mathbf{1}\mathbb{K} = \{a + \mathbf{1}\lambda : a \in A, \lambda \in \mathbb{K}\}$. Moreover, if A is a real algebra without a unit, then $\sigma_m^A(a) = \sigma_m^{(A_1)_\mathbb{C}}(a) = \sigma_m^{(A_\mathbb{C})_1}(a)$ for every $a \in A$; see [29] for details.

On the other hand, as proved in [29], Proposition 2.5, if $(A, \|\cdot\|)$ is a non-associative Banach algebra then $\sigma_m^A(a)$ is a set of complex numbers such that $|\lambda| \leq \|a\|$, for every $a \in A$. In fact, the m-spectrum extends the classical notion of spectrum of an element in an associative algebra to the non-associative framework by keeping a good number of its essential properties.

As proved in [29], Proposition 2.2, for an arbitrary complex algebra A and $a \in A$, we have that, if A has not a unit then

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}, \tag{8}$$

whereas if A has a unit then

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a), \tag{9}$$

where, for a linear operator $T : A \to A$, the set $\sigma^{\mathcal{L}(A)}(T)$ denotes the spectrum of T in the associative algebra $\mathcal{L}(A)$ of all linear operators on A. This is,

$$\sigma^{\mathcal{L}(A)}(T) = \{\lambda \in \mathbb{C} : T - \lambda I : A \to A \text{ is not bijective}\}.$$

Next, we show the behavior of the m-spectrum of surjective homomorphisms.

Recall that an **homomorphism** between two arbitrary algebras A and $\widetilde{A}$ is a linear map $\theta : A \to \widetilde{A}$ such that $\theta(ab) = \theta(a)\theta(b)$, for every $a, b \in A$. Surjective homomorphism are named **epimorphisms**, while the bijective ones are named **isomorphisms.**

Theorem 6. *Let A and $\widetilde{A}$ be algebras and $\theta : A \to \widetilde{A}$ an homomorphism.*

(i) If θ is an isomorphism then $\sigma_m^{\widetilde{A}}(\theta(a)) = \sigma_m^A(a)$, for every $a \in A$.

(ii) If θ is an epimorphism then $\sigma_m^{\widetilde{A}}(\theta(a)) \subseteq \sigma_m^A(a)$, for every $a \in A$.

Proof. It is not restrictive to assume that A and $\widetilde{A}$ are complex algebras. In fact, if A and $\widetilde{A}$ are real algebras, and if θ is replaced by $\theta_{\mathbb{C}} : A_{\mathbb{C}} \to \widetilde{A}_{\mathbb{C}}$ where $\theta_{\mathbb{C}}(a + ib) = \theta(a) + i\theta(b)$, then we obtain an isomorphism $\theta_{\mathbb{C}}$ that extends θ. Since in the real case, by definition, $\sigma_m^A(a) := \sigma_m^{A_{\mathbb{C}}}(a)$ and $\sigma_m^{\widetilde{A}}(\theta(a)) := \sigma_m^{\widetilde{A}_{\mathbb{C}}}(\theta(a))$ for every $a \in A$, we have that, to prove the theorem, we can replace θ by $\theta_{\mathbb{C}}$.

(i) Suppose that θ is an isomorphism. Let $a \in A$. For simplicity, suppose that A is commutative. Then, from Equations (8) and (9), we obtain that $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \{0\}$ if A does not have a unit, whereas $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a)$ if A has a unit. Let $\sigma_p^{\mathcal{L}(A)}(L_a)$ and $\sigma_{su}^{\mathcal{L}(A)}(L_a)$ denote the pointwise spectrum and the surjective spectrum of L_a, respectively. That is,

$$\sigma_p^{\mathcal{L}(A)}(L_a) = \{\lambda \in \mathbb{C} : L_a - \lambda I : A \to A \text{ is not injective}\},$$
$$\sigma_{su}^{\mathcal{L}(A)}(L_a) = \{\lambda \in \mathbb{C} : L_a - \lambda I : A \to A \text{ is not surjective}\}.$$

Since $\dim A < \infty$, we have that, for every $\lambda \in \mathbb{C}$,

$$\lambda \in \sigma^{\mathcal{L}(A)}(L_a) \Longleftrightarrow \lambda \in \sigma_p^{\mathcal{L}(A)}(L_a) \Longleftrightarrow \lambda \in \sigma_{su}^{\mathcal{L}(A)}(L_a). \tag{10}$$

Moreover, $ab = \lambda b$ if and only if $\theta(a)\theta(b) = \lambda\theta(b)$, for every $a, b \in A$, as θ is an isomorphism. Thus $\sigma_p^{\mathcal{L}(A)}(L_a) = \sigma_p^{\mathcal{L}(\widetilde{A})}(L_{\theta(a)})$, and consequently, for $a \in A$,

$$\sigma^{\mathcal{L}(A)}(L_a) = \sigma_p^{\mathcal{L}(A)}(L_a) = \sigma_p^{\mathcal{L}(\widetilde{A})}(L_{\theta(a)}) = \sigma^{\mathcal{L}(\widetilde{A})}(L_{\theta(a)}). \tag{11}$$

Hence, regardless of whether A has a unit or not, we obtain $\sigma_m^{\widetilde{A}}(\theta(a)) = \sigma_m^A(a)$, as desired.

If A is not commutative, then the same reasoning with the operator R_a shows that $\sigma^{\mathcal{L}(A)}(R_a) = \sigma^{\mathcal{L}(\widetilde{A})}(R_{\theta(a)})$, and the result follows from Equations (8) and (9), as

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\} = \sigma_m^{\widetilde{A}}(\theta(a)) = \sigma^{\mathcal{L}(\widetilde{A})}(L_{\theta(a)}) \cup \sigma^{\mathcal{L}(\widetilde{A})}(R_{\theta(a)}) \cup \{0\},$$

if A does not have a unit. Similarly, if A has a unit, then

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) = \sigma_m^{\widetilde{A}}(\theta(a)) = \sigma^{\mathcal{L}(\widetilde{A})}(L_{\theta(a)}) \cup \sigma^{\mathcal{L}(\widetilde{A})}(R_{\theta(a)}).$$

(ii) Since $\ker\theta$ is an ideal of A, from Lemma 19 in [30], we have that

$$\sigma_{su}^{A/\ker\theta}(L_a) \subseteq \sigma_{su}^{A}(L_a), \text{ and } \sigma_{su}^{A/\ker\theta}(R_a) \subseteq \sigma_{su}^{A}(R_a),$$

and it follows from Equation (10) that $\sigma_m^{A/\ker\theta}(a+\ker\theta) \subseteq \sigma_m^A(a)$, for every $a \in A$. □

The spectrum of an element a in an evolution algebra A with a natural basis $B = \{e_1, ..., e_n\}$ was determined in [28]. From Proposition 5.1,5.3 in [28], we obtain the following description of the spectrum of an element in an evolution algebra.

Theorem 7. *Let A be an evolution algebra over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$ with a natural basis $B = \{e_1, ..., e_n\}$ and structure matrix $M_B(A) = (w_{ij})$. Let $a = \sum_{i=1}^n \alpha_i e_i \in A$. Then, $\lambda \in \mathbb{C}\backslash\{0\}$ is such that $\lambda \in \sigma_m^A(a)$ if and only if λ is an eigenvalue of the following matrix*

$$\begin{pmatrix} w_{11} & & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & & w_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}. \tag{12}$$

If A is a non-zero trivial evolution algebra, then $M_B(A) = diag(w_{11}, ..., w_{nn})$, and from the above result we obtain that, if $a = \sum_{i=1}^n \alpha_i e_i$, then

$$\sigma_m^A(a) = \{\alpha_i w_{ii} : i = 1, ..., n\}.$$

Similarly, if A is not a non-zero trivial evolution algebra (that is, $M_B(A)$ is not diagonal with non-zero entries), then $0 \in \sigma_m^A(a)$, for every $a \in A$. Indeed, since in this case A does not have a unit as shown in Theorem 5, it follows that $a \in A$ is not invertible in A_1 as the unit of A_1 cannot belong to A. Thus, if A does not have a unit (which means that A is not a non-zero trivial evolution algebra) then $\sigma_m^A(a)$ is given by the eigenvalues of the product matrix (12) joint with zero.

4. Evolution Algebras and Pulse Process

Every couple (A, B), where A is an evolution algebra and $B = \{e_1, ...e_n\}$ is a natural basis of A, uniquely determines a weighted graph $G_B(A)$ with set of vertices B and adjacency matrix $M_B(A)^T$ (we consider the transposition of the structure matrix of A since, in graph theory, it is usual to determine the i-th row of the adjacency matrix by the weight of the arcs with origin in the vertex e_i). Consequently, and conversely, every weighted graph G with set of vertices B and adjacency matrix M_G uniquely determines an evolution algebra A provided with a natural basis B in which structure matrix is $M_B(A) = M_G^T$ (thus, we have that the associated graph to (A, B) is G).

From now on, if A is an evolution algebra and B is a natural basis of A then, we will denote the graph associated to A relative to B by $G_B(A)$.

Example 3. *Consider the following weighted digraph Figure 2 taken from R-1578-NSF [20].*

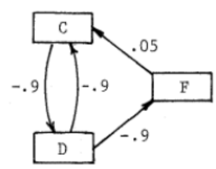

Figure 2. Weighted digraph with vertices $\{C, D, F\}$. Captured from [20], p. 40. Copyright permission from The Rand Corporation

The associated evolution algebra A has a natural basis $B = \{e_C, e_D, e_F\}$ and structure matrix

$$M_B(A) = \begin{pmatrix} 0 & -0.9 & -0.05 \\ -0.9 & 0 & 0 \\ 0 & -0.9 & 0 \end{pmatrix}.$$

Note that $M_B^T(A)$ is the adjacency matrix of the given graph.

From Theorem 7 we obtain the following result.

Corollary 1. *Let A be an evolution algebra, $B = \{e_1, ..., e_n\}$ a natural basis, and $e = e_1 + ... + e_n$. Let $G_B(A)$ be the weighted digraph associated to A relative to B. Let M_G be the adjacency matrix of $G_B(A)$ and $\sigma(M_G)$ its spectrum.*

(i) If A is a non trivial evolution algebra, then $\sigma_m^A(e) = \sigma(M_G) \cup \{0\}$.
(ii) If A is a trivial evolution algebra, then $\sigma_m^A(e) = \sigma(M_G)$.

The above result and the fact that the evolution operator of A relative to B is the multiplication operator by the element $e = e_1 + ... + e_n$, motivate the following definition.

Definition 6. *If A is an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis then, we define the **evolution element** of A relative to B as $e = e_1 + ... + e_n$.*

Next, we translate the theory of pulse processes on weighted digraphs to the framework of evolution algebras.

Definition 7. *Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. We say that A is **pulse** (resp. **value**) **stable relative to** B, under all autonomous pulse processes, if the associated weighted digraph $G_B(A)$ is pulse (resp. value) stable.*

Consequently, a graph G is pulse and/or value stable if, and only if, its associated evolution algebra is pulse and/or value stable.

*If A is not pulse (resp. value) stable relative to a natural basis B, then we say that A is **unstable** relative to B.*

From Theorem 3, Theorem 4 and Corollary 1, we obtain the following result, where $\overline{\mathbb{D}}$ denotes the closed unit disk and $\mathbb{D}$ the open unit disk.

Theorem 8. *Let A be an evolution algebra with a natural basis $B = \{e_1, ..., e_n\}$, and let $e = e_1 + ... + e_n$ be the corresponding evolution element. Then, the following assertions are equivalent:*

(i) A is pulse stable relative to B, under all autonomous pulse processes, if and only if $\sigma_m^A(e) \subseteq \overline{\mathbb{D}}$, and it satisfies that, if $\lambda \in \sigma_m^A(e)$ is an eigenvalue in which algebraic and geometric multiplicities do not coincide, then $\lambda \in \mathbb{D}$.
(ii) A is value stable relative to B, under all autonomous pulse processes, if and only if A is pulse stable relative to B, under all autonomous pulse processes, and $1 \notin \sigma_m^A(e)$.

Moreover, the pulse and/or value stability of A under all autonomous pulse processes is equivalent to that of simple pulse processes.

From the above result, we have that *the value stability of A relative to B is equivalent to the fact that $\sigma_m^A(e) \subseteq \overline{\mathbb{D}} \backslash \{1\}$ and the property that if $\lambda \in \sigma_m^A(e)$ is an eigenvalue in which algebraic and geometric multiplicities do not coincide, then $\lambda \in \mathbb{D}$.*

Corollary 2. *Let A be an evolution algebra with $B = \{e_1, ..., e_n\}$ a natural basis, and let $e = e_1 + ... + e_n$ be the corresponding evolution element. If $\sigma_m^A(e) \subseteq \mathbb{D}$, then A is pulse and value stable relative to B, under all autonomous pulse processes.*

Example 4. *Let A be an evolution algebra with a natural basis $B = \{e_1, ..., e_n\}$, relative to which the associated graph is*

$$e_1 \overset{w_{12}}{\rightarrow} e_2 \overset{w_{23}}{\rightarrow} \cdots \overset{w_{(n-2)(n-1)}}{\rightarrow} e_{n-1} \overset{w_{(n-1)n}}{\rightarrow} e_n.$$

Then, A is pulse and value stable. This is due to the fact that $\sigma_m^A(e) = \{0\}$.

Corollary 3. *Let A be an evolution algebra with $B = \{e_1, ..., e_n\}$ a natural basis, and let $e = e_1 + ... + e_n$ be its evolution element. If $\sigma_m^A(e) \cap (\mathbb{C}\backslash\overline{\mathbb{D}}) \neq \emptyset$ then, A is pulse unstable under some simple pulse process and, consequently, value unstable.*

5. Pulse Processes, Evolution Algebras, Cycles, And Ideals

The ideals, and mainly the basic ideals of evolution algebras associated to a pulse process, play a main role in determining the pulse and/or value stability of the given pulse process, as we show next. In R-1578-NSF (see [20]), the behavior of the so-called "interesting strong connected components" was checked as a "logical" preliminary test. However, the fact of making clear to what extent the behavior of such components determines the stability of the whole pulse process was omitted. By means of the notion of basic ideal we will clarify the role of these "interesting strong connected components", showing that such components often determine the stability of the whole pulse process, and giving the reasons for this.

Recall that a subspace M of a commutative algebra A is said to be an **ideal** if $MA \subseteq M$ (which means that A/M is an algebra with the canonical product $(a + M)(b + M) = ab + M$ for $a, b \in A$). An ideal M of an evolution algebra A is a subalgebra. Nevertheless, an ideal M is not necessarily an evolution subalgebra. In other words, not every ideal of an evolution algebra has a natural basis.

Example 5. *Let A be the evolution algebra determined by the natural basis $B = \{e_1, e_2, e_3\}$, where $e_1^2 = e_1 + e_2 = -e_2^2$ and $e_3^2 = e_1 - e_3$ (and $e_ie_j = 0$ for $i \neq j$). Then, it is easy to check that the ideal M generated by e_1^2 and e_3^2, given by*

$$M = \{(\alpha + \beta)e_1 + \alpha e_2 - \beta e_3 : \alpha, \beta \in \mathbb{K}\}, \tag{13}$$

is an ideal that does not have a natural basis. In fact, it does not exist $v_1, v_2 \in M$ linearly independent and such that $v_1v_2 = 0$, because if $v_1 = (\alpha + \beta)e_1 + \alpha e_2 - \beta e_3$ and $v_2 = (\gamma + \delta)e_1 + \gamma e_2 - \delta e_3$ for $\alpha, \beta, \gamma, \delta \in K$ then it follows that

$$v_1v_2 = ((\alpha + \beta)(\gamma + \delta) - \alpha\gamma)u_1 + \beta\delta u_2.$$

Thus, $v_1v_2 = 0$ if and only if $(\alpha + \beta)(\gamma + \delta) - \alpha\gamma = 0$ and $\beta\delta = 0$. If $\beta = 0$, then $\alpha(\gamma + \delta) - \alpha\gamma = 0$ with $\alpha \neq 0$ (as $u_1 \neq 0$), so that $\delta = 0$ and hence v_1 and v_2 are proportional (and, therefore, linearly dependent). A similar situation is obtained if $\delta = 0$. This proves that M does not have a natural basis.

This motivates the following definition:

Definition 8. *Let A be an evolution algebra. An **evolution ideal** of A is an ideal M having a natural basis B_M (this is an ideal M that, regarded as an algebra, is an evolution algebra).*

If A is an evolution algebra and M is an ideal, then A/M is an evolution algebra. In fact, if $B = \{e_1, ..., e_n\}$ it turns out that $A/M = lin\{e_1 + M, ..., e_n + M\}$, with $(e_i + M)(e_j + M) = 0 + M$, if $i \neq j$. However the set $\{e_1 + M, ..., e_n + M\}$ does not need to be linearly independent, as the next example shows. This means that $\{e_1 + M, ..., e_n + M\}$ contains a natural basis of A/M, but this set needs to be linearly independent to become a natural basis of A/M.

Example 6. *Let A be the evolution algebra from Example 5, and $B = \{e_1, e_2, e_3\}$ the natural basis provided there. If M is the ideal defined in (13), then we have that $e_1 + M = -(e_2 + M) = e_3 + M$, so they are proportional and therefore a natural basis of A/M is given, for instance, by $B_{A/M} = \{e_1 + M\}$.*

An outstanding type of ideals of an evolution algebra are the following ones.

Definition 9. *Let A be an evolution algebra. We say that M is a **basic ideal** of A if M is an evolution ideal having a natural basis B_M such that $B_M \subseteq B_A$, for some natural basis B_A of A. In this case, more explictly, we also say that M is a **basic ideal relative to the natural basis** B_A.*

Not every evolution ideal of an evolution algebra is a basic ideal.

Example 7. *Let A be an evolution algebra and $B = \{e_1, e_2, e_3\}$ a natural basis such that $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$ and $e_3^2 = e_3$. Then, $M = lin\{e_1 + e_2, e_3\}$ is an evolution ideal such that any of its natural basis is contained in (or can be extended to) a natural basis of A; see [22], Example 2.11, for details.*

An interesting property of the basic ideals is the following one.

Proposition 1. *Let A be an evolution algebra and M a basic ideal of A. If $B = \{e_1, ..., e_n\}$ is a natural basis of A such that M is a basic ideal of A relative to B, then the set*

$$B_{A/M} = \{e_1 + M, ..., e_n + M\} \setminus \{0 + M\}$$

is a natural basis of A/M.

Proof. If $M = B$ then the result is obvious. Otherwise, by reordering B if needed, it is not restrictive to assume that $B_M = \{e_1, ..., e_k\}$ is a natural basis of M, with $1 \leq k < n$, and $B = \{e_1, ..., e_k, e_{k+1}, ..., e_n\}$. Note that the set $B_{A/M}$ is a natural basis of A/M if, and only if, it is linearly independent (as it generates A/M and $(e_i + M)(e_j + M) = 0 + M$ if $i \neq j$). To prove that $B_{A/M}$ is linearly independent, let $\alpha_1, ..., \alpha_n \in \mathbb{K}$ be such that $\sum\limits_{i=1}^{n} \alpha_i(e_i + M) = 0 + M$. Since $e_1, ..., e_k \in M$, we obtain that

$$\sum_{i=k+1}^{n} \alpha_i(e_i + M) = (\sum_{i=k+1}^{n} \alpha_i e_i) + M = 0 + M.$$

Therefore, there exists $m = \sum\limits_{\beta=1}^{k} \beta_i e_i \in M$ such that $\sum\limits_{i=k+1}^{n} \alpha_i e_i = m$. This means that

$$\beta_1 e_1 + ... + \beta_k e_k - \alpha_{k+1} e_{k+1} - ... - \alpha_n e_n = 0;$$

hence, $\beta_1 = ... = \beta_k = \alpha_{k+1} = ... = \alpha_n = 0$ as B is a basis of A. This shows that the set

$$\{e_{k+1} + M, ..., e_n + M\}$$

is linearly independent; hence, $B_{A/M}$ is a natural basis of A/M as desired. □

Corollary 4. *Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. Let M be a basic ideal relative to B and*

$$B_{A/M} = \{e_1 + M, ..., e_n + M\} \setminus \{0 + M\}.$$

If A/M is pulse and/or value unstable relative to $B_{A/M}$ then A is pulse and/or value unstable relative to B.

Proof. From the previous proposition, $B_{A/M}$ is a natural basis of A/M. Let $\pi : A \to A/M$ be the canonical projection. Note that π is an epimorphism so that, by Theorem 6,

$$\sigma_m^{A/M}(\pi(a)) \subseteq \sigma_m^A(a), \text{ for every } a \in A.$$

Taking into account that the canonical projection π transforms the evolution element of A relative to B (that is $e = e_1 + ... + e_n$) into the evolution element of A/M relative to $B_{A/M}$, the proof is concluded from Theorem 8. □

Note that the graph $G_{B/M}(A/M)$ (that is the graph associated to A/M relative to the natural basis $B_{A/M} = \{e_1 + M, ..., e_n + M\} \backslash \{0 + M\}$) is the graph that we obtain from $G_B(A)$ by deleting all the nodes $e_i \in B$ such that $e_i \in M$ (that is $e_i + M = 0$), as well as the arcs ending in these nodes.

Theorem 9. *Let A be an evolution algebra and $B_A = \{e_1, ..., e_n\}$ a natural basis of A. Let M be a proper basic ideal of A with a natural basis B_M such that $B_M \subseteq B_A$. Let A/M be the quotient algebra and let e_B, e_{B_M} and $e_{B_{A/M}}$ be the evolution elements associated to B, B_M and $B_{A/M}$, respectively.*

(i) If $\sigma_m^M(e_{B_M})$ or $\sigma_m^{A/M}(e_{B_{A/M}})$ intersect $\mathbb{C}\backslash\overline{\mathbb{D}}$ then A is not pulse and value stable relative to B.
(ii) If $\sigma_m^M(e_{B_M}) \subseteq \mathbb{D}$ and $\sigma_m^{A/M}(e_{B_{A/M}}) \subseteq \mathbb{D}$ then A is pulse and value stable relative to B.

Proof. It is not restrictive to assume that $B = \{e_1, ..., e_k, e_{k+1}, ..., e_n\}$ where $B_M = \{e_1, ..., e_k\}$, by reordering B if needed. Consequently, by Proposition 1 we have that

$$B_{A/M} = \{e_{k+1} + M, ..., e_n + M\}.$$

Moreover, if $M_B(A)$, $M_{B_M}(M)$ and $M_{B_{A/M}}(A/M)$ are, respectively, the structure matrices of A relative to B, of M relative to B_M, and of A/M relative to $B_{A/M}$, then we have

$$M_B(A) = \begin{pmatrix} M_{B_M}(M) & P \\ 0 & M_{B_{A/M}}(A/M) \end{pmatrix}, \tag{14}$$

for a certain $k \times (n-k)$ matrix P. Therefore, from [31], Section 3, the eigenvalues of $M_{B_M}(M)$ and $M_{B_{A/M}}(A/M)$ determine those of $M_B(A)$, and it follows from Corollary 1 that

$$\sigma_m^A(e_B)\backslash\{0\} = \sigma_m^M(e_{B_M})\backslash\{0\} \cup \sigma_m^{A/M}(e_{B_{A/M}})\backslash\{0\},$$

and the result is obtained from Corollaries 2 and 3. □

Note that an application of Theorem 8, whenever $\sigma_m^M(e_{B_M})$ or $\sigma_m^{A/M}(e_{B_{A/M}})$ meets the boundary of $\mathbb{D}$, connects with the well known Carlson Problem [32].

In the next example, we apply the above theorem to a pulse process considered in R-1578-NSF, providing a new approach for the analysis achieved there.

Example 8. *In the report R-1578-NSF [20], the pulse process for the 10% bus case is given by the following weighted digraph (see Figure 3 below).*

The associated evolution algebra A has a natural basis

$$B = \{e_M, e_Y, e_P, e_C, e_R, e_E, e_A, e_D, e_F\},$$

(where $e_M \equiv$ passenger miles, $e_Y \equiv$ fuel economy, $e_P \equiv$ population size, $e_C \equiv$ cost of the bus system, $e_R \equiv$ prize of the ticket, $e_E \equiv$ emissions, $e_A \equiv$ accidents, $e_D \equiv$ average delay, $e_F \equiv$ fuel consumption).

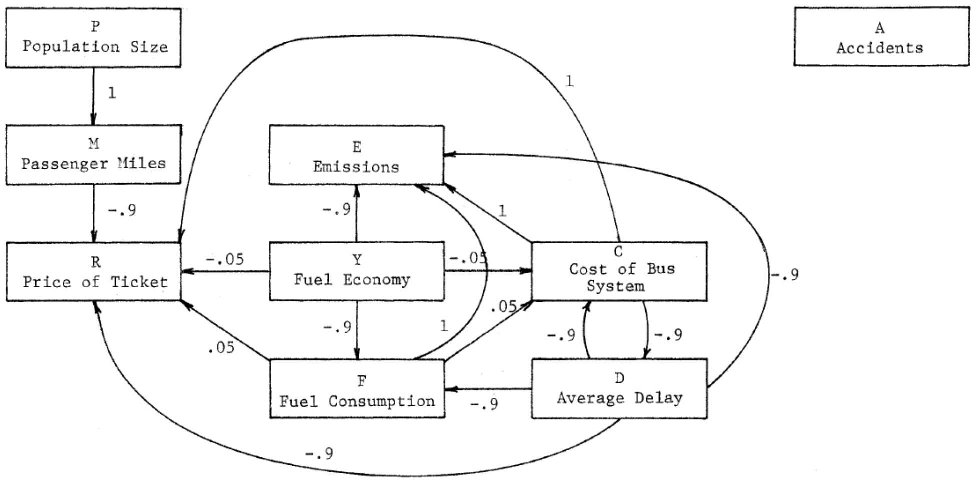

Figure 3. Weighted digraph for the 10% Bus Case. Captured from [20], p. 40. Copyright permission from The Rand Corporation

The associated structure matrix (whose columns are the coefficients of $\{e_M, e_Y, e_P, e_C, e_R, e_E, e_A, e_D, e_F\}$ *respectively) is given by*

$$M_B(A) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.05 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{-0.9} & \mathbf{0.05} \\ -0.9 & -0.05 & 0 & 1 & 0 & 0 & 0 & -0.9 & 0.05 \\ 0 & -0.9 & 0 & 1 & 0 & 0 & 0 & -0.9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-0.9} & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & -0.9 & 0 & \mathbf{0} & 0 & 0 & 0 & \mathbf{-0.9} & \mathbf{0} \end{pmatrix} \tag{15}$$

Note that $M = lin\{e_M, e_P, e_R, e_E, e_A\}$ *is a basic ideal of* A*. The structure matrix of* M *relative to the natural basis* $B_M = \{e_M, e_P, e_R, e_E, e_A\}$ *is given by*

$$M_{B_M}(M) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

whereas the structure matrix of A/M *relative to* $B_{A/M} = \{e_Y + M, e_C + M, e_D + M, e_F + M\}$ *is*

$$M_{B_{A/M}}(A/M) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.05 & 0 & -0.9 & 0.05 \\ 0 & -0.9 & 0 & 0 \\ -0.9 & 0 & -0.9 & 0 \end{pmatrix}. \tag{17}$$

Since $\sigma_m^M(e_M) \subseteq \mathbb{D}$ *and* $\sigma_m^{A/M}(e_{B_{A/M}}) \subseteq \mathbb{D}$*, the pulse and value stability of* A *relative to* B*, and hence the pulse process described by Figure 3, follows from Theorem 9.*

There are some types of basic ideals M, of an evolucion algebra A, with the property that the study of the pulse and/or value stability of A can be reduced to the corresponding study in the quotient algebra A/M, as we show next.

Recall that the **annihilator** of an algebra A is the ideal defined as

$$An(A) = \{a \in A : ab = ba = 0, \text{ for every } b \in A\}.$$

If A is an evolution algebra, then the annihilator of A can be obtained from each natural basis of A, as it is shown in the following proposition ([22], Proposition 2.18).

Proposition 2. *Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. Then*

$$An(A) = lin\{e_i : e_i^2 = 0\}. \tag{18}$$

Consequently, the annihilator of an evolution algebra A is a basic ideal (with respect to any natural basis of A). The following result shows that this basis ideal is very helpful for determining the stability of A.

Theorem 10. *Let A be an evolution algebra with non-zero annihilator, $An(A)$, and let $B = \{e_1, ..., e_n\}$ be a natural basis of A. Then, $An(A) = lin\{e_i : e_i^2 = 0\}$ is a basic ideal of A, and*

$$B_{A/An(A)} := \{e_i + An(A) : e_i \in B \text{ with } e_i^2 \neq 0\} \tag{19}$$

is a natural basis of $A/An(A)$. Moreover,

$$\sigma_m^{A/An(A)}(a + An(A))\backslash\{0\} = \sigma_m^A(a)\backslash\{0\}. \tag{20}$$

Therefore, A is pulse and/or value stable relative to B if and only if $A/An(A)$ is pulse and/or value stable, respectively, relative to $B_{A/An}$.

Proof. As said before, from Proposition 2.18 in [22], we obtain (18) and, consequently, by Proposition 1, we have that (19) is a natural basis of $A/An(A)$. To prove (20), it is not restrictive to assume that A is a complex algebra. Let $a \in A$. From Theorem 6 we obtain that $\sigma_m^{A/An(A)}(a + An(A))\backslash\{0\} \subseteq \sigma_m^A(a)\backslash\{0\}$. On the other hand, if $\lambda \in \mathbb{C}\backslash\{0\}$ is such that $\lambda \in \sigma_m^A(a)$, then $L_a - \lambda I$ is not bijective and, since $\dim A < \infty$, we have that $L_a - \lambda I$ is not injective. Thus, there exists $b \in A$ such that $ab - \lambda b = 0$. Moreover $b \notin An(A)$ because, in this case, $ab = 0$, and hence $\lambda b = 0$, so that $b = 0$. It follows that $(L_{a+An(A)} - \lambda I)(b + An(A)) = 0 + An(A)$ with $b + An(A)$ non-zero, so that $\lambda \in \sigma_m^{A/An(A)}(a + An(A))$ as desired. This proves (20). The rest is clear, from Theorem 8. □

Note that the annihilator of the algebra $A/An(A)$ does not need to be zero, as we show next.

Example 9. *In Example 8 we have that $An(A) = lin\{e_C, e_E, e_A\}$ so that the pulse and/or value stability of A is equivalent to that of $A/An(A)$ relative to the natural basis $B_{A/An(A)}$ given by*

$$\{e_M + An(A), e_Y + An(A), e_P + An(A), e_C + An(A), e_D + An(A), e_F + An(A)\}.$$

Since $e_M^2 + An(A) = 0 + An(A)$, we conclude that the annihilator of the quotient algebra $A/An(A)$ is not zero.

Recall that a *source vertex* of a graph is a node with positive outdegree but zero indegree. This means that the vertex has edges leading from, but not leading to, the node. Conversely, a *sink vertex* is a node with positive indegree but zero outdegree, which means that it has edges leading to, but not from, the node. An *isolated* node is a vertex with zero indegree and zero outdegree (that means that no edge starts or ends in this vertex). According to this, if $B = \{e_1, ..., e_n\}$ is a natural basis of an evolution algebra A, then we split B as follows:

$$B = B_{\text{isol}} \dot{\cup} B_{\text{sink}} \dot{\cup} B_{\text{sour}} \dot{\cup} B_{\text{stan}},$$

where B_{isol} is the set of elements in B that are **isolated** vertices of the associated graph $G_B(A)$. Similarly, B_{sink} (resp. B_{sour}) is the set of elements in B that are **sink** (resp. **source**) vertices of $G_B(A)$, and B_{stan} is the set of **standard** elements in B, which are those nodes in $G_B(A)$ having both positive outdegree and positive indegree.

Looking at the structure matrix $M_B(A)$ we have that $e_i \in B_{\text{isol}}$ if and only if both the $i-$th row and the $i-$th column of $M_B(A)$ are zero, respectively. Similarly, $e_i \in B_{\text{sink}}$ (resp. B_{sour}) if and only if the $i-$th column (resp. arrow) is zero while the $i-$th row (resp. column) is non-zero. Finally, $e_i \in B_{\text{stan}}$ if and only if both the $i-$th row and the $i-$th column of $M_B(A)$ are non-zero.

Note that according to (18) we have $An(A) = lin(B_{\text{isol}} \dot{\cup} B_{\text{sink}})$. Therefore,

$$B_{A/An(A)} := \{e_i + An(A) : e_i \in (B_{\text{sour}} \dot{\cup} B_{\text{stan}})\}.$$

Consequently, the structure matrix $M_{B_{A/An(A)}}$ is nothing but the matrix obtained by removing from $M_B(A)$ the rows and the columns corresponding to the elements in $B_{\text{isol}} \dot{\cup} B_{\text{sink}}$. Moreover, $G_{B_{A/An(A)}}(A/An(A))$ is the graph obtained by removing from $G_B(A)$ its sinks and the arcs leading to them.

Theorem 11. *Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. Let $I_B := lin(B_{sink} \dot{\cup} B_{stan})$. Then, I_B is a basic ideal of A with natural basis*

$$B_I = B_{sink} \dot{\cup} B_{stan}.$$

Moreover, A is pulse and/or value stable relative to B if and only if I is pulse and/or value stable relative to B_I. In fact, if e_B and e_I denote the respective evolution elements (of A relative to B and of I relative to B_I), then

$$\sigma_m^A(e_B) \backslash \{0\} = \sigma_m^I(e_I) \backslash \{0\}.$$

Proof. The fact that $B_I = B_{\text{sink}} \dot{\cup} B_{\text{stan}}$ is a basic ideal of A is obvious, as B is a natural basis of A and $B_I \subseteq B$. Suppose that $B_{\text{sink}} \cup B_{\text{stan}} = \{e_1, ..., e_k\}$ and $B_{\text{isol}} \cup B_{\text{sour}} = \{e_k, ..., e_n\}$ which is not restrictive, by reordering B if needed (note that such a reordering of the elements of B defines an isomorphism on A and that isomorphisms preserve the spectrum of each element, as shown in Theorem 6). Then $e_I = e_k + ... + e_n$ and $e_B = e_1 + ... + e_n$. If $M_B(A)$ is the structure matrix of A relative to B and $M_{B_I}(A)$ is the structure matrix of I_B relative to B_I, then we have that

$$M_B(A) = \begin{pmatrix} M_{B_I}(A) & P_{(n-k)\times k} \\ 0_{k\times(n-k)} & 0_{k\times k} \end{pmatrix}$$

for a certain matrix $P_{(n-k)\times k}$. Since $\sigma_m^A(e_B)\backslash\{0\}$ is given by the non-zero eigenvalues of $M_B(A)$ and $\sigma_m^I(e_I)\backslash\{0\}$ is given by the non-zero eigenvalues of $M_{B_I}(A)$ (see Corollary 1), the result follows from Theorem 8. □

Definition 10. *If A is an evolution algebra and if $B = \{e_1, ..., e_n\}$ is a natural basis, then we define the* ***reduced ideal*** *of A relative to B as the basic ideal*

$$I_B := lin(B_{sink} \dot{\cup} B_{stan}).$$

By combining Theorem 10 and Theorem 11, we obtain the following result.

Corollary 5. *Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. Let $I_B := lin(B_{sink} \dot{\cup} B_{stan})$. Then, $I_B/An(A)$ is an evolution algebra and*

$$B_{I_B/An(A)} = \{e_i + An(A) : e_i \in B_{stan}\}$$

is a natural basis of $I_B/An(A)$. Moreover, A is pulse and/or value stable if and only if $I_B/An(A)$ is pulse and/or value stable. In fact, if e_A is the evolution element of A relative to B and if $e_{B_{I_B/An(A)}}$ is the evolution element of $I_B/An(A)$ relative to $B_{I_B/An(A)}$ then

$$\sigma_m^A(e_B)\backslash\{0\} = \sigma_m^{I_B/An(A)}(e_{B_{I_B/An(A)}})\backslash\{0\}.$$

6. The Reduction Process

Let A be an evolution algebra and $B = \{e_1, ..., e_n\}$ a natural basis. We define the **first spectral reduction** of (A, B) as the couple $(I_B/An(A), B_{I_B/An(A)})$ where $I_B := lin(B_{\text{sink}} \dot{\cup} B_{\text{stan}})$ is the reduced ideal of A relative to B, and $B_{I_B/An(A)}$ is the natural basis of $I_B/An(A)$ given by

$$B_{I_B/An(A)} = \{e_i + An(A) : e_i \in B_{\text{stan}}\}.$$

If some element in $B_{I_B/An(A)}$ is not standard (that is, it is either an isolated, a sink or a source vertex) then, we repeat the process and define the **second spectral reduction** of (A, B) as the first spectral reduction of $(I_B/An(A), B_{I_B/An(A)})$. We reiterate the process until we get a reduced natural basis consisting only of standard elements. Then, we say that this couple is the **spectral reduction** of (A, B) and we denote it by (A_{red}, B_{red}). Note that, from Corollary 5, the study of the pulse and/or value stability of A relative to B is equivalent to the study of the stability of A_{red} relative to B_{red}.

Example 10. *Consider the evolution algebra A with a natural basis B given in Example 8. Then, we have*

$$B_{isol} = \{e_A\},\ B_{sink} = \{e_R, e_E\}, B_{sour} = \{e_Y, e_P\}, B_{stan} = \{e_M, e_C, e_D, e_F\}.$$

Therefore, $I_B := lin(B_{sink} \dot{\cup} B_{stan}) = lin\{e_M, e_C, e_R, e_E, e_D, e_F\}$, and

$$An(A) = lin(B_{isol} \dot{\cup} B_{sink}) = \{e_R, e_E, e_A\}.$$

Consequently, the first spectral reduction of (A, B) is the evolution algebra $I_B/An(A)$ with natural basis

$$B_{I_B/An(A)} = lin\{e_M + An(A), e_C + An(A), e_D + An(A), e_F + An(A)\},$$

and structure matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9 & 0.05 \\ 0 & -0.9 & 0 & 0 \\ 0 & 0 & -0.9 & 0 \end{pmatrix}.$$

Note that $e_M + An(A)$ is an isolated element of $I_B/An(A)$ and that the other elements in $B_{I_B/An(A)}$ are standard. Thus, the spectral reduction of (A, B) is given by the reduced ideal of $I_B/An(A)$. Hence, A_{red} is the evolution algebra with natural basis

$$B_{red} = \{e_C + An(A), e_D + An(A), e_F + An(A)\},$$

and structure matrix

$$\begin{pmatrix} 0 & -0.9 & 0.05 \\ -0.9 & 0 & 0 \\ 0 & -0.9 & 0 \end{pmatrix}. \tag{21}$$

It follows that the pulse and/or value stability of A relative to of B is nothing but the pulse and/or value stability of A_{red} relative to of B_{red}. This means that the pulse stability of this system lies on the values of the variables: cost of the bus system, average delay and fuel consumption (marked in bold in (15)).

We point out that, concerning the pulse and/or value stability of the given system, the role of the remaining variables is the same if we replace the non-zero value of their weights for another non-zero value (therefore, it does not matter even if we change the sign of some of these values). In [20], the weighted digraph of Figure 2 is said to be an "interesting strong component" of the one in Figure 3. Note that the associated graph to the spectral reduction (A_{red}, B_{red}) of (A, B) is precisely such "strong component" $\{C, D, F\}$. The associated structure matrix of (A_{red}, B_{red}) is (21), whose spectrum is

$$\sigma_m^{A_{red}}(e_{red}) = \{-0.87, -0.05,\ 0.92\}.$$

This means that the process is both pulse and value stable. However, since the module of some of these eigenvalues is close to 1, we deduce that small changes in some of these weights may produce an unstable digraph. For instance, this is the case if we replace -0.9 in the first column by -1.1 (as shown in [20]).

Example 11. *In the report R-1578-NSF [20], the pulse process considered for the 20% bus case is the following one. (see Figure 4 below).*

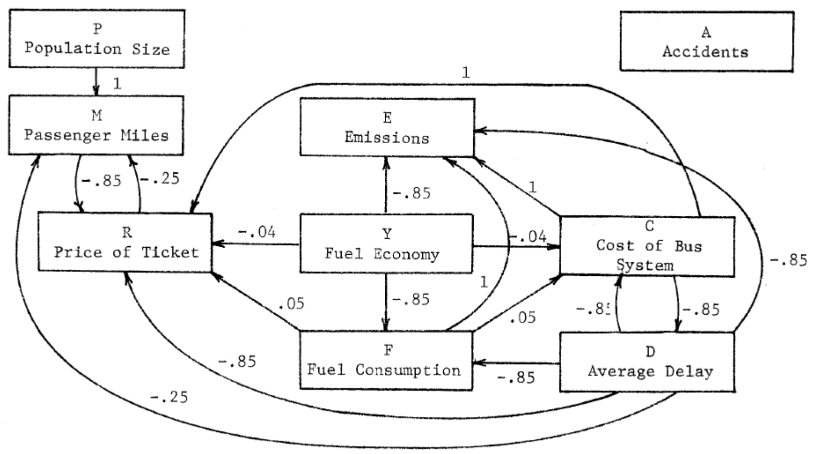

Figure 4. Weighted digraph for the 20% Bus Case. Captured from [20], p. 41. Copyright permission from The Rand Corporation.

The corresponding evolution algebra A is given by the natural basis

$$B = \{e_M, e_Y, e_P, e_C, e_R, e_E, e_A, e_D, e_F\}$$

and structure matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -0.25 & 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & -0.85 & 0.05 \\ -0.85 & -0.04 & 0 & 1 & 0 & 0 & 0 & -0.85 & 0.05 \\ 0 & -0.85 & 0 & 1 & 0 & 0 & 0 & -0.85 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.85 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.85 & 0 & 0 & 0 & 0 & 0 & -0.85 & 0 \end{pmatrix}. \tag{22}$$

In this case, the annihilator of A is $An(A) = lin\{e_E, e_A\}$ and the reduced ideal is given by $I_B = lin\{e_M, e_C, e_R, e_E, e_D, e_F\}$. Therefore, the reduced evolution algebra is $A_{red} = I_B/An(A)$ with natural basis

$$B_{red} = \{e_M + An(A), e_C + An(A), e_R + An(A), e_D + An(A), e_F + An(A)\},$$

and structure matrix

$$\begin{pmatrix} 0 & 0 & -0.25 & -0.25 & 0 \\ 0 & 0 & 0 & -0.85 & 0.05 \\ -0.85 & 1 & 0 & -0.85 & 0.05 \\ 0 & -0.85 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.85 & 0 \end{pmatrix}. \tag{23}$$

Note that $M = lin\{e_M + An(A), e_R + An(A)\}$ is a basic ideal of A_{red} such that

$$A_{red}/M = lin\{e_C + An(A), e_D + An(A), e_F + An(A)\}.$$

The structure matrix of M relative to $B_M = \{e_M + An(A), e_R + An(A)\}$ is

$$\begin{pmatrix} 0 & -0.25 \\ -0.85 & 0 \end{pmatrix}$$

and $B_{A_{red}/M} = \{e_C + An(A), e_D + An(A), e_F + An(A)\}$ is a natural basis of A_{red}/M with structure matrix

$$M_{B_{A_{red}/M}}(A_{red}/M) = \begin{pmatrix} 0 & -0.85 & 0.05 \\ -0.85 & 0 & 0 \\ 0 & -0.85 & 0 \end{pmatrix}.$$

Since the corresponding eigenvalues of these matrices (which determine $\sigma_m^M(e_{B_M})$ and $\sigma_m^{A_{red}/M}(e_{B_{A_{red}/M}})$) are contained in $\mathbb{D}$, the pulse and value stability of the whole pulse process follows from Theorem 9.

7. Conclusions

In this paper, we established the connection between the theory of pulse processes and the theory of evolution algebras. Both theories are enriched with this merged approach. Moreover, since we are simultaneously dealing with two theories, the motivation increases as it comes from two different sources. This would be the case of Proposition 1 (for evolution algebras) and Corollary 4 (for pulse processes).

The approach of Example 8 (also used in the Example 11 when Theorem 9 was applied there) enlightens the theory of pulse processes. The reduction process also gets it. Moreover, we have given a meaning to the study of the "interesting strong components" considered in R-1578-NSF [20], by showing the real role of each one of these components. For instance, in Example 10 we study a pulse process considered in R-1578-NSF [20] that according to this report has one interesting strong component, namely $\{C, D, F\}$, presented in Figure 3. We show that such a component is precisely the weighted digraph associated to the reduced evolution algebra A_{red}, and consequently this component determines the pulse and/or value stability of the whole pulse process (the bus case 10%). However, concerning Example 11, two "interesting strong components" are considered in R-1578-NSF [20], namely $\{M, R\}$ and $\{C, D, F\}$. The first one corresponds to the pulse process associated to the ideal M of the reduced evolution algebra A_{red} described in Example 11, whereas $\{C, D, F\}$ is that of the quotient algebra A_{red}/M. The stability of $\{C, D, F\}$, in Figure 4, is a necessary condition for the pulse and/or value stability of the main process, as deduced from Corollary 4, but it is not sufficient. However, Theorem 9 shows that the pulse and/or value stability of both components, $\{C, D, F\}$ and $\{M, R\}$, is a necessary and a sufficient condition for the pulse and/or value stability of the main pulse process (we gather this new information in Example 11). As we see, not all of the "interesting strong connected components" in the different pulse processes considered in R-1578-NSF play the same role, and our algebraic approach helps us to clarify this.

Note that all the "interesting strong components" mentioned above are evolution algebras of dimension 2 and 3. In [33], all the evolution algebras of dimension 3 were classified in 14 non-isomorphic types of algebras (meanwhile evolution algebras of dimension 2 were classified in 6 non-isomorphic types). To study if some of these types of evolution algebras are in general more stable in pulse and/or value than others, justifying the reason for this, may be a topic for future research. Note that, by Corollary 4, if some quotient algebra A/M of an evolution algebra A, by a basic ideal M, is unstable (in pulse and/or value) then A is unstable.

Anyway, the combination of the theories of pulse processes and evolution algebras opens a window to a new and promising field of research in both frameworks.

Author Contributions: Conceptualization, M.V.V.; methodology, M.V.V. and M.B.; software, M.B. and M.V.V.; validation, M.V.V., M.B. and J.B.; formal analysis, M.V.V. and M.B.; investigation, M.V.V., M.B. and J.B.; resources, M.V.V.; data curation: M.V.V. and M.B.; writing—original draft preparation, M.V.V. and M.B.; writing—review and editing, M.V.V. and M.B.; visualization, M.V.V.; supervision, M.V.V., M.B. and J.B.; project administration, M.V.V., J.B.; funding acquisition, M.V.V. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the Spanish Project MTM2016-76327-C3-2-P (AEI\FEDER, UE).

Acknowledgments: The graphs of these paper have been reprinted with permission expressly granted by the Rand Corporation, Santa Monica, CA. We wish explicitly to show our gratitude for this permission, with special thanks to the people of the Rand Corporation who have helped us with this management.

Conflicts of Interest: The authors declare no conflict of interest.

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