# Inner Model from the Cofinality Quantifier

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Tiivistelmä — Referat — Abstract

This thesis discusses the inner model obtained from the cofinality quantifier introduced in the paper *Inner Models From Extended Logics: Part 1* by Juliette Kennedy, Menachem Magidor and Jouko Väänänen, to appear in the *Journal of Mathematical Logic*. The paper is a contribution to inner model theory, presenting many different inner models obtained by replacing first order logic by extended logics in the definition of the constructible hierarchy $L_\alpha$. We will focus on the model $C^*$ obtained from the logic that extends first order logic by $Q^{cf}_\omega$, the cofinality quantifier for $\omega$. The goal of this thesis is to present two major theorems of the paper and the theory that is needed to understand their proofs. The first theorem states that the Dodd-Jensen core model of $V$ is contained in $C^*$. The second theorem, the Main Theorem of the thesis, is a characterization of $C^*$ assuming $V = L[U]$.

Chapters 2-5 present the theory needed to understand the proofs. Our presentation in these chapters mostly follows standard sources but we present the proofs of many lemmas in much greater detail than our source material. Chapter 2 presents the basics of iterated ultrapowers. If a model $\langle M, U\rangle$ of $ZFC^-$ satisfies "$U$ is a normal ultrafilter on $\kappa$" for some ordinal $\kappa$, then we can construct its ultrapower by $U$. We can take the ultrapower of the resulting model $M_1$ and then continue taking ultrapowers at successor ordinals and direct limits at limit ordinals. If the constructed iterated ultrapowers $M_\alpha$ are well-founded for all ordinals $\alpha$, the model $M$ is called iterable.

Chapter 3 presents $L[A]$, the class of sets constructible relative to a set or class $A$. The hierarchy $L_\alpha[A]$ is a generalization of the constructible hierachy $L_\alpha$. The difference is that the formulas defining the successor level $L_{\alpha+1}[A]$ can use $A \cap L_\alpha[A]$ as a unary predicate. The Main Theorem uses the model $L[U]$, where $U$ is a normal measure on some cardinal $\kappa$. Chapter 4 presents the basics of Prikry forcing, a notion of forcing defined from a measurable cardinal. The sequence of critical points of the iterable ultrapowers of $L[U]$ generates a generic set for the Prikry forcing defined from the critical point of the $\omega$-th iterated ultrapower.

Chapter 5 presents the theory of the Dodd-Jensen core model which is an important inner model. The core model is based on the Jensen hierarchy $J^A_\alpha$ which produces $L[A]$ as the union of all levels. The theory is concerned with so called premice which are levels of the J-hierarchy $J^U_\alpha$ satisfying "$U$ is a normal ultrafilter on $\kappa$" for some ordinal $\kappa \in J^U_\alpha$. A mouse is a premouse satisfying some specific properties and the core model $K$ is the union of all mice.

The last chapter presents the approach of the paper in detail. We present the definition of $C(\mathcal{L}^*)$, the class of sets constructible using an extended logic $\mathcal{L}^*$, and the exact definition of $C^*$. Then we present the proofs of the two major theorems mentioned above. The chapter naturally follows the paper but presents the proofs in greater detail and adds references to lemmas in the previous chapters that are needed for the arguments in the proofs.

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# Contents

# Chapter 1

# Introduction

This thesis discusses the inner model obtained from the cofinality quantifier introduced in the paper *Inner Models From Extended Logics: Part 1* by Juliette Kennedy, Menachem Magidor and Jouko Väänänen [12]. The paper, which we will refer to as the KMV paper, presents many different inner models obtained by replacing first order logic by extended logics in the definition of the constructible hierarchy $L_\alpha$. We will focus on the model $C^*$ obtained from the logic that extends first order logic by $Q^{\mathrm{cf}}_\omega$, the cofinality quantifier for $\omega$.

We will present two major theorems of the paper concerning $C^*$. The first theorem is about the Dodd-Jensen core model:

**Theorem.** The Dodd-Jensen core model of $V$ is contained in $C^*$.

The second theorem is the Main Theorem of this thesis:

**Main Theorem.** Suppose $V = L^\mu$, where $\mu$ is a normal measure on $\kappa$ and $M_\beta$, $\beta \in \mathrm{On}$, are the iterated ultrapowers of $V$ by the measure $\mu$. Then $C^*$ is $M_{\omega^2}[E]$, the Prikry forcing extension of the $\omega^2$-th iterate by the sequence $E = \{\kappa_{\omega\cdot n} : n < \omega\}$ of the critical points of the iterates $M_{\omega\cdot n}$ for $n < \omega$.

The KMV paper is a contribution to a branch of set theory called inner model theory, which we briefly outline below.

## 1.1 Inner model theory

Inner model theory studies inner models that are consistent with large cardinals existing in $V$. An inner model is a transitive model of $ZF$ that contains all the ordinals. The smallest inner model is $L$, the class of all constructible sets. It is included in all other inner models and is a very robust model of $ZFC$. Inner model theorists endeavour to find models that are similar to $L$ in its robustness and well-understood structure but also cover as much of $V$ as possible.

$L$ was introduced by Kurt Gödel in 1938[1] to show the consistency of the Continuum Hypothesis. The construction of $L$ is as follows. For a set $X$, $\mathrm{Def}(X)$ denotes the set of all subsets of $X$ definable in first order logic, i.e.,

$$\mathrm{Def}(X) = \{\{y \in X : (X, \in) \vDash \phi(y, a_1, \ldots, a_n)\} : a_1, \ldots, a_n \in X, \phi \in \mathcal{L}_{\omega\omega}\}. \tag{1.1}$$

The constructible hierarchy is defined recursively as follows:

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\ L_\delta &= \bigcup_{\alpha<\delta} \text{ for limit ordinal } \delta. \end{aligned}$$

$L$ is the union of all levels of the constructible hierarchy: $L = \bigcup_{\alpha \in \mathrm{On}} L_\alpha$.

If there exist large cardinals, then $V$ cannot be $L$. Dana Scott proved in 1961 [2] that if there exists a measurable cardinal, i.e., a cardinal $\kappa$ that has a $\kappa$-complete nonprincipal ultrafilter $U$ on $\kappa$, then $V \neq L$. That ultrafilter allows us to define an ultrapower of $\mathrm{Ult}_U(V)$ of $V$. Scott's construction differs from the usual ultrapower construction of model theory in that the equivalence classes are modified by the so called Scott's trick to make them sets. The $\kappa$-completeness of the ultrafilter $U$ ensures that $\mathrm{Ult}_U(V)$ is well-founded, so it has a transitive collapse $M$ and we can define an elementary embedding $j : V \to M$.

The elementarity of $j$ implies that $M$ must be a model of $ZFC$ and since $j$ is injective and maps ordinals to ordinals, $M$ must contain all ordinals. Hence, $M$ is an inner model. It can be proved that $j(\kappa) > \kappa$. But if $V$ is $L$, the only inner model is $L$, so, in particular, $M$ must be $L$. But if $\kappa$ is the least measurable cardinal, then the elementarity of $j$ implies that $M \vDash$ "$j(\kappa)$ is the least measurable cardinal". That is a contradiction, since $j(\kappa)$ and $\kappa$ cannot both be the least measurable cardinal. This shows that if there is a measurable cardinal, then $V$ cannot be $L$.

Scott's discovery sparked the search for inner models that would be consistent with some large cardinal axioms. One important model is $L^\mu$ which is the class of sets constructible relative to a normal measure $\mu$. That is the smallest inner model consistent with the existence of a measurable cardinal. Another important inner model that will also figure prominently in this thesis is the Dodd-Jensen core model which started the

[1] The result was announced in [8].

[2] Published in [17].

development of different core models. We mention also the model $HOD$ of hereditarily ordinal definable sets which is identical to the model obtained by replacing first order logic by second order logic in the construction of $L$.

The novel contribution of the KMV paper is the systematic study of the models obtained by replacing first order logic by extended logics in the construction of the constructible hierarchy. Although $HOD$ and other inner models based on strong logics had been studied before, the approach of the KMV paper had not been developed systematically.

## 1.2 Goals

The goal of this thesis is to present the theory that is needed to understand the proofs of the two major theorems concerning $C^*$ mentioned above. For this purpose we will present many fundamental concepts and results in set theory. We will mostly limit our discussion to those results that are necessary for the proofs in the last chapter but we will also discuss some basic results that are fundamental in their fields. We will give the proofs of almost all basic lemmas but some proofs have to be omitted to keep the thesis reasonably compact. Our presentation and proofs follow mostly our sources but we add many details to proofs and present some things differently than the sources. Some proofs of lemmas that are not proven in our source literature are our own, including Lemmas 2.13, 5.67 and 5.69.

The first important piece of theory is the theory of iterated ultrapowers discussed in Chapter 2. Iterated ultrapowers are a very central concept and tool in set theory and they have an important role in the proofs concerning $C^*$. If a model $\langle M, U\rangle$ of $ZFC^-$ satisfies $M \models$ "$U$ is a normal ultrafilter on $\kappa$" for some ordinal $\kappa$, then we can construct its ultrapower by $U$. If that ultrapower is well-founded, its transitive closure $\langle M_1, U_1\rangle$ also thinks that $U_1$ is a normal ultrafilter on some $\kappa_1$ and we can continue the process defining iterated ultrapowers $M_\alpha$ for all cardinals $\alpha$ as long as the ultrapowers are well-founded. We will present the basic properties of iterated ultrapowers focusing on those that are needed later on in the thesis. Chapter 2 follows closely Kanamori's presentation in [11].

Chapter 3 presents $L[A]$, the class of sets constructible relative to a set or class $A$. The hierarchy $L_\alpha[A]$ is a generalization of the constructible hierachy $L_\alpha$. The difference is that the formulas defining the successor level $L_{\alpha+1}[A]$ can use $A \cap L_\alpha[A]$ as a unary predicate. Those basic properties of $L[A]$ that are analogous to the properties of $L$ have also analogous proofs, so we do not present their proofs. The most important model obtained from relative constructibility is the model $L[U]$, where $U$ is a normal measure on some cardinal $\kappa$. It is sometimes denoted by $L^\mu$ if the measure is denoted by $\mu$. $L[U]$ satisfies $GCH$ and has the special property that $\kappa$ is the only measurable cardinal in $L[U]$.

$L[U]$ has a central role in the proofs concerning $C^*$. This chapter is also mostly based on Kanamori's book [11].

Chapter 4 presents Prikry forcing, which is a notion of forcing defined for a measurable cardinal $\kappa$. The fundamental property of Prikry forcing is that the forcing extension preserves all cardinalities and all cofinalities except the cofinality of $\kappa$ which has cofinality $\omega$ in the forcing extension. Another important property is the connection to iterated ultrapowers. The sequence of critical points of iterated ultapowers of a model $\{\kappa_n : n < \omega\}$ is Prikry sequence over the $\omega$-th iterate $M_\omega$, i.e., it generates a generic set for the Prikry forcing defined from $\kappa_\omega$. In this chapter we mostly follow Jech's textbook [9].

The largest and most technical piece of theory needed for the proofs concerning $C^*$ is the Dodd-Jensen core model presented in chapter 5. The core model is based on the Jensen hierarchy $J^A_\alpha$ which produces $L[A]$ as the union of all levels: $\bigcup_{\alpha \in \mathrm{On}} J^A_\alpha = L[A]$. The building blocks of the theory are so called premice which are levels of the J-hierarchy satisfying $J^U_\alpha \models$ "$U$ is a normal ultrafilter on $\kappa$" for some ordinal $\kappa \in J^U_\alpha$. A mouse is a premouse satisfying some specific properties and the core model $K$ is the union of all mice. Chapter 5 follows the original paper by Jensen and Dodd [5] and book by Dodd [4] that introduced the core model. To avoid making the thesis excessively long, we will not prove all the fine structure theoretical results of [10] that the core model theory is based on but we will present the proofs of the results concerning the core model.

The last chapter presents the approach of the KMV paper in detail. We present the definition of $C(\mathcal{L}^*)$, the class of sets constructible using an extended logic $\mathcal{L}^*$, and the exact definition of $C^*$. Then we present the proofs of the two major theorems mentioned at the beginning. The chapter naturally follows the KMV paper but presents the proofs in greater detail and adds references to lemmas in the previous chapters that are needed for the arguments in the proofs.

# Chapter 2

# Iterated ultrapowers

Iterated ultrapowers are one of the most fundamental concepts in set theory and they will also figure prominently in our proofs concerning $C^*$ in the last chapter of this thesis.

The intuitive idea of iterated ultrapowers can be described as follows. When we form the ultrapower $M_1$ of $V$ by a $\kappa$-complete ultrafilter $U$ defined on a measurable cardinal $\kappa$, $M_1$ is well-founded due to the $\kappa$-completeness of $U$. Moreover, $\kappa_1 = j(\kappa)$, the image of $\kappa$ under the canonical embedding $j : V \to M_1$, is measurable in $M_1$ and $U_1 = j(U)$ is a $\kappa_1$-complete ultrafilter on $\kappa_1$ by the elementarity of $j$. Thus we can again form the ultrapower of $M_1$ by $U_1$ and then continue this process and define, for all natural numbers $n$, a well-founded model $M_n$, a measurable $\kappa_n$, an ultrafilter $U_n$ on $\kappa_n$ and elementary embeddings $i_{n,m} : M_n \to M_m$ for all $n \leq m$. Finally, at $\omega$ we can take a direct limit of $\langle M_n : n \in \omega \rangle$ and again we get a well-founded model $M_\omega$, a measurable cardinal, an ultrafilter and the embeddings from previous models to $M_\omega$. When we continue by taking ultrapowers at successor ordinals and direct limits at limit ordinals we can define iterated ultrapowers $M_a$ of $V$ for all ordinals $M_\alpha$ and elementary embeddings $i_{\alpha\beta} : M_\alpha \to M_\beta$ for all $\alpha < \beta$.

The details of this idea were first developed by Haim Gaifman in the 1960s[1]. Kenneth Kunen showed in his 1968 dissertation and 1970 article [13] that iterated ultrapowers can be defined for a model $M$ of set theory even in the case that the ultrafilter $U$ is not in the model $M$. In this chapter we present the definition of iterated ultrapowers for a model of $ZFC^-$ along with the basic results that are needed for our proofs concerning $C^*$. We will mostly follow Kanamori's development of the subject in his textbook [11]. Kanamori's definition is slightly different from Kunen's original version but they result in isomorphic models. However, Kanamori's definition follows more explicitly the general idea outlined above, so we find his development more intuitive as an introduction to the subject.

[1]Published in [6], [7].

## 2.1 $M$-ultrafilter and $M$-ultrapower

We begin our discussion of iterated ultrapowers by defining the concept of $M$-ultrafilter.

**Definition 2.1.** [2] Suppose that $M$ is a transitive model of $ZFC^-$, that is, $ZFC$ minus the power set axiom, and $\kappa$ is an infinite cardinal in $M$. Then $U$ is an $M$*-ultrafilter* over $\kappa$ if the following hold:

(i) $U$ is a proper subset of $\mathcal{P}(\kappa) \cap M$ containing no singletons.

(ii) If $X \subset Y \in \mathcal{P}(\kappa) \cap M$ and $X \in U$, then $Y \in U$.

(iii) For all $X \in \mathcal{P}(\kappa) \cap M$, either $X \in U$ or $\kappa - X \in U$.

(iv) If $\eta < \kappa$ and $\langle X_\xi : \xi < \eta \rangle \in M$ and each $X_\xi \in U$, then $\bigcap\{X_\xi : \xi < \eta\} \in U$.

(v) For any $F \in M^\kappa \cap M$, $\{\xi < \kappa : F(\xi) \in U\} \in M$.

Condition (iv) says that $U$ satisfies $\kappa$*-completeness* for tuples from $U$ that are in $M$. Condition (v) is called *weak amenability*. It is important that $U$ need not be in $M$, so $\kappa$ need not be a measurable cardinal in $V$. We are mostly interested in ultrafilters that satisfy the further condition of *normality*, which is defined below. For the rest of this chapter, when we talk of an $M$-ultrafilter for some model $M$, we assume that it is normal.

**Definition 2.2.** An $M$-ultrafilter $U$ is called normal if for any tuple $\{X_\alpha : \alpha < \kappa\} \in M$ such that each $X_\alpha$ is in $U$, the diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha = \{\xi : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$ is in $U$.

As a first step in the construction of iterated ultrapowers, we define the ultrapower of a model $\langle M, \in, U\rangle$[3]. The definition is similar to the ultrapower of $V$ by a measure over a measurable cardinal $\kappa$. Let $U$ be an $M$-ultrafilter over $\kappa$. Let the language of $\langle M, \in U\rangle$ be $\mathcal{L}_\in(\dot{U})$, i.e., the normal language of set theory augmented with the unary predicate symbol $\dot{U}$. For any $f, g \in M^\kappa \cap M$, define the equivalence relation

$$f \sim_U g \quad \text{iff} \quad \{\xi < \kappa : f(\xi) = g(\xi)\} \in U.$$

Because the equivalence classes of $\sim_U$ may not be sets, Scott's trick is applied to them:

$$[f]_U = \{g : g \sim_U f \text{ and for all } h \sim_U f,\ \operatorname{rank}(g) \leq \operatorname{rank}(h)\}.$$

The domain of the ultrapower is the collection of these sets:

$$M^\kappa/U = \{[f]_U : f \in M^\kappa \cap M\}.$$

[2]We modify the definition on pp. 182-183 of [13].

[3]We follow pp. 245-246 of [11].

The membership relation is defined by

$$[f]\, E_U\, [g] \quad \text{iff} \quad \{\xi < \kappa : f(\xi) \in g(\xi)\} \in U.$$

The predicate symbol $\dot{U}$ is interpreted in the ultrapower by

$$[f]_U\, E_U\, \dot{U}_U \quad \text{iff} \quad \{\xi < \kappa : f(\xi) \in U\} \in U.$$

Weak amenability is needed to make the last definition possible. Łoś's theorem holds for $\langle M^\kappa/U, E_U, \dot{U}_U\rangle$ by the usual induction on the complexity of formulas. Since $U$ is $\kappa$-complete only for the sequences that are in $M$, the relation $E_U$ may not be well-founded. If it is, it is isomorphic to its transitive collapse $\langle M_1, \in, U_1\rangle$ by a function $\pi : \langle M^\kappa/U, E_U, \dot{U}_U\rangle \cong \langle M_1, \in, U_1\rangle$. We identify $[f]_U$ with it's image $\pi([f]_U)$ and usually mean $\pi([f]_U)$ when we speak of $[f]_U$. The subscript $U$ is usually omitted if it is clear from the context.

We can define an embedding $j : \langle M, \in, U\rangle \to \langle M_1, \in, U_1\rangle$, called the *canonical embedding*, by $j(x) = [c_x]$, where $c_x$ is the constant function with value $x$. Łoś's theorem implies that $j$ really is an embedding, so $M_1$ is a model of $ZFC^-$ and $\mathrm{On} \cap M \subset \mathrm{On} \cap M_1$.

By induction on $\alpha$, we can see that $j(\alpha) \geq \alpha$ for all ordinals $\alpha$ and $\kappa$-completeness for sequences in $M$ implies that, in fact, $j(\alpha) = \alpha$ for all $\alpha < \kappa$. However, $j(\kappa)$ must be greater than $\kappa$. For any $\xi < \kappa$, we have $c_\kappa(\xi) = \kappa > \xi = \mathrm{id}(\xi)$, where id is the identity on $\kappa$, so $[c_\kappa] > [\mathrm{id}]$. On the other hand, for all $\alpha < \kappa$, $\mathrm{id}(\xi) > \alpha$ for all $\xi > \alpha$, so $[\mathrm{id}] > \alpha$. Thus, $j(\kappa) = [c_\kappa] > [\mathrm{id}] > \alpha$ for all $\alpha < \kappa$, so $j(\kappa) > \kappa$. Thus, $j(\kappa)$ is the least ordinal moved by $j$, and we call it the *critical point* of $j$ and denote it by $crit(j)$.

The following lemma lists some important properties of the ultrapower $\langle M_1, \in, U_1\rangle$. Parts (a) – (e) and their proofs follow Kanamori's Lemma 19.1. We sometimes say that a condition holds for almost all $\alpha$ when the set of $\alpha$'s that satisfy the condition is in $U$.

**Lemma 2.3.** [4]

*(a) The embedding $j$ is cofinal: for any $y \in M_1$, there is $x \in M$ such that $y \in j(x)$. Moreover, if $y$ is an ordinal, $x$ can be taken to be an ordinal as well.*

*(b) If $M$ is a set, $|M| = |M_1|$.*

*(c) $j(x) = x$ for every $x \in V_\kappa \cap M$. Moreover, $V_\kappa \cap M = V_\kappa \cap M_1$ and $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M_1$.*

*(d) $U \notin M_1$.*

*(e) $U_1$ is a normal $M_1$-ultrafilter over $j(\kappa)$.*

[4](a) - (e) follow Lemma 19.1 of [11]. The proof of (f) is our elaboration of a basic fact about normal ($M$-)ultrafilters

(f) *Since $U$ is normal, $\kappa = [id]_U$, where id is the identity function on $\kappa$.*

*Proof.* (a) If $y \in M_1$, then $y = [f]$ for some $f$. Thus, we can take $x = \mathrm{ran}(f)$. If $y$ is ordinal, then $f(\alpha)$ is an ordinal for almost all $\alpha < \kappa$, so we can assume that $\mathrm{ran}(f) \subset \mathrm{On}$. Thus, we can let $x = \sup(\mathrm{ran}(f)) + 1$.

(b) On the one hand, $|M_1| \leq |M^\kappa \cap M| \leq |M|$. On the other hand, $|M| \leq |M_1|$ because $j$ is an embedding.

(c) We show by induction on $\alpha$ that for all $\alpha \leq \kappa$

$$j(x) = x \text{ for every } x \in V_\alpha \cap M \text{ and } V_\alpha \cap M = V_\alpha \cap M_1. \qquad (*)$$

The limit stage is immediate. So suppose that $\alpha < \kappa$ and $(*)$ holds for $\alpha$. Let $x \in M$ and $\mathrm{rank}(x) = \alpha$, so $x \in V_{\alpha+1} \cap M$. The formula saying that $\mathrm{rank}(v_0) = v_1$ can be defined in $ZF$, so $\mathrm{rank}(j(x)) = j(\alpha) = \alpha$. Thus by the induction hypothesis

$$\begin{aligned} j(x) &= \{y \in V_\alpha \cap M_1 : y \in j(x)\} \\ &= \{y \in V_\alpha \cap M : y \in j(x)\} \\ &= \{y \in V_\alpha \cap M : j(y) \in j(x)\} \\ &= \{y : y \in x\} \\ &= x. \end{aligned}$$

This also shows that $V_{\alpha+1} \cap M \subset V_{\alpha+1} \cap M_1$, so we need to show the other direction to complete the successor stage.

Suppose that $x \in M_1$ with $\mathrm{rank}(x) = \alpha$. Let $x = [f]$. By Łoś's theorem we can assume that $\mathrm{rank}(f(\xi)) = \alpha$ for all $\xi < \kappa$. Let $u = \bigcup \mathrm{ran}(f)$, which is in $M$ since $M$ is a model of $ZFC^-$. Then we have $|u| < \kappa$ in $M$. Suppose this is not the case, so there is in $M$ a surjection $s : u \to \kappa$. Then there is in $M$ an injection $g$ such that $s(g(\xi)) = \xi$ for every $\xi < \kappa$. Since $g(\xi)$ is in some $y_\xi \in \mathrm{ran}(f)$, $\mathrm{rank}(g(\xi)) < \alpha$ for every $\xi < \kappa$. Hence, $\mathrm{rank}([g]) < \alpha$, so $[g] \in V_\alpha \cap M_1 = V_\alpha \cap M$. Let $[g] = z \in V_\alpha \cap M$. Now $[c_z] = j(z) = z = [g]$, which is a contradiction since $g$ is an injection.

Since $|u| < \kappa$ in $M$, there is an injection $g_0 : u \to \kappa$ in $M$. Let $g_1 \in M^\kappa \cap M$ be such that $g_0(g_1(\alpha)) = \alpha$ for all $\alpha \in \mathrm{ran}(g_0)$. Let $g_2 \in M^\kappa \cap M$ be defined by $g_2(\alpha) = \{\xi < \kappa : g_1(\alpha) \in f(\xi)\}$. Then $A = \{\alpha < \kappa : g_2(\alpha) \in U\}$ is in $M$, so

$$x' = \{y \in u : \{\xi < \kappa : y \in f(\xi)\} \in U\} = g_1[A \cap \mathrm{ran}(g_0)] \in M.$$

For all $y \in x \cup u$, $\mathrm{rank}(y) < \alpha$, so $y$ is in $V_\alpha \cap M_1 = V_\alpha \cap M$. Since by induction $[c_y] = j(y) = y$ for such $y$, $x' = x$. Thus, $V_{\alpha+1} \cap M_1 \subset V_{\alpha+1} \cap M$, so we have proved $(*)$ for all $\alpha \leq \kappa$.

To prove the last claim of (c), we note that if $X \in \mathcal{P}(\kappa) \cap M$, then $j(X) \cap \kappa = X \in M_1$. Hence, $\mathcal{P}(\kappa) \cap M \subset \mathcal{P}(\kappa) \cap M_1$. For the other direction, let $Y$ be in $\mathcal{P}(\kappa) \cap M_1$ and let $Y = [f]$. Then as with $x'$ and $x$ above, we can see that

$$Y = \{\alpha < \kappa : \{\xi < \kappa : \alpha \in f(\xi)\} \in U\},$$

so $Y$ is in $M$.

(d) Suppose that $U \in M_1$. Then $\mathcal{P}(\kappa) \cap M = U \cup \{\kappa - X : X \in U\}$ is in $M_1$. By (c), we have $\mathcal{P}(\kappa) \cap M_1 = \mathcal{P}(\kappa) \cap M$, so $\mathcal{P}(\kappa) \cap M_1$ is in $M_1$. There is in $M_1$ a surjection $f : \mathcal{P}(\kappa) \cap M \to 2^\kappa \cap M$ defined by $f(X)(\alpha) = 1$ if $\alpha \in f(X)$. There is also in $M_1$ a surjection $g : 2^\kappa \cap M \to \kappa^\kappa \cap M$ defined, e.g., by

$$\begin{aligned} g(h)(\alpha) = &\text{the order type of the } \alpha\text{-th sequence of consecutive 0's} \\ &\text{or consecutive 1's in } h(\alpha) \end{aligned}$$

when we interpret $h$ as a $\kappa$-sequence of 0's and 1's. Hence, there is in $M_1$ a surjection from $\mathcal{P}(\kappa) \cap M_1 = \mathcal{P}(\kappa) \cap M$ onto $\kappa^\kappa \cap M$. Since $U \in M_1$, the function that maps $f \in \kappa^\kappa \cap M$ to $[f]$ is in $M_1$. But $j(\kappa) = \{[f] : f \in \kappa^\kappa \cap M\}$, so we have

$$M_1 \models \exists \alpha < j(\kappa)\, \exists y \exists g\, (y = \mathcal{P}(\alpha) \text{ and } g : y \to j(\kappa) \text{ is surjective}).$$

Since $j$ is an embedding, this implies that

$$M \models \exists \alpha < \kappa\, \exists y \exists g\, (y = \mathcal{P}(\alpha) \text{ and } g : y \to \kappa \text{ is surjective}).$$

Hence, $\alpha$ and $g$ show that $\kappa$ is not strong limit in $M$. But we can see that this is impossible: then there is an injective function $f : \kappa \to 2^\alpha$ in $M$. For all $\beta < \alpha$, there is $i_\beta < 2$ such that $X_\beta = \{\xi < \kappa : f(\xi)(\beta) = i_\beta\}$ is in $U$. Since $\{X_\beta : \beta < \alpha\} \in M$, $X = \bigcap_{\beta<\alpha} X_\beta$ is in $U$ and for $\xi \in X$, $f(\xi)(\beta) = i_\beta$ for all $\beta < \alpha$. But since $f$ is an injection, $X$ can have at most one member, a contradiction.

(e) Denote $\kappa_1 = j(\kappa)$. We show that $U_1$ satisfies the definition of an $M$-ultrafilter for $M = M_1$. The first condition is clear, so suppose for condition (ii) that $x \in U_1$ and $x \subset y \in \mathcal{P}(\kappa_1) \cap M_1$. Let $[f_x] = x$ and $[f_y] = y$. Since $x \subset y$, the set of those $\xi < \kappa$ such that $f_x(\xi) \subset f_y(\xi)$ is in $U$. Hence, $\{\xi < \kappa : f_y(\xi) \in U\}$ is in $U$, so $y \in U_1$. Condition (iii) holds because if $x \in \mathcal{P}(\kappa_1) \cap M_1$ and $x \notin U_1$, the set $\{\xi \in \kappa : \kappa - f_x(\xi) \in U\}$ is in $U$, so $\kappa_1 - x$ is in $U_1$. For condition (iv), suppose that $\gamma < \kappa_1$, $X = \{X_\alpha : \alpha < \gamma\}$ is in $M_1$ and each $X_\alpha$ is in $U_1$. Let $[f] = \bigcap_{\alpha<\gamma} X_\alpha$ and let $[f_X] = X$. Then by Łoś $A_0 = \{\xi < \kappa : \forall x\, (x \in f_X(\xi) \to x \in U)\}$ is in $U$ and $A_1 = \{\xi < \kappa : f(\xi) = \bigcap f_X(\xi)\}$ is in $U$. Thus $A_2 = \{\xi < \kappa : f(\xi) \in U\} \supset A_0 \cap A_1 \in U$, so $A_2$ is in $U$ and $[f] \in U_1$.

To prove weak amenability, suppose that $F \in M_1^{\kappa_1} \cap N$ and $F = [f]$. We can assume that $f(\xi) \in M^\kappa$ for each $\xi < \kappa$. Define the function $f' : \kappa \times \kappa \to M$ by $f'(\xi, \eta) = f(\xi)(\eta)$. Since $f'$ is in $M$, the weak amenability of $\langle M, \in, U \rangle$ implies that $X = \{(\xi, \eta) : f'(\xi, \eta) \in U\}$ is in $M$. Define $g \in M^\kappa$ by $g(\xi) = \{\eta : (\xi, \eta) \in X\}$. Then $g \in M$ and for any $h \in M^\kappa \cap M$ and $\xi < \kappa$, $h(\xi) \in g(\xi)$ holds if and only if $f(\xi)(h(\xi))$ is in $U$. Thus we have for any $h \in M^\kappa \cap M$

$$\begin{aligned} [h] \in [g] &\text{ iff } \{\xi < \kappa : h(\xi) \in g(\xi)\} \in U \\ &\text{ iff } \{\xi < \kappa : f(\xi)(h(\xi)) \in U\} \in U \\ &\text{ iff } F([h]) \in U_1. \end{aligned}$$

Hence, $[g] = \{\alpha < j(\kappa) : F(\alpha) \in U_1\}$. So we have proved that $U_1$ is an $M_1$-ultrafilter on $j(\kappa)$.

We have normality left to prove. Suppose that $X = \{X_\alpha : \alpha < \kappa_1\}$ is in $M_1$ and each $X_\alpha$ is in $U_1$. Let $[f] = X$ and let $[g] = \Delta_{\alpha<\kappa_1} X_\alpha$, the diagonal intersection of $X$. Then $A_0 = \{\xi < \kappa : g(\xi) \text{ is the diagonal intersection of } f(\xi)\}$ is in $U$ and $A_1 = \{\xi < \kappa : \forall x\,(x \in f(\xi) \to x \in U)\}$ is also in $U$. Thus $A_2 = \{\xi < \kappa : g(\xi) \in U\} \supset A_0 \cap A_1 \in U$, so $A_2$ is in $U$ and $\Delta_{\alpha<\kappa_1} X_\alpha$ is in $U_1$.

(f) Let $f \in M^\kappa \cap M$ be such that $A = \{\xi < \kappa : f(\xi) < \xi\} \in U$. Suppose there is no $\alpha < \kappa$ such that $\{\xi < \kappa : f(\xi) = \alpha\} \in U$. Let for all $\alpha < \kappa$, $X_\alpha = \{\xi < \kappa : f(\xi) \neq \alpha\}$. Then each $X_\alpha$ is in $U$ and $\{X_\alpha : \alpha < \kappa\}$ is in $M$, so the diagonal intersection $\Delta_{\alpha<\kappa} X_\alpha$ is in $U$, so its intersection with $A$ is in $U$. If $\xi \in \Delta_{\alpha<\kappa} X_\alpha \cap A$, $f(\xi) = \alpha$ for all $\alpha < \xi$, so $f(\xi) \geq \xi$. However, since $\xi \in A$, $f(\xi) < \xi$. This is a contradiction, so there must be $\alpha < \kappa$ such that $\{\xi < \kappa : f(\xi) = \alpha\} \in U$. Hence, $[f] = \alpha$.

If $[f] < [id]$, $f$ is regressive on a set that is in $U$. Thus, $[f] = \alpha$ for some $\alpha < \kappa$. Hence, $[id] \leq \kappa$. On the other hand, $[id] > \alpha$ for all $\alpha < \kappa$, so $[id] = \kappa$.

□

## 2.2 Iterated ultrapowers

To define iterated ultrapowers, we need the concept of *direct limit* that is employed to define the iterated ultrapowers at limit ordinals[5].

**Definition 2.4.** A *directed set* is a partially ordered set $\langle S, \leq \rangle$ such that for any $i, j \in S$ there is $k \in S$ such that $i \leq k$ and $j \leq k$.

A *directed system* is a pair $\langle \langle \mathcal{M}_i : i \in S \rangle, \langle f_{ij} : i \leq j \rangle \rangle$, where $\langle S, \leq \rangle$ is a directed set, each $\mathcal{M}_i$ is a model in some fixed language $\mathcal{L}$ and every $f_{ij} : \mathcal{M}_i \prec \mathcal{M}_j$ is an elementary embedding satisfying $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. Moreover, $f_{ii}$ is the identity on $\mathcal{M}_i$.

[5]The definitions follow pp. 9-10 of [11].

A *direct limit* of a directed system is an $\mathcal{L}$-model $\mathcal{M}$ such that there are elementary embeddings $f_i : \mathcal{M}_i \prec \mathcal{M}$ for all $i \in S$ satisfying $f_i = f_j \circ f_{ij}$ for all $i \leq j$. Moreover, for each $x \in dom(\mathcal{M})$ there are $i \in S$ and $x' \in \mathcal{M}_i$ such that $x = f_i(x')$.

The following lemma shows that the direct limit of a directed system always exists. The proof works also in the case that the models are classes.

**Lemma 2.5.** [6] *Suppose $\langle\langle \mathcal{M}_i : i \in S\rangle, \langle f_{ij} : i \leq j\rangle\rangle$ is a directed system. Then it has a direct limit.*

*Proof.* Suppose $M_i$ the domain of $\mathcal{M}_i$ for $i \in S$. Let $A = \bigcup_{i \in S}\{i\} \times M_i$, a disjoint union of copies of the $M_i$'s. We define a relation $\sim$ on $A$ as follows:

$$(i, x) \sim (j, y) \quad \text{iff} \quad \exists k \in S\,(i \leq k \text{ and } j \leq k \text{ and } f_{ik}(x) = f_{jk}(y)).$$

Clearly $\sim$ is an equivalence relation and we let the domain of the direct limit be the set of equivalence classes: $M = \{[(i, x)] : (i, x) \in A\}$. To get a structure $\mathcal{M}$ from $M$, the symbols of $\mathcal{L}$ are interpreted as follows. Suppose $[(i_1, x_1)], \ldots, [(i_n, x_n)] \in M$ and pick some $k \in S$ such that $i_1, \ldots, i_n \leq k$. Then we set for relation, function and constant symbols $R$, $f$ and $c$:

$$([(i_1, x_1)], \ldots, [(i_n, x_n)]) \in R^{\mathcal{M}} \text{ iff } (f_{i_1k}(x_1), \ldots f_{i_nk}(x_k)) \in R^{\mathcal{M}_k},$$
$$f^{\mathcal{M}}([(i_1, x_1)], \ldots, [(i_n, x_n)]) = [f^{\mathcal{M}_k}(f_{i_1k}(x_1), \ldots f_{i_nk}(x_k))] \text{ and}$$
$$c^{\mathcal{M}} = [(i, x)] \text{ if } c^{\mathcal{M}_i} = x.$$

It follows easily from the definition of $\sim$ that the above definitions do not depend on the choice of representatives for the equivalence classes. For $i \in S$, the embedding $f_i : \mathcal{M}_i \to \mathcal{M}$ is defined by $f_i(x) = [(i, x)]$. This definition clearly satisfies $f_i = f_j \circ f_{ij}$ for all $i \leq j$.

We can easily see that the $f_i$'s are elementary embeddings. By induction on the length of the formula $\phi$ we can show that for any $i \in S$ and $a_i, \ldots, a_n \in M_i$, $\mathcal{M}_i \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{M} \models \phi(f_i(a_1), \ldots, f_i(a_n))$. For atomic $\phi$ this follows directly from the definition of $f_i$ and the interpretation of the symbols of $\mathcal{L}$. The steps for negation and conjunction are trivial and $\mathcal{M}_i \models \exists x\phi(x, a_1, \ldots, a_n)$ clearly implies that $\mathcal{M} \models \exists x\phi(x, f_i(a_1), \ldots, f_i(a_n))$. For the other direction, if $\mathcal{M} \models \exists x\phi(x, f_i(a_1), \ldots, f_i(a_n))$, then there is some $[(j, b)] \in M$ such that $\mathcal{M} \models \phi([(j, b)], f_i(a_1), \ldots, f_i(a_n))$. Pick any $k$ such that $i, j \leq k$. Then $f_k(f_{jk}(b)) = f_j(b) = [(j, b)]$ and $f_k(f_{ik}(a_s)) = f_i(a_s)$ for all $1 \leq s \leq n$. Thus we have $\mathcal{M} \models \phi(f_k(f_{jk}(b)), f_k(f_{ik}(a_1)), \ldots, f_k(f_{ik}(a_n)))$, whence by the induction hypothesis $\mathcal{M}_k \models \phi(f_{jk}(b), f_{ik}(a_1), \ldots, f_{ik}(a_n))$, so $\mathcal{M}_k \models \exists x\phi(x, f_{ik}(a_1), \ldots, f_{ik}(a_n))$. But this implies that $\mathcal{M}_i \models \exists x\phi(x, a_1, \ldots, a_n)$ since $f_{ik}$ is an elementary embedding. Thus, the $f_i$'s show that $\mathcal{M}$ is a direct limit of $\langle\langle \mathcal{M}_i : i \in S\rangle, \langle f_{ij} : i \leq j\rangle\rangle$. $\square$

[6] The proof follows p. 10 in [11]

The following property of direct limits is needed in the discussion of iterability in the last section of this chapter.

**Lemma 2.6.** [7] *Suppose $\langle\langle \mathcal{M}_i : i \in S\rangle, \langle f_{ij} : i \leq j\rangle\rangle$ is a directed system and $\mathcal{M}$ is a direct limit with embeddings $f_i : \mathcal{M}_i \prec \mathcal{M}$. Suppose $\mathcal{N}$ is a structure such that each $\mathcal{M}_i$ is embeddable into it by $g_i : \mathcal{M}_i \prec \mathcal{N}$ and the embeddings satisfy $g_i = g_j \circ f_{ij}$ for all $i \leq j$. Then there is an elementary embedding $g : \mathcal{M} \prec \mathcal{N}$ such that $g_i = g \circ f_i$.*

*Proof.* The elementary embedding $g$ can be defined as follows: for $x \in \mathrm{dom}(\mathcal{M})$, choose any $i \in S$ and $x' \in \mathrm{dom}(\mathcal{M}_i)$ such that $x = f_i(x')$. Then define $g(x) = g_i(x')$. Since the $g_i$'s satisfy $g_i = g_j \circ f_{ij}$, $g$ is well-defined. It is straightforward to see that $g$ is an elementary embedding. □

Before we can define iterated ultrapowers, we need one more concept. We let $h : \langle X, \in, R\rangle \prec^- \langle X', E, R'\rangle$ mean that $h$ is elementary for $\mathcal{L}_\in$-formulas and $h$ also preserves the unary predicate, that is, $h$ is an embedding of $\langle X, \in, R\rangle$ into $\langle X', E, R'\rangle$. The following lemma shows that if the direct limit of iterated ultrapowers taken at some limit ordinal is well-founded, we can continue the definition of iterated ultrapowers beyond that limit ordinal.

**Lemma 2.7.** [8] *Suppose that for each $\alpha < \delta$, $W_\alpha$ is a normal $N_\alpha$-ultrafilter over $\kappa_\alpha$ and that $\langle\langle\langle N_\alpha, \in, W_\alpha\rangle : \alpha < \delta\rangle, \langle j_{\alpha\beta} : \alpha \leq \beta\rangle\rangle$ is a directed system of $\prec^-$-embeddings with a well-founded direct limit $\langle M, E, U\rangle$. Then it has a transitive collapse $\langle N, \in, W\rangle$ and if for $\alpha < \delta$, $j_{\alpha\delta} : \langle N_\alpha, \in, W_\alpha\rangle \prec^- \langle N, \in W\rangle$ is the direct limit embedding composed with the transitive collapse, then $W$ is a normal $N$-ultrafilter over $\kappa = j_{\alpha\delta}(\kappa_\alpha)$ for some, and thus all, $\alpha < \delta$.*

*Proof.* To prove the first claim, the extensionality of $\langle M, E\rangle$ follows straightforwardly from the extensionality of the $\langle N_\alpha, \in\rangle$'s. Let $i_\alpha : \langle N_\alpha, \in, W_\alpha\rangle \to \langle M, E, U\rangle$ be the direct limit embeddings for $\alpha < \delta$. Let $a \in M$, so $a = i_\alpha(a')$ for some $\alpha < \delta$ and $a' \in N_\alpha$. Then for any $x \in M$,

$$xEa \text{ iff } \exists\beta\exists y\,(\alpha \leq \beta < \delta \text{ and } i_\beta(y) = x \text{ and } y \in j_{\alpha\beta}(b)),$$

so $\{x \in M : xEa\}$ is a set by Replacement. Hence, we can apply Mostowski collapse to $\langle M, E, U\rangle$.

Then we prove that $W$ is an $N$-ultrafilter over $\kappa$. The first condition is obvious, so suppose for condition (ii) that $x \in W$ and $x \subset y \in \mathcal{P}(\kappa) \cap N$. Let $x = i_\alpha(x')$ and $y = i_\beta(y')$

[7] Lemma 0.7 of [11].

[8] Exercise 19.3 of [11]. The proof that Mostovski collapse can be applied and the proof of amenability are given by Kanamori, the rest is our own.

for some $\alpha, \beta < \delta$. Then there is $\gamma < \delta$ such that $\alpha, \beta \leq \gamma$. Now $j_{\alpha\gamma}(x') \subset j_{\beta\gamma}(y')$ so $j_{\beta\gamma}(y') \in W_\gamma$, whence $y \in W$. The proof of condition (iii) is equally straighforward. For condition (iv) suppose that $\lambda < \kappa$, $X = \{X_\eta : \eta < \lambda\}$ is in $N$ and each $X_\eta$ is in $W$. Let $Y \in N$ be the intersection of $X$. Let $\lambda = i_\alpha(\lambda')$ and $X = i_\beta(X')$ for some $\alpha, \beta \leq \delta$. Then $\lambda' < \kappa_\alpha$ so in fact $\lambda' = \lambda$. Pick some $\gamma < \delta$ such that $\alpha, \beta \leq \gamma$. Then every element of $\bar{X} = j_{\beta\gamma}(X')$ is in $W_\gamma$ and $\bar{X}$ has size $\lambda < \kappa_\gamma$ in $N_\gamma$ so $\bar{Y} = \bigcap \bar{X}$ is in $W_\gamma$. Thus $Y = i_\gamma(\bar{Y})$ is in $W$. To prove weak amenability, suppose that $F \in N^\kappa \cap N$, so $F = i_\alpha(F')$ for some $\alpha < \delta$ and $F' \in N_\alpha^{\kappa_\alpha} \cap N_\alpha$. By the weak amenability of $\langle N_\alpha, \in, W_\alpha\rangle$, $X = \{\xi < \kappa_\alpha : F'(\xi) \in W_\alpha\}$ is in $N_\alpha$, so $j_{\alpha\beta}(X) = \{\xi < \kappa : F(\xi) \in W\}$ is in $N$. Hence, $W$ is an $N$-ultrafilter.

To prove normality, suppose that $X = \{X_\eta : \eta < \kappa\}$ is in $N$ and each $X_\eta$ is in $W$. Let $Y = \Delta_{\eta<\kappa} X_\eta$ be the diagonal intersection of $X$. Let $X = i_\alpha(X')$ for some $\alpha < \delta$. Then $X'$ has size $\kappa_\alpha$ in $N_\alpha$ and each element of $X'$ is in $W_\alpha$. Let $Y'$ be the diagonal intersection of $X'$ in $N_\alpha$. Be the normality of $N_\alpha$, $Y'$ is in $W_\alpha$, so $Y = i_\alpha(Y')$ is in $W$.

□

We are now ready to define the iterated ultrapowers of a model $\langle M, \in, U\rangle$[9]. We define recursively for each $\alpha \in \tau$, where $\tau$ will be the length of the iteration, $\mathcal{L}_\in(\dot{U})$-structures $\langle M_\alpha, \in, U_\alpha\rangle$ such that $U_\alpha$ is an $M_\alpha$-ultrafilter over $\kappa_\alpha$, and embeddings $i_{\alpha\beta} : \langle M_\alpha, \in, U_\alpha\rangle \prec^- \langle M_\beta, \in U_\beta\rangle$ for all $\alpha \leq \beta < \tau$. First we let $M_0 = M$, $U_0 = U$, $\kappa_0 = \kappa$ and let $i_{0,0}$ be the identity on $M$.

Suppose $M_\alpha$, $U_\alpha$, $\kappa_\alpha$ and $i_{\alpha\beta}$ have been defined for $\alpha \leq \beta < \delta$. Suppose first that $\delta$ is a successor ordinal, say $\delta = \gamma + 1$. If the ultrapower of $M_\gamma$ by $U_\gamma$ is well-founded, we let $\langle M_\delta, \in, U_\delta\rangle$ be its transitive collapse. Further we set $\kappa_\delta = j(\kappa_\gamma)$, $i_{\gamma\delta} = j$ and $i_{\alpha\delta} = j \circ i_{\alpha\gamma}$ for $\alpha < \gamma$, where $j$ is the canonical embedding from $\langle M_\gamma, \in, U_\gamma\rangle$ into $\langle M_\delta, \in, U_\delta\rangle$. If the ultrapower is not well-founded, we let $\tau = \delta$.

Suppose then that $\delta$ is a limit ordinal. If the direct limit of

$$\langle\langle\langle M_\alpha, \in, U_\alpha\rangle : \alpha < \delta\rangle, \langle i_{\alpha\beta} : \alpha \leq \beta\rangle\rangle$$

is well-founded, we let $\langle M_\delta, \in, U_\delta\rangle$ be its transitive collapse. For each $\alpha < \delta$, the direct limit embedding followed by the collapsing function is an embedding $\langle M_\alpha, \in, U_\alpha\rangle \prec^- \langle M_\delta, \in, U_\delta\rangle$. We let $i_{\alpha\gamma}$ be that embedding for each $\alpha < \delta$ and let $\kappa_\delta = i_{\alpha\delta}(\kappa_\alpha)$ for some, and consequently all, $\alpha < \delta$ and let $i_{\delta\delta}$ be the the identity on $M_\delta$. If the direct limit is not well-founded, we set $\tau = \delta$.

If the definition process goes on through all the ordinals, i.e., no ultrapower or direct limit is ill-founded, we set $\tau = \mathrm{On}$. Otherwise, $\tau$ is defined to be some ordinal, and $\tau$ is the stage at which the iteration encounters ill-founded structures. In this thesis we do not define the ultrapowers beyond $\tau$. The following definition summarizes our terminology:

[9]We follow the definitions on pp. 249-250 of [11].

**Definition 2.8.** $\langle M_\alpha, U_\alpha, \kappa_\alpha, i_{\alpha\beta}\rangle_{\alpha\leq\beta\in\tau}$ is the *iteration* of $\langle M, \in, U\rangle$ and $\tau$ is called the *length* of the iteration. For each $\alpha \in \tau$, $\langle M_\alpha, \in, U_\alpha\rangle$ is called the $\alpha$-th *iterate* or *iterated ultrapower* of $\langle M, \in, U\rangle$.

Naturally, a central question concerning iterated ultrapowers is whether the iterated ultrapowers of a given model are defined for all ordinals or whether they are only defined up to the ordinal $\tau$. We call a model $\langle M, \in, U\rangle$ *iterable*, if the iterates are defined for all ordinals, i.e., $\tau = \mathrm{On}$. If that case we also say that $U$ is an *iterable* $M$-ultrafilter. In the last section of this chapter we will present and prove a sufficient condition for the iterability of a model. Before that we discuss a few lemmas that state some important properties of iterated ultrapowers.

**Lemma 2.9.** [10] *Suppose that $\alpha < \beta < \tau$. Then the following hold:*

*(a)* $\mathrm{crit}(i_{\alpha\beta}) = \kappa_\alpha$ *and* $i_{\alpha\beta}(\kappa_\alpha) = \kappa_\beta$.

*(b)* $i_{\alpha\beta}(x) = x$ *for every* $x \in V_{\kappa_\alpha} \cap M_\alpha$, $V_{\kappa_\alpha} \cap M_\alpha = V_{\kappa_\alpha} \cap M_\beta$ *and* $\mathcal{P}(\kappa_\alpha) \cap M_\alpha = \mathcal{P}(\kappa_\alpha) \cap M_\alpha$.

*(c) If $\beta$ is a limit ordinal, then $\kappa_\beta = sup\{\kappa_\gamma : \gamma < \beta\}$*

*(d) If $M$ is a set, then $|M_\alpha| = |M| \cdot \alpha$.*

*Proof.* (a) Follows directly from the definition of the iteration.

(b) Consequence of Lemma 2.3(b).

(c) If $\xi < \kappa_\beta$, then since $M_\beta$ is a direct limit, $\xi = i_{\gamma\beta}(\xi')$ for some $\gamma < \beta$ and $\xi' \in M_\gamma$. Since $i_{\gamma\beta}$ is an embedding, $\xi' < \kappa_\gamma$. But then $i_{\gamma\beta}(\xi') = \xi'$, so $\xi < \kappa_\gamma$. Thus we have $\sup\{\kappa_\gamma : \gamma < \beta\} \geq \kappa_\beta$, so necessarily $\sup\{\kappa_\gamma : \gamma < \beta\} = \kappa_\beta$.

(d) By induction on $\alpha$ we see that $|M_\alpha| \leq |M| \cdot |\alpha|$. For successors the induction step follows from Lemma 2.3(b) and for limits it follows from the standard construction of a direct limit. On the other hand, $i_{0\alpha}$ is an injection from $M$ to $M_\alpha$ and $\{\kappa_\gamma : \gamma < \alpha\} \subset M_\alpha$, so $|M_\alpha| \geq |M| \cdot |\alpha|$.

□

**Lemma 2.10.** [11] *If $\beta \in \tau$ is a limit ordinal, then for all $X \in \mathcal{P}(\kappa_\beta) \cap M_\beta$,*

$$X \in U_\beta \quad \text{iff} \quad \exists\alpha < \beta \text{ such that } \{\kappa_\gamma : \alpha \leq \gamma < \beta\} \subset X.$$

[10] We follow Lemma 19.4 of [11]

[11] We follow Lemma 19.5 of [11].

*Proof.* Suppose that $X = i_{\alpha\beta}(X')$ for some $\alpha < \beta$ and $X' \in \mathcal{P}(\kappa_\alpha) \cap M_\alpha$. Then

$$X \in U_\beta \text{ iff } X' \in U_\alpha \text{ iff } i_{\alpha\gamma}(X') \in U_\gamma \text{ for all } \alpha \leq \gamma < \beta$$

since $i_{\alpha\beta} = i_{\gamma\beta} \circ i_{\alpha\gamma}$ for any $\alpha \leq \gamma < \beta$. On the other hand we have for all $\gamma$ such that $\alpha \leq \gamma < \beta$,

$$i_{\alpha\gamma}(X') \in U_\gamma \text{ iff } \kappa_\gamma \in i_{\gamma,\gamma+1}(i_{\alpha\gamma}(X')) \text{ iff } \kappa_\gamma \in i_{\alpha\beta}(X'),$$

since $i_{\gamma+1,\beta}(\kappa_\gamma) = \kappa_\gamma$. Therefore, for any $\alpha < \beta$, if $X \in \operatorname{ran}(i_{\alpha\beta})$, then $X \in U_\beta$ iff $\{\kappa_\gamma : \alpha \leq \gamma < \beta\} \subset X$.

Since $M_\beta$ is the transitive collapse of a direct limit, if $X \in \mathcal{P}(\kappa_\beta) \cap M_\beta$, then there are $\alpha < \beta$ and $X' \in \mathcal{P}(\kappa_\alpha) \cap M_\alpha$ such that $X = i_{\alpha\beta}(X')$. Thus for all $X \in \mathcal{P}(\kappa_\beta) \cap M_\beta$, $X \in U_\beta$ if and only if there is $\alpha < \beta$ such that $\{\kappa_\gamma : \alpha \leq \gamma < \beta\} \subset X$.

□

**Lemma 2.11.** [12] *For any $\alpha \in \tau$ and $x \in M_\alpha$, there are $n \in \omega$, $f \in M^{[\kappa]^n} \cap M$ and $\gamma_1, \dots \gamma_n < \alpha$ such that $x = i_{0\alpha}(f)(\kappa_{\gamma_1}, \dots \kappa_{\gamma_n})$.*

*Proof.* We prove the claim by induction on $\alpha$. Suppose that the claim holds for $\alpha, \alpha+1 \in \tau$ and $x \in M_{\alpha+1}$. Then $x = [g]_{U_\alpha}$ for some $g \in M_\alpha^{\kappa_\alpha} \cap M_\alpha$. Since $U_\alpha$ is a normal $M_\alpha$-ultrafilter, $i_{\alpha,\alpha+1}(g)(\kappa_\alpha) = [c_g]_{U_\alpha}([\mathrm{id}_\alpha]_{U_\alpha})$ where $c_g : \kappa_\alpha \to M_\alpha$ is the constant function with value $g$ and $\mathrm{id}_\alpha$ is the identity on $\kappa_\alpha$. Since for all $\xi < \kappa_\alpha$, $c_g(\xi)(\mathrm{id}_\alpha(\xi)) = g(\xi)$, $[c_g]_{U_\alpha}([\mathrm{id}_\alpha]_{U_\alpha}) = [g]_{U_\alpha}$ and, therefore, $i_{\alpha,\alpha+1}(g)(\kappa_\alpha) = x$. If $\alpha = 0$, we have proved the claim.

If $\alpha > 0$, by induction $g = i_{0\alpha}(h)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})$ for some $n \in \omega$, $h \in M^{[\kappa]^n} \cap M$ and $\gamma_1 < \dots < \gamma_n < \alpha$ where it can be assumed that $\operatorname{ran}(h)$ consists of functions. Define $f \in M^{[\kappa]^{n+1}} \cap M$ by $f(\xi_1, \dots, \xi_n, \xi_{n+1}) = h(\xi_1, \dots, \xi_n)(\xi_{n+1})$. Now we have:

$$\begin{aligned} i_{0,\alpha+1}(f)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n}, \kappa_\alpha) &= i_{0,\alpha+1}(h)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})(\kappa_\alpha) \\ &= i_{\alpha,\alpha+1}(i_{0,\alpha}(h)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n}))(\kappa_\alpha) \\ &= i_{\alpha,\alpha+1}(g)(\kappa_\alpha) \\ &= x. \end{aligned}$$

This shows that if the claim holds for $\alpha$, it holds for $\alpha + 1$ as well. We have the limit case left to prove.

So suppose $\alpha < \tau$ is a limit ordinal and the claim holds for all $\beta < \alpha$ and $x \in M_\alpha$. Then $x = i_{\beta\alpha}(x')$ for some $\beta < \alpha$ and $x' \in M_\beta$ since $M_\alpha$ is a direct limit. By induction, $x' = i_{0\beta}(f)(\kappa_{\gamma_1,\dots,\kappa_{\gamma_n}})$ for some $n \in \omega$, $f \in M^{[\kappa]^n} \cap M$ and $\gamma_1, \dots, \gamma_n < \alpha$. Hence, $x = i_{\beta\alpha}(i_{0\beta}(f)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})) = i_{0\alpha}(f)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})$, since $i_{\beta\alpha}(\kappa_{\gamma_i}) = \kappa_{\gamma_i}$ for all $i \leq n$. Thus, the claim holds for limit ordinals as well. □

[12] The proof follows Lemmas 19.6 and 5.13(a) of [11].

Lemma 2.11 shows that $M_\alpha$ is generated from $\mathrm{ran}(i_{0\alpha})$ and $\{\kappa_\gamma : \gamma < \alpha\}$. The lemma is needed in the proof of the Main Theorem and it is also used in the proof of following lemma. The cardinals and the cardinalities of sets mentioned in the lemma are in the sense of $V$.

**Lemma 2.12.** [13]

*(a) If $\xi \in On \cap M$ and $\alpha < \tau$, then $i_{0\alpha}(\xi) < (|\xi^\kappa \cap M| \cdot |\alpha|)^+$*

*(b) If $\theta$ is a cardinal such that $|\kappa^\kappa \cap M| < \theta \in \tau$, then $\kappa_\theta = i_{0\theta}(\kappa_0) = \theta$.*

*(c) If $\theta$ is a cardinal, $\alpha < min(\theta, \tau)$ and $M \models ZFC \wedge$ "$\theta$ is a strong limit" $\wedge cf(\theta) > \kappa$, then $i_{0\alpha}(\theta) = \theta$.*

*Proof.* (a) By Lemma 2.11, $\eta < i_{0\alpha}(\xi)$ if and only if $\eta = i_{0\alpha}(f)(\kappa_{\gamma_1}, \dots, \kappa_{\gamma_n})$ for some $n \in \omega$, $f \in \xi^{[\kappa]^n} \cap M$ and $\gamma_1 < \dots < \gamma_n < \alpha$. The claim holds, since $|\xi^{[\kappa]^n} \cap M| = |\xi^\kappa \cap M|$ and $|[\alpha]^{<\omega}| = |\alpha| \cdot \omega$.

(b) By (a) and Lemma 2.9(c) we have

$$\theta \leq \kappa_\theta = \sup\{\kappa_\alpha : \alpha < \theta\} \leq \sup\{(|\kappa^\kappa \cap M| \cdot |\alpha|)^+ : \alpha < \theta\} \leq \theta.$$

(c) Since $i_{0\alpha}(\theta) \geq \theta$, it suffises to show that $\eta < i_{0\alpha}(\theta)$ implies that $\eta < \theta$. So suppose $\eta < i_{0\alpha}(\theta)$. By Lemma 2.11, $\eta = i_{0\alpha}(f)(\kappa_{\gamma_1,\dots,\gamma_n})$ for some $n \in \omega$, $f \in \theta^{[\kappa]^n} \cap M$, and $\gamma_1 < \dots, < \gamma_n < \alpha$. Since $\mathrm{cf}(\theta) > \kappa$ in $M$, there is a $\xi < \theta$ such that $f \in \xi^{[\kappa]^n}$. Thus, $i_{0\alpha}(f) \in i_{0\xi}(\xi)^{[\kappa_\alpha]^n}$, so $\eta < i_{0\alpha}(\xi)$. By (a) and the assumptions on $\theta$ we get that

$$\eta < i_{0\alpha}(\xi) < (|\xi^\kappa \cap M| \cdot |\alpha|)^+ \leq \theta.$$

□

## 2.3 Iterability

We conclude this chapter with a sufficient condition for the iterability of $\langle M, \in, U\rangle$: $U$ is *countably complete* if for any $\{X_n : n \in \omega\} \subset U$, $\bigcap_n X_n \neq \emptyset$. The condition requires that any countable intersection of members of $U$ is in $U$ while the definition of an $M$-ultrafilter only concerns the intersections of those subsets of $U$ that are in $M$. The sufficiency of this condition was proved by Kunen in [13] but our proofs mostly follow Kanamori's [11] discussion of the matter.

We need the following auxiliary concept. For $n \in \omega$, define

$$U^n = \{X \in \mathcal{P}([\kappa]^n) \cap M \; : \exists H \in U\, ([H]^n \subset X)\}.$$

The following lemma gives a more useful characterization of $U^n$.

[13] We follow Lemma 19.7 of [11].

**Lemma 2.13.** [14] *For $n \in \omega$ and $X \in \mathcal{P}([\kappa]^{n+1}) \cap M$,*

$$X \in U^{n+1} \quad \textit{iff} \quad \{s \in [\kappa]^n : \{\xi < \kappa : s \cup \{\xi\} \in X\} \in U\} \in U^n.$$

To prove the lemma we need the following result that holds also for a normal $M$-ultrafilter and those $f : [\kappa]^n \to 2$ that are in $M$.

**Lemma 2.14.** [15] *Suppose $\kappa$ is a measurable cardinal and $U$ is a normal measure on $\kappa$. If $F$ is a partition of $[\kappa]^{<\omega}$ into less than $\kappa$ parts, then there is a set $H \in U$ homogeneous for $F$.*

*Proof.* It suffices to show that for each $n < \omega$ there is $H_n \in U$ such that $F$ is constant on $[H_n]^n$. Then $H = \bigcap_{n<\omega} H_n$ is in $U$ by the $\kappa$-completeness of $D$ and $H$ is homogeneous for $F$.

We prove by induction on $n$ that for every partition $F$ of $[\kappa]^n$ into fewer than $\kappa$ parts there is some $H \in U$ that is homogeneous for $F$. For $n = 1$, we can assume that a partition $F$ is a function from $\kappa$ to some $\lambda < \kappa$. Define $X_\alpha = F^{-1}\{\alpha\}$ for each $\alpha < \lambda$ . Now $\kappa = \bigcup_{\alpha<\lambda} X_\alpha$, whence some $X_\alpha$ must be in $U$. Otherwise, each $\kappa - X_\alpha$ would be in $U$, so $\bigcap_{\alpha<\lambda}(\kappa - X_\alpha)$ would be in $U$. But that is impossible since $\bigcap_{\alpha<\lambda}(\kappa - X_\alpha) = \emptyset$ as every $\beta < \kappa$ is in some $X_\alpha$. Since $F$ is constant on every $X_\alpha$, the claim holds for $n = 1$.

Suppose then that the claim holds for $n$. We prove that it holds for $n + 1$ as well. Let $F : [\kappa]^{n+1} \to \lambda$ be a partition for some $\lambda < \kappa$. For each $\alpha < \kappa$ define a function $F_\alpha : [\kappa - \{\alpha\}]^n \to \lambda$ by $F_\alpha(X) = F(X \cup \{\alpha\})$. By the induction hypothesis, for each $\alpha < \kappa$, there is $X_\alpha \in U$ such that $F_\alpha$ is constant on $X_\alpha$. Denote for each $\alpha$ the constant value by $i_\alpha$. Defining $Y_\alpha = \{\gamma < \kappa : i_\gamma = \alpha\}$ for all $\alpha < \lambda$, the same argument as in the case $n = 1$ shows that some $Y_\alpha$ is in $U$. Denote that $Y_\alpha$ by $Y$ and the value $i_\alpha$ by $i$. Since $U$ is normal, the diagonal intersection $X = \Delta_{\alpha<\kappa} X_\alpha$ is in $U$. Let $H = X \cap Y$. If $\gamma < \alpha_0 < \cdots < \alpha_n$ are in $X$, then $\alpha_0, \ldots, \alpha_n \in X_\gamma$, so $F\{\gamma, \alpha_0, \ldots, \alpha_n\} = F_\gamma(\alpha_0, \ldots, \alpha_n) = i_\gamma$. Hence, for every $Y' \in [H]^{n+1}$, $F(Y') = i$, so the claim holds for $n + 1$. $\square$

*Proof of lemma 2.13.*

Let $n \in \omega$ and $X \in \mathcal{P}([\kappa]^{n+1}) \cap M$. If $X \in U^{n+1}$, then there is $H \in U$ such that $[H]^{n+1}$ is a subset of $X$. Then for any $s \in [H]^n$, $H - s \in U$, so $\{\xi < \kappa : s \cup \{\xi\} \in X\} \in U$. Since $[H]^n \in U^n$, we have $\{s \in [\kappa]^n : \{\xi < \kappa : s \cup \{\xi\} \in X\} \in U\} \in U^n$.

[14] Exercise 19.8 of [11]. The proof is our own.

[15] We follow Lemma 10.22 of Jech [9].

For the other direction, suppose that the lemma holds for all $k \leq n$ and suppose that $A_0 = \{s \in [\kappa]^n : \{\xi < \kappa : s \cup \{\xi\} \in X\} \in U\} \in U^n$. By the induction assumption we have

$$A_1 = \{s \in [\kappa]^{n-1} : \{\xi < \kappa : s \cup \{\xi\} \in A_0\} \in U\} \in U^{n-1},$$
$$A_2 = \{s \in [\kappa]^{n-2} : \{\xi < \kappa : s \cup \{\xi\} \in A_1\} \in U\} \in U^{n-2},$$

and so on until

$$A_{n-1} = \{s \in [\kappa]^1 : \{\xi < \kappa : s \cup \{\xi\} \in A_{n-2}\} \in U\} \in U^1.$$

Clearly $X \in U^1$ if and only if $\bigcup X \in U$, so $\bigcup A_{n-1} \in U$. Let $f : [\kappa]^{n+1} \to 2$ be such that $f(s) = 1$ if $s \in X$. Since $f$ is in $M$, by Lemma 2.14 there is $H \in U$ homogeneous for $f$. Then $H' = H \cap (\bigcup A_{n-1})$ is in $U$. Choose any $a_1 \in H'$. Since $\{a_1\}$ is in $A_{n-1}$, $\{\xi < \kappa : \{a_1\} \cup \{\xi\} \in A_{n-2}\}$ is in $U$, so there is $a_2$ in $H' \cap \{\xi < \kappa : \{a_1\} \cup \{\xi\} \in A_{n-2}\}$. Now $\{a_1, a_2\} \in A_{n-2}$, so there is again $a_3 \in H'$ such that $\{a_1, a_2, a_3\}$ is in $A_3$. Continuing like this we find $a_1, \ldots, a_n \in H'$ such that $\{a_1, \ldots, a_n\} \in A_0$, so finally there is $a_{n+1} \in H'$ such that $\{a_1, \ldots, a_{n+1}\} \in X$. Hence, $f(s) = 1$ for all $s$ in $[H']^{n+1}$, so $[H']^{n+1} \subset X$ and $X$ is in $U^{n+1}$.

□

The characterization of $U^n$ given by Lemma 2.13 allows us to prove the following lemma.

**Lemma 2.15.** [16] *For any formula $\phi(v_0, \ldots v_n) \in \mathcal{L}_\in(\dot{U})$, $x_0, \ldots, x_k \in M$ and $\gamma_1 < \cdots < \gamma_n < \alpha \in \tau$,*

$$\begin{aligned} &\langle M_\alpha, \in U_\alpha\rangle \vDash \phi[i_{0\alpha}(x_0), \ldots, i_{0\alpha}(x_k), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n}] \\ \text{iff}\quad &\langle M, \in, U\rangle \vDash \{\{\xi_1, \ldots, \xi_n\} \in [\kappa]^n : \phi[x_0, \ldots x_k, \xi_1, \ldots, \xi_n]\} \in U^n. \end{aligned}$$

*Proof.* The proof is by induction on $n$. Since $U^0 = \{\emptyset\}$, the claim holds for $n = 0$. Suppose that the claim holds for $n - 1$. Then we have

$$\begin{aligned} &\langle M_\alpha, \in, U_\alpha\rangle \vDash \phi[i_{0\alpha}(x_0), \ldots, i_{0\alpha}(x_k), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n}] \\ &\quad \text{iff } \langle M_{\gamma_n+1}, \in, U_{\gamma_n+1}\rangle \vDash \phi[i_{0\gamma_n+1}(x_0), \ldots, i_{0\gamma_n+1}(x_k), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_n}] \\ &\quad \text{iff } \langle M_{\gamma_n}, \in, U_{\gamma_n}\rangle \vDash \{\xi < \kappa_{\gamma_n} : \phi[i_{0\gamma_n}(x_0), \ldots, i_{0\gamma_n}(x_k), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_{n-1}}, \xi]\} \in U_{\gamma_n} \\ &\quad \text{iff } \langle M_{\gamma_n}, \in, U_{\gamma_n}\rangle \vDash \{\xi < \bigcup U_{\gamma_n} : \phi[i_{0\gamma_n}(x_0), \ldots, i_{0\gamma_n}(x_k), \kappa_{\gamma_1}, \ldots, \kappa_{\gamma_{n-1}}, \xi]\} \in U_{\gamma_n}. \end{aligned}$$

The second step uses Łoś's theorem. The last step is based on the fact that $\bigcup U_{\gamma_n} = \kappa_{\gamma_n}$. Since $U_{\gamma_n}$ and $U$ are the interpretations of $\dot{U}$ in $M_{\gamma_n}$ and $M$, respectively, when we apply

[16] We follow Lemma 19.9(a) of [11].

the induction assumption to the statement to the right of $\vDash$ on the last line, we get that the last line is equivalent to

$$\langle M, \in, U\rangle \vDash \{\{\xi_1, \dots, \xi_{n-1}\} \in [\kappa]^{n-1} : \\ \{\xi < \bigcup U : \phi[x_0, \dots x_k, \xi_1, \dots, \xi_{n-1}, \xi]\} \in U\} \in U^{n-1}.$$

By lemma 2.13, since $\kappa = \bigcup U$, this is equivalent with

$$\langle M, \in U\rangle \vDash \{\{\xi_1, \dots, \xi_n\} \in [\kappa]^n : \phi[x_0, \dots, x_k, \xi_1, \dots, \xi_n]\} \in U^n.$$

Thus, the claim holds for $n$. □

We need one more concept before we can prove that countable completeness implies iterability.

**Definition 2.16.** [17]

We call $\langle M, \in, U\rangle$ *countably iterable* if for any countable $\langle N, \in, W\rangle$ such that $W$ is an $N$-ultrafilter and $\langle N, \in, W\rangle$ is $\prec^-$-embeddable into $\langle M, \in, U\rangle$, the length of the iteration of $\langle N, \in, W\rangle$ is at least $\omega_1$.

**Lemma 2.17.** [18] *If $U$ is countably complete, then $\langle M, \in, U\rangle$ is countably iterable.*

*Proof.* Suppose that $W$ is an $N$-ultrafilter over $\lambda$, $\langle N, \in, W\rangle$ is countable and $e$ is an $\prec^-$-embedding of $\langle N, \in, W\rangle$ into $\langle M, \in, U\rangle$. We need to show that the iteration $\langle N_\alpha, W_\alpha, \lambda_\alpha, j_{\alpha\beta}\rangle_{\alpha\leq\beta\in\tau}$ of $\langle N, \in, W\rangle$ has length $\geq \omega_1$, i.e., that $\tau \geq \omega_1$.

We show by induction on $\alpha < \omega_1$ that:

(i) $\langle N_\alpha, \in, W_\alpha\rangle$ is defined

(ii) there is $e_\alpha : \langle N_\alpha, \in, W_\alpha\rangle \prec^- \langle M, \in, U\rangle$ such that $e_\gamma = e_\alpha \circ j_{\gamma\alpha}$ for $\gamma < \alpha$.

For $\alpha = 0$, set $e_0 = e$. To prove the induction step from $\alpha$ to $\alpha + 1$, suppose that (i) and (ii) have been proved for all $\beta \leq \alpha$. Then, for any $X \in W_\alpha$, we have $e_\alpha(X) \in U$, and since $\langle N, \in, W\rangle$ is countable, $\langle N_\alpha, \in, W_\alpha\rangle$ is countable by lemma 2.9(d). Hence, the countable completeness of $U$ implies that $\bigcap\{e_\alpha(X) : X \in W_\alpha\}$ is nonempty, so there is some $\eta$ in it. Define an embedding of the ultrapower $\langle N_\alpha^{\lambda_\alpha}/W_\alpha, E_{W_\alpha}, \dot{U}_\alpha\rangle$ into $\langle M, \in, U\rangle$ by $j[f]_{W_\alpha} = e_\alpha(f)(\eta)$.

[17]Definition on p. 254 of [11].
[18]We follow Lemma 19.11 of [11].

We show that $j$ is an elementary embedding. Since $e_\alpha$ is elementary, for any $\xi < \lambda_\alpha$ and $f \in N_\alpha^{\lambda_\alpha} \cap N_\alpha$, $\langle N_\alpha, \in\rangle \models \phi[f(\xi)]$ if and only if $\langle M, \in\rangle \models \phi[e_\alpha(f)(e_\alpha(\xi))]$. Thus we have

$$\begin{aligned} \langle N_\alpha^{\lambda_\alpha}/W_\alpha, E_{W_\alpha}\rangle \models \phi[[f]_{W_\alpha}] \quad &\text{iff} \quad \{\xi < \lambda_\alpha : \langle N_\alpha, \in\rangle \models \phi[f(\xi)]\} \in W_\alpha \\ &\text{iff} \quad \eta \in \{e_\alpha(\xi) < \kappa : \langle M, \in\rangle \models \phi[e_\alpha(f)(e_\alpha(\xi))]\} \\ &\text{iff} \quad \langle M, \in\rangle \models \phi[e_\alpha(f)(\eta)]. \end{aligned}$$

Now $\langle N_\alpha^{\lambda_\alpha}/W_\alpha, E_{W_\alpha}, \dot{U}_\alpha\rangle$ is well-founded because it is embeddable into a well-founded structure. Hence, $\langle N_{\alpha+1}, \in, W_{\alpha+1}\rangle$ is defined as its transitive collapse and $j \circ \pi^{-1}$, where $\pi$ is the collapsing map, gives an embedding $e_{\alpha+1} : \langle N_{\alpha+1}, \in, W_{\alpha+1}\rangle \prec^- \langle M, \in, U\rangle$. Moreover, since $e_{\alpha+1}(j_{\alpha,\alpha+1}(x)) = j[c_x] = e_\alpha(c_x)(\eta) = e_\alpha(x)$ for every $x \in N_\alpha$, we have $e_\alpha = e_{\alpha+1} \circ j_{\alpha,\alpha+1}$. Hence, for every $\gamma < \alpha + 1$, $e_\gamma = e_\alpha \circ j_{\gamma\alpha} = e_{\alpha+1} \circ j_{\gamma,\alpha+1}$. So (i) and (ii) hold for $\alpha + 1$ as well.

Suppose then that $\delta < \omega_1$ is a limit ordinal and (i) and (ii) hold for all $\gamma < \delta$. By Lemma 2.6, the direct limit of

$$\langle\langle\langle N_\alpha, \in, W_\alpha\rangle : \alpha < \beta\rangle, \langle j_{\alpha\beta} : \alpha \le \beta\rangle\rangle$$

is $\prec^-$-embeddable into $\langle M, \in, U\rangle$ due to the embeddings $e_\alpha$, $\alpha < \delta$. Hence, the direct limit is well-founded, and $\langle N_\delta, \in, W_\delta\rangle$ can be defined as its transitive collapse. The embedding of the direct limit composed with the inverse of the collapsing function gives an embedding $e_\delta : \langle N_\delta, \in, W_\delta\rangle \prec^- \langle M, \in U\rangle$. By Lemma 2.6, $e_\gamma = e_\delta \circ j_{\gamma\delta}$ for all $\gamma < \delta$. Thus, (i) and (ii) hold for $\delta$. □

We now prove that countable iterability implies iterability, a result first proved by Gaifman.

**Lemma 2.18.** [19] *If $\langle M, \in U\rangle$ is countably iterable, then it is iterable.*

*Proof.* Suppose that $\langle M, \in, U\rangle$ is not iterable, i.e., $\tau \in \mathrm{On}$. If $\tau$ is a successor, say $\tau = \gamma + 1$, let $\langle M_\tau, \in, U_\tau\rangle$ denote the ultrapower $\langle M_\gamma^{\kappa_\gamma}/U_\gamma, E_{U_\gamma}, \dot{U}_{U_\gamma}\rangle$ of $\langle M_\gamma, \in, U_\gamma\rangle$ and let $i_\tau : \langle M, \in, U\rangle \prec^- \langle M_\tau, \in, U_\tau\rangle$ be the natural embedding into the ultrapower composed with $i_{0\gamma}$.

If $\tau$ is limit, let $\langle M_\tau, \in, U_\tau\rangle$ denote the direct limit of the iteration of $\langle M, \in, U\rangle$ and let $i_\tau : \langle M, \in, U\rangle \prec^- \langle M_\tau, \in, U_\tau\rangle$ be the embedding given by the direct limit.

In either case, $\langle M_\tau, \in, U_\tau\rangle$ is ill-founded by assumption. However, the proofs of Lemmas 2.11 and 2.15 work also with $\alpha = \tau$, $\langle M_\tau, \in, U_\tau\rangle$ substituted for $\langle M_\alpha, \in, U_\alpha\rangle$ and $i_\tau$ substituted for $i_{0\alpha}$. If $\tau = \gamma + 1$, $\kappa_\gamma$ needs to be replaced by $[id]_{\kappa_\gamma}$.

[19] Lemma 19.12 of [11].

So suppose $x_k \in M_\tau$, $k \in \omega$, are such that $x_{k+1} \, E_\tau \, x_k$ for all $k \in \omega$. By Lemma 2.11 at $\tau$, for each $k$, $x_k = i_\tau(f_k)(\kappa_{\gamma_1^k}, \ldots, \kappa_{\gamma_{n(k)}^k})$ for some $n(k) \in \omega$, $f_k \in M^{[\kappa]^{n(k)}} \cap M$ and $\gamma_1^k < \cdots < \gamma_{n(k)}^k < \tau$. Taking the transitive collapse of the Skolem hull of $\{f_k : k \in \omega\}$ in $\langle M, \in, U\rangle$, we get a countable $\langle N, \in, W\rangle$ where $W$ is an $N$-ultrafilter. The inverse of the collapsing map is an embedding $e : \langle N, \in, W\rangle \prec^- \langle M, \in, U\rangle$. By the construction of $N$, for each $k$ there is $f'_k \in N$ such that $e(f'_k) = f_k$.

Since $\langle M, \in, U\rangle$ is countably iterable, the iteration $\langle N_\alpha, W_\alpha, \lambda_\alpha, j_{\alpha\beta}\rangle_{\alpha \leq \beta \in \sigma}$ of $\langle N, \in, W\rangle$ has length $\geq \omega_1$ by the previous lemma. Let $\zeta < \omega_1$ be the order type of the set $S = \{\gamma_m^k : 1 \leq m \leq n(k),\, k \in \omega\}$ and let $h : S \to \zeta$ be the unique order-preserving function. For all $k \in \omega$, define

$$\begin{aligned} &\delta_m^k = h(\gamma_m^k) \text{ for } 1 \leq m \leq n(k), \text{ and} \\ &x'_k = j_{0\zeta}(f'_k)(\lambda_{\delta_1^k}, \ldots, \lambda_{\delta_{n(k)}^k}). \end{aligned}$$

For all $k \in \omega$, let $\phi_\kappa(v_0, v_1, \ldots)$ be a formula such that $\phi_k[i_\tau(f_k), i_\tau(f_{k+1}), \kappa_1^k, \ldots, \kappa_{m(k)}^k]$, where $\kappa_1^k, \ldots, \kappa_{m(k)}^k$ list the elements of the set $\{\kappa_{\gamma_1^k}, \ldots, \kappa_{\gamma_{n(k)}^k}, \kappa_{\gamma_1^{k+1}}, \ldots, \kappa_{\gamma_{n(k+1)}^{k+1}}\}$ in increasing order, says that $i_\tau(f_{k+1})(\kappa_{\gamma_1^{k+1}}, \ldots, \kappa_{\gamma_{n(k+1)}^{k+1}}) \in i_\tau(f_k)(\kappa_{\gamma_1^k}, \ldots, \kappa_{\gamma_{n(k)}^k})$. By Lemma 2.15 at $\tau$, for every $k \in \omega$,

$$\begin{aligned} x_{k+1} E_\tau x_k \ &\text{iff } \langle M_\tau, E_\tau, U_\tau\rangle \vDash \phi_k[i_\tau(f_k), i_\tau(f_{k+1}), \kappa_1^k, \ldots, \kappa_{m(k)}^k] \\ &\text{iff } \langle M, \in, U\rangle \vDash \{\{\xi_1, \ldots, \xi_{m(k)}\} \in [\kappa]^{m(k)} : \phi_k[f_k, f_{k+1}, \xi_1, \ldots, \xi_{m(k)}]\} \in U^{m(k)} \\ &\text{iff } \langle N, \in, W\rangle \vDash \{\{\xi_1, \ldots, \xi_{m(k)}\} \in [\lambda]^{m(k)} : \phi_k[f'_k, f'_{k+1}, \xi_1, \ldots, \xi_{m(k)}]\} \in W^{m(k)} \\ &\text{iff } \langle N_\zeta, \in, W_\zeta\rangle \vDash \phi_k[i_\zeta(f'_k), i_\zeta(f'_{k+1}), \lambda_1^k, \ldots, \lambda_{m(k)}^k] \\ &\text{iff } x'_{k+1} \in x'_k, \end{aligned}$$

where on the second last line, $\lambda_1^k, \ldots, \lambda_{m(k)}^k$ list the elements of the set $\{\lambda_{\delta_1^k}, \ldots, \lambda_{\delta_{n(k)}^k}, \lambda_{\delta_1^{k+1}}, \ldots, \lambda_{\delta_{n(k+1)}^{k+1}}\}$ in increasing order. Hence, $x'_{k+1} \in x'_k$ for each $k$, which is a contradiction since $\langle N_\zeta, \in, W_\zeta\rangle$ is well-founded. □

The two preceding lemmas show that the countably completeness of $U$ is a sufficient condition for $\langle M, \in, U\rangle$ to be iterable.

# Chapter 3

# Relative constructibility and $L[U]$

In this chapter we introduce the definition and basic properties of a generalization of $L$ developed by Azriel Lévy[1]. For a class or set $A$, he defined the inner model $L[A]$ of sets constructible relative to $A$. This is the smallest inner model $M$ such that for every $x \in M$, $x \cap A \in M$. $L[A]$ has many of the same or analogous properties as $L$. Unlike in $L$, $GCH$ is generally true in $L[A]$ only for sufficiently large cardinals. Whether it holds for all cardinals depends on $A$.

In this thesis, the most important inner model obtained from relative constructibility will be $L[U]$. That is constructed from a $\kappa$-complete ultrafilter $U$ over a measurable cardinal $\kappa$. $L[U]$ satisfies $GCH$ for all cardinals and in $L[U]$, $\kappa$ is the only measurable cardinal. Our proofs in this chapter follow Kanamori [11].

## 3.1 Relative constructibility

The idea of the definition of $L[A]$ is that in the definition of the successor level we can make assertions about membership in $A$ of sets defined so far[2]. We let

$$\mathrm{def}_A(x) = \{y \subset x : y \text{ is definable over } \langle x, \in, A \cap x \rangle\}$$

[1]Presented in 1960 in [14].
[2]We follow here pp. 34-35 of [11].

where $A \cap x$ is considered a unary predicate and can be used in the defining formulas. $L[A]$ is defined recursively in analogy with $L$:

$$\begin{aligned} L_0[A] &= \emptyset, \\ L_{\alpha+1}[A] &= \mathrm{def}_A(L_\alpha[A]), \\ L_\delta[A] &= \bigcup_{\alpha<\delta} L_\alpha[A] \text{ for limit } \delta, \\ L[A] &= \bigcup_{\alpha\in\mathrm{On}} L_\alpha[A]. \end{aligned}$$

Like $L$, $L[A]$ is a model of $ZFC$ and has a definable well-ordering:

**Lemma 3.1.** [3] *There is a formula $\phi_1(v_0, v_1)$ of the language $\mathcal{L}_\in(\dot{A})$ such that it defines a well-ordering $<_{L[A]}$ of $L[A]$ in any transitive $\langle L[A], \in, A \cap L[A]\rangle$ and for any limit $\delta > \omega$, any $y \in L_\delta[A]$ and any $x$,*

$$x <_{L[A]} y \textit{ iff } x \in L_\delta[A] \textit{ and } \langle L_\delta[A], \in, A \cap L_\delta[A]\rangle \models \phi(x, y).$$

The existence of the sentence $\sigma$ in the following lemma implies that $L[A]$ satisfies the condensation lemma: if $\langle X, A \cap X\rangle$ is an elementary submodel of $L_\alpha[A]$, then there is a limit $\beta \leq \alpha$ such that $X = L_\beta[A]$. The proofs of these facts and the preceding lemma are analogous to the corresponding proofs for $L$ that can be found, e.g., in [2].

**Lemma 3.2.** [4] *There is a sentence $\sigma$ of $\mathcal{L}_\in(\dot{A})$ with $\dot{A}$ unary such that for any $A$ and any transitive class $N$,*

$$\langle N, \in, A \cap N\rangle \models \sigma \quad \textit{iff} \quad N = L[A] \textit{ or } N = L_\delta[A] \textit{ for some limit } \delta > \omega.$$

The following lemma shows that $L[A]$ satisfies $GCH$ for sufficiently large cardinals. Whether $L[A]$ satisfies $GCH$ for all cardinals depends on $A$.

**Lemma 3.3.** [5] *Suppose $V = L[A]$ and $\lambda$ is a cardinal such that $A \subset \mathcal{P}(\lambda)$. Then $2^\lambda = \lambda^+$.*

*Proof.* We will show that $\mathcal{P}(\lambda) \cap (L[A]) \subset L_{\lambda^+}[A]$. This suffices since $|L_{\lambda^+}[A]| = \lambda^+$.

Suppose that $x \in \mathcal{P}(\lambda) \cap L[A]$. Let $\gamma > \lambda$ be a limit ordinal such that $x$ and $A$ are members of $L_\gamma[A]$. By the Löwenheim-Skolem theorem there is an elementary submodel $\langle H, \in, A\cap H\rangle \prec \langle L_\gamma[A], \in, A\rangle$ such that $\lambda \cup \{x, A\} \subset H$ and $|H| = \lambda$. Let $\langle N, \in, W\rangle$ be the transitive collapse of $\langle H, \in, A\cap H\rangle$ and let $\pi$ be the collapsing isomorphism. Since $\lambda \subset H$, $\pi(y) = y$ for every $y \in \mathcal{P}(\lambda) \cap H$. Hence, $\pi(x) = x$ and $W = \pi``(A \cap H) = A \cap N$. Since $\langle N, \in, W\rangle$ is elementarily equivalent to $\langle L_\gamma[A], \in, A\rangle$, it satisfies the sentence $\sigma$ of Lemma 3.2, so $N = L_\delta[A]$ for some $\delta$. Since $|N| = \lambda$, we have $\delta < \lambda^+$, so $x \in L_\delta[A] \subset L_{\lambda^+}[A]$. □

[3] The analogous result for $L$ can be found e.g. on p. 74 of [2].

[4] The analogous result for $L$ can be found e.g. on p. 70 of [2].

[5] We follow Lemma 20.2(a) of [11].

## 3.2 $L[U]$

Suppose there is a measurable cardinal $\kappa$ and $U$ is a $\kappa$-complete measure on $\kappa$. $L[U]$ is the inner model of sets constructible relative to $U$. It is sometimes denoted $L^\mu$ when the measure is denoted by $\mu$. In this section we present the most important properties of $L[U]$

A fundamental feature of $L[U]$ is that $\kappa$ is measurable in the sense of $L[U]$.

**Lemma 3.4.** [6] *Let $\bar{U} = U \cap L[U]$. Then $L[U] \models$ "$\bar{U}$ is a $\kappa$-complete ultrafilter on $\kappa$". Moreover, if $U$ is normal, then $L[U] \models$ "$\bar{U}$ is normal".*

*Proof.* The proof is straightforward since $X \in L[U]$ is in $\bar{U}$ if and only if $X \in U$. Suppose $X \subset \kappa$ is in $L[U]$. Then $\kappa \setminus X \in L[U]$ so either $X \in \bar{U}$ or $\kappa \setminus X \in \bar{U}$. Suppose $X \subset Y \subset \kappa$, $X \in \bar{U}$ and $Y \in L[U]$. Then if $Y$ is not in $\bar{U}$, $\kappa \setminus Y$ is in $\bar{U}$, so $\kappa \setminus Y \in U$. But then $X \cap (\kappa \setminus Y) = \emptyset$ is in $U$, a contradiction. Hence, $Y \in \bar{U}$.

If $\{X_\alpha : \alpha < \lambda < \kappa\}$ is in $L[U]$, then $A = \bigcap\{X_\alpha : \alpha < \lambda\}$ is in $\mathcal{P}(\kappa) \cap L[U]$. If $A \notin \bar{U}$, then $\kappa \setminus A$ is in $\bar{U}$. But then $\kappa \setminus A \in U$, so $A \notin U$, contradiction. Hence, $A$ is in $\bar{U}$. If $U$ is normal and $f \in L[U]$ is a regressive function on $\kappa$, then there is $\gamma < \kappa$ such that the set $X = \{\alpha : f(\alpha) = \gamma\}$ is in $U$. $X$ is in $L[U]$ by separation, so $L[U] \models$" $f$ is constant on some $X \in \bar{U}$". □

We start proving a few of the important properties of $L[U]$.

**Theorem 3.5.** [7] *If $V = L[U]$, then $\kappa$ is the only measurable cardinal.*

*Proof.* Suppose for reductio that there is a measurable cardinal $\lambda \neq \kappa$. Let $W$ be a $\lambda$-complete ultrafilter over $\lambda$ and let $j : V \prec M \cong Ult(V, W)$ be the canonical embedding, where $Ult(V, W)$ is the ultrapower of $V$ by $W$. Since $V$ is the class $L[U]$, by elementarity and Lemma 3.2, $M = L[j(U)]^M$. Since $j(U)$ is in $M$ and $M$ is an inner model, $L[j(U)]^M$ is the same as $L[j(U)]$. We will show that $M = L[U]$ which is a contradiction since by Lemma 2.3(d), $W \notin M$.

If $\lambda > \kappa$, then $j(U) = U$, so we get the contradiction. So suppose $\lambda < \kappa$. The normality of $U$ implies that every club in $\kappa$ is in $U$, so

$$E = \{\alpha < \kappa : \alpha > \lambda \text{ and } \alpha \text{ is inaccessible}\} \in U.$$

By Lemma 2.12(c), $j(\alpha) = \alpha$ for all $\alpha \in E$ and $j(\kappa) = \kappa$. From this it follows that $j(U) = U \cap M$. Suppose $X \in j(U)$ and let $X = [f]_W$ for $f \in U^\lambda$. Then $Y = \bigcap_{\xi<\lambda} f(\xi)$ is in $U$ by $\kappa$-completeness and $j(Y) \subset X$ by Łoś's theorem. Now $j(Y) \supset j``(Y \cap E) = Y \cap E \in U$ because $j$ is the identity on $E$. Since $j(Y) \subset \kappa$, $j(Y) \in U \cap M$, so as $X$ is a subset of $\kappa$,

[6] Lemma 19.1 of [9]. Normality is proved by Jech, the rest of the proof is our own.

[7] We follow Lemma 20.2(b) of [11].

$X \in U \cap M$. Hence, $j(U) \subset U \cap M$, which implies that $j(U) = U \cap M$ since $j(U)$ is an ultrafilter on $\kappa$ in $M$. Since for any $A$, $L[A] = L[A \cap L[A]]$, we get that

$$M = L[j(U)] = L[U \cap M] = L[U],$$

which concludes the proof. $\square$

To prove $GCH$ in $L[U]$, we need a lemma that uses the following concept.

**Definition 3.6.** [8] Suppose $F$ is a filter over a cardinal $\kappa$ and $\omega < \nu < \kappa$. $F$ is called $\nu$-Rowbottom if for any function $f : [\kappa]^{<\omega} \to \gamma$ with $\gamma < \kappa$, there is a set $H \in F$ such that $|f``[H]^{<\omega}| < \nu$.

By Lemma 2.14 every normal measure $U$ on a measurable cardinal $\kappa$ is $\nu$-Rowbottom for any $\omega < \nu < \kappa$. Another concept needed in the following lemma is that of a *complete set of Skolem functions.* For a model $\mathcal{M}$ of language $\mathcal{L}$, a complete set of Skolem functions is the closure under functional composition of any set $\{f_\phi : \phi \text{ a formula of } \mathcal{L}\}$, where each $f_\phi$ is a Skolem function for $\phi$. Such a set has size $|\mathcal{L}|$. For $X \subset M$ and a complete set of Skolem functions $\{f_\alpha : \alpha < |\mathcal{L}|\}$, the *Skolem hull* of $X$ is $\bigcup_{\alpha<|\mathcal{L}|} f_\alpha``[X]^{k(\alpha)}$ where $k(\alpha)$ is the arity of $f_\alpha$. The Skolem Hull is an elementary submodel of $\mathcal{M}$ by the Tarski-Vaught criterion.

**Lemma 3.7.** [9] *Suppose $U$ is a normal measure on a measurable cardinal $\kappa$ and $\lambda^+ > \omega$ is a successor cardinal smaller than $\kappa$. Suppose that $\mathcal{A} = \langle L_\gamma[U], R, \dots \rangle$, where $\gamma$ is a limit ordinal greater than $\kappa$ and $R$ is a subset of $L_\gamma[U]$, is a structure for a first-order language of cardinality $\lambda$ containing a constant symbol $c_\alpha$ with $c_\alpha^{\mathcal{A}} = \alpha$ for all $\alpha < \lambda$, and suppose $|R| = \lambda^+$. Then there is an elementary submodel $\langle B, R \cap B, \dots \rangle \prec \langle L_\gamma[U], R, \dots \rangle$ such that $|B| = \kappa$, $\lambda \subset B$, $B \cap \kappa \in U$, and $|R \cap B| \leq \lambda$.*

*Proof.* Let $R = \{r_i : i < \lambda^+\}$. Let $\{h_\alpha : \alpha < \lambda\}$, be a complete set of Skolem functions for the language and let each $h_\alpha$ be $k(\alpha)$-ary. Define the functions $f'$ and $f$ with domain $[\kappa]^{<\omega}$ by

$$f'(\xi_1, \dots, \xi_n) = \{i : \alpha < \lambda,\ n = k(\alpha) \text{ and } h_\alpha(\xi_1, \dots, \xi_n) = r_i\},$$
$$f(\xi_1, \dots, \xi_n) = \sup\,(f'(\xi_1, \dots, \xi_n)).$$

Since $\lambda^+$ is regular, each value of $f$ is smaller than $\lambda^+$, so $\operatorname{ran}(f) \subset \lambda^+$. Since $U$ is $\nu$-Rowbottom, there is $H \in U$ such that $|f``[H]^{<\omega}| < \lambda^+$. Let $B = \bigcup_{\alpha<\lambda} h_\alpha``[H]^{k(\alpha)}$ be the Skolem hull of $H$. Then $\langle B, R \cap B, \dots \rangle$ is an elementary submodel of $\langle L_\gamma[U], R, \dots \rangle$ and $|R \cap B| \leq |\bigcup f``[H]^{<\omega}|$. Since every value of $f$ is smaller than $\lambda^+$, the regularity of $\lambda^+$ implies that $|\bigcup f``[H]^{<\omega}| \leq \lambda$, so $|R \cap B| \leq \lambda$. Since the language contains constant symbols for all $\alpha < \lambda$, $\lambda \subset B$. Moreover, clearly $|B| = \kappa$ and $H \subset B$, so $B \cap \kappa \in U$. $\square$

[8] This definition is found on p. 218 in [11].

[9] We adapt the proof of Lemma 8.4 of [11].

Now we can prove $GCH$ in $L[U]$.

**Theorem 3.8.** [10] *If $V = L[U]$, then $GCH$ holds.*

*Proof.* By Lemma 3.3 we only need to show that $GCH$ holds below $\kappa$. Suppose $\lambda < \kappa$. Let $<_{L[U]}$ be the well-ordering of $L[U]$. We will prove that $<_{L[U]} \restriction (\mathcal{P}(\lambda) \times \mathcal{P}(\lambda))$ has order type $\leq \lambda^+$. That means that for any $y \in \mathcal{P}(\lambda) \cap L[U]$,

$$|\{x \in \mathcal{P}(\lambda) \cap L[U] : x <_{L[U]} y\}| \leq \lambda.$$

Suppose that this does not hold, i.e., there is $y \in \mathcal{P}(\lambda) \cap L[U]$ such that $R := \{x \in \mathcal{P}(\lambda) \cap L[U] : x <_{L[U]} y\}$ has size $\lambda^+$. Let $\gamma$ be a limit ordinal greater than $\kappa$ such that $y$ and $U$ are in $L_\gamma[U]$ and let $\mathcal{A} = \langle L_\gamma[U], \in, U, R, \{y\}\rangle$. By augmenting the language of $\mathcal{A}$ with constant symbols for all $\alpha < \lambda$ and taking the reduct of the elementary submodel given by Lemma 3.7 we get an elementary submodel

$$\mathcal{B} = \langle B, \in, U \cap B, R \cap B, \{y\}\rangle \prec \mathcal{A}.$$

such that $|R \cap B| \leq \lambda$, $\lambda \subset B$ and $B \cap \kappa \in U$. Let $\langle N, \in, W\rangle$ be the transitive collapse of $\langle B, \in, U \cap B\rangle$ and let $\pi$ be the collapsing isomorphism. Since $\lambda \subset B$, $\pi(x) = x$ for all $x \in \mathcal{P}(\lambda) \cap B$, so $y \in N$ and $R \cap N = R \cap B$.

We show that $W = U \cap N$. Because $\pi$ is injective, $B \cap \kappa \in U$ and $\pi(\xi) \leq \xi$ for any $\xi \in B$, the set $E = \{\xi \in B \cap \kappa : \pi(\xi) = \xi\}$ is in $U$. Otherwise, the set $\{\xi \in B \cap \kappa : \pi(\xi) < \xi\}$ would be in $U$, so by normality there would be $Z \in U$ such that $\pi$ is constant on $Z$. For any $x \in N$, let $x' \in B$ be such that $x = \pi(x')$. Since $W = \pi``(U \cap B)$, we have for any $x \in N$:

$$\begin{aligned} x \in W &\text{ iff } \exists D \in U\,(D \cap E \subset x') \\ &\text{ iff } \exists D \in U\,(D \cap E \subset x) \\ &\text{ iff } x \in U. \end{aligned}$$

The middle step holds because $\pi$ is the identity on $D \cap E$. Hence, $W = U \cap N$.

But now by elementarity and Lemma 3.2 $N = L_\delta[U]$ for some limit $\delta$ and $R \subset N$ since $y \in N$. Thus we get the contradiction $|R| = |R \cap N| = |R \cap B| \leq \lambda$. □

We conclude the chapter with a result concerning the iterated ultrapowers of $L[U]$.

**Theorem 3.9.** [11] *Suppose $M = L[U]$, where $U$ is a normal ultrafilter on $\kappa$, and let $\langle M_\alpha, U_\alpha, \kappa_\alpha, i_{\alpha\beta}\rangle_{\alpha \leq \beta \in \tau}$ be the iteration of $M$. Let $\lambda$ be a regular cardinal with $\lambda > |\kappa^\kappa \cap M|$ and let $F$ be the filter generated by the end segments of $\{\kappa_\alpha : \alpha < \lambda\}$.*

*Then $M_\lambda = L[U_\lambda] = L[F]$ and $U_\lambda = F \cap L[F]$.*

[10] We follow Lemma 20.3 of [11].

[11] We follow Lemma 20.6 of [11].

*Proof.* By Lemma 2.12(b) we have $\kappa_\lambda = \lambda$ and by Lemma 2.10 $U_\lambda \subset F \cap M_\lambda$. Since $U_\lambda$ is a normal ultrafilter on $\mathcal{P}(\lambda) \cap M_\lambda$, $U_\lambda = F \cap M_\lambda$. Since $M_\lambda$ satisfies the sentence $\sigma$ of Lemma 3.2, $M_\lambda = L[U_\lambda] = L[F]$. $\square$

# Chapter 4

# Prikry forcing

Prikry forcing is notion of forcing for measurable cardinals developed by Karel Prikry [1]. When Prikry forcing is defined for a measurable cardinal $\kappa$, the forcing extension preserves all cardinals and all cofinalities except the cofinality of $\kappa$ which becomes $\omega$ in the forcing extension. Another important property is the connection to iterated ultrapowers. For a model $M$ with an iterable $M$-ultrafilter, the sequence of critical points of the iterates forms a Prikry generic sequence in the $\omega$-th iterated ultrapower, i.e., the sequence $\{\kappa_n : n < \omega\}$ generates a $M_\omega$-generic filter $G$ for the Prikry forcing defined from $\kappa_\omega$. In this chapter we give the definition of Prikry forcing and present the proofs for the central properties mentioned above. Our presentation follows closely Jech's textbook [9] but we present many proofs in more detail.

## 4.1 Definition and the basic properties

**Definition 4.1.** [2] Let $\kappa$ be a measurable cardinal and let $D$ be a normal measure on $\kappa$. Prikry forcing defined from $D$ is the following notion of forcing: The forcing conditions are pairs $p = (s, A)$, where $s \in [\kappa]^{<\omega}$, that is, $s$ is a finite subset of $\kappa$, and $A \in D$. A condition $(s, A)$ is stronger than a condition $(t, B)$ if the following hold:

(i) $t$ is an initial segment of $s$, that is, $t = s \cap \alpha$ for some $\alpha \in \kappa$

(ii) $A \subset B$

(iii) $s - t \subset B$

We first show that the Prikry forcing defined from a normal measure on $\kappa$ preserves all cardinals and cofinalities above $\kappa$ and changes the cofinality of $\kappa$ to $\omega$.

[1] Prikry forcing was introduced in [16].

[2] Definition part of Theorem 21.10 of [9].

**Lemma 4.2.** [3] *Suppose $\kappa$ is a measurable cardinal, $D$ is a normal measure on $\kappa$ and $(P,<)$ is the Prikry forcing defined from $D$. Let $G$ be a generic filter on $P$. Then in $V[G]$ $cf(\kappa)=\omega$ and all cardinals and cofinalities above $\kappa$ are preserved.*

*Proof.* Any two conditions $(s,A)$ and $(s,B)$ with the same first coordinate are compatible since $(s,A\cap B)$ is stronger than either of the two. Thus, any antichain $W\subset P$ has size at most $\kappa$ because $|[\kappa]^{<\omega}|=\kappa$. Thus, $P$ satisfies the $\kappa^+$-chain condition, so all cofinalities and cardinals above $\kappa$ are preserved.

We also notice that if $(s,A)$ and $(t,B)$ are compatible then either $s$ is an initial segment of $t$ or $t$ is an initial segment of $s$. This is because if $(u,C)\le(s,A)$ and $(u,C)\le(t,B)$ then both $s$ and $t$ are initial segments of $u$. Since $G$ is a filter, every two conditions in $G$ are compatible, so $S:=\bigcup\{s:(s,A)\in G \text{ for some } A\}$ is a subset of $\kappa$ of order type $\omega$. It is also easy to see that $S$ is an unbounded subset of $\kappa$ and, therefore, $\kappa$ has cofinality $\omega$ in $V[G]$. For each $\alpha<\kappa$ and condition $(s,A)\in P$, define the condition $(s,A)_\alpha$ as follows: If $s-\alpha\neq\emptyset$, let $(s,A)_\alpha=(s,A)$. If $s-\alpha=\emptyset$, let $s_\alpha=s\cup\{\min(A-\alpha)\}$ and let $(s,A)_\alpha=(s_\alpha,A)$. Define for all $\alpha<\kappa$ the dense set $D_\alpha=\{(s,A)_\alpha:(s,A)\in P\}$. Since $G$ is generic, $D_\alpha\cap G\neq\emptyset$ for all $\alpha<\kappa$, so $G$ is unbounded in $\kappa$.

□

Next we prove the lemmas needed to show that $V[G]$ preserves all the cardinals smaller or equal to $\kappa$ and the cofinality of every ordinal below $\kappa$.

**Lemma 4.3.** [4] *Suppose $\sigma$ is a sentence of the forcing language. There exists a set $A\in D$ such that the condition $(\emptyset,A)$ decides $\sigma$, that is, either $(\emptyset,A)\Vdash\sigma$ or $(\emptyset,A)\Vdash\neg\sigma$.*

*Proof.* Define $S^+=\{s\in[\kappa]^{<\omega}:(s,X)\Vdash\sigma \text{ for some X } \in D\}$ and $S^-=\{s\in[\kappa]^{<\omega}:(s,X)\Vdash\neg\sigma \text{ for some X } \in D\}$ and let $T=[\kappa]^{<\omega}-(S^+\cup S^-)$. If $s\in S^+\cap S^-$, then there are $X$ and $Y$ such that $(s,X)\Vdash\sigma$ and $(s,Y)\Vdash\neg\sigma$. But then $(s,X\cap Y)\le(s,X)$ and $(s,X\cap Y)\le(s,Y)$ so $(s,X\cap Y)\Vdash\sigma$ and $(s,X\cap Y)\Vdash\neg\sigma$, a contradiction. Thus, $S^+$ and $S^-$ are disjoint. By Lemma 2.14, there is $A\in D$ such that for every $n$, either $[A]^n\subset S^+$ or $[A]^n\subset S^-$ or $[A]^n\subset T$. We show that $(\emptyset,A)$ decides $\sigma$.

If $(\emptyset,A)$ does not decide $\sigma$, then there are conditions $(s,X),(t,Y)\le(\emptyset,A)$ such that $(s,X)$ forces $\sigma$ and $(t,Y)$ forces $\neg\sigma$. By extending, if necessary, one of $s$ or $t$ by elements from $X$ or $Y$, respectively, we can assume that $|s|=|t|=n$ for some $n$. Thus, $s\in S^+\cap[A]^n$ and $t\in S^-\cap[A]^n$ so $S^+\cap[A]^n\neq\emptyset$ and $S^-\cap[A]^n\neq\emptyset$, a contradiction. Hence, $(\emptyset,A)$ decides $\sigma$. □

**Lemma 4.4.** [5] *Suppose $\sigma$ is a sentence of the forcing language and $(s_0,A_0)$ is a condition. Then there exists a set $A\subset A_0$ in $D$ such that the condition $(s_0,A)$ decides $\sigma$.*

[3] We follow Theorem 21.10 of [9].

[4] Lemma 21.11 of [9].

[5] We follow Lemma 21.12 of [9].

*Proof.* The proof is similar to the proof of the preceding lemma. We define $S^+ = \{s \in [A_0 - \max(s_0)]^{<\omega} : (s_0 \cup s, X) \Vdash \sigma \text{ for some } X \subset A_0\}$ and $S^- = \{s \in [A_0 - \max(s_0)]^{<\omega} : (s_0 \cup s, X) \Vdash \neg\sigma \text{ for some } X \subset A_0\}$ and let $T = [A_0 - \max(s_0)]^{<\omega} - (S^+ \cup S^-)$. As in the preceding proof, there is some $A \subset (A_0 - max(s_0))$ in $D$ such that for all $n$, $[A]^n$ does not intersect both $S^+$ and $S^-$. The same argument as above shows that $(s_0, A)$ decides $\sigma$. □

We have shown in Lemma 4.2 that the Prikry forcing preserves all cardinals above $\kappa$ and $\kappa$ has cofinality $\omega$ in the forcing extension. With the previous lemma we can show that Prikry forcing preserves all the cardinals less or equal to $\kappa$ as well.

The proof is based on the fact that in the Boolean-value approach to forcing we can add a name $\check{M}$ for the ground model. Then we can define that a condition $p$ forces that $\dot{a} \in \check{M}$ for a name $\dot{a}$ if and only if $p \leq \sum_{x \in M}[[\dot{a} = \check{x}]]$ which is equivalent to $\forall q \leq p \, \exists r \leq q \, \exists x \in M \, (r \Vdash \dot{a} = \check{x})$.

We show that the definition of $p \Vdash \dot{a} \in \check{M}$ gives $\check{M}$ the right interpretation[6]. If $\dot{a}^G = x$ for some generic $G$ and $x \in M$, then some condition $p \in G$ forces that $\dot{a} = \check{x}$ so $p \leq \sum_{x \in M}[[\dot{a} = \check{x}]]$. If $\dot{a}^G \notin M$ but some $p \in G$ forces $\dot{a} \in \check{M}$, then the set $S = \{r \leq p : \exists x \in M \, (r \Vdash \dot{a} = \check{x})\}$ is dense below $p$. Since $G$ is generic, there exists $q_0 \in G \cap S$. Then $q_0 \Vdash \dot{a} = \check{x}$ for some $x \in M$. On the other hand, since $\dot{a}^G \notin M$, there is $q_1 \in G$ such that $q_1 \Vdash \neg \dot{a} = \check{x}$. Since $G$ is a filter, there exists $r \leq q_0, q_1$. But then $r \Vdash \dot{a} = \check{x}$ and $r \Vdash \neg\dot{a} = \check{x}$ which is a contradiction. Hence, for any generic $G$ there is a condition $p$ in $G$ such that $p \Vdash \dot{a} \in \check{M}$ if and only if $\dot{a}^G = x$ for some $x \in M$, so the definition works.

**Theorem 4.5.** [7] *Suppose $\kappa$ is a measurable cardinal, $D$ is a normal measure on $\kappa$ and $(P, <)$ is the Prikry forcing defined from $D$. Let $G$ be a generic filter on $P$. Then in $V[G]$ all cardinals are preserved and all cofinalities except the cofinality of $\kappa$ are preserved.*

*Proof.* We show that if $X$ is a bounded subset of $\kappa$ in $V[G]$, then $X \in V$. So suppose $X \in V[G]$ and $X \subset \lambda < \kappa$ and let $\dot{X}$ be a name such that $\dot{X}^G = X$ and let $p$ be a condition such that $p \Vdash \dot{X} \subset \lambda$. By the definition of $p \Vdash \dot{a} \in \check{M}$ above, it is enough to show that for all $q \leq p$ there are $r \leq q$ and $Z \in V$ such that $r \Vdash \dot{X} = \check{Z}$.

Let $q \leq p$, say $q = (s, A)$. By Lemma 4.4, there is for each $\alpha < \lambda$ a set $A_\alpha \subset A$ such that $(s, A_\alpha)$ decides the sentence $\alpha \in \dot{X}$. Let $B = \bigcap_{\alpha<\lambda} A_\alpha$. Since $r = (s, B)$ is stronger than each $(s, A_\alpha)$, $r$ decides $\alpha \in \dot{X}$ for each $\alpha < \lambda$. Let $Z = \{\alpha < \lambda : r \Vdash \alpha \in \dot{X}\}$. Since $r \Vdash \alpha \in \dot{X}$ can be decided within $V$, $Z$ is in $V$ and we have $q \Vdash \dot{X} = \check{Z}$.

[6] The proof is our own.

[7] We follow Corollary 21.13 of [9].

Since all bounded subsets of $\kappa$ are in $V$, we can show by induction on rank that $V_\kappa^{V[G]} = V_\kappa$, so every cardinal below $\kappa$ is preserved. Since $\kappa$ is a limit cardinal, $\kappa$ is preserved as well. $\square$

## 4.2 Prikry sequences

We show that the sequence of critical points of iterated ultrapowers yields a Prikry generic sequence in the $\omega$-th iterate. When we say that a subset $S \subset \kappa$ is a Prikry generic sequence over $M$, we mean that the filter

$$G = \{(s, A) \in P : s \text{ is an initial segment of } S \text{ and } S - s \subset A\}$$

is $M$-generic.

For the next Theorem, which is due to Adrian Mathias [15], we also need a variant of the diagonal intersection. If $\{A_s : s \in [\kappa]^{<\omega}\}$ is a collection of subsets of $\kappa$, define

$$\Delta_s A_s = \{\alpha < \kappa : \alpha \in \bigcap \{A_s : \max(s) < \alpha\}\}. \tag{4.1}$$

It is easy to see that every normal ultrafilter $D$ on $\kappa$ is closed under these diagonal intersections[8]. Suppose $X_s \in D$ for every $s \in [\kappa]^{<\omega}$. Choose some well-ordering $<'$ of $[\kappa]^{<\omega}$ such that $\max(s) < \max(t)$ implies $s <' t$ and let $\{s_\alpha : \alpha < \kappa\}$ be an increasing enumeration of of $([\kappa]^{<\omega}, <')$. Then for each infinite cardinal $\lambda < \kappa$, $\max(s_\beta)$ is smaller than $\lambda$ if and only if $\beta < \lambda$. Setting $X'_\alpha = X_{s_\alpha}$ for each $\alpha < \kappa$, we see that for all infinite cardinals $\lambda < \kappa$, $\bigcap\{X_s : \max(s) < \lambda\} = \bigcap\{X'_\alpha : \alpha < \lambda\}$. Hence, $\Delta_s X_s \supset \Delta_{\alpha<\kappa} X'_\alpha \cap A$, where $A := \{\lambda < \kappa : \lambda \text{ is an infinite cardinal}\}$. Since $\kappa$ is a limit cardinal, the set $A$ is a club in $\kappa$. The normality of $D$ implies that $A \in D$ and $\Delta_{\alpha<\kappa} X'_\alpha \in D$, so $\Delta_s X_s \in D$.

Now we are ready to present the lemma.

**Theorem 4.6.** [9] *Suppose $M$ is a transitive model of ZFC, $U$ is a normal measure on $\kappa$ in $M$ and $P$ is the Prikry forcing defined from $U$. Then for every $S \subset \kappa$ of order type $\omega$, $S$ is Prikry generic over $M$ if and only if for every $X \in U$, $S \setminus X$ is finite.*

*Proof.* First, suppose that $G$ is a generic filter on $P$ and let $S = \bigcup\{s : (s, A) \in G\}$. Let $X \in U$ and pick any $(s, A) \in G$. The set $D_X = \{(t, B \cap X) : (t, B) \leq (s, A)\}$ is dense below $(s, A)$, so there is some $(t, B \cap X) \in D_X \cap G$. Now, every $\alpha \in S - t$ must be in $X$ because any two conditions in $G$ are compatible. Thus, $S - X$ is finite.

[8]The proof is our own.

[9]We follow Theorem 21.14 of [9].

For the other direction, suppose that $S \subset \kappa$ of order type $\omega$ is such that $S - X$ is finite for all $X \in U$. We want to show that the filter

$$G = \{(s, A) \in P : s \text{ is an initial segment of } S \text{ and } S - s \subset A\}$$

is $M$-generic. So suppose $D \in M$ is an open dense subset of $P$ and we will show that $G \cap D \neq \emptyset$.

For each $s \in [\kappa]^{<\omega}$, let $F_s : [\kappa]^{<\omega} \to \{0, 1\}$ be a partition such that $F_s(t) = 1$ if and only if $\max(s) < \max(t)$ and there is $X$ such that $(s \cup t, X) \in D$. By lemma 2.14 there is $A_s \in U$ that is homogeneous for $F_s$. If there is $X \in U$ such that $(s, X) \in D$, choose $X_s$ to be one such $X$ and let $B_s = A_s \cap X_s$, otherwise we let $B_s = A_s$. Let $A = \Delta_s B_s$ be the diagonal intersection as defined in (4.1). For any $s \in [\kappa]^{<\omega}$, if there is $X$ such that $(s, X) \in D$, then $(s, X_s) \in D$ and $B_s \subset X_s$. By the definition (4.1), $(A \setminus s) \subset X_s$, where $(A \setminus s) = A - (\max(s) + 1)$. Since $A$ and $A \setminus s$ are in $U$ and $D$ is open, $(s, A \setminus s) \in D$. We have shown that for all $s \in [\kappa]^{<\omega}$,

$$\text{If there is } X \text{ such that } (s, X) \in D, \text{ then } (s, A \setminus s) \in D. \tag{4.2}$$

By assumption, $S$ has an initial segment $s$ such that $S - s \subset A$. By the density of $D$ there are $t \in [A \setminus s]^{<\omega}$ and $X$ such that $(s \cup t, X) \in D$. Let $u$ be an initial segment of $S - s$ such that $|u| = |t|$. By the homogeneity of $(A \setminus s) \subset A_s$ for $F_s$, $F_s(u) = F_s(t)$ so there is some $Y$ such that $(s \cup u, Y) \in D$. By (4.2) we have $(s \cup u, A \setminus u) \in D$. Since $(s \cup u, A \setminus u) \in G$ by the definition of $G$, $D \cap G \neq \emptyset$. □

Now we have everything we need to prove that the sequence of critical points forms a Prikry generic sequence over the $\omega$-th iterate $M_\omega$. For the Main Theorem of this thesis we also need to show that the set $\{\kappa_{\omega \cdot n} : n < \omega\}$, where $\kappa_{\omega \cdot n}$ is the $\omega \cdot n$-th critical point, is Prikry generic over the $\omega^2$-th iterate $M_{\omega^2}$.

**Theorem 4.7.** [10]

*Suppose $U$ is a normal measure on $\kappa$ in $M$ and $\langle M_\alpha, U_\alpha, \kappa_\alpha, i_{\alpha\beta} \rangle_{\alpha<\beta\in On}$ is the iteration of $M$. Suppose further that $P_\omega$ is the Prikry forcing for the measure $U_\omega$ on $\kappa_\omega$ in the $\omega$-th iterate $M_\omega$ and $P_{\omega^2}$ is the Prikry forcing for the measure $U_{\omega^2}$ on $\kappa_{\omega^2}$ in $M_{\omega^2}$. Then the set $S_\omega = \{\kappa_n : n < \omega\}$ is $P_\omega$-generic over $M_\omega$ and the set $S_{\omega^2} = \{\kappa_{\omega \cdot n} : n < \omega\}$ is $P_{\omega^2}$-generic over $M_{\omega^2}$.*

*Proof.* By Lemma 2.10, for every $X \in U_\omega$ there is $\beta_X < \omega$ such that $\{\kappa_\gamma : \beta_X \leq \gamma < \omega\}$ is included in $X$. Therefore, $S_\omega - X$ is finite for every $X \in U_\omega$. Similarly, Lemma 2.10 implies that $S_{\omega^2} - X$ is finite for every $X \in U_{\omega^2}$. Hence, by Theorem 4.6 $S_\omega$ is $P_\omega$-generic over $M_\omega$ and $S_{\omega^2}$ is $P_{\omega^2}$-generic over $M_{\omega^2}$. □

[10] We follow a part of the proof of Theorem 21.15 of [9].

# Chapter 5

# The core model

The Dodd-Jensen core model $K$ was developed by Ronald Jensen and Anthony Dodd in the late 1970s. Its development was based on results in fine structure theory that Jensen had been developing. Fine structure theory is the detailed study of the structure of $L$.

A fundamental concept in fine structure theory and for the core model is the Jensen hierarchy which is a modification of the construction of $L$. $J_\alpha^A$, the Jensen hierarchy constructed relative to a set $A$, produces $L[A]$ as the union of all levels: $L[A] = \bigcup_{\alpha \in \mathrm{On}} J_\alpha^A$. The individual levels of the Jensen hierarchy, $J_\alpha^A$, are in general different from the levels $L_\alpha[A]$. The building blocks of the core model theory are premice that are of the form $N = J_\alpha^U$, where $U$ is a normal ultrafilter in $N$. A mouse is a premouse with certain specific properties and the core model is the union of all mice. The core model is an inner model of $ZFC$ and satisfies $GCH$.

One of the major results of the KMV paper is that the core model of $C^*$ is identical to the core model of $V$. We will present the proof of that theorem in the last chapter and the core model is also needed in the proof of the Main Theorem. In this chapter we will present the basics of core model theory and especially those results that are needed in the proofs of the last chapter. Different ways of constructing the core model have been developed after the concept's introduction but we will follow Jensen's and Dodd's original construction. This chapter is mostly based on Dodd's and Jensen's 1981 paper [5] that first introduced the core model. Some of our proofs follow or adapt ideas from Dodd's 1982 book [4] and the first section mostly follows Jensen's 1972 paper about fine structure theory [10].

## 5.1 Jensen hierarchy

The construction of the core model uses concepts from fine structure theory. This section presents those fine structural concepts and results that are needed for the construction of $K$. To avoid making the thesis excessively long, we do not present proofs for all the results. In those cases we indicate where the proof can be found.

We begin with the definition of rudimentary functions on which Jensen hierarchy is based.

**Definition 5.1.** A function $f : V^n \to V$ is *rudimentary* if it is finitely generated from the following schemas:

(a) $f(\bar{x}) = x_i$,

(b) $f(\bar{x}) = x_i \setminus x_j$,

(c) $f(\bar{x}) = \{x_i, x_j\}$,

(d) $f(\bar{x}) = h(g(\bar{x}))$,

(e) $f(y, \bar{x}) = \bigcup_{z \in y} g(z, \bar{x})$.

A function is *rudimentary in $A$* for a set or class $A$ if it is generated from (a)-(e) and the schema: $f^A(\bar{x}) = x_i \cap A$.

$X$ is *rudimentarily closed* if it is closed under rudimentary function. $M = \langle X, A \rangle$ is *rudimentarily closed* if its closed under functions which are rudimentary in $A$.

**Lemma 5.2.** [1] *Every rudimentary function is a composition of the following functions:*

$$\begin{aligned}
&F_0(x, y) = \{x, y\},\\
&F_1(x, y) = x \setminus y,\\
&F_2(x, y) = x \times y,\\
&F_3(x, y) = \{(u, z, v) : z \in x \ \textit{and} \ (u, v) \in y\},\\
&F_4(x, y) = \{(u, v, z) : z \in x \ \textit{and} \ (u, v) \in y\},\\
&F_5(x, y) = \bigcup x,\\
&F_6(x, y) = dom(x),\\
&F_7(x, y) = \in \cap x^2,\\
&F_8(x, y) = \{x``(z) : z \in y\}.
\end{aligned}$$

*Every function rudimentary in $A$ is a composition of $F_0, \ldots, F_8$ and $F^A(x, y) = x \cap A$.*

[1] Lemmas 1.8 and 1.9 of [10].

To define the Jensen hierarchy we need the following concepts:

**Definition 5.3.** $\mathrm{rud}(X)$ denotes the closure of $X \cup \{X\}$ under rudimentary functions. Similarly, $\mathrm{rud}_A(X)$ denotes the closure of $X \cup \{X\}$ under functions rudimentary in $A$.

The functions $s(u)$ and $s_A(u)$ are defined by

$$s(u) = u \cup \bigcup_{i=0}^{8} F_i``(u^2),$$
$$s_A(u) = u \cup \bigcup_{i=0}^{8} F_i``(u^2) \cup F^A(u).$$

By Lemma 5.2 $\bigcup_{n<\omega} s^n(u)$ is the rudimentary closure of $u$. We define further $S(u) = s(u \cup \{u\})$ and $S_A(u) = s_A(u \cup \{u\})$.

An important property of $\mathrm{rud}(X)$ is given by the following lemma. When we say that a subset $B$ of $A$ is $\Sigma_n^C(A)$ we mean that it is defined over $A$ by a $\Sigma_n$ formula with parameters from $C$. $B \in \Sigma_n(A)$ means that $B$ is $\Sigma_n$ over $A$ with parameters from $A$ unless the parameters are specified otherwise. $\Sigma_\omega(A) = \bigcup_{n<\omega} \Sigma_n(A)$.

**Lemma 5.4.** [2] $\mathcal{P}(X) \cap \mathit{rud}(X) = \Sigma_\omega(X)$.

Now we can define the Jensen hierarchy and the finer $S_\alpha$-hierarchy.

**Definition 5.5.** *Jensen hierarchy* is defined recursively as follows:

$$\begin{aligned} J_0 &= \emptyset, \\ J_{\alpha+1} &= \mathrm{rud}(J_\alpha), \\ J_\lambda &= \bigcup_{\alpha<\lambda} J_\alpha \text{ for limit } \lambda. \end{aligned}$$

$S_\alpha$*-hierarchy* is defined by

$$\begin{aligned} S_0 &= \emptyset, \\ S_{\alpha+1} &= S(S_\alpha), \\ S_\lambda &= \bigcup_{\alpha<\lambda} S_\alpha \text{ for limit } \lambda. \end{aligned}$$

[2]Corollary 1.7 of [10].

By induction we can see that both $J_\alpha$ and $S_\alpha$ are transitive for all $\alpha$ and always $J_\alpha = S_{\omega\alpha}$. The $J_\alpha$'s generate $L$: $\bigcup_{\alpha<\mathrm{On}} J_\alpha = L$. Always $\mathrm{On} \cap J^A_\alpha = \omega\alpha$. If an ordinal $\alpha$ satisfies $\alpha = \omega\alpha$, then $J_\alpha = L_\alpha$ [3].

Core model theory is mostly concerned with $J^A_\alpha$, the Jensen hierarchy defined relative to $A$:

**Definition 5.6.** Suppose $A$ is a class or set. Then the hierarchy $J^A_\alpha$ is defined recursively as follows:

$$\begin{aligned} J^A_0 &= \emptyset, \\ J^A_{\alpha+1} &= \mathrm{rud}_A(J_\alpha), \\ J^A_\lambda &= \bigcup_{\alpha<\lambda} J^A_\alpha \text{ for limit } \lambda. \end{aligned}$$

The finer $S^A_\alpha$-hierarchy is defined as follows:

$$\begin{aligned} S^A_0 &= \emptyset, \\ S^A_{\alpha+1} &= S_A(S^A_\alpha), \\ S^A_\lambda &= \bigcup_{\alpha<\lambda} S^A_\alpha \text{ for limit } \lambda. \end{aligned}$$

Again, the $J^A_\alpha$'s and $S^A_\alpha$'s are transitive and $J^A_\alpha = S^A_{\omega\alpha}$. The hierarchy $J^A_\alpha$ generates $L[A]$ as the union of all levels: $\bigcup_{\alpha\in\mathrm{On}} J^A_\alpha = L[A]$. In our discussion of core model theory, we conceive of $J^A_\alpha$ as the structure $\langle J^A_\alpha, A \cap J^A_\alpha\rangle$. It will always be clear from the context whether we mean just the level of the hierarchy or the structure. Function $F^A$ in the definition of $S^A_{\alpha+1}$ guarantees that for all $\alpha$, $\langle J^A_\alpha, A \cap J^A_\alpha\rangle$ is amenable, which is defined as follows:

**Definition 5.7.** We call a structure $M = \langle M, A\rangle$ *amenable* if for all $x \in M$, $A \cap x \in M$.

An important feature of $J^A_\alpha$ is that it has a $\Sigma_1$ Skolem function [4].

**Definition 5.8.** Suppose $M = \langle M, A\rangle$ is amenable and $\omega \subset M$ and suppose $\langle \phi_i\rangle_{i<\omega}$ is a recursive enumeration of $\Sigma_1$ formulas with 2 free variables. The $\Sigma_1$ *Skolem function* for $M$ is a $\Sigma_1(M)$ function $h$ such that $\mathrm{dom}(h) \subset \omega \times M$ and for every $A \in \Sigma_1(M)$ defined over $M$ by $\phi_i$ and parameter $x$, if $A \neq \emptyset$, then $h(i, x) \in A$.

The $\Sigma_1$ *Skolem hull* of $X \subset M$ in $M$ is $h_M``[\omega \times X^{<\omega}]$.

[3] See note on p. 255 of [10].

[4] The existence is proved in Lemma 2.8 of [10].

When we speak of a Skolem function in this chapter, we mean the $\Sigma_1$ Skolem function. We will usually denote the Skolem hull $h_M``[\omega \times X^{<\omega}]$ by just $h_M(X)$ if there is no risk of confusion.

We now present concepts that are needed for the core model. From now on, we are mostly following [5]. The following lemma lists some useful properties of $J_\alpha^A = \langle J_\alpha^A, A \cap J_\alpha^A \rangle$.

**Lemma 5.9.** *1.* [5] *There is a $\Sigma_1(J_\alpha^A)$ function from $\omega\alpha$ onto $J_\alpha^A$.*

*2.* [6] *There is a $\Sigma_1(J_\alpha^A)$ map of $\omega\alpha$ onto $(\omega\alpha)^2$.*

*3.* [7] *$\langle J_\beta^A : \beta < \alpha \rangle$ is uniformly parameter free $\Sigma_1(J_\alpha^A)$.*

*4.* [8] *There is a well-ordering relation $<_{J_\alpha^A}$ on $J_\alpha^A$. The relation is $\Sigma_1(J_\alpha^A)$.*

In some definitions we use the following well-ordering[9] on $[\mathrm{On}]^{<\omega}$:

$$\begin{aligned} p \leq_* q \quad \text{iff} \quad & \exists\alpha\, [p \setminus \alpha = q \setminus \alpha \wedge q \cap \alpha \neq \emptyset \\ & \wedge (p \cap \alpha = \emptyset \vee \max(p \cap \alpha) < \max(q \cap \alpha)]. \end{aligned}$$

**Lemma 5.10.** [10]

*(a) If $x \in J_\alpha^A$, then $TC(x) \in J_\alpha^A$.*

*(b) $\langle J_\alpha^A, TC \restriction J_\alpha^A \rangle$ is amenable.*

*(c) The relations $y = TC \restriction x$ and $y = TC(x)$ are uniformly $\Sigma_1(J_\alpha^A)$.*

*Proof.* For any $\nu$ we have

$$\begin{aligned} f = \mathrm{TC} \restriction S_\nu^A \quad \text{iff} \quad & f \text{ is a function} \wedge \mathrm{dom}(f) = S_\nu^A \\ & \wedge (\forall x \in S_\nu^A)[f(x) = x \cup \bigcup_{z \in x} f(z)]. \end{aligned}$$

Every $x \in J_\alpha^A$ is in $S_\nu^A$ for some $\nu < \omega\alpha$. Hence, to show (a) it suffices to show

(1) $\mathrm{TC} \restriction S_\nu^A \in J_\alpha^A$ for all $\nu < \omega\alpha$.

[5] Lemma 2.36 of [4].
[6] Lemma 2.30 of [4].
[7] Corollary 2.3 of [10] gives the result for $J_\alpha$.
[8] Corollary 2.5 of [10] gives the analogous result for $J_\alpha$.
[9] That this really is a well-ordering is proved e.g. in Lemma 9.2 of [4].
[10] We follow Lemma 2.1 of [5].

Then (b) holds, too, since $\mathrm{TC} \upharpoonright J^A_\alpha \cap x = \mathrm{TC} \upharpoonright S^A_\nu \cap x$. Moreover,

$$y = \mathrm{TC} \upharpoonright x \Leftrightarrow \exists\nu(x \in S^A_\nu \wedge y = (\mathrm{TC} \upharpoonright S^A_\nu) \cap (x \times \mathrm{TC}(x)), \text{ and}$$
$$y = \mathrm{TC}(x) \Leftrightarrow \exists\nu(x \in S^A_\nu \wedge y = (\mathrm{TC} \upharpoonright S^A_\nu)(x),$$

so (c) holds as well.

We prove (1) by induction on $\alpha$. Case $\alpha = 0$ is trivial and the limit case follows immediately from the induction hypothesis. Suppose $\alpha = \beta + 1$. Then $\mathrm{TC} \upharpoonright J^A_\beta \in J^A_\alpha$ since $\mathrm{TC} \upharpoonright J^A_\beta = \bigcup_{\nu<\omega\beta} \mathrm{TC} \upharpoonright S^A_\nu$ is $\Sigma_1(J^A_\beta)$. The definition of the functions $F_0, \ldots, F_8, F^A$ implies that for every $n < \omega$ there is $m_n < \omega$ such that $\bigcup^{m_n} S^A_{\omega\beta+n} \subset J^A_\beta$.

For $i \leq m_n$ we define $t^n_i$ as follows:

$$t^n_0 = \mathrm{TC} \upharpoonright J^A_\beta,$$
$$t^n_{i+1} = \{(x, y) : x \in S_{\omega\beta+n},\, x \subset \mathrm{dom}(t^n_i) \text{ and } y = x \cup \bigcup_{x \in x} t_i(z)\}$$

Since $\mathrm{TC} \upharpoonright J^A_\beta$ is in $J^A_\alpha$, each $t^n_i$ is in $J^A_\alpha$. For all $n$, $t^n_{m_n} = \mathrm{TC} \upharpoonright S^A_{\omega\beta+n}$, so (1) holds. □

The following lemma can be proved in a similar way as the previous one using the definition of the rank function. We omit the proof for brevity.

**Lemma 5.11.** [11]

*(a) If $x \in J^A_\alpha$, then $rank(x) \in J^A_\alpha$.*

*(b) $\langle J^A_\alpha, rank \upharpoonright J^A_\alpha\rangle$ is amenable.*

*(c) $y = rank \upharpoonright x$ and $y = rank(x)$ are uniformly $\Sigma_1(J^A_\alpha)$.*

The following concept is needed to get a useful characterization of set of hereditarily small elements of $N = J^A_\alpha$

**Definition 5.12.** $N = J^A_\alpha$ is *$\delta$-tidy* if for all $\nu < \delta$ and all $a \in \mathcal{P}(\nu) \cap N \subset$, $J^a_\delta \subset N$.

**Lemma 5.13.** [12] *Suppose $N = J^A_\alpha$ is $\delta$-tidy, and either $\omega\delta$ is a cardinal in $N$ or $\delta = \alpha$ and $\delta = \omega\delta$. Then*

$$H^N_{\omega\delta} = \bigcup_{\substack{\nu<\delta \\ a\subset\nu,\, a\in N}} J^a_\delta.$$

[11] Lemma 2.2 of [5].

[12] We combine the proofs of Lemma 2.5 of [5] and Lemma 3.17 of [4].

*Proof.* There is always a $\Sigma_1(J_\delta^A)$ function from $\delta$ onto $\omega\delta$. If $\omega\delta$ is in $N$, then necessarily $\delta = \omega\delta$ since $\omega\delta$ is a cardinal in $N$. Hence, in either case of the assumption, $\delta = \omega\delta$.

To prove inclusion from right to left, suppose $\nu < \delta$ and $a \in \mathcal{P}(\nu) \cap N$. By $\delta$-tidiness $J_\delta^a \subset N$. Every $x \in J_\delta^a$ is in $J_{\xi+1}^a$ for some $\xi < \delta$. Since there is a $\Sigma_1(J_\xi^a)$ surjection from $\omega\xi$ onto $S_{\omega\xi+k}^a$ for every $k \in \omega$, there is a function $f_k \in J_{\xi+1}^a$ from $\omega\xi$ onto $S_{\omega\xi+k}^a$ for every $k \in \omega$. Since every $S_{\omega\xi+k}^a$ is transitive and $J_{\xi+1}^a = \bigcup_{k<\omega} S_{\omega\nu+k}^a$, $|\mathrm{TC}(x)|^N < \omega\xi < \omega\delta$ for every $x \in J_\delta^a$.

For the other direction, suppose $x \in H_{\omega\delta}^N$ and let $u = \mathrm{TC}(\{x\})$. Then there are a $\nu < \omega\delta$ and $f \in N$ such that $\nu$ is a cardinal in $N$ and $f$ is a bijection from $\nu$ to $u$. Let $e = \{(\xi_1, \xi_2) \in \nu^2 : f(\xi_1) \in f(\xi_2)\}$. Since $\nu$ is a cardinal in $N$, it is a limit ordinal, say $\nu = \omega\xi$. There is a $\Sigma_1(J_\xi^A)$ function $g$ from $\nu^2$ onto $\nu$, so $e' = g``(e)$ is in $N$. Now $e'$ is a subset of $\nu < \delta$ and since $g^{-1}$ is $\Sigma_1(J_\xi^A)$, $J_\delta^{e'} = J_\delta^e$. Hence, it is enough to show that $x \in J_\delta^e$.

We define functions $f_i$, $i \leq \mathrm{rank}(\{x\})$, as follows

$$\begin{aligned} &f_0 = \emptyset, \\ &f_{i+1} = \{\langle \xi, f_i``\{\tau : (\tau, \xi) \in e\}\rangle : \xi < \nu \text{ and } \{\tau : (\tau, \xi) \in e\} \subset \mathrm{dom}(f_i)\}, \\ &f_\lambda = \bigcup_{i<\lambda} f_i \text{ for limit } \lambda. \end{aligned}$$

By induction on $i$ we can see that if $\xi \in \mathrm{dom}(f_i)$, then $f_i(\xi) = f(\xi)$. In particular, the limit case is well-defined. If $i$ is a limit ordinal, the claim is clear by the induction hypothesis. If $i = j + 1$ and $\xi \in \mathrm{dom}(f_i)$, then

$$f_i(\xi) = f_j``\{\tau : f(\tau) \in f(\xi)\} = \{f(\tau) : f(\tau) \in f(\xi)\} = f(\xi),$$

where the last equation holds since $f$ is a surjection onto $\mathrm{TC}(\{x\})$. We can also see by induction that if $\mathrm{rank}(f(\xi)) < i$, then $\xi \in \mathrm{dom}(f_i)$. The limit case is again clear by the induction hypothesis. If $i = j + 1$, and $\mathrm{rank}(f(\xi)) < i$, then $f(\tau) < f(\xi)$ implies that $\mathrm{rank}(f(\tau)) < j$. Hence, by the induction hypothesis $\{\tau : f(\tau) \in f(\xi)\} \subset \mathrm{dom}(f_j)$, so $\xi$ is in $\mathrm{dom}(f_i)$. Then $f_{\mathrm{rank}(\{x\})} = f$, so we need to show that $f_{\mathrm{rank}(\{x\})} \in J_\delta^e$.

By Lemma 5.11(a), $\mathrm{rank}(\{x\}) = \gamma$ for some $\gamma < \omega\alpha$. Since $\mathrm{rank} \restriction u = (\mathrm{rank} \restriction N) \cap u \times \gamma$ is in $N$ by Lemma 5.11(b), $f' = (\mathrm{rank} \restriction u) \circ f$ is in $N$ and $f'$ is an onto function from $\nu$ onto $\gamma$. If $\gamma \geq \delta$, then $\delta = \omega\delta$ is in $N$ but is not a cardinal in $N$, a contradiction. Hence, $\mathrm{rank}(\{x\}) < \delta$.

For any $\alpha' < \delta$, we have

$$\nu + \alpha' \leq 2\max\{\nu, \alpha'\} \leq \omega \max\{\nu, \alpha'\} < \omega\delta = \delta.$$

Thus, to show $f_{\mathrm{rank}(\{x\})} \in J_\delta^e$ it suffices to prove by induction on $\alpha' \leq \mathrm{rank}(\{x\})$ the following:

$$f_{\alpha'} \in J_{\nu+\alpha'+1}^e \quad \text{and} \quad \langle f_i : i < \alpha'\rangle \in J_{\nu+\alpha'+1}^e.$$

The case $\alpha' = 0$ is trivial. The case $\alpha' = \beta + 1$ follows from the induction assumption since $f_{\beta+1}$ is definable in $J^e_{\nu+\beta+1}$ from $f_\beta$. If $\alpha'$ is a limit, then $\langle f_i : i < \alpha' \rangle$ is $\Sigma_1(J^e_{\nu+\alpha'})$ by the definition of the $f_i$'s. Hence, $\langle f_i : i < \alpha' \rangle$ and $f_{\alpha'}$ are in $J^e_{\nu+\alpha'+1}$. □

**Definition 5.14.** $N = J^A_\alpha$ is *acceptable* if whenever $\nu < \alpha$ and $\delta < \omega\nu$ with $P(\delta) \cap J^A_{\nu+1} \not\subset J^A_\alpha$, then for each $u \in J_{\nu+1}$ there is a sequence of functions $\langle f_\xi : \delta \leq \xi < \omega\nu \rangle \in J^A_{\nu+1}$ such that each $f_\xi : \xi \to \{\xi\} \cup (\mathcal{P}(\xi) \cap u)$ is onto.

For the rest of this section we suppose that $N = J^A_\alpha$ is acceptable.

**Lemma 5.15.** [13] *Suppose $\omega \in N$. For every $\nu \in On \cap N$ we let $\nu^+ = (\nu^+)^N$ denote the least cardinal in $N$ greater than $\nu$, or $\omega\alpha$ if there is no such cardinal. There is a uniformly, without parameters, $\Sigma_1(N)$ sequence*

$$\langle a^\nu_{ji} : \omega \leq \nu \in N,\, j < \nu,\, \omega i < \nu^+ \rangle$$

*satisfying*

*(i)* $\{a^\nu_{ji} : j < \nu,\, \omega i < \nu^+\} = \mathcal{P}(\nu) \cap N$,

*(ii)* $\langle a^\nu_{ji} : j < \nu,\, \omega i < \tau \rangle \in N$ *for* $\tau < \nu^+$.

*Proof.* For $\nu \geq \omega$ we let

$$B_\nu = \{\xi < \alpha : \nu \leq \omega\xi \text{ and } \mathcal{P}(\nu) \cap J^A_{\xi+1} \not\subset J^A_\xi\}.$$

$B_\nu$ is $\Sigma_1(N)$ so there is a $\Sigma_1(N)$ increasing enumeration of $\langle \xi^\nu_i : i < \nu^* \rangle$ of $B_\nu$. For $n < \omega$ and $i < \nu^*$, let $f^\nu_{i,n}$ be the $<_N$-least surjection from $\nu$ onto $\mathcal{P}(\nu) \cap S^A_{\omega\xi^\nu_i+n}$. The acceptability of $N$ guarantees that $f^\nu_{i,n}$ exists. Let $g^\nu$ be the $<_N$-least bijection between $\omega\nu$ and $\nu$. Such a bijection exists since $\nu \geq \omega$. Finally let

$$a^\nu_{g^\nu(\omega j+n),i} = f^\nu_{i,n}(j)$$

for $j < \nu$, $n < \omega$ and $i < \nu^*$.

The sequence $\langle a^\nu_{ji} : \omega \leq \nu \in N,\, j < \nu,\, \omega i < \nu^* \rangle$ is uniformly $\Sigma_1(N)$. Every member of $\mathcal{P}(\nu) \cap N$ is in some $S^A_{\omega\xi^\nu_i+n}$, so the bijectivity of $g^\nu$ implies that

$$\mathcal{P}(\nu) \cap N = \{a^\nu_{ji} : j < \nu,\, i < \nu^*\} = \{a^\nu_{ji} : j < \nu,\, \omega i < \omega\nu^*\}.$$

If $\omega i < \tau < \omega\nu^*$, then $i < \tau'$ for some $\tau' < \nu^*$. For $\tau' > \nu$, $\{a^\nu_{ji} : j < \nu,\, i < \tau'\}$ is definable in $J^A_{\tau'}$ so $\{a^\nu_{ji} : j < \nu,\, i < \tau'\}$ is in $N$. Hence, $\{a^\nu_{ji} : j < \nu,\, \omega i < \tau\}$ is in $N$ for all $\tau < \omega\nu^*$. Thus, we only need to show that $\omega\nu^* = \nu^+$.

[13] We follow Lemma 2.12 of [5].

Suppose first that $\omega\nu^* > \nu^+$. Then $\nu^+ < \omega\alpha$, so $\nu^+$ is a cardinal in $N$ which implies that $\omega\nu^+ = \nu^+$ and $\nu^* > \nu^+$. Define an injection $b : \nu^+ \to \mathcal{P}(\nu)$ by

$$b(i) = \text{the } <_N \text{-least } a \in \mathcal{P}(\nu) \cap J^A_{\xi^\nu_i+1} \setminus J^A_{\xi^\nu_i}.$$

$\{b(i) : i < \nu^+\}$ is in $J^A_{\xi_{\nu^+}+1}$ so it is in some $S^A_{\omega\xi_{\nu^+}+n}$, and hence by acceptability there is a surjection $f \in N$ from $\nu$ onto $\{b(i) : i < \nu^+\}$. But then $b^{-1} \circ f$ is in $N$ and $b^{-1} \circ f$ is a surjection from $\nu$ onto $\nu^+$, a contradiction. Hence, $\omega\nu^* \leq \nu^+$.

Suppose then that $\omega\nu^* < \nu^+$. Now $\omega\nu^* < \omega\alpha$ so $\mathcal{P}(\nu) \cap N = \{a^\nu_{ji} : j < \nu, i < \nu^*\}$ is in $N$. As $\nu^* < \nu^+$, we have $|\nu \times \nu^*|^N < \nu^+$, so $|\mathcal{P}(\nu)|^N \leq \nu$, a contradiction. Hence, $\omega\nu^* = \nu^+$. □

A fundamentally important concept in core model theory is that of a projectum.

**Definition 5.16.** [14] The *projectum* of $N$ is the least $\rho \leq \alpha$ such that there is a $\Sigma_1(N)$ subset of $\omega\rho$ that is not in $N$, in other words, $\mathcal{P}(\omega\rho) \cap \Sigma_1(N) \not\subset N$. The projectum is denoted by $\rho_N$.

**Lemma 5.17.** [15] *$N$ is $\rho_N$-tidy.*

*Proof.* Since $\rho_N$ is a limit ordinal, it suffices to show the following:

(1) For any infinite $\nu < \rho_N$ and $a \in \mathcal{P}(\nu) \cap N$, $J^a_\nu$ is in $N$.

Suppose (1) does not hold for some $a$ and $\nu$. Let $\tau$ be the least ordinal such that $a \in J^A_\tau$. Then for all $\tau + \xi < \alpha$, $J^a_\xi \in J^A_{\tau+\xi+1}$ and $\langle S^\alpha_\eta : \eta < \omega\xi \rangle$ is $\Sigma_1(J^A_{\tau+\xi})$. Thus, $\alpha$ must be less than $\tau + \nu + 1$.

Suppose $\tau = \eta + 1$ and $\alpha = \eta + \gamma$ for some $\gamma \leq \nu$. Define $S_{\xi n} = S^A_{\omega(\eta+\xi)+n}$ for all $\xi < \gamma$ and $n < \omega$. Then we have

(2) $|\mathcal{P}(\nu) \cap S_{\xi n}|^{J^A_{\eta+\xi+1}} \leq \nu$ for all $\xi < \gamma$ and $n < \omega$.

Suppose (2) does not hold and let $\xi$ be the least such that (2) fails for $\xi$ and some $n$. Since $N$ is acceptable, (2) holds for every $\xi < \gamma$ such that $\mathcal{P}(\nu) \cap J^A_{\eta+\xi+1} \not\subset J^A_{\eta+\xi}$. Hence, $\xi > 0$ and $\mathcal{P}(\nu) \cap J^A_{\eta+\xi+1} \subset J^A_{\eta+\xi}$. In particular, $\mathcal{P}(\nu) \cap S_{\xi n} = \mathcal{P}(\nu) \cap J^A_{\eta+\xi}$. For $\beta < \xi$, let $f_{\beta n}$ be the $<_{J^A_{\eta+\beta+1}}$-least surjection from $\nu$ onto $\mathcal{P}(\nu) \cap S_{\beta n}$. Then the sequence $\langle f_{\beta n} : \beta < \xi, n < \omega \rangle$ is $\Sigma_1(J_{\eta+\xi})$. Since $1 \leq \xi < \nu$, there is a $\Sigma_1(J^A_{\eta+\xi})$ function from $\nu$ onto $\xi \times \omega$, and hence there is a $\Sigma_1(J^A_{\eta+\xi})$ onto function $g : \nu \to \mathcal{P}(\nu) \cap J^A_{\eta+\xi}$. But then $g \in J^A_{\eta+\xi+1}$, a contradiction. Hence, (2) holds.

[14] Definition 2.14 of [5].
[15] We follow Lemma 2.16 of [5].

Since (2) holds, we can let $f_{\xi n}$ be the $<_{J^A_{\eta+\xi+1}}$-least function from $\nu$ onto $\mathcal{P}(\nu) \cap S_{\xi n}$ for every $\xi < \gamma$ and $n < \omega$. As in the proof of (2) we get a $\Sigma_1(N)$ onto function $g : \nu \to \mathcal{P}(\nu) \cap N$. If $g \notin N$, then there is a $\Sigma_1(N)$ subset of $\nu$ that is not in $N$, which is impossible as $\nu < \omega\rho_N$. Hence, $g$ and $\mathcal{P}(N) \cap N$ are in $N$, and $|\mathcal{P}(\nu)|^N \leq \nu$, a contradiction. Hence, (1) holds. $\square$

**Definition 5.18.** [16]

(i) Suppose $N = J^A_\alpha$. Then $p_N$ is the $<_*$-least p $\in [\mathrm{On}]^{<\omega}$ such that there is $A \subset \mathrm{On}$ such that $A$ is $\Sigma_1(N)$ in parameters from $\omega\rho_N \cup p$ and $A \cap (\omega\rho_N) \notin N$.

(ii) $A_N = \{(i, x) \in J_{\rho_N} : N \vDash \phi_i(x, p_N)\}$ where $\{\phi_i : i < \omega\}$ is a fixed recursive enumeration of the $\Sigma_1$ formulas with two free variables.

(iii) $N^* = J^{A_N}_{\rho_N}$.

**Lemma 5.19.** [17] $N^* = H^N_{\omega\rho_N}$.

*Proof.* Since for every $\beta < \alpha$ there is a uniformly $\Sigma_1(J_\beta)$ map of $\omega\beta$ onto $J_\beta$, there is a $\Sigma_1(N)$ set $\bar{A} \subset \omega\rho_N$ such that $J^{\bar{A}}_{\rho_N} = J^{A_N}_{\rho_N} = N^*$. For every $\nu < \omega\rho_N$, $\bar{A} \cap \nu$ is in $N$, and hence $J^{\bar{A}\cap\nu}_{\rho_N} \subset N$ by the $\rho_N$-tidiness of $N$. But

$$J^{\bar{A}}_{\rho_N} = \bigcup_{\nu<\omega\rho_N} J^{\bar{A}\cap\nu}_{\rho_N},$$

so $J^{\bar{A}}_{\rho_N} \subset H^N_{\omega\rho_N}$ by Lemma 5.17.

To prove the other direction, it suffices by Lemma 5.13 to show that $J^a_{\rho_N} \subset N^*$ for every $a \subset \nu < \omega\rho_N$ such that $a \in N$. Since $N$ is acceptable, $J^a_{\rho_N} \subset N$ by Lemma 5.17. If $a \in N^*$, then $a \in J^{A_N}_\beta$ for some $\beta < \rho_N$. Since $\rho_N$ is a $\Sigma_1$ cardinal in $N$, $\beta + \nu + 1 < \rho_N$, so $J^a_\nu \in N^*$. Similarly $J^a_{\nu'} \subset N^*$ for every $\nu'$ such that $\nu \leq \nu' < \rho_N$, so $J^a_{\rho_N} \subset N^*$ since $\rho_N$ is a limit ordinal. Hence, it suffices to show that every $a \subset \nu < \omega\rho_N$, $a \in N$, is in $N^*$.

Fix such $a$ and $\nu$. Since the sequence in the statement of Lemma 5.15 has a uniform $\Sigma_1(N)$ definition without parameters, and since $\nu^+ \leq \omega\rho_N$ in $N$, there are $i < \omega$, $i' < \nu$, $\xi < \omega\rho_N$ such that $a = h_N(i, \langle i', \xi, \nu\rangle)$. Let $\phi_j$ be a $\Sigma_1$ formula satisfying

$$N \vDash \forall y \forall x_1 \forall x_2 \forall x_3\, [\phi_j(\langle y, x_1, x_2, x_3\rangle, p_N) \leftrightarrow y \in h_N(i, \langle x_1, x_2, x_3\rangle)].$$

Then

$$a = \{\gamma < \nu : (j, \langle i', \xi, \nu\rangle)\} \in N^*.$$

$\square$

[16]Definition 2.17 of [5].

[17]We follow Lemma 2.19 of [5].

This result is needed mostly for the following easy but important corollary.

**Corollary 5.20.** [18] *Suppose $N$ is a premouse and $\rho_N > \rho_{N^*}$. If $a \subset \omega\rho_{N^*}$ is not in $N^*$, then $a$ is not in $N$.*

*Proof.* Since $a$ is a subset of $\omega\rho_{N^*} < \omega\rho_N$, if $a$ is in $N$, it is in $H^N_{\omega\rho_N}$. But $N^* = H^N_{\omega\rho_N}$, so then $a$ is in $N^*$. $\square$

## 5.2 Premice and their iterations

Premice and their iterations are an essential part of the core model theory. A premouse is just a level of the Jensen hierarchy. The iteration of a premouse is very similar to the iteration of a model by an $M$-ultrafilter and premouse iterations share many properties with iterated ultrapowers.

**Definition 5.21.** [19] $N = J^U_\alpha$ is called a *premouse* if $N \models$ "$U$ is a normal measure on $\kappa$" for some $\kappa < \alpha$. Then $N$ is said to be a premouse *at* $\kappa$.

If $J^U_\alpha$ is a premouse at $\kappa$, the ordinal $\alpha$ must be greater than $\kappa$ because otherwise no member of $U$ can be in $J^U_\alpha$. If we set $\bar{U} = U \cap J^U_\alpha$, then $J^U_\alpha = J^{\bar{U}}_\alpha$ since in the construction up to $J^U_\alpha$ the function $F^U(x, y)$ is only applied to sets that already belong to some previous level of the $S$-hierarchy.

The definition of the iteration of a premouse is very similar to the definition of iterated ultrapowers for a model of $ZFC^-$. First we define the ultrapower of a premouse. Scott's trick is not needed because premice are sets. As in Chapter 2, let the language of premice be $\mathcal{L}_\in(\dot{U})$.

**Definition 5.22.** [20] Let $N = J^U_\alpha$ be premouse at $\kappa$. Define the equivalence relation $\sim$ on $N^\kappa \cap N$ by

$$f \sim g \text{ iff } \{\xi : f(\xi) = g(\xi)\} \in U.$$

Let the domain of the ultrapower be the set of equivalence classes:

$$\tilde{N} = \{[f]_\sim : f \in N^\kappa \cap N\}.$$

Define the interpretations of $\in$ and $\dot{U}$ by

$$[f]E_{\tilde{N}}[g] \text{ iff } \{\xi : f(\xi) \in g(\xi)\} \in U \text{ and}$$
$$U_{\tilde{N}}([f]) \text{ iff } \{\xi : f(\xi) \in U\} \in U.$$

Then $\tilde{N} = \langle \tilde{N}, E_{\tilde{N}}, U_{\tilde{N}} \rangle$ is the ultrapower of $N$ by $U$.

[18]The proof is our own.
[19]Definition 3.1 of [5].
[20]Definition 3.2 of [5].

Unlike in the case of an $M$-ultrapower by an $M$-ultrafilter, for an ultrapower of a premouse, Łoś's theorem holds only for $\Sigma_0$ formulas. The proof is by induction on the length of the formula.

**Lemma 5.23.** [21] *For all $\Sigma_0$ formulas $\phi$, $\tilde{N} \models \phi([f])$ if and only if $\{\xi : N \models \phi(f(\xi))\} \in U$.*

We are only interested in ultrapowers of premice that are well-founded. The canonical embedding is defined similarly as in chapter 2.

**Definition 5.24.** [22] If $\tilde{N}$ is well-founded, let $N^+$ be its transitive collapse and let $g_N : \tilde{N} \cong N^+$ be the collapsing function. Define the embedding $\pi_N : N \to N^+$ by $\pi_N(x) = g_N([c_x])$, where $c_x \in N^\kappa \cap N$ is the constant function with value $x$.

By Lemma 5.23 $\pi_N$ is $\Sigma_0$-elementary. The proof of the next lemma is identical to the beginning of the proof of Lemma 2.11.

**Lemma 5.25.** [23] *If $x \in N^+$, then $x = \pi_N(f)(\kappa)$ for some $f \in N^\kappa \cap N$.*

This allows us to prove the next lemma.

**Lemma 5.26.** [24] *The range of the embedding $\pi_N$ is cofinal in $On \cap N^+$.*

*Proof.* Suppose $x \in \mathrm{On} \cap N^+$. Then $x = \pi_N(f)(\kappa)$ for some $f \in N^\kappa \cap N$. Let $\beta \geq \sup(\mathrm{ran}(f))$. Since $N \models (\forall x \in \kappa)\,(f(x) \leq \beta)$, the $\Sigma_0$-elementarity of $\pi_N$ implies that $N^+ \models \forall x \in \pi_N(\kappa)(\pi_N(f)(x) \leq \pi_N(\beta))$, so $\pi_N(\beta) \geq x$. $\square$

From this it follows that $\pi_N$ is actually $\Sigma_1$-elementary. However, this does not imply full elementarity since a premouse is not necessarily a model of $ZF$.

**Lemma 5.27.** [25] *$\pi_N$ is $\Sigma_1$-elementary.*

*Proof.* Let $N = J^U_\alpha$. Let $S'_\nu = \pi_N(S_\nu)$ for all $\nu < \omega\alpha$. Suppose $x \in N^+$, say $\pi_N(f)(\kappa)$. If $f \in S_\nu$, then $f(\xi) \in S_\nu$ for all $\xi < \kappa$, so $\pi_N(f)(\kappa)$ is in $S'_\nu$. Thus $N^+ = \bigcup_{\nu<\omega\alpha} S'_\nu$.

Let $\phi(y, \bar{x})$ be $\Sigma_0$. Then we have by $\Sigma_0$-elementarity

$$\begin{aligned} N \models \exists y\, \phi(y, \bar{a}) &\text{ iff } \exists \nu \text{ such that } N \models (\exists y \in S_\nu)\, \phi(y, \bar{a}) \\ &\text{ iff } \exists \nu \text{ such that } N^+ \models (\exists y \in S'_\nu)\, \phi(y, \pi_N(\bar{a})) \\ &\text{ iff } \exists \nu \exists y \in S'_\nu \text{ such that } N^+ \models \phi(y, \pi_N(\bar{a})) \\ &\text{ iff } N^+ \models \exists y\, \phi(y, \pi_N(\bar{a})). \end{aligned}$$

$\square$

[21] Lemma 3.3 of [5].
[22] Definition 3.4 of [5].
[23] Lemma 3.5 of [5].
[24] Lemma 3.6 of [5].
[25] We follow Lemma 3.7 [5].

This allows us to prove that $N^+$ is a premouse as well.

**Lemma 5.28.** [26] *$N^+$ is a premouse.*

*Proof.* $S'_\nu = S^{U^+}_{\pi_N(\nu)}$ where $U^+ = g_N``(U_{\tilde{N}})$. So $N^+ = \bigcup_{\nu<\omega\alpha} S^{U^+}_{\pi_N(\nu)} = S^{U^+}_{\alpha'}$, where $\alpha' = \sup(\pi_N``(\omega\alpha))$. Moreover, since '$U$ is normal' is $\Pi_1(N)$, the $\Sigma_1$-elementarity of $\pi_N$ implies that $N^+ \models$ '$U^+$ is a normal measure on $\kappa$'. □

The proof of the following lemma is identical to the proof for an $M$-ultrafilter in Chapter 2.

**Lemma 5.29.** [27] *$\pi_N \restriction \kappa = id \restriction \kappa$ and $\mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap N^+$.*

Next we define the full iteration of a premouse through all ordinals. The construction and the main results are very similar to the ones for iterated ultrapowers. The main difference is that the embeddings are not necessarily more than $\Sigma_1$-elementary.

**Definition 5.30.** [28]
Let $N = N_0$ be a premouse. $N_\alpha$, $\pi_{ij}$ and *$\alpha$-iterability* are defined recursively for $i, j, \alpha \in \mathrm{On}$ as follows:

1. If $N$ is $\alpha$-iterable and $\tilde{N}_\alpha$ is well-founded, then $N$ is $\alpha+1$-iterable, $N_{\alpha+1} = N^+_\alpha$ and $\pi_{i,\alpha+1} = \pi_{N_\alpha} \circ \pi_{i\alpha}$.
2. If $\lambda$ is a limit ordinal, $N$ is $\alpha$-iterable for all $\alpha < \lambda$ and the direct limit of $\langle\langle N_\alpha : \alpha < \lambda\rangle, \langle\pi_{\alpha\beta} : \alpha \leq \beta < \lambda\rangle\rangle$ with $\Sigma_1$-elementary limit maps is well-founded, then $N$ is $\lambda$-iterable and $N_\lambda$ is the transitive collapse of the limit. For all $\alpha < \lambda$, $\pi_{\alpha\lambda}$ is the direct limit embedding composed with the collapsing map.

$N$ is *iterable* if it is $\alpha$-iterable for all ordinals $\alpha$. If $N$ is a premouse at $\kappa$, we denote $\kappa_i = \pi_{0i}(\kappa)$. Then $\langle N_i, \pi_{ij}, \kappa_i\rangle$ is called the *iteration* of $N$.

The following lemma is proved in the same way as the corresponding results for iterated ultrapowers in Chapter 2.

**Lemma 5.31.** [29] *Suppose $N$ is iterable. Then*

*(a) $\pi_{ij}$ is $\Sigma_1$-elementary and cofinal,*

*(b) $\pi_{ij} \restriction \kappa_i = id \restriction \kappa_i$ and $\pi_{ij}(\kappa_i) = \kappa_j > \kappa_i$,*

[26] Lemma 3.8 of [5].
[27] Lemmas 3.9 and 3.10 of [5].
[28] Definition 3.11 of [5].
[29] Lemma 3.12 of [5].

(c) *For any $x \in N_j$, there are $n < \omega$, $f \in N_i$, $f : [\kappa]^n \to N_i$ and $i \leq \kappa_{\gamma_1} < \cdots < \kappa_{\gamma_n} < j$ such that $x = \pi_{ij}(f)(\kappa_{\gamma_1} < \cdots < \kappa_{\gamma_n})$. Hence, $N_j$ is $\Sigma_0$-generated from $ran(\pi_{ij}) \cup \{\kappa_h : i \leq h < j\}$,*

(d) *$\mathcal{P}(\kappa_i) \cap N_i = \mathcal{P}(\kappa_i) \cap N_j$ for $i \leq j$.*

The following two lemmas will be useful in later proofs.

**Lemma 5.32.** [30] *Suppose $\bar{N} = J_{\bar{\alpha}}^{\bar{U}}$ and $N = J_{\alpha}^{U}$ are iterable premice at $\bar{\kappa}$ and $\kappa$, respectively. Suppose $\sigma : \bar{N} \to N$ is a $\Sigma_1$-embedding. Then there is a unique $\sigma^+ : \bar{N}^+ \to_{\Sigma_1} N^+$ such that $\sigma^+ \circ \pi_{\bar{N}} = \pi_N \circ \sigma$ and $\sigma^+(\bar{\kappa}) = \kappa$.*

*Proof.* Uniqueness follows from Lemma 5.25 and the conditions $\sigma^+ \circ \pi_{\bar{N}} = \pi_N \circ \sigma$ and $\sigma^+(\bar{\kappa}) = \kappa$. Let $\sigma^+$ be defined by $\sigma^+(\pi_{\bar{N}}(f)(\bar{\kappa})) = \pi_N(\sigma(f))(\kappa)$, so the conditions $\sigma^+ \circ \pi_{\bar{N}} = \pi_N \circ \sigma$ and $\sigma^+(\bar{\kappa}) = \kappa$ are immediately satisfied. Suppose $\phi(x)$ is a $\Sigma_1$-formula. Then we have by Łoś's theorem and the assumption on $\sigma$,

$$\begin{aligned} N^+ \vDash \phi(\sigma^+(\pi_{\bar{N}}(f)(\bar{\kappa}))) &\text{ iff } N^+ \vDash \phi(\pi_N(\sigma(f))(\kappa)) \\ &\text{ iff } \{\xi : N \vDash \phi(\sigma(f)(\xi))\} \in U \\ &\text{ iff } \{\xi : \bar{N} \vDash \phi(f(\xi)) \in \bar{U} \\ &\text{ iff } \bar{N}^+ \vDash \phi(\pi_{\bar{N}}(f)(\bar{\kappa})). \end{aligned}$$

□

**Lemma 5.33.** [31] *Let $\bar{N}$, $N$ and $\sigma$ be as in the preceding lemma. Let $f : On \to On$ be monotone. Then there are unique $\sigma_i : \bar{N}_i \to_{\Sigma_1} N_{f(i)}$ such that $\sigma_0 = \pi_{0f(0)} \circ \sigma$, $\sigma_j \circ \bar{\pi}_{ij} = \pi_{f(i)f(j)} \circ \sigma_i$ and $\sigma_i(\bar{\kappa}_i) = \kappa_{f(i)}$ for all $i \leq j$.*

*Proof.* Uniqueness is again clear by the assumptions and 5.31(c). We define $\sigma_i$ by induction on $i$. First we set $\sigma_0 = \pi_{0f(0)} \circ \sigma$. Suppose $\sigma_i$ has been defined. Set $\sigma_{i+1} = \pi_{f(i)+1,f(i+1)} \circ \sigma_i^+$, where $\sigma_i^+$ is given by the preceding lemma. Then by the requirements on $\sigma^+$ in the preceding lemma,

$$\begin{aligned} \sigma_{i+1} \circ \bar{\pi}_{i,i+1} &= \pi_{f(i)+1,f(i+1)} \circ \sigma_i^+ \circ \bar{\pi}_{i,i+1} = \pi_{f(i)+1,f(i+1)} \circ \pi_{f(i),f(i)+1} \circ \sigma_i \\ &= \pi_{f(i),f(i+1)} \circ \sigma_i, \end{aligned}$$

from which condition $\sigma_{i+1} \circ \bar{\pi}_{j,i+1} = \pi_{f(j)f(i+1)} \circ \sigma_i$ follows for all $j \leq i+1$. That $\sigma_{i+1}(\bar{\kappa_j}) = \kappa_{f(j)}$ for $j \leq i+1$ follows from the induction hypothesis and the preceding lemma.

[30] We follow Lemma 3.13 of [5].
[31] We follow Lemma 3.14 of [5].

Suppose then that $\lambda$ is a limit and $\sigma_i$ has been defined for all $i < \lambda$. Let $\tilde{\lambda} = \sup\{f(i) : i < \lambda\}$. Define $\sigma^* : \bar{N}_\lambda \to N_{\tilde{\lambda}}$ so that it satisfies $\sigma^* \circ \bar{\pi}_{i\lambda} = \pi_{f(i)\tilde{\lambda}} \circ \sigma_i$ for $i < \lambda$. Then set $\sigma_\lambda = \pi_{\tilde{\lambda}f(\lambda)} \circ \sigma^*$. Then $\sigma_\lambda$ satisfies $\sigma_\lambda \circ \bar{\pi}_{i\lambda} = \pi_{f(i)f(\lambda)} \circ \sigma_i$ for all $i \leq \lambda$.

To prove the last requirement, pick any $i \leq \lambda$. If $i < \lambda$, then choose any $i'$ such that $i < i' < \lambda$. Then by the requirement on $\sigma^*$ for $i'$, we have $\sigma^*(\bar{\kappa}_i) = \kappa_{f(i)}$ since $\bar{\pi}_{i'\lambda}(\bar{\kappa}_i) = \bar{\kappa}_i$ and $\pi_{f(i')\tilde{\lambda}}(\sigma_{i'}(\bar{\kappa}_i)) = \kappa_{f(i)}$. Thus, $\sigma_\lambda(\bar{\kappa}_i) = \pi_{\tilde{\lambda}f(\lambda)}(\sigma^*(\bar{\kappa}_i)) = \kappa_{f(i)}$. If $i = \lambda$, then again by the requirement on $\sigma^*$, $\sigma^*(\bar{\kappa}_\lambda) = \kappa_{\tilde{\lambda}}$, so $\sigma_\lambda(\bar{\kappa}_\lambda) = \pi_{\tilde{\lambda}f(\lambda)}(\sigma^*(\bar{\kappa}_\lambda)) = \kappa_{f(\lambda)}$. $\square$

We use this result first to show that the set of critical points $\{\kappa_i : i < j\}$ are $\Sigma_1$ indiscernibles in $N_j$.

**Lemma 5.34.** [32] *$\{\kappa_h : i \leq h < j\}$ is a set of $\Sigma_1$ indiscernibles for $\langle N_j, x\rangle_{x \in ran(\pi_{ij})}$, i.e., for any $\Sigma_1$-formula $\phi$, $\bar{x} \in ran(\pi_{ij})^{<\omega}$ and $\kappa_{h_1}, \ldots, \kappa_{h_{2n}}$, $i \leq h_k < j$, $N_j \models \phi(\bar{x}, \kappa_{h_1}, \ldots, \kappa_{h_n})$ holds if and only if $N_j \models \phi(\bar{x}, \kappa_{h_{n+1}}, \ldots, \kappa_{h_{2n}})$ holds.*

*Proof.* Let $\phi$ be a $\Sigma_1$-formula. Let $x \in N_i^{<\omega}$ and $\kappa_{h_0} < \ldots, \kappa_{h_{n-1}} < i \leq h_k < j$ be arbitrary. Define $f : \mathrm{On} \to \mathrm{On}$ by

$$\begin{aligned} f(k) &= k \quad \text{if } k < i, \\ f(i+m) &= h_m \quad \text{if } m \leq n-1, \\ f(i+n+m) &= j+m \quad . \end{aligned}$$

Let $\sigma_i$, $i \in \mathrm{On}$, be the functions given by Lemma 5.33. Since we start from $\mathrm{id} : N_i \to N_i$, the functions are simply $\sigma_i = \pi_{if(i)}$. Then using $\sigma_{i+n}$ we get

$$\begin{aligned} N_j \models \phi(\pi_{ij}(\bar{x}), \kappa_{h_0}, \ldots, \kappa_{h_{n-1}}) &\text{ iff } N_{f(i+n)} \models \phi(\pi_{i,f(i+n)}(\bar{x}), \kappa_{h_0}, \ldots, \kappa_{h_{n-1}}) \\ &\text{ iff } N_{i+n} \models \phi(\pi_{i,i+n}(\bar{x}), \kappa_i, \ldots, \kappa_{i+n-1}). \end{aligned}$$

This equivalence if independent of the choice $h_0, \ldots, h_{n-1}$, so the claim concerning $\Sigma_1$ formulas holds. $\square$

Next we show that a subset of $\{\kappa_i : i < j\}$ is a set of $\Sigma_n$ indiscernibles in $N_j$. We need the following auxiliary definition and lemmas.

**Definition 5.35.** [33] Suppose $i$ and $j$ are ordinal multiples of $\omega^\omega$ and suppose $i_1 < \cdots < i_p < i$ and $j_1 < \cdots < j_p < j$. Let $i_k = \omega^n \bar{\alpha}_k + \bar{\beta}_k$ and $j_k = \omega^n \alpha_k + \beta_k$ for all $k \leq p$. Then $(i_1, \ldots, i_p) \sim_n (j_1, \ldots, j_p) \sim_n$ if and only if for $k, l \leq p$

1. $\bar{\beta}_k = \beta_k$,

[32]We follow Lemma 3.15 of [5].
[33]Definition 3.16 of [5].

2. $\bar{\alpha}_k = \bar{\alpha}_l$ iff $\alpha_k = \alpha_l$.

**Lemma 5.36.** [34] *Suppose $\bar{\imath} \sim_{n+1} \bar{\jmath}$ where $\bar{\imath} < i$ and $\bar{\jmath} < j$, i.e., each $i_k < i$ and $j_k < j$. Suppose $\bar{\jmath}'$ extends $\bar{\jmath}$, i.e., each $j_k$ is $j'_l$ for some $l$, and suppose further that $\bar{\jmath}' < j$. Then there is $\bar{\imath}'$ extending such that $\bar{\imath} \sim_n \bar{\jmath}$ and $\bar{\imath}' < i$.*

*Proof.* Let $\bar{\jmath} = (j'_1, \ldots, j'_{p'})$. Suppose first that $j_k < j'_{l+1} < \ldots j'_{l+r} < j_{k+1}$. Let

$$\begin{aligned} j_k &= \omega^{n+1}\alpha + \omega^n k + \beta, \\ j_{k+1} &= \omega^{n+1}\alpha' + \omega^n k' + \beta', \\ i_k &= \omega^{n+1}\bar{\alpha} + \omega^n k + \beta, \\ i_{k+1} &= \omega^{n+1}\bar{\alpha}' + \omega^n k' + \beta'. \end{aligned}$$

*Case 1*: $\alpha = \alpha'$. Then $\bar{\alpha} = \bar{\alpha}'$ must hold by Definition 5.36. Define $i'_{l+m} = \omega^{n+1}\bar{\alpha} + \omega^n \tilde{k} + \tilde{\beta}$ where $j'_{l+m} = \omega^{n+1}\alpha + \omega^n \tilde{k} + \tilde{\beta}$..

*Case 2*: $\alpha < \alpha'$. Then $\bar{\alpha} < \bar{\alpha}'$ must hold again. Suppose $j'_{l+m} = \omega^{n+1}\alpha_m + \omega^n k_m + \beta_m$. Let $m'$ be greatest such that $\alpha_{m'} \neq \alpha'$. $m'$ must exist by the case hypothesis.

*Case 2a*: $m > m'$. Define $i'_{l+m} = \omega^{n+1}\bar{\alpha}' + \omega^n k_m + \beta_m$.

*Case 2b*: $m \leq m'$. We handle this by induction. Let $k'_0 = k$ and suppose $i'_{l+m}$ is defined. I $k_m = k_{m+1}$, then $i'_{l+m+1} = \omega^{n+1}\bar{\alpha} + \omega^n k'_m + \beta_{m+1}$. Otherwise $i'_{l+m+1} = \omega^{n+1}\bar{\alpha} + \omega^n(k'_m + 1) + \beta_{m+1}$.

If $j_p < j'_{l+1} < \cdots < j'_{p'}$, we use the same definition. The only case that can arise is the case 2b. If $j'_1 \cdots < j_{l'} < j_1$, we can first extend $\bar{\jmath}'$ and $\bar{\imath}$ to $(0, \bar{\jmath}')$ and $(0, \bar{\imath})$, then argue as in the case $j_k < j'_{l+1} < \ldots j'_{l+r} < j_{k+1}$, and finally delete the first component from the resulting tuples $\bar{\imath}''$ and $\bar{\jmath}''$ to get $\bar{\imath}'$ and $\bar{\jmath}'$. □

**Lemma 5.37.** [35] *Suppose $i$ and $j$ are multiples of $\omega^\omega$. If $N_i$ and $N_j$ are iterates of $N$ and $(i_1, \ldots, i_p) = \bar{\imath} \sim_n = \bar{\jmath} = (j_1, \ldots, j_n)$, then*

$$N_i \vDash \phi(\kappa_{i_1}, \ldots, \kappa_{i_p}, \pi_{0i}(\bar{x})) \quad \textit{iff} \quad N_j \vDash \phi(\kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0j}(\bar{x})), \tag{5.1}$$

*where $\bar{x} \in N^{<\omega}$ and $\phi$ is $\Sigma_{i+1}$.*

*Proof.* We prove the claim by induction on $n$.For $n = 0$, pick any $k$ such that $p < k < i, j$. Then there are monotone functions $f^i : \mathrm{On} \to \mathrm{On}$ and $f^j : \mathrm{On} \to \mathrm{On}$ such that $f^i(m) = i_m$ and $f^j(m) = j_m$ for $1 \leq m \leq p$. As in Lemma 5.34 we get

$$\begin{aligned} N_i \vDash \phi(\kappa_{i_1}, \ldots, \kappa_{i_p}, \pi_{0i}(\bar{x})) &\text{ iff } N_k \vDash \phi(\kappa_1, \ldots, \kappa_p, \pi_{0k}(\bar{x})) \\ &\text{ iff } N_j \vDash \phi(\kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0j}(\bar{x})). \end{aligned}$$

[34]We follow Lemma 3.17 of [5].
[35]We follow Lemma 3.18 of [5].

For $n > 0$, suppose $N_j \vDash \exists y \phi(y, \kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0j}(\bar{x}))$, where $\phi$ is $\Pi_n$. Then for some $\kappa_{j_1^*}, \ldots, \kappa_{j_r^*}$ and $f \in \kappa^{[r]} \to N$

$$N_j \vDash \phi(\pi_{0j}(f)(\kappa_{j_1^*}, \ldots, \kappa_{j_r^*}), \kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0j}(\bar{x})),$$

so there is a rudimentary function $t$ such that

$$N_j \vDash \phi(t(\pi_{0j}(f), \kappa_{j_1^*}, \ldots, \kappa_{j_r^*}), \kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0j}(\bar{x}))$$

Let $\bar{j}' = (j_1', \ldots, j_l')$ list $\{\kappa_{j_1^*}, \ldots, \kappa_{j_r^*}, \kappa_{j_1}, \ldots, \kappa_{j_p}\}$ in ascending order. Then $\bar{j}'$ extends $\bar{j}$ so by Lemma 5.36 there is $\bar{i}'$ extending $\bar{i}$ such that $\bar{i}' \sim_{n-1} \bar{j}'$. For $1 \leq k \leq r$, let $i_k^*$ be the $i_h$ such that $j_h = j_k^*$. The induction hypothesis holds also for $\Pi_n$ formulas since the negation of a $\Pi_n$ formula is $\Sigma_n$. So by the induction hypothesis we get

$$N_i \vDash \phi(t(\pi_{0i}(f), \kappa_{i_1^*}, \ldots, \kappa_{i_r^*}), \kappa_{i_1}, \ldots, \kappa_{i_p}, \pi_{0i}(\bar{x})).$$

Hence,

$$N_i \vDash \exists y \phi(y, \kappa_{i_1}, \ldots, \kappa_{i_p}, \pi_{0i}(\bar{x})),$$

which proves the induction step by symmetry. $\square$

As an immediate corollary we get:

**Corollary 5.38.** [36] *Let $i$ be a multiple of $\omega^\omega$ and let $\phi$ be a $\Sigma_{n+1}$ formula. If $(i_1, \ldots, i_p) \sim_n (j_1, \ldots, j_p)$ and $i_p, j_p < i$, then*

$$N_i \vDash \phi(\kappa_{i_1}, \ldots, \kappa_{i_p}, \pi_{0i}(\bar{x})) \leftrightarrow \phi(\kappa_{j_1}, \ldots, \kappa_{j_p}, \pi_{0i}(\bar{x})).$$

The following corollary is useful to us Sections 5 and 6.

**Corollary 5.39.** [37] *If $i$ is a multiple of $\omega^\omega$ and $C = \{\kappa_j : j < i$ and $j$ is a multiple of $\omega^n\}$, then $C$ is a set of $\Sigma_n$ indiscernibles for $\langle N_i, x\rangle_{x \in ran(\pi_{0i})}$.*

*Proof.* Suppose $i_1, \ldots, i_p, j_1, \ldots, j_p$ are all multiples of $\omega^n$. If $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$, then necessarily $(i_1, \ldots, i_p) \sim_n (j_1, \ldots, j_p)$. Hence the claim follows from the previous lemma. $\square$

As in the case of iterated ultrapowers, we can show that if $U$ is countably closed, then $J_\alpha^U$ is iterable. This could be proved similarly as in Chapter 2 but we will follow the proof in [5] which is based on a modification of Kunen's original definition of iterated ultrapowers in [13].

[36] Corollary 3.19 of [5].
[37] Corollary 3.20 of [5]. The short proof is our own.

**Lemma 5.40.** [38] *Suppose $N = J^U_\alpha$ is a premouse at $\kappa$ and $U$ is countably closed, i.e., for any $X_i$, $i < \omega$, such that each $X_i$ is in $U$, their intersection $\bigcap_{i<\omega} X_i$ is in $U$. Then $N$ is iterable.*

*Proof.* First we define $U^k$ for all $k < \omega$ by:

$$U^0 = \{\emptyset\},$$
$$U^{k+1} = \{X \in \mathcal{P}(\kappa^{n+1}) \cap N : \{\xi : \{\bar{v} : (\xi, \bar{v}) \in X\} \in U^k\} \in U\}.$$

For every ordinal $i$, let $V_i = \{f : \kappa^u \to N : u \subset i \text{ and } u \text{ is finite}\}$. Suppose $\phi$ is a $\Sigma_1$- formula and suppose $u_1, \dots, u_m$ are finite subsets of $i$ and $f_h : \kappa^{u_h} \to N$ for $h \leq m$. Let $u = \bigcup_{h=1}^m u_h$ and let $\{j_1, \dots, j_n\}$ enumerate $u$ in increasing order. Suppose $u_h = \{j_{q_1}, \dots, j_{q_r}\}$ where $q_1 < \dots < q_r$. For $\bar{v} = (v_1, \dots, v_n) \in \kappa^n$, define

$$f_h^{*u}(v_1, \dots, v_n) = f_h((j_{q_1}, v_{q_1}), \dots, (j_{q_r}, v_{q_r})).$$

Then we define the relation

$$T_i^\phi(f_1, \dots, f_m) \quad \text{iff} \quad \{\bar{v} : N \vDash \phi(f_1^{*u}(\bar{v}), \dots, f_m^{*u}(\bar{v}))\} \in U^k,$$

where $\bar{v} = (v_1, \dots, v_k)$.

We define an equivalence relation $\sim$ on $V_i$ by

$$f \sim g \quad \text{iff} \quad T_i^{'x=y'}(f, g).$$

We let $\tilde{V_i}$ be the set of equivalence classes and say the relation $E_i$ by

$$[f] E_i [g] \quad \text{iff} \quad T_i^{'x\in y'}(f, g).$$

Now $\tilde{V_i}$ is just a slight modification of Kunen's definition of the $i$-th iterated ultrapower and the definition of premouse iteration is a modification of the iterated ultrapowers. Thus, the corresponding modification of Theorem 2.11 in [13] shows that if $E_i$ is well-founded, then $\langle \tilde{V_i}, E_i \rangle$ is isomorphic to $N_i$, the $i$-th premouse iterate of $N$. Hence, if $\langle N_i', \in \rangle$ is the transitive collapse of $\langle \tilde{V_i}, E_i \rangle$, then $\langle N_i', \in \rangle = \langle N_i, \in \rangle$ by the uniqueness of the transitive collapse.

So we have to show that $E_i$ is well-founded. Suppose it is not. Then there are $[f_h]$, $h < \omega$, such that $[f_{h+1}] E_i [f_h]$ for all $h$. Then $T_i^{'x\in y'}(f_{h+1}, f_h)$ for all $h$. Define for $h < \omega$

$$Y_h = \{\bar{v} : N \vDash f_{h+1}^{*\hat{u}_h}(\bar{v}) \in f_h^{*\hat{u}_h}(\bar{v})\} \in U^{l(h)}$$

[38] We follow Lemma 3.22 of [5]. The connection to Kunen's definition is not mentioned in [5].

where $f_h : \kappa^{u_h} \to N$, $\hat{u}_h = u_h \cup u_{h+1}$ and $\hat{u}_h = \{j_1, \ldots, j_{l(h)}\}$. By the definition of $U^{l(h)}$ there are $Y_h^1, \ldots, Y_h^{l(h)} \in U$ so that if $v_k \in Y_h^k$ for all $1 \leq k \leq l(h)$, then $\bar{v} \in Y_h$. Now $Y_h^* = \bigcap_{k=1}^{l(h)}$ is in $U$, so $Y = \bigcap_{n \in \omega} Y_h^*$ is nonempty.

So we can pick some $\delta \in Y$. Then for all $h$ $f_{h+1}^{*\hat{u}_h}(\delta \ldots \delta) \in f_h^{*\hat{u}_h}(\delta \ldots \delta)$. Because $f_{h+1}^{*\hat{u}_h}(\delta \ldots \delta) = f_{h+1}^{*\hat{u}_{h+1}}(\delta \ldots \delta)$, there is an infinite descending sequence in $V$, a contradiction. □

The first consequence of the preceding lemma is the following useful result.

**Lemma 5.41.** [39] *Suppose $N = J_\alpha^U$ is a premouse and $U$ is countably closed. Suppose $B$ is a rudimentary relation over $N$ and well-founded. If $B_i$ is defined over $N_i$ with the same rudimentary definition, then $B_i$ is well-founded.*

*Proof.* Let $\phi$ be the formula defining $B$ and the $B_i$'s. The proof of well-foundedness of $E_i$ in the previous proof works for $B_i$ with $T_i^\phi$ in place of $T_i^{\prime x \in y^\prime}$. □

**Lemma 5.42.** [40] *Suppose $N$ is iterable, $\bar{N}$ a premouse and $\sigma : \bar{N} \to_{\Sigma_0} N$. Then $\bar{N}$ is iterable and the conclusion of Lemma 5.33 holds for $\Sigma_0$ functions.*

*Proof.* We prove by induction that $\bar{N}$ is $i$-iterable for all $i$ and there is a $\Sigma_0$ embedding $\sigma_i : \bar{N}_i \to N_i$. Let $\sigma_0 = \sigma$.

*Claim.* Suppose $\bar{N}_i$ exists and $\sigma_i : \bar{N}_i \to_{\Sigma_0} N_i$. Then $\bar{N}_i^+$ exists and there is a unique $\sigma_i^+ : \bar{N}_i^+ \to_{\Sigma_0} N_i^+$ such that $\sigma_i^+ \circ \pi_{\bar{N}_i} = \pi_{N_i} \circ \sigma_i$ and $\sigma_i^+(\bar{\kappa}_i) = \kappa_i$.

*Proof.* For any $\Sigma_0$ formula $\phi$, $\tilde{N}_i \vDash \phi([f])$ if and only if $\{\nu : N_i \vDash \phi(f(\nu))\} \in U_i$. Define $\tilde{\sigma}_i : \tilde{\bar{N}} \to \tilde{N}$ by $\tilde{\sigma}_i([f]_{\tilde{\bar{N}}}) = [\sigma_i(f)]_{\tilde{N}}$. Then $\tilde{\bar{N}}_i$ must be well-founded because otherwise $\tilde{\sigma}_i$ shows that $\tilde{N}_i$ is not well-founded. Hence $\bar{N}_i^+$ exists. Define $\sigma_i^+ : \bar{N}_i^+ \to_{\Sigma_0} N_i^+$ by $\sigma_i^+ = g_{N_i} \circ \tilde{\sigma}_i \circ g_{\bar{N}_i}^{-1}$. □ *Claim.*

If $\sigma_i : \bar{N}_i \to_{\Sigma_0} N_i$ is defined, then we define $\sigma_{i+1} : \bar{N}_{i+1} \to_{\Sigma_0} N_{i+1}$ by $\sigma_{i+1} = \sigma_i^+$.

Now suppose $\lambda$ is a limit, $\bar{N}$ is $i$-iterable and $\sigma_i$ exists for all $i < \lambda$. Then $N_\lambda$ is the direct limit of $\langle N_i, \pi_{ij} \rangle_{i<j<\lambda}$. Since for all $i < j < \lambda$, $\sigma_j \circ \bar{\pi}_{ij} = \pi_{ij} \circ \sigma_i$, the direct limit of $\langle \bar{N}_i, \pi_{ij} \rangle_{i<j<\lambda}$ must be well founded. Otherwise the infinite descending sequence of elements of $N_\lambda$ would yield an infinite descending sequence of elements of some $\bar{N}_i$. Hence, $\bar{N}_\lambda$ exists. Then $\sigma_\lambda$ can be defined by $\sigma_\lambda \circ \bar{\pi}_{i\lambda} = \pi_{i\lambda} \circ \sigma_i$. The definition works since $\bar{N}_\lambda$ and $N_\lambda$ are direct limits.

The conclusion of Lemma 5.33 is proved by a similar argument and is omitted for brevity.

□

[39] Lemma 3.23 of [5].

[40] We follow Lemma 3.24 of [5].

The following important lemma allows as to define a prewellordering on the class of all premice. Its proof uses the same argument as to the proof of Lemma 3.9 so we omit it.

**Lemma 5.43.** [41] *Let $N$ be an iterable premouse at $\kappa$. Suppose $\theta$ is a regular cardinal in $V$ and $\theta > |\kappa^\kappa \cap N|$. Then $N_\theta = J_\alpha^{F_\theta}$ where $F_\theta$ is the club filter on $\theta$.*

An immediate consequence is the following:

**Corollary 5.44.** [42] *Suppose $M$ and $N$ are iterable premice. Then there are iterates $\bar{M}$ and $\bar{N}$ of $M$ and $N$, respectively such that either $\bar{M} \in \bar{N}$ or $\bar{M} = \bar{N}$ or $\bar{N} \in \bar{M}$.*

**Definition 5.45.** [43] The partial order $<_{pm}$ is defined on the class of all premice by $M <_{pm} N$ if for some iterates $\bar{M}, \bar{N}$ of $M, N$, respectively, it holds that $\bar{M} \in \bar{N}$. The equivalence relation $\approx$ is defined on iterable premice by $M \approx N$ if for some $\theta$, $M_\theta = N_\theta$.

If $M, N$ are iterable premice with iterates $\bar{M}, \bar{N}$ satisfying $\bar{M} \in \bar{N}$, then for all $\alpha$, $\bar{M}_\alpha$ is a proper subset of $\bar{N}_\alpha$. Thus, by lemma 5.43, for any regular $\theta > |\bar{N}^\kappa \cap \bar{N}|$, $\bar{M}_\theta \in \bar{N}_\theta$. Hence the above definition makes sense and $<_{pm}$ is a well-ordering on the equivalence classes of $\approx$.

We conclude the section with a lemma that is needed in the proofs concerning mice.

**Lemma 5.46.** [44] *Suppose $M \approx N$, say $Q = N_\theta = M_\theta$. Let $\langle M_i, \bar{\kappa}_i, \bar{\pi}_{ij} \rangle$ and $\langle N_i, \kappa_i, \pi_{ij} \rangle$ be the respective iterations. If ran$(\bar{\pi}_{0\theta}) \subset$ ran$(\pi_{0\theta})$, then $N$ is an iterate of $M$.*

*Proof.* Suppose $N = J_\alpha^U$ is a premouse at $\kappa$. Suppose $\kappa \neq \bar{\kappa}_\xi$ for all $\xi$. We show that $\kappa \in \mathrm{ran}(\pi_{0\theta})$. If $\kappa < \bar{\kappa}$, then immediately $\kappa \in \mathrm{ran}(\bar{\pi}_{0\theta}) \subset \mathrm{ran}(\pi_{0\theta})$. So suppose $\kappa > \bar{\kappa}$. Then there is $\xi < \theta$ such that $\bar{\kappa}_\xi < \kappa < \bar{\kappa}_{\xi+1}$.

By Lemma 5.31(c) there are $f : \bar{\kappa}^{[n]} \to M$ in $M$ and $\bar{\kappa}_{\xi_1} < \cdots < \bar{\kappa}_{\xi_n} < \bar{\kappa}_{\xi+1}$ such that $M_{\xi+1} \vDash \kappa = \bar{\pi}_{0,\xi+1}(f)(\bar{\kappa}_{\xi_1}, \ldots, \bar{\kappa}_{\xi_n})$. Then

$$Q \vDash \kappa = \bar{\pi}_{0\theta}(f)(\bar{\kappa}_{\xi_1}, \ldots, \bar{\kappa}_{\xi_n}). \tag{5.2}$$

Because $\mathrm{ran}(\bar{\pi}_{0\theta}) \subset \mathrm{ran}(\pi_{0\theta})$, there must be in $N$ a function $f' : \kappa^{[n]} \to N$ such that $\pi_{0\theta}(f') = \bar{\pi}_{0\theta}(f)$. But $\bar{\kappa}_{\xi_1}, \ldots, \bar{\kappa}_{\xi_n} < \kappa$, so $N \vDash y = f'(\bar{\kappa}_{\xi_1}, \ldots, \bar{\kappa}_{\xi_n})$ for some $y \in N$. Hence, $\pi_{0\theta}(y) = \pi_{0\theta}(f')(\bar{\kappa}_{\xi_1}, \ldots, \bar{\kappa}_{\xi_n})$ , so $\kappa \in \mathrm{ran}(\pi_{0\theta})$, which is a contradiction. Thus, $\kappa = \bar{\kappa}_\xi$ for some $\xi < \theta$.

[41] Lemma 3.25 of [5].
[42] Corollary 3.26 of [5].
[43] Definition 3.27 of [5].
[44] We follow lemma 3.28 of [5].

We show that $N = M_\xi$. Let $M_\xi = J^{U_\xi}_{\alpha'}$. Then, as with iterated ultrapowers, we have

$$X \in N \cap U \text{ iff } \kappa \in \pi_{0\theta}(X)$$

and

$$X' \in M_\xi \cap U_\xi \text{ iff } \bar{\kappa}_\xi \in \bar{\pi}_{\xi\theta}(X').$$

But if $X' \in M_\xi \cap U_\xi$, then $\bar{\pi}_{\xi\theta}(X') \in \operatorname{ran}(\pi_{0\theta})$, so there is $Y \in N$ such that $\bar{\pi}_{\xi\theta}(X') = \pi_{0\theta}(Y)$. But then $Y \in N \cap U$, so

$$X' = \bar{\pi}_{\xi\theta}(X') \cap \kappa = \pi_{0\theta}(Y) \cap \kappa = Y.$$

Hence, $M_\xi \cap U_\xi \subset N \cap U$, so $M_\xi \cap U_\xi \subset M \cap U$, which implies that $M_\xi \cap U_\xi = M \cap U$ since $U_\xi$ is an ultrafilter on $\kappa$ in $M_\xi$. Thus, in fact, $M_\xi = J^U_{\alpha'}$. Then we must have $\alpha' = \alpha$ because otherwise either $N \in M_\xi$ or $M_\xi \in N$ contradicting the assumption $M \approx N$. Hence, $N = M_\xi$. □

## 5.3 Soundness

This section presents some lemmas that are related to soundness or $\Sigma_1$ Skolem hulls of premice. They will be needed in the definition of mice and the core model in the last two sections.

**Definition 5.47.** [45] Suppose $N = J^U_\alpha$ is a premouse. $N$ is *sound* if $N = h_N(j_{\rho_N} \cup \{p_N\})$. $\delta_N$ is the $<_N$-least $\delta \leq \alpha$ such that $U \subset J^U_\delta$.

We can assume that for a premouse $N = J^U_\alpha$, $U = U \cap N$, so the definition of $\delta_N$ makes sense. The following lemma is important in the proofs of this section.

**Lemma 5.48.** [46] *If $\rho_N \geq \delta_N$, then $N^* = J^U_{\rho_N}$.*

*Proof.* By Lemmas 5.13 and 5.19 we have

$$N^* = H^N_{\omega\rho_N} = \bigcup_{\substack{\nu < \rho_N \\ a \subset \nu,\ a \in N}} J^a_{\rho_N}.$$

Because $\rho_N \geq \delta_N$ and $\rho_N$ is a limit ordinal, every member of $U$ is a subset of some $\nu < \rho_N$. Every member of $U$ is in $J^U_\beta$ for some $\beta < \rho_N$. Since $\rho_N$ is a $\Sigma_1$-cardinal in $N$, $\beta + \rho_N = \rho_N$. This implies that $J^U_{\rho_N} \subset H^N_{\omega\rho_N}$.

On the other hand, if $a \subset \nu < \rho_N$ and $a$ is in $N$, then either $\kappa \cap a$ or $\kappa \setminus a$ must be in $U$ as $N$ thinks that $U$ is a normal ultrafilter on $\kappa$. Hence, $\kappa \cap a$ or $\kappa \setminus a$ is in $J^U_\beta$ for some $\beta < \rho_N$, so $a \in J^U_{\beta+1}$. Since $\rho_N$ is a $\Sigma_1$-cardinal in $N$, $J^a_{\rho_N}$ is included in $J^U_{\rho_N}$. □

[45] Definition 4.1 of [5].

[46] Result mentioned on p. 60 of [5]. The proof is our own.

**Definition 5.49.** [47] Suppose $N = J^U_\alpha$ is a premouse. Then we define

(i) $N^{(0)} = N$; $N^{(i+1)} = (N^{(i)})^*$,

(ii) $\rho^0_N = \alpha$; $\rho^{i+1}_N = \rho_{N^{(n)}}$,

(iii) $p^0_n = \emptyset$ ; $p^{i+1}_N = p_{N^{(i)}}$,

(iv) $A^0_N = U$ ; $A^{i+1}_N = A_{N^{(i)}}$.

$N$ is *n-sound* if $N^{(i)}$ is sound for every $i < n$.

By Lemma 5.19, $N^{(j)} \subset N^i$ holds for all $i \leq j$.

For $\alpha > \delta_N$, the function $F^U$ is not needed in the construction of the levels $J^U_\alpha$ so the following lemmas can be proved as in the case of the $J_\alpha$-hierarchy. The proofs can be found in the standard fine-structure theoretical sources [10], [2] and [3] and they are omitted for brevity. The lemmas use the concept of a $\Sigma_n$ master code, defined below:

**Definition 5.50.** [48] A $\Sigma_n$ *master code* for $J^A_\alpha$ is a set $B \in \Sigma_n(J^A_\alpha)$ such that, setting $\rho = \rho^n_{J^A_\alpha}$, $B \subset J_\rho$ and

$$\Sigma_m(J^B_\rho) = \mathcal{P}(J^B_\rho) \cap \Sigma_{n+m}(J^A_\alpha).$$

for $m \geq 1$ and $n, \alpha \geq 0$.

**Lemma 5.51.** [49] *Suppose $N = J^U_\alpha$ and $\rho_N \geq \delta_N$. Then $N$ is sound and $A_N$ is a $\Sigma_1$ master code for $N$.*

**Lemma 5.52.** [50] *Suppose $N$ is as in Lemma 5.51 and $\pi : M \to_{\Sigma_1} N^*$. Then*

*(i) There is a unique $\bar{N}$ such that $\bar{N}$ is sound and $M = \bar{N}^*$.*

*(ii) There is a unique $\tilde{\pi} \supset \pi$ such that $\tilde{\pi} : \bar{N} \to_{\Sigma_1} N$ and $\tilde{\pi}(p_{\bar{N}}) = p_N$.*

*(iii) If $\pi : M \to_{\Sigma_i} N^*$, then $\tilde{\pi} : \bar{N} \to_{\Sigma_{i+1}} N$.*

*(iv) $\rho_{\bar{N}} \geq \delta_{\bar{N}}$.*

**Lemma 5.53.** [51] *Suppose $\bar{N} = J^{\bar{U}}_{\bar{\alpha}}$ is a premouse with $\rho_{\bar{N}} \geq \delta_{\bar{N}}$. Then there is a well-founded relation $\bar{E} \subset J_{\rho_{\bar{N}}}$ uniformly rudimentary in $A_{\bar{N}}$ such that for $\pi : \bar{N}^* \to_{\Sigma_1} M$, $E$ defined over $M$ with the same rudimentary definition as $\bar{E}$ over $\bar{N}^*$, if $E$ is well-founded, then*

[47] Definition 4.5 of [5].

[48] We modify the definition for $J_\alpha$ on p. 260 of [10].

[49] Lemma 4.2 of [5]. Soundness follows from the proof of Lemma 11, Chapter 7 of [1]. Master code is proved in Theorem 14 of Chapter 7 of [1] and in Lemma 3.4 of [10].

[50] Lemma 4.3 of [5]. Part of Theorem 4.1 of [10] and part of Lemma 20 in Chapter 7 of [1].

[51] Lemma 4.4 of [5]. This follows from the proof of Lemma 3 in [3].

*(i) There is a unique $N$ such that $N$ is sound and $M = N^*$.*

*(ii) There is a unique $\tilde{\pi} \supset \pi$ such that $\tilde{\pi} : \bar{N} \to_{\Sigma_1} N$ and $\tilde{\pi}(p_{\bar{N}}) = p_N$.*

*(iii) If $\pi : \bar{N}^* \to_{\Sigma_i} M$, then $\tilde{\pi} : \bar{N} \to_{\Sigma_{i+1}} N$.*

*(iv) $\rho_N \geq \delta_N$.*

The following lemma iterates the results of Lemmas 5.51-5.53.

**Lemma 5.54.** [52] *Suppose $N = J^U_\alpha$ is a premouse with $\rho^n_N \geq \delta_N$. Then*

*(a)* [53] *$N$ is $n$-sound and $A^n_N$ is a $\Sigma_n$ master code for $N$.*

*(b)* [54]*Suppose $\pi : M \to_{\Sigma_1} N^{(n)}$. Then*

*(i) There is a unique $\bar{N}$ such that $\bar{N}$ is $n$-sound and $M = \bar{N}^{(n)}$.*

*(ii) There is a unique $\tilde{\pi} \supset \pi$ such that*

$$\tilde{\pi} : \bar{N} \to_{\Sigma_1} N, \quad \tilde{\pi} \restriction \bar{N}^j : \bar{N}^j \to_{\Sigma_1} N^{(j)} \quad \text{and} \quad \tilde{\pi}(p^j_{\bar{N}}) = p^j_N$$

*for $j \leq n$.*

*(iii) If $\pi : M \to_{\Sigma_i} N^{(n)}$, then $\tilde{\pi} \restriction \bar{N}^{(j)} : \bar{N}^{(j)} \to_{\Sigma_{i+n-j}} N^{(j)}$, for $j \leq n$.*

*(iv) $\rho^n_{\bar{N}} \geq \delta_{\bar{N}}$.*

*(c)* [55] *There are relations $\bar{E}_1, \ldots, \bar{E}_n \subset J_{\rho^n_N}$ uniformly rudimentary in $A^n_N$ such that for $\pi : N^{(n)} \to_{\Sigma_1} M$ and $E_i$ defined over $M$ with the same rudimentary definition, if $E_1, \ldots, E_n$ are well-founded, then*

*(i) There is a unique $\bar{N}$ such that $\bar{N}$ is $n$-sound and $M = \bar{N}^{(n)}$.*

*(ii) There is a unique $\tilde{\pi} \supset \pi$ such that*

$$\tilde{\pi} : N \to_{\Sigma_1} \bar{N},\ \tilde{\pi} \restriction N^{(j)} \to_{\Sigma_1} \bar{N}^{(j)} \text{ and } \tilde{\pi}(p^j_N) = p^j_{\bar{N}} \text{ for } j \leq n.$$

*(iii) If $\pi : N^{(n)} \to_{\Sigma_i} M$, then $\tilde{\pi} \restriction N^{(j)} : N^{(j)} \to_{\Sigma_{i+n-j}} \bar{N}^{(j)}$ for $j \leq n$.*

*(iv) $\rho^n_{\bar{N}} \geq \delta_{\bar{N}}$.*

The following two lemmas are useful in the proofs of Section 5.4.

**Lemma 5.55.** [56] *Suppose $N = J^U_\alpha$ is a premouse at $\kappa$, $\rho^n_N \geq \delta_N > \kappa \geq \rho^{n+1}_N$ and $J^U_{\delta_N} \models \forall \nu\, (|\nu| \leq \kappa)$. Then $h_{N^{(n)}}(J_\kappa \cup p^{n+1}_N) = N^{(n)}$.*

[52] Lemma 4.6 of [5].
[53] Proved in Lemma 3.4 of [10] and Theorem 14 of [1].
[54] Theorem 4.1 of [10] and Lemma 20 in Chapter 7 of [1].
[55] This is based on an iteration of Lemma 3 of [3].
[56] We follow Lemma 4.7 of [5].

*Proof.* To simplify notation we let $h = h_{N^{(n)}}$, $p = p_N^{n+1}$, $\rho = \rho_N^n$ and $A = A_N^n$. Let $X = h(J_\kappa \cup p)$ and let $\pi : M \cong \langle X, U \cap X\rangle$, where $M$ is transitive. Then $M = J_{\bar\beta}^{\bar U}$ for some $\bar\beta, \bar U$. Then by Lemma 5.54(b) there is a unique $\bar N = J_{\bar\alpha}^{\bar U}$ such that $\bar N$ is $n$-sound and $\bar N^{(n)} = M$. Since $\rho_N^n \geq \delta_N$, $N^{(n)} = J_{\rho_N^n}^U$ by Lemma 5.48.

Since $J_{\delta_N}^U \models \forall\nu\,(|\nu| \leq \kappa)$, for each $\nu < \omega\delta_N$ there is a surjection $f_\nu \in N^{(n)}$ from $\kappa$ onto $\nu$. Thus, if $\nu \in \omega\delta_N \cap X$, the $\Sigma_1$-definability of the $<_{N^{(n)}}$-least surjection from $\kappa$ onto $\nu$ shows that every ordinal below $\nu$ must be in $\omega\delta_N \cap X$. Hence, $\omega\delta_N \cap X$ is transitive. Let $\omega\bar\delta$ be $\omega\delta_N \cap X$. Then transitivity implies that $\pi \restriction \omega\bar\delta = \mathrm{id} \restriction \omega\bar\delta$, so $\pi(\omega\bar\delta) \geq \omega\delta_N$. Hence, $\pi \restriction J_{\bar\delta}^{\bar U} = \mathrm{id} \restriction J_{\bar\delta}^{\bar U}$. Since $\pi(\omega\bar\delta) \geq \omega\delta_N$, $\bar U \subset U \cap J_{\bar\delta}^{\bar U}$. For every $x \in M$, $x \in \bar U$ if and only if $\pi(x) \in U$, so $\bar U = U \cap J_{\bar\delta}^{\bar U}$. Hence, $J_{\bar\delta}^{\bar U} = J_{\bar\delta}^{U}$, so $\bar U = U \cap J_{\bar\delta}^{U}$.

Suppose $\bar\alpha < \alpha$. Since $M$ is isomorphic to $X$ which is a $\Sigma_1$ elementary submodel of $N^{(n)}$, $M \models \phi(x)$ if and only if $N^{(n)} \models \phi(\pi(x))$ for any $x \in M$ and $\Sigma_1$ $\phi$. Hence $A_{N^{(n)}} \subset J_{\rho_N^{n+1}}$ is $\Sigma_1$ definable over $M$ and thus $\Sigma_{n+1}$ definable over $\bar N$. Because $\bar N \in N$, $A_{N^{(n)}} \in N$. Since $A_{N^{(n)}} \subset \kappa$, the definition of $\delta_N$ implies that $A_{N^{(n)}} \in J_{\delta_N}^U \subset N^{(n)}$, a contradiction. If $\alpha < \bar\alpha$, a similar argument for $A_{\bar N^{(n)}}$ yields a contradiction. Hence, $\bar\alpha = \alpha$, so $\bar N = J_\alpha^{\bar U}$.

Next we show that $\bar U \cap \bar N = U \cap N$. If not, then there is $\gamma \geq \bar\delta$ such that $\gamma < \delta_N$ and $\mathcal{P}(\kappa) \cap J_{\gamma+1}^U \not\subset J_\gamma^U$. Let $\gamma$ be the least such ordinal. Suppose $\mathcal{P}(\kappa) \cap \Sigma_\omega(J_\gamma^U) \subset J_\gamma^U$. Let $\tilde M$ be $J_{\gamma+1}^{U \cap J_\gamma^U} = \Sigma_\omega(J_\gamma^U)$. $\tilde M$ is transitive, so for any $x \in \tilde M$, $x \cap (\tilde M \cap U) = x \cap U$. If $y \in x \in \tilde M$, then $y \in \Sigma_\omega(J_\gamma^U) \subset J_\gamma^U$. Hence, $x \cap U = x \cap (U \cap J_\gamma^U)$. Since $x \in S_{\omega\gamma+k}^{U \cap J_\gamma^U}$ for some $k$, $x \cap (U \cap J_\gamma^U) \in \tilde M$. Hence, for any $x \in \tilde M$, $x \cap (U \cap \tilde M) \in \tilde M$, i.e., $\langle \tilde M, U \cap \tilde M\rangle$ is amenable. If $x$ is in $S_{\omega\gamma+k}^U$ and $x \in \tilde M$, then by amenability $x \cap U \in \tilde M$. Thus, since $J_{\gamma+1}^U = \bigcup_{k<\omega} S_{\omega\gamma+k}^U$, we see by induction on $k$ that every element of $J_{\beta+1}^U$ must be in $\tilde M$. On the other hand, $\tilde M = \mathrm{rud}_{U \cap M}(M)$ is obviously a subset of $\mathrm{rud}_U(M) = J_{\gamma+1}^U$, so $J_{\gamma+1}^{U \cap J_\gamma^U} = J_{\gamma+1}^U$. But then $\mathcal{P}(\kappa) \cap J_{\gamma+1}^U \subset \mathcal{P}(\kappa) \cap \Sigma_\omega(J_\gamma^U) \subset J_\gamma^U$, which contradicts the definition of $\gamma$.

So there is $a \subset \kappa$ such that $a \in \Sigma_\omega(J_\gamma^U) \setminus J_\gamma^U$. But because $\gamma+1$ is the least ordinal $\geq \bar\delta$ such that $J_{\gamma+1}^U$ has a new subset of $\kappa$, $J_\gamma^U = J_\gamma^{\bar U}$. Hence, $a \in \Sigma_\omega(J_\gamma^{\bar U})$, so $a \in J_\alpha^{\bar U}$. Thus, either $a$ or $\kappa \setminus a$ is in $\bar U \cap \bar N \setminus J_\gamma^{\bar U}$. But $\gamma \geq \bar\delta$, a contradiction. Hence, $\bar U \cap \bar N = U \cap N$, so $\bar N = N$. In particular, $\bar N^{(n)} = N^{(n)}$ and $\bar p = \pi^{-1}(p) = p$. Hence, $\pi(h_{\bar N}(i, x, \bar p)) = h_N(i, x, p)$ for all $x \in J_\kappa$, so $\pi$ is the identity and $X = N^{(n)}$.

□

**Lemma 5.56.** [57] *Let $N = J_\alpha^U$ be a premouse at $\kappa$.*

*(a) $\delta_N > \kappa$ and $J_{\delta_N}^U \models \forall\nu\,(|\nu| \leq \kappa)$.*

[57] The proof follows Lemma 4.8 of [5].

*(b) If $\rho_N^n > \kappa$, then $N$ is $n$-sound.*

*(c) If $\rho_N^n > \kappa$, then $\rho_N^n \geq \delta_N$ and $N^{(n)} = J^U_{\rho_{N^n}}$.*

*(d) If $\rho_N^{n+1} \leq \kappa < \rho_N^n$, then $h_{N^{(n)}}(\kappa \cup p_N^{n+1}) = N^{(n)}$.*

*Proof.* Suppose (a) holds. If $\rho_N^n < \delta_N$, then $J^U_{\delta_N} \models |\rho_N^n| \leq \kappa$, so $\rho_N^n \leq \kappa$. Thus, if $\rho_N^n > \kappa$, then $\rho_N^n \geq \delta_N$. Hence, (a) implies (c), so by Lemma 5.54(a), (a) implies (b). Moreover, if $\rho_N^n > \kappa$, since there is a $\Sigma_1$ surjection from $\kappa$ to $J_\kappa$ in $N^{(n)} = J^U_{\rho_N^n}$, $h_{N^{(n)}}(\kappa \cup p_N^{n+1}) = h_{N^{(n)}}(J_\kappa \cup p_n^{n+1}) = N^{(n)}$ by the previous lemma. Hence, (a) implies (d) as well, so we only need to prove (a).

$\delta_N > \kappa$ is clear since $J^U_\kappa$ only has bounded subsets of $\kappa$. We prove the other claim in (a) by induction on $\alpha$. So suppose the claim holds for all $\beta < \alpha$. Suppose $\nu < \delta_N$. We show that $J^U_{\delta_N} \models |\nu| \leq \kappa$. Pick the least $\gamma \geq \nu$ such that $\mathcal{P}(\kappa) \cap J^U_{\gamma+1} \not\subset J^U_\gamma$. Then $\gamma < \delta_N$. As in the previous lemma $\mathcal{P}(\kappa) \cap \Sigma_\omega(J^U_\gamma) \not\subset J^U_\gamma$. Then setting $M = J^U_\gamma$, there is $n$ such that $\rho_M^n > \kappa \geq \rho_M^{n+1}$. By the induction hypothesis $|\nu|^{J^U_{\delta_N}} \leq |\nu|^{J^U_{\gamma+1}} \leq \kappa$ since $\nu < \gamma + 1$. $\square$

The following lemma shows that if there is a bounded subset of $\kappa$ in $J^U_{\alpha+1} \setminus J^U_\alpha$, then $\rho^n_{J^U_\alpha} < \kappa$ for some $n$.

**Lemma 5.57.** [58] *Let $N = J^U_{\beta+1}$ be iterable, so $M = J^U_\beta$ is iterable by Lemma 5.42, too. If $\rho_M^n = \kappa$ for some $n$, then $H^M_\kappa = H^N_\kappa$.*

*Proof.* Let $\langle N_i, \pi_{ij}, \kappa_i \rangle$ be the iteration of $N$. Then $N_i = J^{U_i}_{\beta_i+1}$ for some $\beta_i$. Let $M_i = J^{U_i}_{\beta_i}$ and let $B_i = A^n_{M_i}$ and $H_i = M_i^n = J^{B_i}_{\rho^n_{M_i}}$. Since $\rho_M^n = \kappa$ is $\Sigma_1$ definable in $N$, $\rho^n_{M_i} = \kappa_i$. Hence, $H_i = J^{B_i}_{\kappa_i}$. In this proof, the notation for sets $H_i$ is distinguished from the notation for the collection of hereditarily small sets $H^X_\lambda$ because we only consider hereditarily small sets in some model $X$. We do not claim that $M_i$ is an iterate of $M$.

Since $H_i = H^{M_i}_{\kappa_i}$ is $\Sigma_1(N_i)$ definable from $\kappa_i$, $\pi_{ij}(H_i) = H_j$. Since by Lemma 5.13

$$H^{J^{U_i}_{\beta_i}}_{\kappa_i} = \bigcup_{\substack{\nu < \kappa_i \\ a \subset \nu,\ a \in J^{U_i}_{\beta_i}}} J^a_{\kappa_i},$$

from Lemmas 5.9(3) and 5.31(b) it follows that $\pi_{ij} \restriction H_i = id \restriction H_i$. The embedding $\pi_{ij} \restriction H_i : H_i \to H_j$ is fully elementary:[59] because $\pi_{ij}(H_i) = H_j$, we have for any $\phi$ and

[58] We combine the proof of Lemma 4.9 of [5] and Lemmas 11.11 - 11.20 on pp. 88-91 of [4].

[59] The proof of the fact follows Lemma 11.11 of [4].

$x_1, \ldots, x_n \in H_i$,

$$\begin{aligned} H_j \models \phi(\pi_{ij}(x_n), \ldots, \pi_{ij}(x_n)) &\text{ iff } N_j \models \phi^{H_j}(\pi_{ij}(x_1), \ldots, \pi_{ij}(x_n)) \\ &\text{ iff } N_i \models \phi^{H_i}(x_1, \ldots, x_n) \\ &\text{ iff } H_i \models \phi(x_1, \ldots, x_n). \end{aligned}$$

Hence, $H_i \prec H_j$ for $i \leq j$.

To proof proceeds through a series of claims:

*Claim 1.* $H_i \models ZF$.

*Proof.* Clearly $H_i$ satisfies pairing, union, extensionality and foundation. $H_i$ satisfies $\Sigma_0$ separation, i.e., $H_i \models \forall u \forall \bar{x} \exists y \forall z\, (z \in y \leftrightarrow z \in u \wedge \phi(z, \bar{x}))$ for any $\Sigma_0$ $\phi$. Fixing $u, \bar{x} \in H_i$, $\Sigma_0$ separation for $H_j$, $i < j$, implies that $H_j \models \exists y \forall z\, (z \in y \leftrightarrow z \in u \wedge \phi^{H_i}(z, \bar{x}))$. Since $H_i \prec H_j$, $H_i \models \phi(z, \bar{x})$ implies $H_j \models \phi(z, \bar{x})$. Hence, $H_j \models \exists y \forall z\, (z \in y \leftrightarrow z \in u \wedge \phi(z, \bar{x}))$, so by elementarity $H_i \models \exists y \forall z\, (z \in y \leftrightarrow z \in u \wedge \phi(z, \bar{x}))$. Hence, $H_i$ satisfies full separation.

To prove replacement, we show that

$$H_i \models \forall x \exists y\, \phi(x, y) \rightarrow \forall u \exists v (\forall x \in u)(\exists y \in v)\, \phi(x, y).$$

Then replacement follows by separation. Suppose $H_i \models \forall x \exists y\, \phi(x, y)$ and $u \in H_i$. Then since $H_i \prec H_j$ for $j > i$, we have

$$\begin{aligned} &H_j \models (\forall x \in u)(\exists y \in H_i)\, \phi(x, y), \text{ so} \\ &H_j \models \exists v (\forall x \in u)(\exists y \in v)\, \phi(x, y), \text{ so by elementarity} \\ &H_i \models \exists v (\forall x \in u)(\exists y \in v)\, \phi(x, y). \end{aligned}$$

To prove power set, let $x \in H_i$. Take a regular cardinal $\theta > \kappa_i$ such that $\theta > 2^{|x|^V}$. Then $\kappa_\theta = \theta$ and $|\mathcal{P}(x) \cap H_\theta|^V < \theta$, so there is $\gamma < \theta$ such that $\mathcal{P}(x) \cap H_\theta \subset J_\gamma^{B_\theta}$. Thus, $H_\theta \models \exists y (\mathcal{P}(x) \subset y)$, so by separation $H_\theta \models \exists y (y = \mathcal{P}(x))$. Hence, $H_i \models \exists y (y = \mathcal{P}(x))$. $\square$ *Claim 1.*

*Claim 2.* $H_i = H_{\kappa_i}^{H_j}$ for $i \leq j$.

*Proof.* $H_i \subset H_j$, so $H_i \subset H_{\kappa_i}^{H_j}$. For the other direction, suppose that $a \subset \gamma < \kappa_i$ and $a \in H_j$. Since $H_i$ is a model of $ZF$, there is $x \in H_i$ such that $H_i \models x = \mathcal{P}(\gamma)$. Because $\pi_{ij} \restriction H_i = id$, $H_j \models x = \mathcal{P}(\gamma)$. Hence, $a \in x \in H_i$. $\square$ *Claim 2.*

*Claim 3.* $\{\kappa_i : i < j\}$ is a set of $\Sigma_\omega$ indiscernibles for $H_j$.

*Proof.* $\{\kappa_i : i < j\}$ are $\Sigma_1$ indiscernibles for $\langle N_j, x\rangle_{x \in \mathrm{ran}(\pi_{0i})}$ by Lemma 5.34. $H_j = \pi_{0j}(H_0) \in \mathrm{ran}(\pi_{0j})$ $\square$ *Claim 3.*

*Claim 4.* If $i < j$, then $M_i \in H_j$ and is uniformly $\Sigma_\omega(H_j)$ from $\kappa_i$.

*Proof.* $B_i = B_j \cap J_{\kappa_i}$ since $x \in B_i$ iff $\pi_{ij}(x) \in B_j$. Let $U_i' = U_i \cap J_{\beta_i}^{U_i}$. Then $\beta_i$ and $U_i'$ are the unique $\beta$ and $U$ such that $U \subset J_\beta^U$, $\kappa_i = \rho^n_{J_\beta^U}$ and $B_i = A^n_{J_\beta^U}$. Hence, $J_{\beta_i}^{U_i} = J_{\beta_i}^{U_i'}$ is in $H_j$ since $H_j$ is a model of $ZF$. $\square$ *Claim 4.*

[60] Let $X_i^m$ be the smallest elementary substructure of $H_{i+m}$ containing $\kappa_i \cup \{\kappa_{i+1}, \ldots, \kappa_{i+m-1}\}$, i.e., the $\Sigma_\omega$ Skolem hull of $\kappa_i \cup \{\kappa_{i+1}, \ldots, \kappa_{i+m-1}\}$ in $H_{i+m}$. Let $M_i^m$ be the transitive collapse of $X_i^m$ and let $\pi_i^m : M_i^m \cong H_{i+m} \upharpoonright X_i^m$. Let $K_i^m = H_{\kappa_i^+}^{M_i^m}$ where $\kappa_i^+$ is the least cardinal greater than $\kappa_i$ in $H_{i+m}$. So $K_i^m$ is transitive by definition.

*Claim 5.* [61] $\pi_i^m \upharpoonright K_i^m = id \upharpoonright K_i^m$.

*Proof.* We show that $X_i^m \cap H_{\kappa_i^+}^{H_{i+m}}$ is transitive. So suppose $x \in X_i^m \cap H_{\kappa_i^+}^{H_{i+m}}$. Since $J_{\beta_{i+m}}^{B_{i+m}}$ is transitive, $x \subset H_{\kappa_i^+}^{H_{i+m}}$. Since $H_{i+m} \models |x| \leq \kappa_i$, there is a function $f \in H_{i+m}$ from $\kappa_i$ onto $x$. Let $f_x$ be the $<_{H_{i+m}}$-least such function. Then $f_x$ is definable over $H_{i+m}$ from $x$ and every $y \in x$ is definable over $H_{i+m}$ from $f_x$ and some $\gamma < \kappa_i$. Hence, $x \subset X_i^m$, so $X_i^m \cap H_{\kappa_i^+}^{H_{i+m}}$ is transitive. $\square$ *Claim 5.*

$X_i^0 = H_i$ so $K_i^0 = H_i$. Claim 5 implies that $K_i^m = X_i^m \cap h_{\kappa_i^+}^{H_{i+m}}$, and Claim 4 implies that $X_i^m$ and $H_{i+m}$ are in $X_i^{m+1}$. From the definition of $X_i^m$ it follows that $H_{i+m+1} \models |X_i^m| = \kappa_i$, so $H_{i+m+1} \models |K_i^m| = \kappa_i$. Hence, $K_i^m \in K_i^{m+1}$. Claim 4 implies that $M_i \in X_i^1$ and $|M_i|^{H_{i+1}} = \kappa_i$. Thus, $M_i \in K_i^1$.

Let $f_i^m$ be the $<_{H_{i+m+1}}$-least function from $\kappa_i$ onto $\mathcal{P}(\kappa_i) \cap K_i^m$. Such a function exist since $H_{i+m+1} \models |K_i^m| = \kappa_i$. $f_i^m$ is definable over $H_{i+m+1}$ from $\kappa_i, \kappa_{i+1}, \ldots, \kappa_{i+m}$ and the definition is uniform for all $i$.

*Claim 6.*[62] $\pi_{ij}(f_i^m(\gamma)) = f_j^m(\gamma)$ for all $i \leq j$ and $\gamma < \kappa_i$.

*Proof.* Let $\sigma : N_{i+m+1} \to N_{j+m+1}$ be the $\Sigma_1$ embedding given by Lemma 5.33 satisfying $\sigma \circ \pi_{i,i+m+1} = \pi_{i,j+m+1}$ and $\sigma(\kappa_{i+p}) = \kappa_{j+p}$ for $p \leq m+1$. Since $H_i \in N_i$ for all $i$, the uniform definability of $f_i^m$ implies that $\sigma(f_i^m(\gamma)) = f_j^m(\gamma)$ for all $\gamma < \kappa_i$. Then by the properties of $\sigma$,

$$\begin{aligned} f_j^m(\gamma) = \sigma(f_i^m(\gamma)) &= \sigma(\pi_{i,i+m+1}(f_i^m(\gamma)) \cap \kappa_i) \\ &= \pi_{i,j+m+1}(f_i^m(\gamma)) \cap \kappa_j \\ &= \pi_{ij}(f_i^m(\gamma)). \qquad \square \textit{Claim 6.} \end{aligned}$$

[60] The rest of the proof follows the presentation on pp. 90-91 of [4].

[61] Lemma 11.17 of [4].

[62] Lemma 11.18 of [4].

*Claim 7.*[63] $U_i \cap K_i^m \in K_i^{m+2}$.

*Proof.* By Claim 6

$$f_i^m(\gamma) \in U_i \quad \text{iff} \quad \kappa_i \in \pi_{i,i+1}(f_i^m(\gamma)) \quad \text{iff} \quad \kappa_i \in f_{i+1}^m(\gamma).$$

Hence, $U_i \cap K_i^m = \{f_i^m(\gamma) : \gamma < \kappa_i \text{ and } \kappa_i \in f_{i+1}^m(\gamma)\}$. Since $f_{i+1}^m$ is definable over $H_{i+m+2}$ from $\kappa_i, \dots, \kappa_{i+m+1}$, $U_i \cap K_i^m \in K_i^{m+2}$. □ *Claim 7.*

Define $K_i = \bigcup_{m<\omega} K_i^m$. $K_i$ is transitive and rudimentarily closed since every $K_i^m$ is. Claim 7 implies that $\langle K_i, U_i \rangle$ is amenable. Since $M_i \cup \{M_i\}$ is a subset of $K_i$, $N_i \subset K_i$. But $K_i \subset H_{\kappa_i^+}^{H_{i+\omega}} \subset H_{i+\omega}$, so $N_i \subset H_{i+\omega}$. By Claim 2 $H_i = H_{\kappa_i}^{H_{i+\omega}}$, so $H_{\kappa_i}^{N_i} \subset H_i$. On the other hand, $H_i = H_{\kappa_i}^{M_i} \subset H_{\kappa_i}^{N_i}$ because $M_i \subset N_i$. Hence, $H_{\kappa_i}^{N_i} = H_{\kappa_i}^{M_i}$, so in particular $H_\kappa^N = H_\kappa^M$. □

## 5.4 Mice

In this section we present the definition of a mouse and those central properties that are needed to prove the basic properties of the core model. We begin with the definition of a critical premouse.

**Definition 5.58.** [64] A premouse $N$ at $\kappa$ is *critical* if $N$ is acceptable and $\mathcal{P}(\kappa) \cap \Sigma_1(N) \not\subset N$, i.e., there is a $\Sigma_1$-definable subset of $\kappa$ that is not in $N$.

By Lemma 5.54(a), criticality implies that there is $n$ such that $\rho_N^n > \kappa \geq \rho_{N^n}$. This $n$ is called the *critical number* of $N$ and is denoted by $n(N)$. The following definition gives the concept of $N'$ that we need to define mice.

**Definition 5.59.** [65] Suppose $N$ is a critical premouse. Then we define

$$\begin{aligned} \rho' &= \rho_N^n \\ A' &= A_N^n \\ N' &= \langle N^{(n)}, U \rangle \end{aligned}$$

where $n = n(N)$.

Now we can define a mouse.

**Definition 5.60.** [66] Let $N$ be a critical premouse. Then $N$ is a *mouse* if $N'$ is iterable

[63] Lemma 11.19 of [4].
[64] Definition 5.1 of [5].
[65] Definition 5.2 of [5].
[66] Definition 5.4 of [5].

and for each $i \in \mathrm{On}$ there is a critical premouse $N_i$ such that $(N_i)' = N_i'$ where $\langle N_i', \pi_{ij}', \kappa_i \rangle$ is the iteration of $N'$ and $n(N_i) = n(N)$ for each $i \in \mathrm{On}$.

By Lemma 5.54(b) the embedding $\pi_{ij}'$ can be extended to $\pi_{ij} : N_i \to_{\Sigma_1} N_j$. Then $\langle N_i, \pi_{ij}, \kappa_i \rangle$ is an iteration of $N$, called the *mouse iteration* of $N$.

If a mouse $N$ is a critical premouse at $\kappa$, $N$ is called a *mouse at $\kappa$*.

The following lemma gives a useful sufficient condition for the mouseness of a premouse.

**Lemma 5.61.** [67] *Suppose $N'$ is an iterable premouse and the iteration maps are strong, i.e., for any $T$ rudimentary over $N'$ in parameter $r$ and $\bar{T}$ rudimentary over $N_i$ in parameter $\pi_{0i}(r)$, if $T$ is well-founded, then $\bar{T}$ is well-founded. Then $N$ is a mouse.*

*Proof.* The result is immediate by Lemma 5.54(c). □

**Lemma 5.62.** [68] *Suppose $N$ is a mouse and $\sigma : M \to_{\Sigma_1} N'$. (So $M = \langle J_{\bar{\beta}}^{\bar{U}}, \bar{A} \rangle$ is amenable and $\rho_M \leq \bar{\kappa}$ where $M \models$ '$\bar{U}$ is normal at $\bar{\kappa}$'). Then*

*(a) there is a unique $\bar{N} = J_{\bar{\alpha}}^{\bar{U}}$ such that $\bar{N}' = M$,*

*(b) $n(\bar{N}) = n(N)$ and*

*(c) $\bar{N}$ is a mouse.*

*Proof.* Let $\langle N_i, \pi_{ij}, \kappa_i \rangle$ be the mouse iteration of $N$. Then $\pi_{ij}' = \pi_{ij} \restriction N_i'$ is the iteration of $N'$. Since $N'$ is iterable, by Lemma 5.42 $M$ is iterable as well. Let $\langle M_i, \bar{\pi}_{ij}', \bar{\kappa}_i \rangle$ be its iteration. By Lemma 5.33 there are $\sigma_i : M_i \to_{\Sigma_1} N_i'$ such that $\sigma_0 = \sigma$, $\sigma_j \circ \bar{\pi}_{ij}' = \pi_{ij}' \circ \sigma_i$ and $\sigma_j(\bar{\kappa}_i) = \kappa_i$ for all $i \leq j$. Thus, by Lemma 5.54(b) there are $\bar{N}_i$ such that $\bar{N}_i' = M_i$. Hence, $\bar{N}$ is a mouse and $n(\bar{N}) = n(N)$. □

For the next lemma we make the following definitions [69] for a mouse $N$ at $\kappa$ with $n(N) = n$, let $r_N = p_N^{n+1} \setminus \kappa$ and $q_N = p_N^{n+1} \cap \kappa$.

**Lemma 5.63.** [70] *Let $N$ be a mouse with $n(N) = n$ and the mouse iteration $\langle N_i, \pi_{ij}, \kappa_i \rangle$. Then $n(N_i) = n$, $\rho_{N_i'} = \rho_{N'}$, $A_{N_i'} = A_{N'}$ and $p_{N_i'} = p_{N'}$.*

*Proof.* $n = n(N_i)$ follows from the definition of a mouse. Let $\pi_{0i}' = \pi_{0i} \restriction N'$. By the definition of a mouse, $\pi_{0i}'$ is a $\Sigma_1$ embedding of $N'$ to $N_i'$. From $\pi_{0i} \restriction \kappa = \mathrm{id}$ it follows that if $A \subset \kappa$ is $\Sigma_1(N')$ with parameter $\bar{b}$, then $A$ is $\Sigma_1(N_i')$ with parameter $\pi_{0i}'(\bar{b})$. Hence, since $\mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap N_i$, $\rho_{N_i'} = \rho_{N'}$. Thus, $A_{N'} = A_{N_i'}$ follows from $p_{N_i'} = \pi_{0i}(p_{N'})$.

[67] We follow Lemma 5.5 of [5].
[68] The proof follows Lemma 5.6 of [5].
[69] These definitions are made in the proof of Lemma 10.5 of [4].
[70] We follow Lemma 10.5 of [4] and Lemma 5.7 of [5].

We show that $\pi_{0i}(r_N) = r_{N_i}$. We let $r = r_N$ and $\bar{r} = \pi_{0i}(r)$. Suppose $r_{N_i} <_* \bar{r}$. Then we have

$N'_i \models (\exists r' <_* \bar{r})(\exists \bar{x} < \pi_{0i}(\kappa))(\exists j < \omega)\,(\bar{r} = h_{N'_i}(j, \bar{x}, r'))$, so by elementarity
$N' \models (\exists r' <_* r)(\exists \bar{x} < \kappa)(\exists j < \omega)\,(r = h_{N'}(j, \bar{x}, r'))$.

Then every $\Sigma_1^{N'}(\omega\rho_{N'} \cup p_{N'})$ set is a $\Sigma_1^{N'}(\omega\rho_{N'} \cup q_N \cup r')$, but $q_N \cup r' <_* p_{N'}$, a contradiction. Hence, $r_{N_i} \geq_* \bar{r}$. Again, if $A \subset \kappa$ is $\Sigma_1(N')$ with parameter $p_{N'}$, then $A$ is $\Sigma_1(N'_i)$ with parameter $\pi_{0i}(p_{N'})$. If $A \notin N'$, then $A \notin N'_i$. Thus, $p_{N'_i} \leq_* \pi_{0i}(p_{N'})$, so $r_{N_i} \leq_* \bar{r}$. Hence, $\bar{r} = r_{N'_i}$.

Then we show that $q_N = q_{N_i}$. Since $\pi_{0i}(p_{N'}) \geq_* p_{N'_i}$ and $\pi_{0i}(p_{N'}) \setminus \kappa_i = p_{N'_i} \setminus \kappa_i$, $\pi_{0i}(p_{N'}) \cap \kappa_i \geq_* p_{N'_i} \cap \kappa_i = q_{N'_i}$. Because

$$\pi_{0i}(p_{N'}) \cap \kappa_i = \pi_{0i}(p_{N'} \cap \kappa) = \pi_{0i}(q_N) = q_N \subset \kappa,$$

we have $q_N \geq_* q_{N_i}$ and $q_{N_i} \subset \kappa$. Thus, $q_{N_i} = \pi_{0i}(q_{N_i})$. Let $A$ be $\Sigma_1(N'_i)$ with parameters from $\omega\rho_{N'} \cup p_{N'_i}$ such that $A \notin N'_i$. Because $p_{N'_i} = q_{N_i} \cup r_{N_i}$ and $\pi_{0i}(r_N) = r_{N_i}$, $A$ is $\Sigma_1(N')$ with parameters from $q_{N_i} \cup r_N$. Hence, $q_{N_i}$ must be $\geq_* q_N$ since otherwise $q_{N_i} \cup r_N <_* p_{N'}$. Thus, $p_{N'_i} = q_N \cup \pi_{0i}(r_N) = \pi_{0i}(p_{N'})$. $\square$

Next we define the important concept of the core of a mouse. This will be needed in the definition of the core model and in the proof of Theorem 6.2 in the last chapter.

**Definition 5.64.** [71] Suppose $N$ is a mouse and $X = h_{N'}(J_{\rho_{N'}} \cup p_{N'})$. Then $\langle X, A' \cap X\rangle \prec_{\Sigma_1} N'$. Let $M \cong X$ be the transitive collapse of $X$ and $\pi : M \cong X$ an isomorphism. By Lemma 5.62, there is a mouse $\bar{N}$ such that $M = \bar{N}'$. The mouse $\bar{N}$ is called the *core* of $N$.

The following lemmas prove that a mouse is an iterate of its core.

**Lemma 5.65.** [72] *Suppose $\bar{N}$ and $N$ are as in the preceding definition and $\bar{N}$ is the core of $N$. Then there is a mouse $Q$ that is an iterate of both $\bar{N}'$ and $N'$.*

*Proof.* Let $\langle \bar{N}'_i, \bar{\pi}_{ij}, \bar{\kappa}_i\rangle$ and $\langle N'_i, \pi_{ij}, \kappa_i\rangle$ be the iterations of $\bar{N}'$ and $N$, respectively. Let $\theta$ be a regular cardinal above $\kappa_0$. By Lemma 5.43, $\bar{N}'_\theta = J_{\bar{\alpha}}^{F_\theta}$ and $N_\theta = J_\alpha^{F_\theta}$ for some $\bar{\alpha}$ and $\alpha$, where $F_\theta$ is the club filter over $\theta$. We show that $\bar{\alpha}$ and $\alpha$ must be in fact be identical.

Suppose $\bar{\alpha} < \alpha$. Then $\bar{N}'_\theta$ is in $N'_\theta$. $A_{N'}$ is $\Sigma_1(N')$ with parameters from $\{p_{N'}\} \cup J_{\rho_{N'}}$. Since $A_{N'}$ is a subset of $J_{\rho_{N'}} \subset \bar{N}'$ and $\bar{N}' \cong X \prec_{\Sigma_1} N'$, $A_{N'}$ is also $\Sigma_1(\bar{N}')$ with parameters from $J_{\rho_{N'}} \cup \{\pi^{-1}(p_{N'})\}$. By the $\Sigma_1$-elementarity of $\bar{\pi}_{0\theta}$, $A_{N'}$ is $\Sigma_1(\bar{N}'_\theta)$. Since $\bar{\alpha} < \alpha$, $A_{N'}$

[71] Definition 5.9 of [5].
[72] We follow Lemma 5.10 of [5].

is in $N'_\theta$. Since the iterates have the same subsets of $\kappa_0$ as $N'$, $A_{N'}$ is in $N'$. But that is a contradiction. If $\bar{\alpha} > \alpha$, we get a similar contradiction.

Hence, $\bar{\alpha} = \alpha$, so $Q = N'_\theta = \bar{N}'_\theta$ is a common iterate of $\bar{N}'$ and $N'$.

□

**Lemma 5.66.** [73] *Suppose $\bar{N}$ and $N$ are as in the preceding lemma. Then $N'$ is an iterate of $\bar{N}'$.*

*Proof.* Let $Q = N'_\theta = \bar{N}'_\theta$ be the common iterate from the preceding lemma. Since $\bar{N}$ is the core of $N$, the definition of core implies that $\bar{N}' = h_{\bar{N}'}(J_{\rho_{N'}} \cup \{\pi^{-1}(p_{N'})\})$. Since the $\Sigma_1$ Skolem function is $\Sigma_1$ definable, we have

$$\begin{aligned}
\operatorname{ran}(\bar{\pi}_{0\theta}) &= \bar{\pi}_{0\theta}``(\bar{N}') \\
&= \bar{\pi}_{0\theta}``(h_{\bar{N}'}(J_{\rho'_N} \cup \{\pi^{-1}(p_{N'})\})) \\
&= h_Q(J_{\rho_{N'}} \cup \{\bar{\pi}_{0\theta}(\pi^{-1}(p_{N'}))\}).
\end{aligned}$$

We show that $\pi^{-1}(p_{N'}) = p_{\bar{N}'}$. As in the proof of the preceding lemma, $A_{N'}$ is $\Sigma_1(\bar{N}')$ with parameters from $J_{\rho_{N'}} \cup \{\pi^{-1}(p_{N'})\}$. But $A_{N'}$ cannot be in $\bar{N}'$ because otherwise it would be in $N'$. Thus, $\pi^{-1}(p_{N'}) \geq_* p_{\bar{N}'}$. If $\pi^{-1}(p_{N'}) >_* p_{\bar{N}'}$, then $p_{N'} >_* \pi(p_{\bar{N}'})$, so $\pi_{0\theta}(p_{N'}) >_* \pi_{0\theta}(\pi(p_{\bar{N}'}))$. But $\pi_{0\theta}(p_{N'}) = p_Q$, so now $\pi_{0\theta}(\pi(p_{\bar{N}'})) <_* p_Q$. Because $\bar{N}'$ is isomorphic to $X$ by $\pi$, $A_{\bar{N}'}$ is $\Sigma_1(N')$ with parameters in $J_{\rho_{N'}} \cup \{\pi(p_{\bar{N}'})\}$. Thus, $A_{\bar{N}'}$ is $\Sigma_1(Q)$ with parameters in $J_{\rho_{N'}} \cup \{\pi_{0\theta}(\pi(p_{\bar{N}'}))\}$. Hence, $A_{\bar{N}'}$ is in $Q$, so since $A_{\bar{N}'} \subset J_{\rho_{\bar{N}'}}$, it is in $\bar{N}'$. That is a contradiction. Hence, $\pi^{-1}(p_{N'}) = p_{\bar{N}'}$.

Thus we have

$$\begin{aligned}
\operatorname{ran}(\bar{\pi}_{0\theta}) &= h_Q(J_{\rho_{N'}} \cup \{\bar{\pi}_{0\theta}(p_{\bar{N}'})\}) \\
&= h_Q(J_{\rho_{N'}} \cup \{p_Q\}).
\end{aligned}$$

Since by lemma 5.55 $N' = h_{N'}(J_\kappa \cup \{p_{N'}\})$, we have

$$\begin{aligned}
\operatorname{ran}(\bar{\pi}_{0\theta}) &= h_Q(J_{\rho_{N'}} \cup \{p_Q\}) \\
&\subset h_Q(J_\kappa \cup \{p_Q\}) \\
&= \operatorname{ran}(\pi_{0\theta}).
\end{aligned}$$

Hence by lemma 5.46, $N'$ is an iterate of $\bar{N}'$. □

**Lemma 5.67.** [74] *Suppose $\bar{N}$ is the core of $N$. Then $\bar{N} = core(\bar{N})$.*

[73] We follow Lemma 5.11 of [5].

[74] The proof is our own.

*Proof.* Let $\langle \bar{N}_i, \bar{\pi}_{ij}, \bar{\kappa}_i \rangle$ be the iteration of $\bar{N}$. By lemma 5.66, $N = \bar{N}_\xi$ for some $\xi$. Let $M$ be the transitive collapse of $X = h_{N'}(J_{\rho_{N'}} \cup \{p_{N'}\})$. Since $N = \bar{N}_\xi$, we have $X = h_{\bar{N}'_\xi}(J_{\rho_{\bar{N}'_\xi}} \cup \{p_{\bar{N}'_\xi}\}) = \bar{\pi}_{0\xi}``(h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\}))$. As $\bar{\pi}_{0\xi}$ is injective, $\bar{\pi}_{0\xi} \restriction h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$ is a bijection between $h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$ and $X$. Hence, we can easily see by induction on $\Sigma_n$ that $\bar{\pi}_{0\xi}$ is actually an isomorphism between and $h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$ and $X$. Hence, $M$ is also the transitive collapse of $h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$ by the uniqueness of the transitive collapse. Thus, $\bar{N} = \text{core}(\bar{N})$.

□

**Definition 5.68.** A mouse $N$ is a *core mouse* if $N = \text{core}(N)$.

The following lemma is important in the proof of Theorem 6.2.

**Lemma 5.69.** [75] *Suppose $\bar{N}$ is a core mouse and $N$ is an iterate of $\bar{N}$. Then $\bar{N}$ is the core of $N$.*

*Proof.* Let $\langle \bar{N}_i, \bar{\pi}_{ij}, \bar{\kappa}_i \rangle$ be the iteration of $\bar{N}$. Suppose $N = \bar{N}_\xi$. Let $M$ be the transitive collapse of $X = h_{N'}(J_{N'} \cup \{p_{N'}\}) = \bar{\pi}_{0\xi}``(h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\}))$. Again we get that $h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$ is isomorphic to $h_{N'}(J_{N'} \cup \{p_{N'}\})$, so since by Lemma 5.67 $\bar{N}' = h_{\bar{N}'}(J_{\rho_{\bar{N}'}} \cup \{p_{\bar{N}'}\})$, $\bar{N}'$ is isomorphic to $M$. Since $\bar{N}'$ is transitive, the uniqueness of the transitive collapse implies that $\bar{N}' = M$. Hence, $\bar{N} = \text{core}(N)$. □

**Definition 5.70.** Suppose $N$ is a mouse with core $M$. If $N = M_\xi$ and the iteration of $M$ is $\langle M_i, \pi_{ij}, \kappa_i \rangle$, we define $C_N = \{\kappa_i : i < \xi\}$.

From the following lemma it follows that $C_N = \Pi_1(N')$.

**Lemma 5.71.** [76] *Suppose $N$ and $M$ are as above. Then $\lambda < \kappa_\xi$ is in $C_N$ if and only if $\lambda \notin h_{N'}(\lambda \cup p_{N'})$ and $\lambda > \omega\rho_{N'}$.*

*Proof.* Suppose $\lambda \in C_N$, say $\lambda = \kappa_\gamma$. If $\kappa_\gamma \in h_{N'}(\lambda \cup p_{N'})$, then $\kappa_\gamma = h_{N'}(i, \langle \bar{a}, p_{N'} \rangle)$ for some $i \in \omega$ and $\bar{a} \in (\kappa_\gamma)^{<\omega}$. Thus, $\kappa_\gamma = \pi_{\gamma\xi}(h_{M_\gamma}(i, \langle \bar{a}, p_{M'_\gamma} \rangle))$, so $\lambda \in \text{ran}(\pi_{\gamma\xi})$, which is impossible. Hence, $\lambda \notin h_{N'}(\lambda \cup p_{N'})$. Since $\rho_{N'} = \rho_{M'}$, we also have $\lambda > \omega\rho_{N'}$.

For the other direction, suppose that $\lambda \notin C_N$ and $\kappa > \omega\rho_{N'}$. Let $\gamma$ be the least ordinal such that $\lambda < \kappa_\gamma$. We have $\rho_{N'} = \rho_{M'} = \rho_{M'_\gamma}$ for all $\gamma$. Thus, if $\gamma = 0$, then $\lambda \in M$, so there is $\bar{a} \in \rho_{N'}^{<\omega} \subset \lambda^{<\omega}$ such that $\kappa = h_{M'}(i, \langle \bar{a}, p_{M'} \rangle = h_{M'_\gamma}(i, \langle \bar{a}, p_{M'_\gamma} \rangle)$ for some $i < \omega$. If $\gamma > 0$, we can show that $M'_\gamma = h_{M'_\gamma}(\omega\rho_{N'} \cup \{\kappa_i : i < \gamma\} \cup p_{M'_\gamma})$. By 5.31(c) we have $M'_\gamma = h_{M'_\gamma}(\pi_{0\gamma}``(M') \cup \{\kappa_i : i < \gamma\})$. Since $M' = h_{M'}(J_{M'} \cup p_{M'})$ and there is a uniformly $\Sigma_1(J_{\rho_{M'}})$ function from $\omega\rho_{M'}$ onto $J_{\rho_{M'}}$, $J_{\rho_{M'}}$ is a subset of $h_{M'}(\omega\rho_{M'})$. Thus, we have

$$\pi_{0\gamma}``(M') = \pi_{0\gamma}``(h_{M'}(\omega\rho_{M'} \cup p_{M'})) = h_{M'_\gamma}(\omega\rho_{N'} \cup p_{M'_\gamma}),$$

[75] The proof is our own.

[76] The proof follows Lemmas 10.23 and 10.19 of [4].

so, indeed, $M'_\gamma = h_{M'_\gamma}(\omega\rho_{N'} \cup \{\kappa_i : i < \gamma\} \cup p_{M'_\gamma})$. Hence, there is again $\bar{a} \in \kappa^{<\omega}$ such that $\kappa = h_{M'_\gamma}(i, \langle \bar{a}, p_{M'_\gamma}\rangle)$ for some $i < \omega$. Thus, in either case, $\kappa = h_{N'}(i, \langle \bar{a}, p_{N'}\rangle)$, so $\kappa \in h_{N'}(\kappa \cup p_{N'})$. □

Next we show that for $N = J^U_\alpha$, a mouse at $\kappa$, $C_N = \bigcap(U \cap h_{N'}(J_{\rho_{N'}} \cup p_{N'}))$[77]. So let $D_N = \bigcap(U \cap h_{N'}(J_{\rho_{N'}} \cup p_{N'}))$ in the following lemmas.

**Lemma 5.72.** [78] *$C_N$ is a $\Sigma_1$-generating set of $\Sigma_1$ indiscernibles for $\langle N'_i, p_{N'_i}, x\rangle_{x\in J_\kappa}$, i.e., $N'_i = h_{N'_i}(J_\kappa \cup p_{N'_i})$ and for every $\Sigma_1$ formula $\phi$, $x \in J_\kappa^{<\omega}$ and $\kappa_{i_1} < \cdots < \kappa_{i_{2n}} < \kappa_i$, $N'_i \models \phi(x, p_{N'_i}, \kappa_{i_1}, \ldots, \kappa_{i_n})$ if and only if $N'_i \models \phi(x, p_{N'_i}, \kappa_{i_{n+1}}, \ldots, \kappa_{i_{2n}})$.*

*Proof.* By Lemma 5.56(d), $N'_i = h_{N'_i}(J_\kappa \cup p_{N'_i})$. By Lemma 5.34 $C_N$ is a set of $\Sigma_1$ indiscernibles for $\langle N'_i, x\rangle_{x\in \mathrm{ran}(\pi_{0i}\restriction N')}$. But $\pi_{0i}(p_{N'}) = p_{N'_i}$ and $\pi_{0i}$ keeps every element of $J_\kappa$ unchanged, so $J_\kappa \cup p_{N'_i} \subset \mathrm{ran}(\pi_{0i})$. □

**Lemma 5.73.** [79] *$D_N$ is a set of $\Sigma_1$ indiscernibles for $\langle N', p_{N'}, x\rangle_{x\in J_{\rho_{N'}}}$*

*Proof.* Let $\phi := \exists y\psi$ be a $\Sigma_1$ formula. Let $K = h_{N'}(J_{\rho_{N'}} \cup p_{N'})$.

*Claim.* $K \cap \rho'$ is cofinal in $\rho'$.

*Proof.* Suppose $K \cap \rho'$ is bounded in $\rho'$. $A_{N'}$ is definable in over $K$. If $K \cap \rho'$ is bounded in $\rho'$, then $K \subset J^{A'}_\beta$ for some $\beta < \rho'$. Suppose there is no such $\beta$. For every $x \in K$ the $<_{N'}$-least $\beta_x$ such that $x \in J^{A'}_{\beta_x}$ is definable from $x$ without parameters, so $\beta_X \in K$. Thus, $K \cap \rho'$ is cofinal in $\rho'$, a contradiction. Hence, $K \subset J^{A'}_\beta$ for some $\beta < \rho'$. But $J^{A'}_\beta$ is in $N'$, so $A_{N'}$ is in $N'$, a contradiction. □ *Claim.*

For $\nu \in K \cap \rho'$ and $x \in J^{<\omega}_{\rho_{N'}}$ set

$$f_{\nu,x}(\bar{a}) = \begin{cases} 1 & \text{if } N' \models (\exists y \in S^U_\nu)\, \psi(y, \bar{a}, p_{N'}, x) \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{a} \in [\kappa]^n$. Every $f_{\nu,x}$ is in $K$ because $f_{\nu,x}$ is $\Sigma_1(N')$ with parameters in $J_{\rho_{N'}} \cup p_{N'}$. Lemma 2.14 works for $f_{\nu,x}$ since $f_{\nu,x}$ and $\kappa$ are in $N'$. There is a $\Sigma_1$ formula $\phi'$ such that $N' \models \phi'(X, f_{\nu,xi})$ if and only if $X$ is in $U$ and is homogeneous for $f_{\nu,x}$. Hence, some $X_{\nu,x}$ with $N' \models \phi'(X_{\nu,x}, f_{\nu,x})$ is in $K$. Since $\{X_{\nu,x} : \nu \in K \cap \rho', x \in J^{<\omega}_{\rho_{N'}}\}$ is a subset of $U \cap h_{N'}(J_{\rho'} \cup p_{N'})$, $D_N \subset \bigcap_{\nu,x} X_{\nu,x}$.

[77][5] takes this as the definition of $C_N$

[78]We follow Lemma 5.8 of [5].

[79]We follow Lemma 5.13 of [5].

To prove the claim of the lemma, let $\bar{a} \in (D_N)^n$. If $N' \models \phi(\bar{a}, p_{N'}, x)$, then for some $\nu$ $f_{\nu,x}(\bar{a}) = 1$ since $K \cap \rho'$ is cofinal in $\rho'$. Thus, $f_{\nu,x}(\bar{a}') = 1$ for all $\bar{a}' \in (X_{\nu,x})^n$, so in particular $N' \models \phi(\bar{a}', p_{N'}, x)$ for all $\bar{a}' \in (D_N)^n$. If $N' \not\models \phi(\bar{a}, p_{N'}, x)$, then for all $\nu \in K \cap \rho'$, $f_{\nu,x}(\bar{a}) = 0$, so $f_{\nu,x}(\bar{a}') = 0$ for all $\nu$ and $\bar{a}' \in D_N$. Hence, $N' \not\models \phi(\bar{a}', p_{N'}, x)$ for all $\bar{a}' \in D_N$. $\square$

**Lemma 5.74.** [80] $C_N = D_N$.

*Proof.* We show first that $C_N \subset D_N$. Let $\bar{N} = \mathrm{core}(N)$ and let $\langle \bar{N}_i, \kappa_i, \pi_{ij} \rangle$ be the iteration of $\bar{N}$. Suppose $N = \bar{N}_\lambda$, so $\kappa = \kappa_\lambda$. Suppose $x \in U \cap h_{N'}(J_{\rho_{N'}} \cup p_{N'})$. $\bar{N}'$ is the transitive collapse of $h_{\bar{N}'_i}(J_{\rho_{\bar{N}'_i}} \cup p_{\bar{N}'_i})$ for every $i$. Hence, for every $i < \lambda$, $x = \pi_{i\lambda}(x^i)$ for some $x^i \in h_{\bar{N}'_i}(J_{\rho_{\bar{N}'_i}} \cup p_{\bar{N}'_i})$. Thus we have

$$x \in U \quad \text{iff} \quad x^i \in \bar{U}_i \quad \text{iff} \quad \kappa_i \in \pi_{i\lambda}(x^i) \quad \text{iff} \quad \kappa_i \in x.$$

where $\bar{U}_i$ is the ultrafilter of $\bar{N}_i$. Hence, $C_N = \{\kappa_i : i < \lambda\} \subset x$ for every $x$ in $U \cap h_{N'}(J_{\rho_{N'}} \cup p_{N'})$, so $C_N \subset D_N$.

$C_N$ is a set of $\Sigma_1$ generating indiscernibles for $\langle N', p_{N'}, x \rangle_{x \in J_{\kappa_0}}$ by Lemma 5.72. Since $\rho_{N'} = \rho_{\bar{N}'} \leq \kappa_0$, $C_N$ are also $\Sigma_1$ indiscernibles for $\langle N', p_{N'}, x \rangle_{x \in J_{\rho_{N'}}}$. There is a $\Sigma_1(\bar{N}')$ mapping from $\kappa_0$ onto $J_{\kappa_0}$. The same formula defines a $\Sigma_1(N')$ mapping from $\kappa_0$ onto $J_{\kappa_0}$. Since $\pi_{0\lambda} \upharpoonright \kappa_0 = \mathrm{id} \upharpoonright \kappa_0$, for each $x \in J_{\kappa_0}$, $\pi_{i\lambda}(x) = x$. Thus, since $\bar{N}' = h_{\bar{N}'}(J_{\rho_{N'}} \cup p_{\bar{N}'})$ and $h_{\bar{N}'}(i, x) = h_{N'}(i, \pi_{0\lambda}(x))$, we have $J_{\kappa_0} \subset h_{N'}(J_{\rho_{N'}} \cup p_{\mathbb{N}'})$, so $N'$ is $\Sigma_1$ generated by $C_N \cup p_{N'} \cup J_{\rho_{N'}}$. Hence, $C_N$ is a $\Sigma_1$ generating set of indiscernibles for $\langle N, p_{N'}, x \rangle_{x \in J_{\rho_{N'}}}$.

If $C_N$ is a proper subset of $D_N$, then there is $a \in D_N \setminus C_N$. But then $a \in N'$ so $N' \models a = x \leftrightarrow \phi(x, \kappa_{i_1}, \ldots, \kappa_{i_n}, z, p_{N'})$ for some $\Sigma_1$ formula $\phi$, $z \in (J_{\rho_{N'}})^{<\omega}$ and $\kappa_{i_1}, \ldots, \kappa_{i_n} \in C_N$. Hence, $N' \not\models \phi(\kappa_{j_1}, \ldots, \kappa_{j_{n+1}}, z, p_{N'})$ for all $\kappa_{j_1}, \ldots, \kappa_{j_{n+1}} \in C_N$. But on the other hand $N' \models \phi(a, \kappa_{i_1}, \ldots, \kappa_{i_n}, z, p_{N'})$, which is a contradiction since $D_N$ are $\Sigma_1$ indiscernibles and $C_N \subset D_N$. Hence, $C_N = D_N$. $\square$

The following two lemmas are needed for their corollary which we will use in the proof of Theorem 6.2. The argument of the proof of Lemma 5.76 is also used in the proof of Lemma 5.84.

**Lemma 5.75.** [81] *Suppose $N = J^U_\alpha$ is an iterable premouse and $\kappa < \beta < \alpha$. If $M = J^U_\beta$ is critical, then it is a mouse.*

*Proof.* Clearly $M$ is a premouse. $M'$ is $J^U_\lambda$ for some $\lambda$ so $M'$ is iterable by Lemma 5.42. Let $\langle M'_i, \pi_{ij}, \kappa_i \rangle$ be the iteration of $M'$.

[80] We follow Lemma 5.14 of [5] and the paragraph immediately following it.

[81] We follow Lemma 5.18 of [5].

Let $n = n(M)$. $M' = H^{M^{(n-1)}}_{\omega\rho^n_M}$ and $\rho^n_M$ is a $\Sigma_1$ cardinal in $M^{n-1}$, so $M'$ satisfies replacement[82] Claim 1. Suppose $T$ is a rudimentary relation over $M'$ in parameter $p$. If $T$ is well-founded, then replacement guarantees that there is a $\Sigma_1(M')$ ordinal-valued function $f$ with domain $\mathrm{dom}(T) \cup \mathrm{ran}(T)$ such that for all $x \in \mathrm{dom}(T) \cup \mathrm{ran}(T)$, $f(x) = \sup\{f(y) + 1 : yTx\}$.

Suppose $T_i$ is defined over $M'_i$ with the same rudimentary definition in parameter $\pi_{0i}(p)$. Then since $\pi_{ij}$ is $\Sigma_1$-elementary, there is also $\Sigma_1(M'_i)$ function $f_i : \mathrm{dom}(T_i) \cup \mathrm{ran}(T_i) \to \mathrm{On}$ such that for all $x \in \mathrm{dom}(T_i) \cup \mathrm{ran}(T_i)$, $f_i(x) = \sup\{f(y) + 1 : yT_ix\}$. Hence, the relation $T_i$ is well-founded, so the iteration maps are strong. By Lemma 5.61 $M$ is a mouse. □

**Lemma 5.76.** [83] *Suppose $N = J^U_\alpha$ is a mouse at $\kappa$ and $\kappa < \beta < \alpha$. Then if $M = J^U_\beta$ is critical with $\rho^{n+1}_M < \kappa$, then $J^U_{\beta+1} = J^{U\cap M}_{\beta+1}$.*

*Proof.* $M$ is acceptable because $N$ is a mouse, so $M$ is critical. Hence, $M$ is a mouse by the previous lemma. Let $\tilde{M} = \mathrm{rud}_{U\cap M}(M) = J^{U\cap M}_{\beta+1}$. Suppose $\langle \tilde{M}, U \cap \tilde{M}\rangle$ is amenable. If $x$ is in $S^U_{\omega\beta+k}$ and $x \in \tilde{M}$, then by amenability $x \cap U \in \tilde{M}$. Thus, since $J^U_{\beta+1} = \bigcup_{k<\omega} S^U_{\omega\beta+k}$, we see by induction on $k$ that every element of $J^U_{\beta+1}$ must be in $\tilde{M}$. On the other hand $\tilde{M} = \mathrm{rud}_{U\cap M}(M)$ is obviously a subset of $\mathrm{rud}_U(M) = J^U_{\beta+1}$. Hence, it is enough to show that $\langle \tilde{M}, \tilde{M} \cap U\rangle$ is amenable.

Suppose $\bar{M}$ is the core of $M$ and $\langle \bar{M}_i, \pi_{ij}, \kappa_i\rangle$ is the iteration of $\bar{M}$. Let $M = \bar{M}_\lambda$, so $C_M = \{\kappa_i : i < \lambda\}$. By Lemma 5.71, $C_M \in \Sigma_\omega(M)$, so $C_M$ is in $\tilde{M}$. Because $C_M = \bigcap(U \cap h_{M'}(\omega\rho_{M'} \cup p_{M'}))$ and $h_{M'}(\omega\rho_{M'} \cup p_{M'})$ is a subset of $\tilde{M}$, $C_M$ is the intersection of at most $|\rho^{n(M)+1}_M|$ members of $U \cap \tilde{M}$. Since

$$J^U_{\beta+1} \models \text{“}U \text{ is a normal ultrafilter on } \kappa\text{”}.$$

and $\rho^{n(M)+1}_M < \kappa$, we have

$$J^U_{\beta+1} \models \text{“}U \text{ is } |\rho^{n(M)+1}_M|\text{-complete”}.$$

Hence $C_M \in U$, so $C_M \in U \cap \tilde{M}$. We define $C_k$ for $k < \omega$ as follows: $C_0 = C_M$ and $C_{k+1} =$ the limit points of $C_k$. For all $k$, $C_k \in U$ and $C_k \in \Sigma_\omega(M)$, so $C_k \in U \cap \tilde{M}$. Hence, $\lambda$ is a multiple of $\omega^\omega$. By Corollary 5.39 $C_k \setminus \kappa_i$ are $\Sigma_{k+1}$ indiscernibles for $\langle M, x\rangle_{x\in\mathrm{ran}(\pi_{i\lambda})}$. Because $\lambda$ is a limit, $M = \bigcup_{i<\lambda} \mathrm{ran}(\pi_{i\lambda})$. If $X \in \Sigma_{k+1}(M)$ and $X$ is in $U$, then $C_k \setminus \kappa_i \subset X$ for some $i$ since $C_k \setminus \kappa_i$ are $\Sigma_1$ indiscernibles. On the other hand,

[82] See the proof of Lemma 5.57

[83] We follow Lemma 5.19 of [5].

if $C_k \setminus X$ is bounded in $\kappa$, then $X \in U$. Hence, a $\Sigma_{k+1}(M)$ set $X$ is in $U$ if and only if $C_k \setminus X$ is bounded in $\kappa$. Thus, $U \cap \Sigma_{k+1}(M)$ is a $\Sigma_\omega(M)$ set, so $U \cap \Sigma_{k+1}(M) \in \tilde{M}$.

To prove the amenability of $\langle \tilde{M}, U \cap \tilde{M} \rangle$, suppose $x \in \tilde{M}$. Then $x \subset S^U_{\omega\beta+k}$ for some $k$ and there is a $\Sigma_\omega(M)$ map from $\beta$ onto $S^U_{\omega\beta+k}$. Hence, $x \subset \Sigma_p(M)$ for some $p$, so $U \cap x = (U \cap \Sigma_p(M)) \cap x$ which is in $\tilde{M}$. □

From Lemma 5.57 it follows that if $a$ is a bounded subset of $\kappa$ such that $a \in J^U_{\beta+1} \setminus J^U_\beta$, then $\rho^{n(M)+1}_M < \kappa$ where $M = J^U_\beta$. But then from the previous lemma it follows that $a$ is in $\Sigma_\omega(M)$. This yields the following corollary.

**Corollary 5.77.** [84] *If $N$ is a mouse at $\kappa$ and $\beta \in N$, then for $\gamma < \kappa$, $\mathcal{P}(\gamma) \cap J^U_{\beta+1} = \mathcal{P}(\gamma) \cap \Sigma_\omega(J^U_\beta)$.*

We conclude the section by mentioning without proof an important result that we will use in the last section. The proof is too long to be presented here in complete detail.

**Lemma 5.78.** [85] *If a premouse $M = J^U_\alpha$ is iterable, then it is acceptable.*

## 5.5 The core model

In this section we define the core model and present its most important properties.

**Lemma 5.79.** [86] *There is at most one mouse $N$ such that $N$ is a mouse at $\kappa$ and the order type of $C_N$ is $\omega$.*

*Proof.* Suppose that $N$ and $\bar{N}$ are two such mice. Then we can show that $N \approx \bar{N}$. Let $M = \mathrm{core}(N)$ and $\bar{M} = \mathrm{core}(\bar{N})$ and let $\langle M_i, \pi_{ij}, \kappa_i \rangle$ and $\langle \bar{M}, \bar{\pi}_{ij}, \bar{\kappa}_i \rangle$ be the respective enumerations. Then $N = M_\omega$, $\bar{N} = \bar{M}_\omega$ and $\kappa = \kappa_\omega = \bar{\kappa}_\omega$. Suppose $N <_{pm} \bar{N}$. Take large enough regular $\theta > \kappa$ as given by Lemma 5.43. Then $N_\theta \in \bar{N}_\theta$, and since $C_N = C_{N_\theta} \cap \kappa \in \Pi_1(N'_\theta)$, $C_N$ is in $\bar{N}_\theta$. As $\mathcal{P}(\kappa) \cap \bar{N}_\theta = \mathcal{P}(\kappa) \cap \bar{N}$, we have $C_N \in \bar{N}$. Thus, $\bar{N} \models \mathrm{cf}(\kappa) = \omega$, which is a contradiction. If $\bar{N} <_{pm} N$, we get the same contradiction. Hence, $\mathrm{core}(N) = \mathrm{core}(\bar{N})$, so $N = M_\omega = \bar{M}_\omega = \bar{N}$. □

Now we are ready to define the core model $K$.

**Definition 5.80.**

(i) We define $D = \{\langle \xi, \kappa \rangle : \xi \in C_N, N \text{ a mouse at } \kappa \text{ with } \mathrm{ot}(C_N) = \omega\}$.

[84] Corollary 5.20 of [5]. The above argument follows the lines immediately before the corollary on p. 71 of [5].

[85] The proof can be found e.g. in Lemmas 11.24-11.26 of [4].

[86] The proof follows Lemma 6.2 of [5].

(ii) The core model is defined by $K = L[D]$.

(iii) The $\alpha$-th level of the core model is defined by $K_\alpha = J^D_\alpha$.

We now start to prove that $K$ is the union of all mice[87]. Towards that end we define the following concept. Suppose $\kappa$ is a regular uncountable cardinal. We let

$$Q = Q_\kappa = \bigcup\{N_\kappa : N \in K_\kappa, N \text{ is a core mouse}\}.$$

By Lemma 5.43 each $N_\kappa$ is $J^F_\alpha$ for some $\alpha$, where $F = F_\kappa$ is the club filter on $\kappa$. Since $\kappa$ is regular and $J^F_\alpha$ is a direct limit, we must have $\alpha < \kappa^+$. By Lemma 5.79 there can be at most $\kappa$-many core mice in $K_\kappa$, so $Q = J^F_\theta$ with $\theta = \theta_\kappa < \kappa^+$.

**Lemma 5.81.** [88] *$\theta$ is a limit ordinal.*

*Proof.* Suppose $N_\kappa = J^F_\alpha$ where $N \in K_\kappa$ is a core mouse at $\lambda < \kappa$. Then by Lemma 5.63 $\rho_{N'_\kappa} = \rho_{N'} \leq \lambda < \kappa$. Hence, there is $a \in \mathcal{P}(\lambda) \cap \Sigma_{n(N)+1}(N_\kappa) \setminus N_\kappa$, so $a \in N^+ := J^F_{\alpha+1}$. Since $F$ is countably complete, $N^+$ is iterable and acceptable. Suppose $\rho_{N^+} > \lambda$. Acceptability implies that for all $k$ there is a surjection $f_k : \lambda \to \mathcal{P}(\lambda) \cap S^F_{\omega\alpha+k}$ in $N^+$. Let $f_k$ be the $<_{N^+}$-least such map for all $k$. Then $\langle f_k : k < \omega\rangle$ is $\Sigma_1(N^+)$. There is in $N^+$ a bijection $g : \lambda \times \omega \to \lambda$. Thus, we can define a surjection $f : \lambda \to \mathcal{P}(\lambda) \cap N^+$ by $f(g(\delta, i)) = f_i(\delta)$. But since $\rho_{N^+} > \lambda$, the $\Sigma_1(\lambda)$ set $b = \{\delta < \lambda : \delta \notin f(\delta)\}$ is in $N^+$, a contradiction. Hence, $\rho_{N^+} \leq \lambda < \kappa$ so $N^+$ is critical. Since $F$ is countably closed, the iteration maps are strong by Lemma 5.41, so $N^+$ is a mouse. Let $\bar{N} = \text{core}(N^+)$. Since core mice are sound, $\bar{N} \neq N^+$, so $\bar{N}$ is a mouse at some $\kappa' < \kappa$. Hence, $\bar{N} \in K_\kappa$ and $N^+ = \bar{N}_\kappa$, so $N^+ \subset Q$. Consequently, $\theta$ is a limit. $\square$

**Lemma 5.82.** [89] *$\rho_Q \leq \kappa$.*

*Proof.* We define

$$B = \{\gamma < \theta : \kappa < \gamma \text{ and } \mathcal{P}(\delta) \cap J^F_{\gamma+1} \not\subset J^F_\gamma \text{ for some } \delta < \kappa\}.$$

Since $\langle J^F_\gamma : \gamma < \theta\rangle$ is $\Sigma_1(J^F_\theta)$, $B$ is $\Sigma_1(Q)$. If a core mouse $N$ is in $K_\kappa$, then $\mathrm{On} \cap N \subset \alpha$ for some $\alpha < \kappa$. Hence, $\rho_{N'} < \kappa$ so $\rho_{N'_\kappa} < \kappa$. Thus, there is $\delta$ such that $\rho_{N'_\kappa} \leq \delta < \kappa$, so there is a $\Sigma_{n(N)+1}(N_\kappa)$-subset of $\delta$ that is not in $N_\kappa$. If $N_\kappa = J^F_\gamma$, $\mathcal{P}(\delta) \cap J^F_{\gamma+1} \not\subset J^F_\gamma$. Hence, $B$ is cofinal in $\theta$.

If $\gamma$ is in $B$, then $\rho^n_{J^F_\gamma} < \kappa$ for some $n$ because otherwise Lemma 5.57 implies that $H^{J^F_\gamma}_\kappa = H^{J^F_{\gamma+1}}_\kappa$. Hence, $J^F_\gamma$ is a critical premouse. Since $F$ is countably closed, $J^F_\gamma$ is iterable

[87] Our presentation follows mostly pp. 74-75 of [5].

[88] The proof adapts Claim 2 from the proof of Lemma 3.16 of [4].

[89] We follow Lemma 6.5 of [5].

and the iteration map is strong. Thus, $J^F_\gamma$ is a mouse. Define the function $f : B \to \kappa$ by $f(\gamma) =$ the $\omega$-th point of $C_{J^F_\gamma}$. Then $f$ is $\Sigma_1(Q)$ and $f$ is injective by Lemma 5.79. Let $A = f``(B)$. If $A \in Q$, then we can define similarly as in the proof of the previous lemma a $\Sigma_1(Q)$ surjection $g$ from $\kappa$ onto $\mathcal{P}(\kappa) \cap Q$. If $\rho_Q > \kappa$, then $\{\alpha < \kappa : \alpha \notin g(\alpha)\}$ is in $Q$, a contradiction. Hence, either $A \notin Q$ or $\rho_Q \leq \kappa$, so necessarily $\rho_Q \leq \kappa$. □

**Lemma 5.83.** [90] *$Q$ is a mouse.*

*Proof.* $Q$ is a premouse by definition. It is iterable and acceptable since $F$ is countably closed. The previous lemma shows that $Q$ is also critical. Since the iteration maps are strong, $Q$ is a mouse. □

**Lemma 5.84.** [91] *$\rho_Q = \kappa$.*

*Proof.* Suppose $\rho_Q < \kappa$. Then there is $a \subset \gamma < \kappa$ such that $a \in \Sigma_1(Q) \setminus Q$. Let $M = J^F_{\theta+1}$. Then $a \in M$, so the argument from the proof of Lemma 5.81 shows that $\rho_M < \kappa$. Moreover, $M$ is iterable and hence acceptable, so $M$ is a critical premouse. Since the ieration maps are strong, $M$ is a mouse. But then core$(M)$ is in $K_\kappa$, so $M \subset Q$, a contradiction. □

**Lemma 5.85.** [92] *For every regular uncountable $\kappa$, $K_\kappa = H^{Q_\kappa}_\kappa$.*

*Proof.* We show first that $K_\kappa \subset H^{Q_\kappa}_\kappa$. $K_\kappa = J^D_\kappa = \bigcup_{\gamma<\kappa} J^{D\cap\gamma^2}_\gamma$. Thus, it suffices to show that for all $\gamma$, $D \cap \gamma^2 \in Q_\kappa$. For $\lambda < \gamma$, if there is a mouse $M$ at $\lambda$ with ot$(C_M) = \omega$, let $\xi_\lambda = \alpha$ where $M_\kappa = J^F_\alpha$. Let $\xi = \sup\{\xi_\lambda : \lambda < \gamma\}$. Then clearly $\xi \leq \theta_\kappa$. We show that $\xi < \theta_\kappa$. Suppose for a contradiction that $\xi = \theta_\kappa$. For every $\lambda < \kappa$, acceptability implies that there is a map in $J^F_{\xi_\lambda+1}$ from $\gamma$ onto $\mathcal{P}(\gamma) \cap J^F_{\xi_\lambda}$. Since $C_{J^F_{\xi_\lambda}}$ is definable in $J^F_{\xi_\lambda}$, $\{\xi_\lambda : \lambda < \gamma\}$ is $\Sigma_1(Q_\kappa)$. Hence, we can define a $\Sigma_1(Q_\kappa)$ map from $\gamma$ onto $\mathcal{P}(\gamma) \cap Q_\kappa$. If $\rho_{Q_\kappa} > \gamma$, we get the same contradiction as in Lemma 5.81. Hence, $\rho_{Q_\kappa} \leq \lambda$, a contradiction. Thus, $\xi < \theta_\kappa$. Since $D \cap \gamma^2$ is $\Sigma_1(J^F_\xi)$, $D \cap \gamma^2$ is in $Q_\kappa$.

For the other direction, since $H^{Q_\kappa}_\kappa = \bigcup\{J^a_\kappa : a \subset \gamma < \kappa, a \in Q_\kappa\}$, it suffices to show that if $a \subset \gamma < \kappa$ and $a \in Q_\kappa$, then $a \in K_\kappa$. Suppose $\xi$ is the least such that $\xi \geq \kappa$ and $a \in N = J^{F_\kappa}_{\xi+1}$. Then $\rho_N \leq \gamma$, so $N$ is critical and a mouse since $F_\kappa$ is countably closed. Let $M$ be the core of $N$ with iteration $\langle M_i, \kappa_i, \pi_{ij}\rangle$. Since $\kappa = \kappa_\kappa$, there is $i < \kappa$ such that $\kappa_i > \gamma$. Since $\mathcal{P}(\kappa_i) \cap M_i = \mathcal{P}(\kappa_i) \cap N$, $a$ is in $M_i$. $M$ is in $K_\kappa$ so all iterates $M_j$, $j < \kappa$, are in $K_\kappa$. Hence, in particular, $M_i \in K_\kappa$, so $a$ is in $K_\kappa$. □

This gives the following fundamental property of the core model.

[90] Lemma 6.6 of [5].
[91] We follow Lemma 6.7 of [5].
[92] We follow Lemmas 14.12 and 14.13 of [4].

**Corollary 5.86.** [93] *The core model is the union of all mice.*

*Proof.* We show that every mouse is in $K$. Then the previous lemma implies that $K$ contains exactly all mice. Suppose $N = J^U_\alpha$ is a mouse at $\kappa$ with $\mathrm{ot}(C_N) = \omega$. Then for all $x \in \mathcal{P}(\kappa) \cap N$, $x \in U$ if and only if $x$ contains an end segment of $C_N$. Let $F$ be the filter on $\kappa$ generated by the end segments of $C_N$. Since $F \in K$, also $U = F \cap N$ is in $K$. Hence, $N$ is in $K$. Because $K$ is a model of $ZFC$, the core of $N$ and all the iterates of the core are in $K$. This shows that every mouse is in $K$. $\square$

**Lemma 5.87.** [94] *If $\beta \geq \omega$ is a cardinal in $K$, then $K_\beta = H^K_\beta$.*

*Proof.* The case $\beta = \omega$ is clear, so suppose $\beta > \omega$. Let $a \subset \gamma < \beta$ be in $K$. We show that $a \in K_\beta$. This is clear if $a \in L$ so we may assume $a \notin L$. Let $\kappa$ be the least regular cardinal $\geq \beta$ such that $a \in K_\kappa$. Then $a \in Q_\kappa$ since $K_\kappa = H^{Q_\kappa}_\kappa$. Since $a \notin L$ and $Q_\kappa = J^F_{\theta_\kappa}$, there is a least $\delta \geq \kappa$ such that $a \in J^F_{\delta+1} \setminus J^F_\delta$. Then, as in previous proofs in this section, $N = J^F_{\delta+1}$ is a mouse and $\rho_N \leq \gamma < \beta$. Let $M$ be the core of $N$ and let $\langle M_i, \bar{\kappa}_i, \pi_{ij} \rangle$ be the iteration of $M$. Suppose that $N = M_\lambda$. Since core mice are sound, $M = h_M(J_{\rho_M} \cup \{p_M\})$. Thus, $|M|^K \leq \rho_N < \beta$, so $N = M_\kappa$ and $M_i \in K_\beta$ for $i < \beta$. Let $i < \beta$ be such that $\bar{\kappa}_i \geq \gamma$. Then $\mathcal{P}(\bar{\kappa}_i) \cap M_i = \mathcal{P}(\bar{\kappa}_i) \cap N$, so $a \in \mathcal{P}(\bar{\kappa}_i) \cap M_i \subset K_\beta$. $\square$

This immediately gives the corollary.

**Corollary 5.88.** *$K$ satisfies $GCH$.*

We end the chapter with a result that is useful in the proof of the Main Theorem.

**Lemma 5.89.** *Let $U$ be a normal measure at $\kappa$ and let $\langle M_i, U_i, \kappa_i, \pi_{ij} \rangle$ be the iteration of $L[U]$. Then $K = \bigcup_{i \in On} H^{L[U_i]}_{\kappa_i}$.*

*Proof.* Every $x \in H^{L[U_i]}_{\kappa_i}$ is in $H^{J^{U_i}_\beta}_{\kappa_i}$ for some $\beta$. Hence, by Lemma 5.13

$$H^{L[U_i]}_{\kappa_i} = \bigcup_{\substack{\nu < \kappa_i \\ a \subset \nu,\ a \in L[U_i]}} J^a_{\kappa_i},$$

Thus, it is enough to show $x \in K$ for every $x \subset \gamma < \kappa_i$ such that $x \in L[U_i]$. If $x \in L$, then $x \in K$. If $x \notin L$, then there is $\beta > \kappa_i$ such that $\beta$ is the least ordinal such that $x \in J^{U_i}_{\beta+1}$. Then as in the proof of, e.g., Lemma 5.81 we can see that $J^{U_i}_{\beta+1}$ is a mouse. Hence, $x \in K$.

For the other direction, suppose that $M$ is a mouse. Let $\theta$ be a regular cardinal in $V$ such that $\theta > \max\{|M|, |\kappa^\kappa \cap L[U]|\}$. Then by Lemmas 3.9 and 5.43, $L[U_\theta] = L[F]$, where

[93] Corollary 14.14 of [4]. The proof uses Lemma 14.4 of [4].
[94] We follow Lemma 6.9 of [5].

$F$ is the club filter on $\theta$, and $M_\theta = J^F_\alpha$ for some $\alpha$. Thus, an iterate of $M$ is in $L[U_\theta]$, so every iterate of the core of $M$ is in $L[U_\theta]$ since $L[U_\theta]$ is a model of $ZFC$. In particular, $M$ is in $L[U_\theta]$ and $M \in H^{L[U_\theta]}_{\kappa_\theta}$. $\square$

# Chapter 6

# Inner model from the cofinality quantifier

In this final chapter we will present in detail the KMV paper's definition of the hierarchy of sets constructible using an extended logic $\mathcal{L}^*$, and, in particular, the definition of $C^*$. Then we will present the proofs of two major theorems of the paper concerning $C^*$, the second one being the Main Theorem of this thesis. This chapter is entirely based on the KMV paper but we often refer to lemmas presented in the previous chapters that are needed to understand the proofs.

## 6.1 Inner models from extended logics and $C^*$

The authors of the paper conceive of a logic $\mathcal{L}^*$ as having two essential components: $S^*$, the set of sentences of $\mathcal{L}^*$, and $T^*$, the the truth predicate for $\mathcal{L}^*$ [1]. Every logic considered in the paper has first order logic as a sublogic. The logic $\mathcal{L}(Q)$ with a generalized quantifier $Q$ is the logic $(S^*, Q^*)$ where $S^*$ is obtained by extending first order logic with the new quantifier $Q$. The truth predicate is defined by fixing the defining model class $\mathcal{K}_Q$ of $Q$ and then defining $T^*$ by induction on formulas using the following clause for $Q$:

$$\begin{aligned}&\mathcal{M} \models Qx_1, \ldots, x_n\phi(x_1, \ldots, x_n, \bar{b})\\ &\Leftrightarrow (M, \{(a_1, \ldots, a_n) \in M^n : \mathcal{M} \models \phi(a_1, \ldots, a_n, \bar{b})\}) \in \mathcal{K}_Q.\end{aligned}$$

For an extended logic $\mathcal{L}^*$, the hierarchy $(L'_\alpha)$ of *sets constructible using* $\mathcal{L}^*$ is defined as follows. For a set $M$, $\mathrm{Def}_{\mathcal{L}^*}(M)$ denotes the set of all sets of the form $x = \{a \in M : (M, \in) \models \phi(a, \bar{b})\}$, where $\phi$ is a formula of $\mathcal{L}^*$ and $\bar{b} \in M$. The hierarchy $(L'_\alpha)$ is defined by induction:

[1] We follow the discussion on pp. 4-7 of KMV.

$$L'_0 = \emptyset$$
$$L'_{\alpha+1} = \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha)$$
$$L'_\delta = \bigcup_{\alpha<\delta} L'_\alpha \text{ for limit } \delta.$$

The class $U_{\alpha\in\mathrm{On}} L'_\alpha$ is denoted by $C(\mathcal{L}^*)$. A set of a successor level has the form $x = \{a \in L'_\alpha : (L'_\alpha, \in) \models \phi(a, \bar{b})\}$, where $(L'_\alpha, \in) \models \phi(a, \bar{b})$ means $T^*$ in the sense of $V$, not in the sense of $C(\mathcal{L}^*)$.

The usual proof of $ZF$ in $L$ shows that for any logic $\mathcal{L}^*$ the class $C(\mathcal{L}^*)$ is a transitive model of $ZF$ containing all the ordinals, i.e., an inner model. For a logic $\mathcal{L}^*$ that is adequate for truth in itself, as most logics considered in literature are, $C(\mathcal{L}^*)$ satisfies the Axiom of Choice as well.

We now present the inner model obtained by extending first order logic with the cofinality quantifier[2]. The quantifier was introduced by Saharon Shelah in [18] and the logic it gives satisfies the compactness theorem for a vocabulary of any cardinality. The cofinality quantifier $Q^{\mathrm{cf}}_\kappa$ for a regular cardinal $\kappa$ is defined as follows:

$$\mathcal{M} \models Q^{\mathrm{cf}}_\kappa xy\phi(x, y, \bar{a}) \quad \Leftrightarrow \quad \{(c, d) : \mathcal{M} \models \phi(c, d, \bar{a})\} \text{ is a linear order of cofinality } \kappa.$$

The inner model $C(L(Q^{\mathrm{cf}}_\kappa))$ is denoted by $C^*_\kappa$ and $C^*_\omega$ is denoted by $C^*$. The model $C^*_\kappa$ knows which ordinals have cofinality $\kappa$ in $V$ but ordinals need not have the same cofinality in $C^*_\kappa$ as in $V$. Thus, even though an ordinal does not have cofinality $\kappa$ in $C^*_\kappa$, the fact that its cofinality in $V$ is $\kappa$ is recognized by in $C^*_\kappa$ in the sense that for all $\beta$ and $A, R \in C^*_\kappa$:

(i) $\{\alpha < \beta : \mathrm{cf}^V(\alpha) = \kappa\} \in C^*_\kappa$

(ii) $\{\alpha < \beta : \mathrm{cf}^V(\alpha) \neq \kappa\} \in C^*_\kappa$

(iii) $\{a \in A : \{(b, c) : (a, b, c) \in R\}$ is a linear order on $A$ with cofinality $\kappa$ in $V\} \in C^*_\kappa$.

**Lemma 6.1.** [3] *$C^* = L[On_\omega]$ where $On_\omega$ is the class of all ordinals of cofinality $\omega$.*

*Proof.* Clearly $L[\mathrm{On}_\omega]$ is included in $C^*$ so we need to show that $C^*$ is included in $L[\mathrm{On}_\omega]$. For any $\alpha$, a subset of $L_\alpha[\mathrm{On}_\omega]$ of the form (i) or (ii) above is obviously definable in $L_\alpha[\mathrm{On}_\omega]$ using $\mathrm{On}_\omega \cap L_\alpha[\mathrm{On}_\omega]$ as a predicate in the defining formula.

A subset of $L_\alpha[\mathrm{On}_\omega]$ of the form (iii) is also definable in some $L_\lambda[\mathrm{On}_\omega]$, $\lambda \geq \alpha$. For each $a \in A$, let $R_a = \{(b, c) : (a, b, c) \in R\}$. Since $L[\mathrm{On}_\omega]$ is a model of $ZFC$, there are

[2]This follows p. 20 of KMV.

[3]The result stated on p. 20 of KMV, the proof is our own.

$\lambda \geq \alpha$ and $\beta_a, f_a \in L_\lambda[\mathrm{On}_\omega]$ such that each $f_a$ is an increasing function from $\beta_a$ to $R_\alpha$. If $\beta_a \notin \mathrm{On}_\omega$, then $\mathrm{cf}(R_a)^V > \omega$. Otherwise there is a cofinal increasing function $g : \omega \to R_a$. But then $g' : \omega \to \beta_a$ defined by $g'(n) = \sup\{\gamma < \beta_a : f_a(\gamma) < g(n)\}$ shows that $\mathrm{cf}(\beta_a)^V = \omega$, a contradiction. On the other hand, if $\beta_a \in \mathrm{On}_\omega$, then $\mathrm{cf}(R_a)^V = \omega$. Hence, the set $B = \{a \in A : \{(b,c) : (a,b,c) \in R\}$ is a linear order on $A$ with cofinality $\kappa$ in $V\}$ is definable in $L_\lambda[\mathrm{On}_\omega]$ by

$$\begin{aligned} a \in B \text{ iff } a \in A \wedge \exists f_a \exists \beta_a\, (\beta_a \in \mathrm{On}_\omega \cap L_\lambda[\mathrm{On}_\omega] \\ \wedge\ f_a \text{ is a cofinal increasing function from } \beta_a \text{ onto } R_a). \end{aligned}$$

Hence, $B \in L_{\lambda+1}[\mathrm{On}_\omega]$, so $B \in L[\mathrm{On}_\omega]$. □

The remainder of this chapter presents the proofs of two major theorems about $C^*$.

## 6.2 The core model and $C^*$

One of the major results of the KMV paper is that $C^*$ and $V$ have the same core model. This section presents the proof of that theorem.

**Theorem 6.2.** [4] *The Dodd-Jensen core model is contained in $C^*$.*

*Proof.* We denote the core model of $C^*$ by $K^*$ and the core model of $V$ by just $K$. For a contradiction we suppose that $K$ is not contained in $C^*$. Since the core model is the union of all mice, if $K^*$ is not $K$, $C^*$ does not contain all the mice of $V$. Let $M_0$ be the minimal mouse of $V$ missing from $C^*$. Suppose $M_0 = J_\alpha^{U_0}$ is a mouse at $\kappa$. Let $\langle M_\alpha, j_{\alpha\beta}, \kappa_\alpha\rangle$ be the iteration of $M$. We will show that an iterate of $M_0$, say $M_\alpha$, is in $C^*$. Then the core of $M_\alpha$ must be in $C^*$ since $C^*$ is a model of $ZFC$ and the existence of the core is a theorem of $ZFC$. Hence, all the iterates of the core are in $C^*$, so in particular $M_0$ is in $C^*$. That is a contradiction, so $K^*$ must be the whole $K$.

We start proving that an iterate of $M_0$ is in $C^*$. Define $\xi_0 = (\kappa^+)^{M_0}$ and let $\delta = \mathrm{cf}^M(\xi_0)$. If $(\kappa^+)^{M_0}$ does not exist in $M_0$, we let $\xi_0$ be $\mathrm{On} \cap M_0$. For $\beta > 0$, we let $\xi_\beta = j_{\alpha\beta}(\xi_0)$.

*Claim 1:* For all $\beta$, $\xi_\beta = j_{0\beta}``(\xi_0)$. Hence, $\mathrm{cf}^V(\xi_\beta) = \delta$.

*Proof.* By Lemma 5.31(c) every $x \in M_\beta$ is of the form $j_{0\beta}(f)(\kappa_{i_1}, \dots, \kappa_{i_n})$ for some function $f : \kappa^n \to M_0$, $f \in M_0$ and some $i_1 < \cdots < i_n < \beta$. If $\eta < \xi_\beta$ and $\eta = j_{0\beta}(f)(\kappa_{i_1}, \dots, \kappa_{i_n})$, we can assume that $f(a_1, \dots, a_n) < \xi_0$ for all $(a_1, \dots, a_n) \in \kappa^n$. Since $\xi_0$ is regular in $M_0$, there is $\rho$ such that $f(a_1, \dots, a_n) < \rho$ for all $(a_1, \dots, a_n) \in \kappa^n$. Hence, $j_{0\beta}(f)(\kappa_{i_1}, \dots, \kappa_{i_n}) < j_{0\beta}(\rho)$ for all $i_1 < \cdots < i_n < \beta$. Thus, $\xi_\beta = j_{0\beta}``(\xi_0)$, so $\mathrm{cf}^V(\xi_\beta) \leq \delta$.

[4] Theorem 5.5 of KMV.

To show that $\mathrm{cf}^V(\xi_\beta)$ must be exactly $\delta$, suppose it is smaller, say $\gamma < \delta$. Then there is a cofinal function in $V$ from $\gamma$ to $\xi_\beta$. Define $g : \gamma \to \xi_0$ by $g(\eta) = \sup\{\alpha < \xi_0 : j_{0\beta}(\alpha) < f(\eta)\}$. Then $g \in V$ is cofinal in $\xi_0$, which contradicts the assumption $\mathrm{cf}^V(\xi_0) = \gamma$. Hence, $\mathrm{cf}^V(\xi_\beta)$ must be $\delta$. $\square$ *Claim 1.*

Since $\kappa_\beta$ and $\xi_\beta$ are cardinals in $M_\beta$, $\omega\kappa_\beta = \kappa_\beta$ and $\omega\xi_\beta = \xi_\beta$. Therefore, $J^{U_\beta}_{\kappa_\beta} = L_{\kappa_\beta}[U_\beta]$ and $J^{U_\beta}_{\xi_\beta} = L_{\xi_\beta}[U_\beta]$. Hence, the proof of Lemma 3.3 implies that $\kappa_\beta^{\kappa_\beta} \cap M_\beta \subset J^{U_\beta}_{\xi_\beta}$. Moreover, $J^{U_\beta}_{\xi_\beta}$ is the increasing union of $\delta$ members of $M_\beta$, each one having size $\kappa_\beta$ in $M_\beta$.

*Claim 2:* Let $\kappa_0 < \eta < \kappa_\beta$ be such that $\eta$ is regular in $M_\beta$. Then either there is $\gamma < \beta$ such that $\eta = \kappa_\gamma$ or $\mathrm{cf}^V(\eta) = \delta$.

*Proof.* We prove the claim by induction on $\beta$. The case $\beta = 0$ is impossible. If $\beta$ is a limit, then $\kappa_\beta = \sup\{\kappa_\gamma : \gamma < \beta\}$. Hence, there is $\alpha < \beta$ such that $\eta < \kappa_\alpha$. Since $j_{\alpha\beta} \restriction \kappa_\alpha$ is the identity, $j_{\alpha\beta}(\eta) = \eta$. Thus, the $\Sigma_1$-elementarity of $j_{\alpha\beta}$ implies that $\eta$ is regular in $M_\alpha$. Then the claim holds by the induction assumption on $\alpha$.

We have the successor case left to prove, so suppose $\beta = \alpha + 1$. If $\eta \leq \kappa_\alpha$, the claim follows as in the limit case. So suppose $\kappa_\alpha < \eta < \kappa_\beta$. Then $\eta$ is represented in the ultrapower of $M_\alpha$ by a function $f \in M_\alpha$ whose domain is $\kappa_\alpha$. Since $\eta < \kappa_\beta = j_{\alpha\beta}(\kappa_\alpha)$, we an assume that $f(\gamma) < \kappa_\alpha$ for all $\gamma < \kappa_\alpha$. Since $\kappa_\alpha = [id]$ in the ultrapower and $\eta > \kappa_\alpha$, we can assume that $\gamma < f(\gamma)$ for all $\gamma < \kappa_\alpha$. Finally, since $\eta$ is regular in $M_\beta$, we can assume that $f(\gamma)$ is regular in $M_\alpha$ for all $\gamma < \kappa_\alpha$. To simplify notation, we set temporarily $M = M_\alpha$, $\kappa = \kappa_\alpha$, $U = U_\alpha$ and $\xi = \xi_\alpha$.

To show that $\mathrm{cf}^V(\eta) = \delta$, it suffices to define in $V$ a sequence $\langle g_\nu : \nu < \delta\rangle$ of functions from $\kappa^\kappa \cap M$ satisying the following conditions:

1. The sequence is increasing modulo $U$, i.e., for all $\nu_1, \nu_2$, the set $\{\gamma < \kappa : g_{\nu_1}(\gamma) < g_{\nu_2}(\gamma)\}$ is in $U$.
2. For all $\gamma < \kappa$, $g_\nu(\gamma) < f(\gamma)$.
3. The ordinals represented by these functions in the ultrapower are cofinal in $\eta$.

By the remark before the claim, $\kappa^\kappa \cap M = \bigcup_{\psi<\delta} F_\psi$, where $F_\psi$ is in $M$ and has size $\kappa$ in $M$. For $\psi < \delta$, we fix in $M$ an enumeration $\langle h^\psi_\gamma : \gamma < \kappa\rangle$ of the set $G_\psi = \{h \in F_\psi : \forall\gamma < \kappa\,(h(\gamma) < f(\gamma))\}$. Define the function $f_\psi \in \kappa^\kappa$ by $f_\psi(\gamma) = \sup\{h^\psi_\mu(\gamma) : \mu < \gamma\}$. Then $f_\psi$ is in $M$ and $f_\psi$ bounds all functions in $G_\psi$ modulo $U$. Since $h(\gamma) < f(\gamma)$ for all $\gamma < \kappa$ and $h \in G_\psi$ and $f(\gamma)$ is regular in $M$ for all $\gamma$, we get also that $f_\psi(\gamma) < f(\psi)$ for all $\psi$.

Now we can define by induction on $\nu < \delta$ $g_\nu$ and $\psi_\nu$ such that $\psi_\nu < \delta$ and $g_\nu \in G_{\psi_\nu}$. Given $\langle \psi_\mu : \mu < \nu\rangle$, let $\sigma$ be their supremum. Then let $g_\nu$ be $f_\sigma$ and let $\psi_\nu$ be the least member of $\delta - \sigma$ such that $f_\sigma \in G_{\psi_\nu}$. We show that the sequence of ordinals represented by $\langle g_\nu : \nu < \delta \rangle$ is cofinal in $\eta$. Every ordinal below $\eta$ is represented by some function $h$ bounded everywhere by $f$. Since $h$ belongs to $G_{\psi'}$ for some $\psi' < \delta$, there is $\psi_\nu < \delta$ such that $\psi_\nu > \psi'$. Hence, $g_{\psi_\nu+1}$ bounds $h$ modulo $U$. $\square$ *Claim 2.*

$M_0$ and thus every $M_\beta$ are minimal mice missing from $K^*$. For any mouse $N \in K^*$ and large enough regular $\theta$, $N_\theta$ must be in $M_\theta$. Otherwise, as in the beginning of the proof, $M_0$ would be in $K^*$. Thus, $\mathcal{P}(\kappa_\beta) \cap N \subset \mathcal{P}(\kappa_\beta \cap M_\beta)$. Hence, any subset of $\kappa_\beta$ that is in $K^*$ must be in $M_\beta$, that is, $\mathcal{P}(\kappa_\beta) \cap K \subset \mathcal{P}(\kappa_\beta) \cap M_\beta$. This implies that if $\rho \leq \kappa_\beta$ is regular in $M_\beta$ it is regular in $K^*$.

On the other hand, if $a \in M_\beta$ is a bounded subset of $\kappa_\beta$, then by Corollary 5.77 $a$ is definable in a mouse smaller than $M_\beta$, so $a \in K^*$. Thus, if $\rho < \kappa_\beta$ is regular in $K^*$, it is regular in $M_\beta$. Since $\kappa_\beta$ is always regular in $M_\beta$, every $\rho \leq \kappa$ is regular in $M_\beta$ if and only if $\rho$ is regular in $K^*$. In particular, every $\kappa_\beta$ is regular in $K^*$ since it is regular in $M_\beta$.

*Claim 3:* Let $\theta$ be a regular cardinal greater than $\max(|M_0|, \delta)$. Then there is $D \in C^*$ such that $D$ is a subset of $E = \{\kappa_\beta : \beta < \theta\}$ and $D$ is cofinal in $\theta$.

*Proof.* By Lemma 5.43 $\kappa_\theta = \theta$. Then $E$ is a club in $\theta$. Let $S_0^\theta = \{\alpha < \theta : \mathrm{cf}^V(\alpha) = \omega\}$. Since $E$ is closed, both $E \cap S_0^\theta$ and $E - S_0^\theta$ are unbounded in $\theta$. Define $C = \{\alpha \in \theta \setminus \kappa_0 : \alpha \text{ regular in } K^*\}$. By the definition of $C^*$, $S_0^\theta$ is in $C^*$. Since $K^*$ is the union of all mice in $C^*$ and a mouse can be defined in first-order logic, $C$ is also in $C^*$. Since $\kappa_\beta$ is regular in $M_\beta$ and hence in $K^*$, $E$ is a subset of $C$.

If $\delta \neq \omega$, we can let $D = C \cap S_0^\theta$. By Claim 2, every element of $C \setminus E$ has cofinality $\delta$ in $V$, so $D \subset E$. Also, $D$ is unbounded in $\theta$ because $\kappa_\beta$ is in $D$ if $\mathrm{cf}^V(\beta) = \omega$. If $\delta = \omega$, we let $D = C \setminus S_0^\theta$. Again, $D \subset E$ and $D$ is cofinal in $\theta$. In either case, $D$ is in $C^*$, so the claim has been proved. $\square$ *Claim 3.*

Now we can prove that an iterate of $M_0$ is in $C^*$. Pick $\theta$ and $E$ as in Claim 3 and let $D \subset \theta$ witness the claim. Let $F_E$ be the filter generated by the end segments of $E$. Since every end segment of $E$ is a club in $\theta$, Lemma 5.43 implies that $U_\theta = F_E \cap M_\theta$. But $U_\theta$ is an ultrafilter on $\theta$ in $M_\theta$, so every $x$ in $U_\theta$ must contain an end segment of $D$. Hence, $F_E \cap M_\theta = F_D \cap M_\theta$, where $F_D$ is the filter generated by the end segments of $D$. Thus, $M_\theta = J_\alpha^{F_D}$ for some $\alpha$. Since $D$ is in $C^*$, $L[D]$ is included in $C^*$, so $M_\theta$ is in $C^*$.

$\square$

## 6.3 The Main Theorem

For the proof we need the following lemma which is proved in exactly the same way as Claim 2 in the proof of Theorem 6.2.

**Lemma 6.3.** [5] *Let $M = L^\mu$ and let $\kappa$ be the cardinal on which $L^\mu$ has the normal measure. Let $M_\beta$ be the iterated ultrapowers of $M$. If $\kappa < \eta < \kappa_\beta$ and $\eta$ is regular in $M_\beta$, then either* $\mathrm{cf}^V(\eta) = \kappa^+$ *or there is $\gamma < \beta$ such that $\eta = \kappa_\gamma$.*

**Main Theorem.** [6] If $V = L^\mu$, then $C^*$ is exactly the inner model $M_{\omega^2}[E]$ where $M_{\omega^2}$ is the $\omega^2$-th iterate of $V$ and $E = \{\kappa_{\omega\cdot n} : n < \omega\}$.

To prove the Main Theorem we need to be able to know inside $M_{\omega^2}$ which ordinals have cofinality $\omega$ in $V$. That is established by the following lemma and its corollary.

**Lemma 6.4.** [7] *Let $M$ be a transitive model of $ZFC + GCH$ with a measurable cardinal $\kappa$. For $\beta \in On$, let $M_\beta$ be the $\beta$-th iterate of $M$ and let $\kappa_\beta$ be the image of $\kappa$ under the canonical embedding $j_{0\beta}$ from $M$ to $M_\beta$. Then for every ordinal $\delta \in M_\beta$, if* $\mathrm{cf}^M(\delta) < \kappa$, *then either* $\mathrm{cf}^{M_\beta}(\delta) = \mathrm{cf}^M(\delta)$ *or there is a limit $\gamma \leq \beta$ such that* $\mathrm{cf}^{M_\beta}(\delta) = \kappa_\gamma$.

*Proof.* For each ordinal $\beta$, let $\xi_\beta = (\kappa_\beta^+)^{M_\beta}$, and let $\eta = \mathrm{cf}^M(\xi_0)$. We prove first the following claims:

*Claim 1*: For every $\beta$, $\mathrm{cf}^M(\xi_\beta) = \eta$.

*Proof.* Let $\nu < \xi_\beta$. By lemma 2.11, $\nu = j_{0\beta}(f)(\kappa_{\gamma_0}, \ldots, \kappa_{\gamma_{n-1}})$ for some $\gamma_0, \ldots, \gamma_n - 1 < \beta$ and some $f : \kappa_0^n \to \xi_0$ that is in $M$. Since $\xi_0$ is a successor cardinal in $M_0$, it is regular in $M_0$, so there is $\rho < \xi_0$ such that $f(\alpha_0, \ldots, \alpha_{n-1}) < \rho$ for all $(\alpha_0, \ldots, \alpha_{n-1}) \in \kappa_0^n$. Hence, the elementarity of $j_{0\beta}$ implies that every value of $j_{0\beta}(f)$ is smaller than $j_{0\beta}(\rho)$. Thus, $\nu < j_{0\beta}(\rho)$, so $\xi_\beta = \sup j''_{0\beta}(\xi_0)$ and $\mathrm{cf}^M(\xi_\beta) \leq \mathrm{cf}^M(\xi_0) = \eta$. If $\mathrm{cf}^M(\xi_\beta) < \eta$, then $\xi_\beta = \sup j''_{0\beta}(\xi_0)$ implies that $\mathrm{cf}^M(\xi_0) < \eta$, which is a contradiction. Hence, $\mathrm{cf}^M(\xi_\beta) = \eta$. $\square$ *Claim 1.*

*Claim 2*: For every $\beta$, $\mathrm{cf}^M(\kappa_{\beta+1}) = \eta$.

*Proof.* Since $M_\beta \vDash GCH$, we have $\kappa_{\beta+1} = j_{\beta,\beta+1}(\kappa_\beta) < (\kappa_\beta^{++})^{M_\beta}$, whence $\mathrm{cf}^{M_\beta}(\kappa_{\beta+1}) \leq (\kappa_\beta^+)^{M_\beta} = \xi_\beta$. Since $M_{\beta+1}$ is the ultrapower of $M_\beta$ by the ultrafilter $U_\beta \in M_\beta$, $M_{\beta+1}$ closed under $\kappa_\beta$-sequences. Thus, if $\mathrm{cf}^{M_\beta}(\kappa_{\beta+1}) < \xi_\beta$, then there is a sequence $\langle \alpha_\gamma : \gamma < \mathrm{cf}^{M_\beta}(\kappa_{\beta+1}) \leq \kappa_\beta \rangle$ in $M_{\beta+1}$ cofinal in $\kappa_{\beta+1}$, so $\kappa_{\beta+1}$ is not regular in $M_{\beta+1}$, which is a contradiction. Thus, $\mathrm{cf}^{M_\beta}(\kappa_{\beta+1}) = \xi_\beta$.

Now if $\mathrm{cf}^M(\kappa_{\beta+1}) < \eta$, then $\mathrm{cf}^M(\kappa_{\beta+1}) \leq \kappa$ and there is again a sequence $\langle \alpha_\gamma : \gamma < \mathrm{cf}^M(\kappa_{\beta+1}) \rangle \in M_{\beta+1}$ that is cofinal in $\kappa_{\beta+1}$, whence $\kappa_{\beta+1}$ is singular in $M_{\beta+1}$, which is a contradiction. Thus, $\mathrm{cf}^M(\kappa_{\beta+1}) \geq \eta$. On the other hand, since $\mathrm{cf}^{M_\beta}(\kappa_{\beta+1}) = \xi_\beta$, there is in $M_\beta$ a sequence $\langle \alpha_\gamma : \gamma < \xi_\beta \rangle$ cofinal in $\kappa_{\beta+1}$. As $M$ is a model of $ZFC$ and $M_\beta$

[5] Claim 2 in the proof of Lemma 5.6 of KMV.

[6] Theorem 5.14 of KMV.

[7] Lemma 5.15 of KMV.

is an iterated ultrapower of $M$ by an $M$-ultrafilter on $\kappa$ that is in $M$, we have $M \supset M_\beta$. Hence $\langle \alpha_\gamma : \gamma < \xi_\beta \rangle$ is in $M$, and as $\mathrm{cf}^M(\xi_\beta) = \eta$, we have $\mathrm{cf}^M(\kappa_{\beta+1}) \leq \eta$. Hence, $\mathrm{cf}^M(\kappa_{\beta+1}) = \eta$. $\square$ *Claim 2.*

Now consider the $\delta$ in the formulation of the lemma. Let $\delta' = \mathrm{cf}^{M_\beta}(\delta)$. Since $M_\beta \subset M$, we can show that $\mathrm{cf}^M(\delta') = \mathrm{cf}^M(\delta)$. There is in $M$ a sequence $\langle a_i < \delta : i < \mathrm{cf}^M(\delta) \rangle$ cofinal in $\delta$. Since $M_\beta \subset M$, there is in $M$ also a sequence $\langle b_j < \delta : j < \delta' \rangle$ cofinal in $\delta$. Define for all $i < cf^M(\delta)$, $c_i = \bigcup \{ j : b_j < a_i \}$. Now the sequence $\langle c_i : i < \mathrm{cf}^M(\delta) \rangle$ is in $M$ and is cofinal in $\delta'$. Thus, $\mathrm{cf}^M(\delta') \leq \mathrm{cf}^M(\delta)$. On the other hand, because the sequence $\langle b_j \rangle$ is in $M$, we have $\mathrm{cf}^M(\delta) \leq \mathrm{cf}^M(\delta')$. Hence, $\mathrm{cf}^M(\delta') = \mathrm{cf}^M(\delta)$.

We prove the lemma in several cases. Since $\delta'$ is regular in $M_\beta$ and $\mathrm{cf}^M(\delta') = \mathrm{cf}^M(\delta)$, we can assume that $\delta$ is regular in $M_\beta$ in the following cases:

(i) $\delta \leq \kappa$
In this case the iterated ultrapowers do not change the cofinality of $\delta$. Hence, $\mathrm{cf}^M(\delta) = \mathrm{cf}^{M_\beta}(\delta)$.

(ii) $\kappa < \delta' \leq \kappa_\beta$
Since $M_\beta \vDash GCH$ for all $\beta$, the argument of Claim 2 in the proof of Theorem 6.2 shows that either $\mathrm{cf}^M(\delta) = \eta$ or there is $\gamma \leq \beta$ such that $\delta = \kappa_\gamma$. The first case cannot occur since $\mathrm{cf}^M(\delta) < \kappa < \eta$. In the second case, if $\gamma$ is 0 or successor, we get by Claim 2 that $\mathrm{cf}^M(\delta) \geq \kappa$, which is a contradiction. If $\gamma$ is limit, the claim of the lemma holds.

(iii) $\kappa_\beta < \delta$
Again by lemma 2.11 every ordinal in $M_\beta$ is of the form $j_{0\beta}(f)(\kappa_{\gamma_0}, \ldots, \kappa_{\gamma_{k-1}})$ for some $k \in \omega$, $\gamma_0, \ldots, \gamma_{k-1} < \beta$ and $f \in M$ some ordinal valued function defined on $\kappa^k$. Since $\mathrm{cf}^M(\delta) < \kappa$, there is in $M$ an ordinal $\mu < \kappa$ and a sequence $\langle \alpha_\nu : \nu < \mu \rangle$ that is cofinal in $\delta$. For each $\alpha_\nu$ there is a function $f_\nu \in M$ defined on $\kappa^{k_\nu}$ such that $\alpha_\nu \in j_{0\beta}(f)\text{"}j(\kappa)^{k_\nu}$. Now $\langle f_\nu : \nu < \mu \rangle \in M$ and the union of the ranges $\bigcup_{\nu<\mu} \big( ran(j_{0\beta}(f_\nu)) \cap \delta \big)$ is cofinal in $\delta$. But on the other hand $\langle j_{0\beta}(f_\nu) : \nu < \mu \rangle$ $= j_{0\beta}(\langle f_\nu : \nu < \mu \rangle) \in M_\beta$ so the union of the ranges $\bigcup_{\nu<\mu} \big( ran(j_{0\beta}(f_\nu)) \cap \delta \big)$ is a union of $\mu$ sets of size at most $j_{0\beta}(\kappa) = \kappa_\beta$. Since $\delta$ is regular in $M_\beta$ and $\kappa_\beta < \delta$, the union is bounded in $\delta$, which is a contradiction.

$\square$

**Corollary 6.5.** [8] *If $V \vDash GCH$ and $\kappa$ is measurable, then an ordinal has cofinality $\omega$ in $V$ iff its cofinality in $M_{\omega^2}$ is either $\omega$ or of the form $\kappa_\gamma$ for some limit $\gamma \leq \omega^2$.*

[8] Corollary on p. 34 of KMV.

*Proof.* If $\alpha$ has cofinality $\omega$ in $V$, then $\mathrm{cf}^V(\alpha) < \kappa$ and by the above lemma either $\mathrm{cf}^{M_{\omega^2}}(\alpha) = \omega$ or there is a limit $\gamma \leq \omega^2$ such that $\alpha = \kappa_\gamma$. On the other hand, by lemma 2.9, for all limit $\gamma \leq \omega^2$, $\kappa_\gamma = \sup\{\kappa_\beta : \beta < \gamma\}$, so $\mathrm{cf}^V(\kappa_\gamma) = \omega$. Since each $\kappa_\gamma$, $\gamma \leq \omega^2$, is regular in $M_{\omega^2}$, we have proved the corollary.

□

***Proof of the Main Theorem***[9]

Let $\mu' = j_{0\omega^2}(\mu)$ be the image of $\mu$. By lemma 4.7, $E$ is a Prikry generic sequence over $M_{\omega^2}$ with respect to $\mu'$. By the properties of Prikry forcing, all the cardinals have the same cofinality in $M_{\omega^2}[E]$ as in $M_{\omega^2}$ except $\kappa_{\omega^2}$ which has cofinality $\omega$ in $M_{\omega^2}[E]$. So an ordinal has cofinality $\omega$ in $V$ if its cofinality in $M_{\omega^2}[E]$ is in $\{\omega\} \cup E$. By the properties of forcing, $M_{\omega^2}$ already knows if the cofinality of an ordinal in $M_{\omega^2}[E]$ is in $\{\omega\} \cup E$. Hence, $C^* = L[On_\omega] \subset M_{\omega^2}[E]$.

Now we prove the other direction, i.e., that $M_{\omega^2}[E] \subset C^*$. By theorem 6.2, the Dodd-Jensen core model $K$ of $V$ is the same as the Dodd-Jensen core model of $C^*$. IF $\eta < \kappa_{\omega^2}$ is regular in $K$, then by Lemma 5.89 it is regular in $M_{\omega^2}$. Thus, by lemma 6.3, if $\eta$ is regular in $K$ and $\kappa < \eta < \kappa_{\omega^2}$, then either $\mathrm{cf}^V(\eta) = \kappa^+$ or $\eta = \kappa_\gamma$ for some $\gamma < \omega^2$. By Claim 1 in the proof of lemma 6.4, for successor $\gamma$ the ordinal $\kappa_\gamma$ has cofinality $\kappa^+$ in $V$. Thus, we have showed that $E$ is exactly the set of ordinals $\eta$ which are regular in $K$, $\kappa \leq \eta \leq \kappa_{\omega^2}$ and $\mathrm{cf}(\eta) = \omega$. This shows that $E \in C^*$.

By lemma 3.9, if $F$ is the filter generated by the end segments of $E$, then $M_{\omega^2} = L^{\mu'} = L[F]$. Therefore $M_{\omega^2} \subset C^*$, and since $E \in C^*$, we have $M_{\omega^2}[E] \subset C^*$.

□

[9]Follows the proof on p. 34 of KMV.

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