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RENEWAL THEORY AND ITS APPLICATIONS

Lectures notes
for the course "Stochastic modelling" taken by most Mathematics students and Economics students (directions of training 01.03.01 – Mathematics and 38.04.01 – Economics)

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The goal of this course is to study the main tools of the renewal theory and their applications to some problems of the actuarial analysis for insurance companies in the framework of the Cremér - Lundberg models. We consider such important problems in the renewal theory as limit theorems for the renewal processes and the ruin problems for the insurance companies with investments in the stochastic financial markets. The notes are intended for students of the Mathematics and Economics Faculties.

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Contents

1 Poisson processes

1.1 Definition and main properties

In this section we study the principal properties of the Poisson process. Let $(\tau_j)_{\geq 1}$ be i.i.d. exponential random variables with some parameter $\lambda > 0$. We set $T_n = \sum_{j=1}^{n} \tau_j$ for $n \geq 1$ and $T_0 = 0$.

Definition 1.1. *The random* $\mathbb{R}_+ \to \mathbb{N}$ *function*

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{(T_n \leq t)} \tag{1.1}$$

is called the homogeneous Poisson process of the intensity $\lambda > 0$.

Proposition 1.1. *If* $(\tau_j)_{j\geq 1}$ *is i.i.d. exponential random variables with a parameter* $\lambda > 0$, *then the vector* $(T_1, \ldots, T_n)$ *has the distribution density with respect to the Lebesgue measure in* $\mathbb{R}^n$ *defined as*

$$f_n(x_1, \ldots x_n) \, = \, \lambda^n \, e^{-\lambda x_n} \, \mathbf{1}_{\{0 < x_1 < \ldots < x_n\}} \, . \tag{1.2}$$

Proposition 1.2. *If* $(N_t)_{t\geq 0}$ *is a homogeneous Poisson process with an intensity* $\lambda > 0$, *then for any* $t > 0$ *the random variable* N_t *has the Poisson distribution, i.e. for any integer* $n \geq 0$

$$\mathbf{P}(N_t = n) \, = \, \frac{(\lambda t)^n}{n!} \, e^{-\lambda t} \, . \tag{1.3}$$

Now we study the main properties of the Poisson processes.

Proposition 1.3. *Let* $(N_t)_{t\geq 0}$ *be a homogeneous Poisson process of an intensity* $\lambda > 0$. *Then*

1. *almost sure the function* $(N_t)_{t\geq 0}$ *is increasing, with integer values and right continuous;*

2. *conditionally with respect to* $N_t = n$, *the vector* $(T_1, \ldots, T_n)$ *has the same distribution as* n *order statistics uniformly distributed on the interval* $[0, t]$*;*

3. *the Poisson process* $(N_t)_{t\geq 0}$ *has homogeneous increments, i.e. for all* $0 < s < t$ *and any integer* $n \geq 0$

$$\mathbf{P}(N_t - N_s = n) = \mathbf{P}(N_{t-s} = n);$$

4. *the Poisson process* $(N_t)_{t\geq 0}$ *has independent increments, i.e. for any time moments* $0 = t_0 < t_1 < \ldots < t_m$ *and any integer numbers* $n_1, \ldots, n_m$

$$\mathbf{P}\left(N_{t_1} = n_1\,, N_{t_2} - N_{t_1} = n_2\,, \ldots\,, N_{t_m} - N_{t_{m-1}} = n_m\right) = \prod_{j=1}^{m} \mathbf{P}\left(N_{t_j} - N_{t_{j-1}} = n_j\right)\,;$$

5. *the Poisson process* $(N_t)_{t\geq 0}$ *is a process of rare events, i.e. for any* $t \geq 0$ *and* $\Delta > 0$

$$\mathbf{P}\left(N_{t+\Delta} - N_t \,=\, 1\right) \,=\, \lambda\,\Delta \,+\, o(\Delta)\,,$$
$$\mathbf{P}\left(N_{t+\Delta} - N_t \,>\, 1\right) \,=\, o(\Delta)\,, \tag{1.4}$$

as $\Delta \to 0$.

Remark 1.1. *As we will see later all these properties are very useful in the actuarial mathematics for the constructing the principal insurance models. Indeed, the Poisson process is used to model the number of claims on the time interval* $[0, t]$. *Especially, the independent increments and rare events properties are very natural for the insurance models.*

1.2 Principal features

Proposition 1.4. *Let* $(N_t)_{t\geq 0}$ *be a stochastic process that satisfies the following conditions:*

- *for almost every* ω, *the trajectory* $(N_t(\omega))_{t\geq 0}$ *is zero in* 0, *increasing, right continuous and with integer values;*

- *the process* $(N_t)_{t\ge 0}$ *has independent and homogeneous increments;*
- $(N_t)_{t\ge 0}$ *is a process of rare events, i.e. there exists* $\lambda > 0$, *for which the asymptotic properties* (1.4) *hold.*

Then $(N_t)_{t\ge 0}$ *is the Poisson process of the intensity* $\lambda > 0$.

Proof. Firstly, we show that

$$\mathbf{P}(N_t = 0) \,=\, e^{-\lambda t}\,. \tag{1.5}$$

We denote by $f(t) = \mathbf{P}(N_t = 0)$. Indeed, due to the independence N_t and $N_{t+s} - N_t$ we obtain

$$f(t+s) = \mathbf{P}(N_{t+s} = 0) = \mathbf{P}(N_{t+s} = 0\,,\, N_t = 0)$$
$$= \mathbf{P}(N_{t+s} - N_t = 0\,,\, N_t = 0) = f(t)f(s)\,.$$

Using here the rare events property, we get (1.5). Now we find the distribution of N_t for arbitrary fixed $t > 0$. To this end, we set $G(t) = \mathbf{E}z^{N_t}$ for $0 < z < 1$. Taking into account that the increments are independence and homogeneous, we can represent the function $G(t+s)$ as

$$G(t+s) = \mathbf{E}\, z^{N_{t+s}-N_t}\, z^{N_t} = G(t)G(s)\,.$$

Moreover, for all $t > 0$

$$G(t) \geq \mathbf{E} z^{N_t} \mathbf{1}_{\{N_t = 0\}} = e^{\lambda t} > 0\,.$$

Therefore, $G(t) = e^{tg(z)}$ and

$$g(z) = \lim_{t \to 0} \frac{G(t) - 1}{t}\,.$$

Then the rare events property directly implies that for $t \to 0$

$$G(t) = \mathbf{P}(N_t = 0) + z\mathbf{P}(N_t = 1) + \mathbf{E} z^{N_t} \mathbf{1}_{\{N_t \geq 2\}} = e^{-\lambda t} + z\lambda t + \mathrm{o}(t)\,.$$

Therefore, $g(z) = \lambda(t - 1)$ and

$$G(t) = e^{-\lambda t} e^{\lambda z t} = \sum_{n=0}^{\infty} z^n \left(e^{-\lambda t} \frac{(\lambda t)^n}{n!} \right)\,.$$

This directly implies that for all $t > 0$

$$\mathbf{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}\,.$$

Next, note that we can represent the process $(N_t)_{t \geq 0}$ as

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}}\,,$$

where $T_n = \inf\{t \geq 0 \,:\, N_t \geq n\}$. From here it follows that

$$\mathbf{P}(T_n > t) = \mathbf{P}(N_t \leq n-1) = \sum_{j=0}^{n-1} \mathbf{P}(N_t = j)$$

$$= \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} = \lambda^n \int_t^{\infty} \left(v^{n-1} e^{\lambda v}\right) \, \mathrm{d}v \,.$$

This implies that the distribution of T_n coincides with the distribution of a sum i.i.d. exponential random variables of the parameter $\lambda > 0$. Thus, in view of the definition (1.1), the random function $(N_t)_{t\geq 0}$ is a homogeneous Poisson process. □

1.3 The last jump of the Poisson process

Let's study the properties of the delay between the present time moment $t > 0$ and the last jumping moment T_{N_t}. Putting $T_0 = 0$, we get

$$T_{N_t} = \sum_{k=0}^{\infty} T_k \, \mathbf{1}_{\{N_t = k\}} \,.$$

Therefore, T_{N_t} is a random variable. We will study the properties of the two random variables $V_t = T_{N_t+1} - t$ and $V_t^* = t - T_{N_t}$.

Proposition 1.5. *The random variable V_t is independent of the σ-field, generated by the variables $\{N_s \,,\, s \leq t\}$ and has the exponential distribution with the parameter $\lambda > 0$.*

Proof. We note that for $u > 0$

$$\begin{aligned}
\{V_t \le\; u\} \;&=\; \{T_{N_t+1} \;-\; t \le u\} \;=\; \cup_{n=0}^{\infty} \{T_{n+1} - t \le u\,,\, N_t = n\} \\
&= \cup_{n=0}^{\infty} \{N_{t+u} \;\ge\; n+1\,,\, N_t = n\} \\
&= \cup_{n=0}^{\infty} \{N_{t+u} \;-\; N_t \;\ge\; 1\,,\, N_t = n\} \\
&= \{N_{t+u} \;-\; N_t \;\ge\; 1\,\}\,.
\end{aligned}$$

This immediately implies Proposition 1.5. $\square$

Proposition 1.6. *For any Borelian sets $A \subseteq \mathbb{R}$ and for any $t > 0$*

$$\mathbf{P}(V_t^* \in A) \;=\; e^{-\lambda t}\,\mathbf{1}_{\{t\in A\}} \;+\; \lambda \int_{A\cap[0,t]} e^{-\lambda v}\,\mathrm{d}v\,. \qquad (1.6)$$

Proof. It is clear that for this proposition it suffices to show (1.6) for the sets of the form $A = [0, u[$ with $u > 0$. We note that $V_t^* \le t$

a.s., i.e. for $u \geq t$ the equation (1.6) is true. For $u < t$ we get that

$$\mathbf{P}(V_t^* \in A) = \mathbf{P}(V_t^* < u) = \sum_{n=0}^{\infty} \mathbf{P}(t - T_n < u\,,\, N_t = n)$$

$$= \sum_{n=0}^{\infty} \mathbf{P}(N_{t-u} < n\,,\, N_t = n)$$

$$= \sum_{n=0}^{\infty} \mathbf{P}(N_t - N_{t-u} > 0\,,\, N_t = n)$$

$$= \mathbf{P}(N_t - N_{t-u} > 0) = \mathbf{P}(N_u > 0)\,.$$

Therefore,

$$\mathbf{P}(V_t^* \in A) = 1 - e^{-\lambda u} = \lambda \int_{A \cap [0,t]} e^{-\lambda v}\, \mathrm{d}v\,.$$

To finish this proof we note that

$$\mathbf{P}(V_t^* = t) = \mathbf{P}(N_t = 0) = e^{-\lambda t}\,.$$

Hence Proposition 1.6. □

Propositions 1.5 and 1.6 imply that

$$\mathbf{E}\,(T_{N_t+1} - T_{N_t}) = \frac{2}{\lambda}(1 - e^{-\lambda t})\,. \tag{1.7}$$

Remark 1.2. *The equation* (1.7) *is called the "bus paradox". If we associate the jump moments of a Poisson process with the time moments of passages of a bus through a station, then according to* (1.7) *for sufficiently large t the bus waiting time interval* $[T_{N_t}, T_{N_t+1}]$ *is twice as long on average as an interval* $[T_n, T_{n+1}]$ *since* $\mathbf{E}(T_{n+1} - T_n) = 1/\lambda$.

1.4 Exercises I

1. Let $(N_t)_{t\geq 0}$ be a Poisson process of an intensity $\lambda > 0$ and $(T_n)_{n\geq 1}$ be his jumping moments.

   (a) Calculate $\mathbf{E}N_t$ and $\mathbf{Var}(N_t)$ for $t > 0$.

   (b) Calculate the distribution of T_n for $n \geq 1$.

   (c) Show that for all $A \in \mathcal{B}(\mathbf{R}^n)$

$$\mathbf{P}\left((T_1, \cdots, T_n) \in A \,|\, N_t = n\right) = \frac{n!}{t^n} \int_A \mathbf{1}_{\{0<s_1<s_2<\cdots<s_n\leq t\}} \mathrm{d}s_1 \cdots \mathrm{d}s_n. \quad (1.8)$$

   (d) Let $X_1, \ldots, X_n$ be i.i.d. random variables uniformly distributed on the interval $[0, t]$. Let $Z_1, \ldots, Z_n$ be the ordinal statistics of $X_1, \ldots, X_n$. Show that for all $A \in \mathcal{B}(\mathbf{R}^n)$

$$\begin{aligned} &\mathbf{P}\left((Z_1, \cdots, Z_n) \in A\right) \\ &\qquad = \frac{n!}{t^n} \int_A \mathbf{1}_{\{0<s_1<s_2<\cdots<s_n \le t\}} \mathrm{d}s_1 \cdots \mathrm{d}s_n \,. \quad (1.9) \end{aligned}$$

Deduce that conditionally with respect to $\{N_t = n\}$ the random variables $(T_1, \ldots, T_n)$ has same distribution as the order statistics of n uniform independent random variables on the interval $[0, t]$.

(e) Show that for $0 < s < t$

$$\mathbf{P}\left(N_s = k | N_t\right) = \left(\begin{array}{c} N_t \\ \\ k \end{array} \right) \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{N_t - k} \mathbf{1}_{\{k > N_t\}} \,.$$

(f) Show that $(N_t)_{t \ge 0}$ is a process with homogeneous increments in the sense that for all $0 < s < t$ the increment $N_t - N_s$ has the same distribution as N_{t-s}.

(g) Show that $(N_t)_{t \ge 0}$ has independent increments, i.e. for any increasing time moments $0 = t_0 < t_1 < \cdots < t_k$ the random variables

$$N_{t_1} = N_{t_1} - N_{t_0},\ N_{t_2} - N_{t_1}, \ldots,\ N_{t_k} - N_{t_{k-1}}$$

are independent.

(h) Show that $(N_t)_{t\geq 0}$ is a rare events process, i.e. for any $t \geq 0$ and $\Delta > 0$

$$\mathbf{P}(N_{t+\Delta}-N_t = 1) = \lambda\Delta+\mathrm{o}(\Delta)\,, \mathbf{P}(N_{t+\Delta}-N_t > 1) = \mathrm{o}(\Delta)$$

as $\Delta \to 0$.

2. Let $(N_t^1)_{t\geq 0}$ and $(N_t^2)_{t\geq 0}$ be two independent Poisson processes of the intensities λ and μ. Denote by $(T_n)_{n\geq 1}$ the renewal moments of $(N_t^1)_{t\geq 0}$.

   (a) Calculate the distribution of the random variable $N^2_{T_{n+1}} - N^2_{T_n}$.

   (b) Extend the result of (a) to random variables $N^2_{T_{n+k}} - N^2_{T_n}$, $k > 1$.

3. Let θ be positive a.s. random variable with the finite variance $\sigma_\theta^2 > 0$ and independent of $(N_t)_{t\geq 0}$. It is said that the process

$$\tilde{N}_t = N_{\theta t}, \quad t \geq 0,$$

is *mixed Poisson process of the mixed variable θ.*

   (a) Calculate $\mathbf{P}(\tilde{N}_t = n)$. Deduce that $\tilde{N}_t$ has not usually

Poisson distribution.

(b) Shaw that $\mathbf{Var}(\tilde{N}_t) > \mathbf{E}\tilde{N}_t$ for all $t > 0$, while we have the equality for the Poisson processes.

(c) Calculate the distribution of $\tilde{N}_t$, when $\lambda = 1$ and θ has the Gamma distribution.

2 Asymptotic theory

2.1 Renewal equation

Let $(\eta_j)_{j\geq 1}$ be i.i.d. positive random variables with a distribution function G. Now we consider the *counting process* for this sequence defined as

$$N_t = \sum_{j=1}^{\infty} \mathbf{1}_{\{S_j \leq t\}}, \tag{2.1}$$

where $S_0 = 0$ and $S_j = \sum_{l=1}^{j} \eta_l$ for $j \geq 1$. Note that if the distribution G is exponential, then $(N_t)_{t\geq 0}$ is the Poisson process. Using the large numbers law (Theorem A.1), one can establish that

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbf{E}\eta_1} \quad \text{a.s.} \tag{2.2}$$

Definition 2.1. *We say that a random variable ξ is arithmetic if there exists $d > 0$ such that*

$$\mathbf{P}(\xi \in \Gamma_d) = 1,$$

where $\Gamma_d = \{(kd)_{-\infty<k<\infty}\}$ is the grid of size $d > 0$. A random variable ξ is called non-arithmetic if $\mathbf{P}(\xi \in \Gamma_d) < 1$ for any $d > 0$.

In this section we need the Blackwell Renewal Theorem (see, for example, in [3]):

Theorem 2.1. *Assume that* η_1 *is non-arithmetic and* $0 < \mathbf{E}\,\eta_1 < \infty$. *Then the expectation of the counting function has the following asymptotic properties:*

$$\lim_{t\to\infty} \frac{\mathbf{E}\,N_t}{t} = \frac{1}{\mathbf{E}\,\eta_1}$$

and for any $h > 0$

$$\lim_{t\to\infty} \mathbf{E}\,(N_{t+h} - N_t) = \frac{h}{\mathbf{E}\,\eta_1}\,.$$

We will use this theorem to study the *renewal function*

$$Q(t) = \mathbf{E}\sum_{j=0}^{\infty} V(t - S_j)\mathbf{1}_{\{S_j \le t\}}\,, \tag{2.3}$$

where $V : \mathbb{R}_+ \to \mathbb{R}$ is bounded over all the finite intervals function. One can check directly that this function satisfies the following *renewal equation*

$$Q(u) = V(u) + \int_0^u Q(u - z)\,\mathrm{d}G(z)\,. \tag{2.4}$$

Now we study this equation.

Theorem 2.2. *Assume that the distribution* G *is non-arithmetic* *and the function* V *is bounded over all finite intervals. Then the renewal function* Q *is the unique solution of the renewal equation* (2.4) *among the functions which are bounded over all finite intervals.*

Proof. Note that the Blackwell theorem implies that $\mathbf{E}\,N_t < \infty$ for any $t \geq 0$. Thus, if V is bounded on each finite interval, then the renewal function is bounded by

$$\sup_{0\le u\le t} |Q(u)| \le \sup_{0\le u\le t} |V(u)|\,(\mathbf{E}\,N_t + 1) < \infty$$

on each finite interval $[0,t]$.

Moreover, let $\mathcal{B}(\mathbb{R}_+)$ be a linear space of $\mathbb{R}_+ \to \mathbb{R}$ bounded on each finite interval functions. We will introduce the following linear $\mathcal{B}(\mathbb{R}_+) \to \mathcal{B}(\mathbb{R}_+)$ operator

$$T(f)(u) = \int_0^t f(u-z)\,\mathrm{d}G(z)\,.$$

In this case we can rewrite the renewal equation as

$$f = V + T(f)\,.$$

This implies that for all $n \geq 1$

$$f = \sum_{j=0}^{n} T^j(V) + T^{n+1}(f) . \tag{2.5}$$

To study this equation one needs to know how to calculate the n-th power of T. Let's show by induction that for each $n \geq 1$

$$T^n(f)(u) = \mathbf{E} f(u - S_n) \mathbf{1}_{\{S_n \leq u\}} . \tag{2.6}$$

For $n = 1$ this is the definition. Assume now that this equality holds for some fixed $n > 1$. We set

$$\tilde{f}(u) = T^n(f)(u) = \mathbf{E} f(u - S_n) \mathbf{1}_{\{S_n \leq u\}}$$
$$= \int_0^{+\infty} f(u-y) \mathbf{1}_{\{y \leq u\}} \mathrm{d}F_{S_n}(y) ,$$

where $F_{S_n}(y) = \mathbf{P}(S_n \leq y)$. Using this function, we can represent the $(n+1)$-th power as

$$T^{n+1}(f)(u) = T(\tilde{f})(u) = \int_0^u \mathbf{E} f(u - z - S_n) \mathbf{1}_{\{S_n \leq u - z\}} \mathrm{d}G(z)$$
$$= \mathbf{E} f(u - \eta_{n+1} - S_n) \mathbf{1}_{\{S_n \leq u - \eta_{n+1}\}}$$
$$= \mathbf{E} f(u - S_{n+1}) \mathbf{1}_{\{S_{n+1} \leq u\}} .$$

It means that equality (2.6) is true for any $n \geq 1$. Using it in (2.5), we get that

$$f(u) \;=\; \sum_{j=0}^{n} \mathbf{E}\, V(u-S_j)\, \mathbf{1}_{\{S_j \leq u\}} + \mathbf{E}\, f(u-S_{n+1})\, \mathbf{1}_{\{S_{n+1} \leq u\}}\,. \quad (2.7)$$

According to our condition, we try to solve the equation (2.4) among the functions which are bounded on each finite interval. So, the last term in (2.7) is bounded by

$$|\, \mathbf{E}\, f(u - S_{n+1})\, \mathbf{1}_{\{S_{n+1} \leq u\}}\,| \;\leq\; \sup_{0 \leq s \leq u} |f(s)|\, \mathbf{P}(S_{n+1} \leq u)$$

and, by the large numbers law (Theorem A.1), for any fixed $u > 0$ this term tends to zero as $n \to \infty$. So, taking the limit in (2.7) as $n \to \infty$, we obtain that any solution of the equation (2.4) which is bounded on every finite interval is equal to the renewal function (2.3). $\square$

2.2 Smith theorem

Now we study the asymptotic properties of the function (2.3). To this end one needs the following definition.

Definition 2.2. *We say that a* $\mathbb{R}_+ \to \mathbb{R}$ *function* V *is directly* *integrable by Riemann on* $[0, \infty[$ *if*

$$\sum_{k=1}^{\infty} \sup_{k-1 \le x \le k} |V(x)| < \infty . \tag{2.8}$$

Using this definition, we will study the asymptotic properties of the function (2.3) as $t \to \infty$.

Theorem 2.3. *Let F be a right or left continuous $\mathbb{R}_+ \to \mathbb{R}$ function directly integrable by Riemann and on each finite interval it has a finite number of discontinuity points. Moreover we suppose that η_1 is non-arithmetic and $0 < \mathbf{E}\eta_1 < \infty$. Then the function (2.3) has the following limit*

$$\lim_{u \to \infty} Q(u) = \frac{1}{\mathbf{E}\,\eta_1} \int_0^{\infty} V(z)\,\mathrm{d}z . \tag{2.9}$$

Proof. First, we show this theorem for linear combinations of indicator functions, i.e. we assume that

$$V(x) = \alpha_1\,\mathbf{1}_{[t_0,t_1]}(x) + \sum_{k=2}^{m} \alpha_k\,\mathbf{1}_{(t_{k-1},t_k]}(x) , \tag{2.10}$$

where $0 = t_0 < t_1 < \ldots < t_m < \infty$. It's easy to see that this function for $u \geq t_m$

$$Q(u) \;=\; \alpha_1 \, \mathbf{E} \sum_{j=0}^{\infty} \mathbf{1}_{\{u-t_1 \leq S_j \leq u\}} \;+\; \sum_{k=2}^{m} \alpha_k \, \mathbf{E} \sum_{j=0}^{\infty} \mathbf{1}_{\{u-t_k \leq S_j < u-t_{k-1}\}}$$

$$= \sum_{k=1}^{m} \alpha_k \, \mathbf{E} \, (N_{u-t_{k-1}} \;-\; N_{u-t_k}) \;-\; \alpha_1 \, \mathbf{E} \, \Delta N_{u-t_1}$$

$$- \sum_{k=2}^{m} \alpha_k \, \mathbf{E} \, (\Delta N_{u-t_{k-1}} \;-\; \Delta N_{u-t_k}) \, ,$$

where $\Delta N_t = \sum_{j=1}^{\infty} \mathbf{1}_{\{S_j = t\}}$. Note that for any $h > 0$

$$\Delta N_t \;\leq\; N_{t+h} \;-\; N_{t-h}$$

and, by the Blackwell theorem,

$$\limsup_{t \to \infty} \, \mathbf{E} \, \Delta N_t \;\leq\; \frac{2h}{\mathbf{E} \eta_1} \, .$$

Therefore,

$$\lim_{t \to \infty} \, \mathbf{E} \, \Delta N_t \;=\; 0$$

and

$$\lim_{u \to \infty} \, Q(u) \;=\; \frac{1}{\mathbf{E} \, \eta_1} \sum_{k=1}^{m} \alpha_k \, (t_k - t_{k-1}) \;=\; \frac{1}{\mathbf{E} \eta_1} \int_0^{\infty} V(z) \, \mathrm{d}z \, .$$

Let now V be a function that satisfies the conditions of this theorem, i.e. it is directly integrable by Riemann and has a finite number of the jumps on all finite intervals. In this case, for each $L > 0$ we can find a sequence of functions $(V_m)_{m\ge1}$ of the form (2.10) such that

$$\lim_{m\to\infty} \sup_{0\le x\le L} |V(x) - V_m(x)| = 0 .$$

So, we can represent the function Q as

$$Q(u) = I_1(u) + I_2(u) + I_3(u) , \qquad (2.11)$$

where $I_1(u) = \mathbf{E} \sum_{j=0}^{\infty} V_m(u - S_j) \, \mathbf{1}_{\{u-L\le S_j\le u\}}$,

$$I_2(u) = \mathbf{E} \sum_{j=0}^{\infty} (V(u - S_j) - V_m(u - S_j)) \, \mathbf{1}_{\{u-L\le S_j\le u\}}$$

and

$$I_3(u) = \mathbf{E} \sum_{j=0}^{\infty} V(u - S_j) \, \mathbf{1}_{\{S_j\le u-L\}} .$$

Taking into account that $V_m(z) = 0$ for $z > L$, we find

$$I_1(u) = \mathbf{E} \sum_{j=0}^{\infty} V_m(u - S_j \mathbf{1}_{S_j\le u\}} ,$$

and, therefore,

$$\lim_{u\to\infty} I_1(u) \,=\, \frac{1}{\mathbf{E}\,\eta_1}\int_0^\infty V_m(z)\,\mathrm{d}z \,=\, \frac{1}{\mathbf{E}\,\eta_1}\int_0^L V_m(z)\,\mathrm{d}z\,. \quad (2.12)$$

Moreover,

$$|I_2(u)| \,\le\, \sup_{0\le z\le L}|V(z)\,-\,V_m(z)|\,\left(\mathbf{E}\,(N_u\,-\,N_{u-L})\,+\,\mathbf{E}\,\Delta\,N_{u-L}\right)\,.$$

And we get that

$$\limsup_{u\to\infty}|I_2(u)| \,\le\, \sup_{0\le z\le L}|V(z)\,-\,V_m(z)|\,\frac{L}{\mathbf{E}\,\eta_1}\,.$$

This implies that for any $L>0$

$$\lim_{m\to\infty}\limsup_{u\to\infty}|I_2(u)| \,=\, 0\,. \quad (2.13)$$

Now we consider the last term in (2.11). Setting

$$v_k^* = \sup_{k-1\le x\le k}|V(x)|,$$

we can estimate it from above as

$$
\begin{aligned}
|I_3(u)| &\le \mathbf{E} \sum_{j=0}^{\infty} \sum_{k=L+1}^{\infty} |V(u-S_j)| \mathbf{1}_{\{u-k\le S_j \le u-k+1\}} \\
&\le \mathbf{E} \sum_{j=0}^{\infty} \sum_{k=L+1}^{\infty} v_k^* \, \mathbf{1}_{\{u-k\le S_j \le u-k+1\}} \\
&\le \sum_{k=L+1}^{\infty} v_k^* \left(1 + \mathbf{E}(N_{(u-k)_+ +1} - N_{(u-k)_+}) + \mathbf{E}\,\Delta N_{u-k)_+}\right) \\
&\le \sup_{x\ge 0} \left(1 + \mathbf{E}(N_{x+1} - N_x) + \mathbf{E}\,\Delta N_x\right) \sum_{k=L+1}^{\infty} v_k^* \,.
\end{aligned}
$$

Thus,

$$
\lim_{L\to\infty} \limsup_{u\to\infty} |I_3(u)| = 0 \,. \tag{2.14}
$$

From here, taking into account (2.11), we have

$$
\begin{aligned}
\left| Q(u) - \frac{1}{\mathbf{E}\eta_1} \int_0^{\infty} V(y)\mathrm{d}y \right| &\le \left| I_1(u) - \frac{1}{\mathbf{E}\eta_1} \int_0^{L} V_m(y)\mathrm{d}y \right| \\
&\quad + \frac{1}{\mathbf{E}\eta_1} \int_0^{L} |V_m(y) - V(y)|\mathrm{d}y \\
&\quad + \frac{1}{\mathbf{E}\eta_1} \int_L^{\infty} |V(y)|\mathrm{d}y + |I_2(u)| + |I_3(u)| \,.
\end{aligned}
$$

Taking in this inequality the limit as

$$\limsup_{L\to\infty} \limsup_{m\to\infty} \limsup_{u\to\infty} ,$$

we get (2.11). Hence Theorem 2.3. $\square$

2.3 Exercises II

1. Let $(N_t)_{t\geq 0}$ be counting function, that is

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\{\eta_1+\ldots+\eta_n\leq t\}} ,$$

where $(\eta_j)_{j\geq 1}$ are i.i.d. random variables uniformly distributed on the interval $[0, z]$ with a fixed $z > 0$. Calculate the following limits

(a)

$$\lim_{t\to\infty} \frac{\mathbf{E} N_t}{1+2t} ;$$

(b)

$$\lim_{t\to\infty} \mathbf{E}\left(N_{3t} - N_{3t+4}\right) ;$$

(c)

$$\lim_{t\to\infty} \frac{\mathbf{E} N_{2t}}{\sqrt{1+t^2}} ;$$

(d)

$$\lim_{t\to\infty} \mathbf{E}\left(N_t - N_{t-1/3}\right) ;$$

(e)

$$\lim_{t\to\infty} \sin(1/t)\,\mathbf{E}\,N_{10t} ;$$

(f)

$$\lim_{t\to\infty} \left(1 - e^{1/t}\right) \mathbf{E}\,N_{4t} .$$

2. Are the following functions directly integrable by Riemann

$$\frac{1}{1+x^2}, \quad e^{-x}, \quad \frac{\sin(x)}{1+x^4} ?$$

3. Calculate the limit

$$\lim_{t\to\infty} \left(\frac{1}{1+t^2} + \mathbf{E} \sum_{j=1}^{\infty} \frac{1}{1+(t-T_j)^2} \mathbf{1}_{\{T_j \le t\}} \right) ,$$

where $T_j = \sum_{i=1}^{j} \xi_i^2$ and $(\xi_j)_{j\ge1}$ are i.i.d. Gaussian random variables with the parameters $(0,1)$.

3 Cramér - Lundberg models

3.1 Main definitions and results

In this section we consider non-life insurance models in which the claim sizes are defined by i.i.d. positive random variables $(Y_j)_{j\ge 1}$ with

$$\mu = \mathbf{E}\, Y_1 < \infty\,. \tag{3.1}$$

Moreover, we assume that the claims number on the time interval $[0,t]$ is a homogeneous Poisson process $(N_t)_{t\ge 0}$ of intensity $\lambda > 0$ defined in (1.1). This means that the time moments for claims occurrence $(T_n)_{n\ge 1}$ are the jumps of the Poisson process $(N_t)_{t\ge 0}$ and the inter-arrival times

$$\tau_1 = T_1\,, \quad \tau_k = T_k - T_{k-1}\,, \quad k \ge 2\,, \tag{3.2}$$

are i.i.d. exponentially distributed random variables with $\mathbf{E}\tau_1 = 1/\lambda$. We define the *total claim amount process* as

$$X_t = \sum_{j=1}^{N_t} Y_j \tag{3.3}$$

and $X_t = 0$ for $N_t = 0$. In the theory of stochastic processes such process is called *a compound Poisson process*. Moreover we assume that a continuous stream of revenue brings in ct during the time interval $[0,t]$, where $c > 0$ is *the premium income rate*. In this case *the risk process* is defined as

$$U_t = u + ct - X_t \,, \tag{3.4}$$

where $u > 0$ is *the initial endowment* of the insurance company.

Definition 3.1. *The event*

$$A^- = \{\exists t > 0 \quad \textit{such that} \quad U_t < 0\} = \cup_{t>0} \{U_t < 0\} \tag{3.5}$$

is called the ruin.

The definition of the risk process (3.4) immediately implies that

$$A^- = \cup_{k\ge 1} \{U_{T_k} < 0\} \,. \tag{3.6}$$

This means that this set is measurable. The moment τ^u when the risk process goes below zero is called the *ruin time*:

$$\tau^u = \inf\{t > 0 \,:\, U_t < 0\} \,. \tag{3.7}$$

The *ruin probability* or *ruin function* is given by

$$\psi(u) \,=\, \mathbf{P}(A^- \,|\, U_0 = u) \,=\, \mathbf{P}(\,\tau^u \,<\, \infty)\,. \tag{3.8}$$

Setting

$$\sigma^u \,=\, \inf\{k \geq 1 \,:\, U_{T_k} \,<\, 0\} \tag{3.9}$$

and taking into account the definition (3.8), we obtain

$$\psi(u) \,=\, \mathbf{P}(\,\sigma^u \,<\, \infty)\,. \tag{3.10}$$

Firstly, we study the properties of the total claim amount process (3.3).

Theorem 3.1. *For the process* (3.3) *the following law of large numbers holds*

$$\lim_{t\to\infty} \frac{1}{t}\, X_t \,=\, \lambda\, \mu \quad a.s. \tag{3.11}$$

Moreover, if $\mathbf{E}\, Y_1^2 \,<\, \infty$, *then for the process* (3.3) *the limit theorem holds also, i.e.*

$$\frac{X_t \,-\, \lambda\, \mu t}{\sqrt{t}} \Longrightarrow \mathcal{N}(0,\, \lambda\, \mathbf{E}\, Y_1^2) \quad as \quad t \to \infty\,. \tag{3.12}$$

Proof. To show (3.11) we note that, in view of the definition of the Poisson process in (1.1), for any $t > 0$

$$T_{N_t} \le t < T_{N_t+1} \,. \tag{3.13}$$

Therefore, taking into account that $N_t \to \infty$ a.s. as $t \to \infty$, we obtain through the large numbers law that

$$\lim_{t\to\infty} \frac{T_{N_t}}{N_t} = \mathbf{E}\tau_1 = \frac{1}{\lambda} \quad \text{a.s.}$$

Therefore, from the inequalities (3.13) it follows that

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \quad \text{a.s.}$$

and, using again the large numbers law given in Theorem A.1, we come to the limit (3.11). As to the second equality, note that the deviation $X_t - \lambda\mu t$ can be represented as

$$X_t - \lambda\mu t = S_{N_t} + \lambda\mu(T_{N_t} - t) \,, \tag{3.14}$$

where

$$S_n = \sum_{j=1}^{n} \eta_j \quad \text{and} \quad \eta_j = Y_j - \mu + \mu(1 - \lambda\tau_j) \,.$$

Note that

$$\mathbf{E}\,\eta_j = 0 \quad \text{and} \quad \mathbf{E}\,\eta_j^2 = \mathbf{E}\,Y_1^2 \,,$$

and, in view of (3.13),

$$0 \le t - T_{N_t} \le \tau_{N_t+1}\,.$$

Moreover, we have

$$\mathbf{E}\,\tau_{N_t+1} = \sum_{k=0}^{\infty} \mathbf{E}\,\tau_{k+1}\,\mathbf{1}_{\{N_t=k\}} \le \mathbf{E}\tau_1 + \lambda \int_0^{+\infty} z\,\Upsilon(t,z)\,e^{-\lambda z}\,\mathrm{d}z\,, \tag{3.15}$$

where

$$\Upsilon(t,z) = \sum_{k=1}^{\infty} \mathbf{P}(T_k \le t < T_k + z) = \lambda(t - (t-z)_+)\,,$$

and $(x)_+ = \max(0,x)$. Therefore, the bound (3.15) yields

$$\mathbf{E}\,\tau_{N_t+1} \le \frac{1}{\lambda} + \lambda^2 \int_0^t z^2 e^{-\lambda z}\,\mathrm{d}z + \lambda^2 t \int_t^{\infty} z^2 e^{-\lambda z}\,\mathrm{d}z$$

i.e.

$$\sup_{t\ge 0} \mathbf{E}\,\tau_{N_t+1} < \infty$$

and, therefore,

$$\mathbf{P} - \lim_{t\to\infty} \frac{T_{N_t} - t}{\sqrt{t}} = 0\,.$$

Using this equality in (3.14), we obtain the asymptotic representation

$$\frac{X_t - \lambda\mu t}{\sqrt{t}} = \frac{S_{N_t}}{\sqrt{t}} + \mathrm{o}_P(1)\,, \tag{3.16}$$

where $\mathrm{o}_P(1)$ is a term going to zero in probability as $t \to \infty$. Moreover, let now $\mathbf{m} = [\lambda t]$ and $[x]$ be the integer part of the number x. Then

$$\mathbf{E}\left(S_{N_t} - S_{\mathbf{m}}\right)^2 = \mathbf{E}\,S_{N_t}^2 - 2\mathbf{E}\,S_{N_t}S_{\mathbf{m}} + \mathbf{E}\,S_{\mathbf{m}}^2 = \mathbf{E}\,\eta_1^2\mathbf{E}|N_t - \mathbf{m}|\,,$$

i.e.

$$\frac{\mathbf{E}\left(S_{N_t} - S_{\mathbf{m}}\right)^2}{t} \le \frac{1}{t} + \frac{\sqrt{\mathbf{E}(N_t - \mathbf{E}\,N_t)^2}}{t} = \frac{1}{t} + \frac{\sqrt{\lambda}}{\sqrt{t}}\,.$$

Using this in (3.16), we get

$$\frac{X_t - \lambda\mu t}{\sqrt{t}} = \frac{S_{\mathbf{m}}}{\sqrt{t}} + \mathrm{o}_P(1)\,.$$

Now, applying to the sequence $(S_n)_{n\ge 1}$ the central limit Theorem A.4, we come to the limit property (3.12). Hence Theorem 3.1. □

Now we come back to the ruin problem, i.e. we study the properties for the ruin probability (3.10).

Proposition 3.1. ***(Almost sure ruin)*** *If $c \le \mu\lambda$, then $\psi(u) = 1$*

for all $u > 0$.

Proof. Let $c < \lambda\mu$. We can represent the sequence $(U_{T_k})_{k\geq 1}$ as

$$U_{T_k} = u - \sum_{j=1}^{k} \xi_j \,, \tag{3.17}$$

where $\xi_j = Y_j - c\tau_j$. In this case, by applying the strong large numbers law (Theorem A.1) for $S_k = \sum_{j=1}^{k} \xi_j$ in the equality (3.10), we find that

$$\lim_{n\to\infty} \frac{U_{T_n}}{n} = -\lim_{n\to\infty} \frac{S_n}{n} = -\mathbf{E}\,\xi_1 = \frac{c}{\lambda} - \mu < 0 \quad \text{a.s.}$$

So, taking into account (3.9) and (3.10), we obtain that $\psi(u) = 1$ for all $u \geq 0$. Let now $c = \lambda\mu$, i.e. $\mathbf{E}\,\xi_1 = 0$. In this case note that for any $k \geq 1$ and $\epsilon > 0$

$$\mathbf{P}(|\xi_k| > \epsilon) = \mathbf{P}(|\xi_1| > \epsilon) > 0\,.$$

Using Kolmogorov three-series theorem and Kolmogorov zero-one law (Theorems A.2 - A.3), we obtain that

$$\limsup_{k\to\infty} S_k = +\infty \quad \text{a.s.}$$

From the equalities (3.9) and (3.10) it follows that

$$1 = \mathbf{P}(\limsup_{k\to\infty} S_k = +\infty) \le \mathbf{P}(\sigma^u < \infty)\,.$$

Thus, $\psi(u) = 1$. Hence Proposition 3.1. $\square$

Remark 3.1. *Proposition 3.1 means that insurance companies have to choose the premium rate $c > 0$ such that $\mathbf{E}\xi_1 < 0$. This is the only possibility to avoid being bankrupt almost sure in the framework of the Cramér - Lundberg model. So, if $\mathbf{E}\xi_1 < 0$, then we can hope that the ruin function $\psi(u)$ will be less then* 1.

Definition 3.2. *The Cramér-Lundberg model satisfies "net profit condition" if*

$$\mathbf{E}\,\xi_1 = \mathbf{E}\,(Y_1 - c\,\tau_1) = \mu - c\,\frac{1}{\lambda} < 0\,. \tag{3.18}$$

In the sequel we will assume that the *premium rate* is equal to

$$c = (1+\rho)\,\lambda\,\mu\,, \tag{3.19}$$

where ρ is a positive constant, which provides the net profit condition.

3.2 Exercises III

Let $(Y_j)_{j\geq 1}$ be i.i.d. random variables with values in $\mathbb{R}_+$ and with the finite on a neighborhood around 0 generator function defined as

$$m_Y(h) = \mathbf{E}\, e^{hY_j} \,.$$

Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process of an intensity $\lambda > 0$ independent of $(Y_j)_{j\geq 1}$. For any $t \geq 0$ we set

$$X_t = \sum_{j=1}^{N_t} Y_j \quad \text{and} \quad U_t = u + ct - X_t$$

with $u > 0$ and $c > 0$.

1. Calculate expectation and variance of U_t.
2. Calculate the generator function for X_t.
3. Let $\alpha > 0$. Show that there is only one solution c_α for the equation
$$\mathbf{E}\, e^{-\alpha(ct - X_t)} = 1\,, \quad \text{for any } t > 0.$$
4. Show that $\mathbf{E}\, U_t > u$ for $c = c_\alpha$. What is the limit of c_α as $\alpha \to 0$?

3.3 Lundberg inequality

In this section we will study the behavior of the function $\psi(u)$ under the condition (3.18). Moreover, we assume that the sequence of claims amounts $(Y_j)_{j\geq 1}$ satisfies the following condition, called the *Lundberg condition*,

$\mathbf{H}_1$) *There exists* $\delta > 0$ *such that*

$$\mathbf{E}\, e^{\delta Y_1} < \infty \,. \tag{3.20}$$

Also we define the *Lundberg function* as

$$\mathbf{L}(x) = \ln \mathbf{E}\, e^{x\xi_1} \,. \tag{3.21}$$

The condition $\mathbf{H}_1$) implies that the function $\mathbf{L}(x)$ is finite in absolute value for any $0 \leq x \leq \delta$.

Proposition 3.2. *We assume that the condition* $\mathbf{H}_1$) *holds. If the equation* $\mathbf{L}(x) = 0$ *has a strictly positive root, then this root is unique.*

Proof. First, we note that the function $\mathbf{L}$ is convex. Indeed, by Holder's inequality for $0 < \alpha < 1$ and for $0 \leq x, y \leq \delta$ we obtain that

$$\begin{aligned}\mathbf{L}(\alpha x + (1-\alpha)y) &= \ln\left(\mathbf{E}\, e^{\alpha x \xi_1}\, e^{(1-\alpha)y\xi_1}\right)\\ &\le \ln\left((\mathbf{E}\, e^{x\xi_1})^{\alpha}\, (\mathbf{E}\, e^{y\xi_1})^{1-\alpha}\right)\\ &= \ln\,(\mathbf{E}\, e^{x\xi_1})^{\alpha} + \ln\,(\mathbf{E}\, e^{y\xi_1})^{1-\alpha}\\ &= \alpha\,\mathbf{L}(x) + (1-\alpha)\,\mathbf{L}(y)\,.\end{aligned}$$

We assume that there is $0 < r_1 < r_2$ such that $\mathbf{L}(r_1) = \mathbf{L}(r_2) = 0$. Then for all $z \in [r_1, r_2]$ we obtain

$$\mathbf{L}(z) = \mathbf{L}(\alpha r_1 + (1-\alpha) r_2) \le \alpha \mathbf{L}(r_1) + (1-\alpha)\mathbf{L}(r_2) = 0\,,$$

where $\alpha = (r_2 - z)/(r_2 - r_1)$. If $\mathbf{L}(z) = 0$ (i.e. $\mathbf{E}\, e^{z\xi_1} = 1$) for all $r_1 \le z \le r_2$, then we would have $\mathbf{E}\, \xi_1^2\, e^{z\xi_1} = 0$ and, so $\xi_1 = Y_1 - c\tau_1 = 0$ a.s. But this is not possible since the random variables Y_1 and τ_1 are independent. Therefore, it exists $0 < r_1 < z_1 < r_2$ such that $\mathbf{L}(z_1) < 0$. Similar, as $\mathbf{L}(0) = 0$, we get that it exists $z_0 \in [0, r_1]$ such that $\mathbf{L}(z_0) < 0$. Setting $\alpha = (z_1 - r_1)/(z_1 - z_0)$, we find that

$$0 = \mathbf{L}(r_1) = \mathbf{L}(\alpha z_0 + (1-\alpha) z_1) \le \alpha \mathbf{L}(z_0) + (1-\alpha)\mathbf{L}(z_1) < 0\,.$$

This implies the uniqueness of the positive root. Hence Proposition 3.2. □

Definition 3.3. *If the equation* $\mathbf{L}(x) = 0$ *admits a root* $r > 0$*, then this root is called the Lundberg coefficient.*

We will assume the following condition.

$\mathbf{H}_2$) *The equation* $\mathbf{L}(x) = 0$ *admits a root* $r > 0$.

Remark 3.2. *It is easy to see that the assumptions* $\mathbf{H}_1$)–$\mathbf{H}_2$) *imply the net profit condition* (3.18). *Indeed, if* $\mathbf{E}\,\xi_1 \ge 0$*, then by Jensen inequality we obtain that*

$$\mathbf{L}(x) \,=\, \ln \mathbf{E}\, e^{x\xi_1} \,>\, \ln e^{x\mathbf{E}\xi_1} \,\ge\, 0$$

for any $x > 0$. *So, the function* $\mathbf{L}$ *has no strictly positive root.*

Theorem 3.2. ***(Lundberg inequality)*** *Under the conditions* $\mathbf{H}_1$)–$\mathbf{H}_2$) *for all* $u \ge 0$ *the ruin function admits the exponential upper bound*

$$\psi(u) \,\le\, e^{-ru}\,. \tag{3.22}$$

Proof. First, one notes that according to (3.10), we can represent the ruin probability as the distribution tail of the extreme value for a sequence of sums of i.i.d. random variables:

$$\psi(u) = \mathbf{P}(\inf_{k\geq 1} U_{T_k} < 0) = \mathbf{P}(\max_{k\geq 1} S_k > u)\,,$$

where $S_k = \sum_{j=1}^{k} \xi_j$ and $\xi_j = Y_j - c\tau_j$. Let now

$$\psi_n(u) = \mathbf{P}(\max_{1\leq k\leq n} S_k > u)\,.$$

It's obvious that

$$\psi(u) = \lim_{n\to\infty} \psi_n(u)\,.$$

So, for this theorem it suffices to show the inequality (3.22) for the functions $\psi_n(u)$ for all $n \geq 1$. We will do it by the induction. We start with $n = 1$. In this case $S_1 = \xi_1$ and by the Markov inequality

$$\psi_1(u) = \mathbf{P}(\xi_1 > u) \leq \mathbf{E}\, e^{r\xi_1}\, e^{-ru} = e^{-ru}\,.$$

Moreover, if the inequality (3.22) holds for some fixed $n \geq 1$, then for $n+1$ we get

$$\psi_{n+1}(u) = \mathbf{P}(\max_{1\le k\le n+1} S_k > u\,,\, \xi_1 > u)$$
$$+\,\mathbf{P}(\max_{1\le k\le n+1} S_k > u\,,\, \xi_1 \le u)$$
$$= \mathbf{P}(\xi_1 > u) + \mathbf{P}\left(\max_{2\le k\le n+1} S_k > u\,,\, \xi_1 \le u\right). \quad (3.23)$$

We estimate now the first term in (3.23) more precisely, i.e.

$$\mathbf{P}(\xi_1 > u) \le e^{-ru}\,\mathbf{E}\,e^{r\xi_1}\,\mathbf{1}_{\{\xi_1>u\}}\,. \quad (3.24)$$

Taking into account that S_k is the sum of i.i.d. random variables and using the inequality (3.22) for $\psi_n(\cdot)$, we can estimate the second term in (3.23) as

$$\mathbf{P}\left(\max_{2\le k\le n+1} S_k > u\,,\, \xi_1 \le u\right)$$
$$= \mathbf{P}\left(\max_{2\le k\le n+1} \sum_{j=1}^{n} \xi_{j+1} > u-\xi_1\,,\, \xi_1 \le u\right)$$
$$= \mathbf{E}\,\mathbf{1}_{\{\xi_1\le u\}}\,\psi_n(u-\xi_1) \le e^{-ru}\mathbf{E}\,\mathbf{1}_{\{\xi_1\le u\}}\,e^{r\xi_1}\,.$$

Using this inequality and the upper bound (3.24) in (3.23), we obtain that

$$\psi_{n+1}(u) \le e^{-ru}\,(\mathbf{E}\,e^{r\xi_1}\,\mathbf{1}_{\{\xi_1>u\}} \,+\, \mathbf{E}\,e^{r\xi_1}\,\mathbf{1}_{\{\xi_1\le u\}})$$

$$= e^{-ru}\mathbf{E}\,e^{r\xi_1} = e^{-ru}\,.$$

So, for all $n \ge 1$ the functions $\psi_n(u) \le e^{-ru}$. Taking here the limit as $n \to \infty$, we get the bound (3.22). Hence Theorem 3.2. $\square$

Example 3.1. *We consider the Cramér-Lundberg model in which the random variables* $(Y_j)_{j\ge 1}$ *are exponential with a parameter* $\gamma > 0$. *In this case the net profit condition* (3.19) *takes the form*

$$c \,=\, (1+\rho)\,\lambda\,/\gamma\,,$$

where ρ *is a positive constant. Note, that the condition* $\mathbf{H}_1$) *holds for* $\delta < \gamma$. *Moreover, it is easy to see that the Lundberg coefficient in this case is*

$$r = \gamma - \frac{\lambda}{c} = \gamma\,\frac{\rho}{1+\rho}\,.$$

So, in view of the Lundberg inequality, we get for all $u \ge 0$

$$\psi(u) \le e^{-\gamma\frac{\rho}{1+\rho}u}\,. \tag{3.25}$$

3.4 Exercises IV

We consider the risk process $U_t = u + ct - X_t$ for a reinsurance company, where $X_t = \sum_{i=j}^{N_t}(Y_j - K)_+$ with $K > 0$, $(N_t)_{t\ge 0}$ is a homogeneous Poisson process of intensity $\lambda > 0$ independent of the i.i.d. sequence $(Y_j)_{j\ge 1}$ random exponential variables of parameter $\gamma > 0$. We choose the premium rate as

$$c = (1+\rho)\lambda \mathbf{E}\,(Y_1 - K)_+ \quad \text{with} \quad \rho > 0\,.$$

1. Calculate c.

2. Show that

$$\mathbf{E}\, e^{it(Y_1-K)_+} = 1 + \frac{it}{\gamma - it} e^{-K\gamma}\,, \quad t \in \mathbf{R}.$$

3. Show that X_t has the same distribution as $\tilde{X}_t = \sum_{i=1}^{\tilde{N}_t} Y_j$, where $(\tilde{N}_t)_{t\ge 0}$ is a homogeneous Poisson process of the intensity $\widetilde{\lambda} = \lambda e^{-K\gamma}$ independent of $(Y_j)_{j\ge 1}$.

3.5 Fundamental equation for the non-ruin probability

Denote by $\phi(u) = 1 - \psi(u)$ the non-ruin probability.

Theorem 3.3. *We assume that the Cramér-Lundberg model satisfies the net profit condition* (3.18) *and the distribution function* $F_Y(\cdot)$ *of the random amounts* (Y_j) *has a density* f_Y*. Then the non-ruin probability* $\phi(u)$ *satisfies the following integral equation*

$$\phi(u) \, = \, \frac{\rho}{1+\rho} + \frac{1}{1+\rho} \int_0^u \phi(u-y) \, \mathrm{d}F_{Y,I}(y) \,, \tag{3.26}$$

where

$$F_{Y,I}(y) \, = \, \frac{1}{\mu} \int_0^y \overline{F}_Y(z) \, \mathrm{d}z \text{ and } \overline{F}_Y(y) = 1 - F_Y(y) = \mathbf{P}(Y_1 > y) \,.$$

Proof. Taking into account that $S_n = \sum_{j=1}^n \xi_j$ and $(\xi_j)_{j \ge 1}$ are i.i.d. random variables, one has

$$\begin{aligned}
\phi(u) &= \mathbf{P}(\sup_{n \ge 1} S_n \le u) = \mathbf{P}(\xi_1 \le u \,, \sup_{n \ge 2} S_n \le u) \\
&= \mathbf{P}(\xi_1 \le u, \sup_{n \ge 2} \sum_{j=2}^n \xi_j \le u - \xi_1) = \mathbf{E} \mathbf{1}_{\{\xi_1 \le u\}} \phi(u - \xi_1) \\
&= \mathbf{E}\, \mathbf{1}_{\{Y_1 - c\tau_1 \le u\}} \, \phi(u - Y_1 + c\tau_1) \,,
\end{aligned}$$

i.e.

$$\begin{aligned}\phi(u) &= \lambda \int_0^\infty \int_0^{u+cv} \phi(u-y+cv)\,\mathrm{d}F_Y(y)\,e^{-\lambda v}\,\mathrm{d}v \\ &= \frac{\lambda}{c}\,e^{u\lambda/c} \int_u^\infty e^{-\lambda z/c} \int_0^z \phi(z-y)\,\mathrm{d}F_Y(y)\,\mathrm{d}z\,.\end{aligned}$$

Taking the derivatives in this equality, we find that

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-y)\,\mathrm{d}F_Y(y)$$

and, therefore,

$$\phi(t)-\phi(0) = \frac{\lambda}{c}\int_0^t \phi(u)\,\mathrm{d}u - \frac{\lambda}{c}\int_0^t \int_0^u \phi(u-y)\,\mathrm{d}F_Y(y)\,\mathrm{d}u\,. \quad (3.27)$$

Moreover, the integration by parts yields

$$\begin{aligned}&\int_0^t \int_0^u \phi(u-y)\,\mathrm{d}F_Y(y)\,\mathrm{d}u \\ &\qquad = \int_0^t \left(\phi(0)\,F_Y(u) + \int_0^u F_Y(y)\,\phi'(u-y)\,\mathrm{d}y\right)\mathrm{d}u \\ &\qquad = \phi(0)\int_0^t F_Y(u)\mathrm{d}u + \int_0^t F_Y(y)\left(\int_y^t \phi'(u-y)\mathrm{d}u\right)\mathrm{d}y \\ &\qquad\qquad = \int_0^t F_Y(y)\phi(t-y)\mathrm{d}y\,.\end{aligned}$$

Using now the condition (3.19), we obtain from (3.27) that

$$\phi(t) - \phi(0) = \frac{1}{(1+\rho)\mu} \int_0^t \phi(t-y)\,\overline{F}_Y(y)\,\mathrm{d}y$$
$$= \frac{1}{1+\rho} \int_0^t \phi(t-y)\,\mathrm{d}F_{Y,I}(y)\,. \qquad (3.28)$$

It should be noted now that $\phi(\infty) = 1$. Therefore, the passing here to the limit as $t \to \infty$ yields

$$\phi(0) = \frac{\rho}{1+\rho}$$

and we obtain from (3.28) the equality (3.26). Hence Theorem 3.3. □

Note that (3.26) immediately implies the equation for the ruin probability $\psi(u) = 1 - \phi(u)$:

$$\psi(u) = \frac{\overline{F}_{Y,I}(u)}{1+\rho} + \frac{1}{1+\rho} \int_0^u \psi(u-y)\,\mathrm{d}F_{Y,I}(y)\,, \qquad (3.29)$$

where $\overline{F}_{Y,I}(y) = 1 - F_{Y,I}(y)$.

Example 3.2. *In the case, when distribution of* $(Y_j)_{j\geq 1}$ *is exponential, as in the example 3.1, i.e.* $F_Y(y) = 1 - e^{-\gamma y}$, *this equation has* *the following form*

$$\psi(u) \;=\; \frac{e^{-\gamma u}}{1+\rho} + \frac{\gamma}{1+\rho}\int_0^u \psi(u-y)\, e^{-\gamma y}\mathrm{d}y\,. \tag{3.30}$$

We can resolve this equation directly and get that the solution is

$$\psi(u) = \frac{1}{1+\rho}\, e^{-\gamma \frac{\rho}{1+\rho} u}\,. \tag{3.31}$$

Remark 3.3. *Note that, if we compare the form* (3.31) *with the upper bound* (3.25)*, then one can see that the Lundberg inequality gives sharp upper bound for the coefficient* $(1+\rho)^{-1}$.

3.6 Exercises V

Let $(Y_j)_{j\geq 1}$ be i.i.d. random variables with values in $\mathbb{N}$ and N a random variable with values in $\mathbb{N}$ independent of $(Y_j)_j$ whose distribution is of the form

$$q_n := \mathbf{P}(N=n) = \left(a + \frac{b}{n}\right) q_{n-1}, \quad n = 1, 2, \ldots\,,$$

where $q_0 = \mathbf{P}(N = 0)$, for $a < 1$ and $b \in \mathbb{R}$ are fixed constants. Moreover let

$$X = \sum_{j=1}^{N} Y_j \quad \text{and} \quad p_k := \mathbf{P}(X = k)\,.$$

1. Show that the Poisson and binomial distributions verify the previous hypotheses on N.

2. Let $S_n = \sum_{j=1}^{n} Y_j$. Show that for $i \geq 1$

$$\mathbf{E}\left(\frac{Y_1}{S_i}\middle| S_i\right) = \frac{1}{i}\,.$$

3. Show that

$$\mathbf{E}\left(a + b\frac{Y_1}{n}\middle| S_i = n\right) = \sum_{k=0}^{n}\left(a + \frac{k}{n}\right)\frac{\mathbf{P}(Y_1 = k)\mathbf{P}(S_{i-1} = n-k)}{\mathbf{P}(S_i = n)}\,.$$

4. Show that

$$p_0 = \begin{cases} q_0\,, & \text{if} \quad \mathbf{P}(Y_1 = 0) = 0\,; \\ \mathbf{E}\left(\mathbf{P}(Y_1 = 0)\right)^N\,, & \text{else}\,. \end{cases}$$

5. Show that for $n \geq 1$

$$p_n = \sum_{i=1}^{\infty} \mathbf{P}(S_i = n) q_i \,.$$

6. Show that the probabilities p_k can be calculated recursively (Panjer's algorithm):

$$p_k = \frac{1}{1 - a\mathbf{P}(Y_1 = 0)} \sum_{i=1}^{k} \left(a + \frac{bi}{k}\right) \mathbf{P}(Y_1 = i) p_{k-i}, \quad k \geq 1.$$

3.7 Cramér bound

In this section we will study the limit of $\psi(u)$ when $u \to \infty$ for small claims, i.e. for claims that verify the condition $\mathbf{H}_1$).

Theorem 3.4. *We assume that the conditions* $\mathbf{H}_1$)–$\mathbf{H}_2$) *hold with* $0 < r < \delta$ *and the random variable* Y_1 *has a density* f_Y*. Then*

$$\lim_{u \to \infty} e^{ru} \, \psi(u) \; = \; \psi_* > 0 \,, \tag{3.32}$$

where

$$\psi_* \; = \; \frac{\rho \, \mu}{r \int_0^{\infty} \; z \, e^{rz} \, \mathbf{P}(Y_1 > z) \, \mathrm{d}z}$$

and the parameter $\rho > 0$ *is given in the net profit condition* (3.19).

Proof. First, note that Remark 3.2 implies the condition (3.19). Moreover, by the Theorem 3.3, we can write the equation for ψ

$$\psi(u) \,=\, q\,\overline{F}_{Y,I}(u) \,+\, q\int_0^u \psi(u-y)\,\mathrm{d}F_{Y,I}(y)\,, \tag{3.33}$$

where $q=(1+\rho)^{-1}$. From here we directly get the equation for the function $Q(u)=e^{ru}\psi(u)$, i.e.

$$Q(u) \,=\, V(u) \,+\, \int_0^u Q(u-y)\,\mathrm{d}G(y)\,, \tag{3.34}$$

where $V(u)=qe^{ru}\overline{F}_{Y,I}(u)$ and

$$G(u) \,=\, \frac{q}{\mu}\int_0^u e^{rz}\,\mathbf{P}(Y_1>z)\,\mathrm{d}z\,.$$

Now let us to show that G is a distribution function, i.e. $G(+\infty)=1$. Indeed, the integrating by parts yields

$$\begin{aligned} G(+\infty) \,&=\, \frac{q}{\mu}\int_0^\infty e^{rz}\,\mathbf{P}(Y_1>z)\,\mathrm{d}z \\ &=\, \frac{q}{r\mu}\,e^{rz}\,\mathbf{P}(Y_1>z)\,\Big|_0^\infty + \frac{q}{r\mu}\,\mathbf{E}\,e^{rY_1}\,. \end{aligned}$$

Taking into account that $0<r<\delta$, we obtain

$$\mathbf{P}(Y_1>z) \,\le\, \mathbf{E}\,e^{\delta Y_1}\,e^{-(\delta-r)z} \,\to\, 0 \quad \text{when} \quad z\to\infty\,. \tag{3.35}$$

Therefore,

$$G(+\infty) = -\frac{q}{r\mu} + \frac{q}{r\mu}\,\mathbf{E}\,e^{rY_1}\,.$$

Note here that the definition of r and (3.19) imply

$$\mathbf{E}\,e^{rY_1} = \frac{\lambda + rc}{\lambda} = 1 + \frac{r\mu}{q}\,,$$

i.e. $G(+\infty) = 1$. This means that the equation (3.34) is a renewal equation and the solution Q is a renewal function for i.i.d. random variables $(\eta_j)_{j\geq 1}$ with the distribution function G. Let's study now the function V. We note that the inequality (3.35) implies the following upper bound for V

$$V(u) = \frac{q}{\mu}\int_u^\infty e^{rz}\,\mathbf{P}(Y_1 > z)\,\mathrm{d}z$$

$$\leq \frac{q}{\delta\mu}\,\mathbf{E}\,e^{\delta Y_1}\,e^{-(\delta-r)u}\,,$$

i.e. the function V satisfies the Riemann direct integrability condition. This means that we can apply Smith theorem to the function Q, i.e.

$$\lim_{u\to\infty} Q(u) = \frac{1}{\mathbf{E}\eta_1}\int_0^\infty V(z)\,\mathrm{d}z\,,$$

where

$$\mathbf{E}\eta_1 = \frac{q}{\mu}\int_0^\infty z\,e^{rz}\,\mathbf{P}(Y_1 > z)\,\mathrm{d}z$$

and

$$\begin{aligned}
\int_0^\infty V(z)\,\mathrm{d}z &= q\int_0^\infty e^{rz}\,\overline{F}_{Y,I}(z)\mathrm{d}z \\
&= \frac{q}{r}\,e^{rz}\,\overline{F}_{Y,I}(z)\,|_0^\infty + \frac{q}{\mu r}\int_0^\infty e^{rz}\,\mathbf{P}(Y_1 > z)\,\mathrm{d}z \\
&= -\frac{q}{r} + \frac{1}{r}G(+\infty) = \frac{1-q}{r}\,.
\end{aligned}$$

This directly implies (3.32). Hence Theorem 3.32. □

3.8 Exercises VI

1. We consider a Cramér-Lundberg model with the risk process

$$U_t = u + ct - X_t\,,$$

where the total claim amount process $X_t = \sum_{j=1}^{N_t} Y_j$.

   (a) Show that the random variables $X_t - X_s$ and X_s are independent for $0 < s < t$.

   (b) Show that the random variables $X_t - X_s$ and X_{t-s} have the same distribution for $0 < s < t$.

   (c) Calculate

$$\mathbf{E}(e^{-hU_t}|X_s)\,.$$

(d) Assuming that the Lundberg coefficient $r > 0$ exists, show that

$$\mathbf{E}(e^{-rU_t}|X_s) = e^{-rU_s}\,.$$

(e) Show that $\mathbf{E}e^{-rU_t}$ independent of t.

2. Assume that in a Cramér-Lundberg model the distribution of the claim amounts Y_j is given by the density

$$f_n(x) = \alpha^n \frac{1}{\Gamma(n)} x^{n-1} e^{-\alpha x} \quad \text{for } x > 0 \quad (\alpha > 0\,, n \geq 1).$$

(a) Calculate the generator function $\mathbf{E}e^{hY_1}$. For which values $h > 0$ this function is well defined? Calculate $\mathbf{E}Y_1$.

(b) Find the net profit condition for this model.

(c) Calculate the Lundberg coefficient for $n = 1$ and $n = 2$.

(d) Write the integral equation for the ruin function $\psi_n(u)$. Find this function for $n = 1$.

3.9 Large claims

In this section we study the problem of ruin for the claims $(Y_j)_{j\geq 1}$ which do not hold the condition $\mathbf{H}_1$), i.e., $\mathbf{E}\,e^{\delta Y_1} = +\infty$ for all $\delta > 0$. We replace the condition $\mathbf{H}_1$) by a weaker condition, i.e. we assume that the distribution of $(Y_j)_{j\geq 1}$ is *subexponential*.

Definition 3.4. *We say that a random variable Y is subexponential if i.i.d. random variables $(Y_j)_{j\ge1}$ having the same distribution as Y for all $n \ge 1$ satisfy the following condition*

$$\lim_{z\to\infty} \frac{\mathbf{P}(\sum_{j=1}^n Y_j > z)}{\mathbf{P}(Y_1 > z)} = n\,. \tag{3.36}$$

Example 3.3. *Let Y a positive random variable such that for any $z \ge 0$*

$$\mathbf{P}(Y > z) = \frac{1}{(1+z)^\alpha} \quad \textit{and} \quad \alpha > 0\,.$$

Let's show, by the induction, that Y satisfies the condition (3.36). Assuming that the property (3.36) holds for $n-1$, we will check this condition for n. To this end we set

$$\overline{F}_n(z) = \mathbf{P}(S_n > z)\,,$$

where $S_n = \sum_{j=1}^n Y_j$. We have

$$\begin{aligned}\overline{F}_n(z) &= \mathbf{P}(\sum_{j=1}^{n-1} Y_j > z - Y_n)\\ &= \mathbf{E}\,\overline{F}_{n-1}(z-Y_n)\,\mathbf{1}_{\{Y_n\le z\}} + \mathbf{P}(Y_n > z)\\ &= \int_0^z \overline{F}_{n-1}(z-t)\,\mathrm{d}F(t) + \overline{F}(z)\,,\end{aligned}$$

where F is the distribution function of Y and $\overline{F}(z) = \overline{F}_1(z)$. So, we obtain that

$$\frac{\overline{F}_n(z)}{\overline{F}(z)} = 1 + \int_0^z \frac{\overline{F}_{n-1}(z-t)}{\overline{F}(z)} \, \mathrm{d}F(t) \,. \tag{3.37}$$

Then we can represent the last term in this equality as

$$\int_0^{rz} \frac{\overline{F}_{n-1}(z-t)}{\overline{F}(z)} \, \mathrm{d}F(t) + \int_{rz}^{z} \frac{\overline{F}_{n-1}(z-t)}{\overline{F}(z)} \, \mathrm{d}F(t)$$
$$= I_{1,r}(z) + I_{2,r}(z) \,,$$

where $0 < r < 1$. Note that for the function $F(z) = 1 - (1+z)^{-\alpha}$ for all $t \geq 0$ we have

$$\lim_{z\to\infty} \frac{\overline{F}(z-t)}{\overline{F}(z)} = 1 \,.$$

Thus, in view of the induction hypothesis for all $t > 0$

$$\lim_{z\to\infty} \frac{\overline{F}_{n-1}(z-t)}{\overline{F}(z)} = n - 1 \,.$$

Moreover, for $t \leq rz$ with $0 < r < 1$ we obtain the following upper bound

$$\limsup_{z\to\infty} \frac{\overline{F}_{n-1}(z-t)}{\overline{F}(z)} \leq \limsup_{z\to\infty} \frac{\overline{F}_{n-1}((1-r)z)}{\overline{F}(z)} \leq \frac{n-1}{(1-r)^{\alpha}} \,.$$

Therefore, by the dominated convergence theorem,

$$\lim_{z\to\infty} I_{1,r}(z) \,=\, n-1$$

for all $0 < r < 1$*. As to the function* $I_{2,r}(z)$*, we obtain that for any* $z > 0$

$$\begin{aligned} I_{2,r}(z) \,&\le\, \frac{1}{\overline{F}(z)}\,(F(z)\,-\,F(rz)) \\ &\le\, \left(\frac{1}{(1+rz)^{\alpha}}\,-\,\frac{1}{(1+z)^{\alpha}}\right)(1+z)^{\alpha}\,. \end{aligned}$$

This means that for any $0 < r < 1$

$$\limsup_{z\to\infty} I_{2,r}(z) \,\le\, r^{-\alpha}\,-\,1$$

and, passing here to the limit as $r \to 1$*, we find*

$$\limsup_{r\to 1}\,\limsup_{z\to\infty} I_{2,r}(z) \,=\, 0\,.$$

Therefore, the equality (3.37) *implies directly* (3.36)*.*

Proposition 3.3. *Let* Y *be a subexponential random variable. Then* *for any $\varepsilon > 0$ it exists $K = K(\varepsilon) > 0$ such that for any $n \geq 1$*

$$\sup_{z\geq 0} \frac{\overline{F}_n(z)}{\overline{F}(z)} \leq K\,(1+\varepsilon)^n\,, \tag{3.38}$$

where $\overline{F}_n(z) = \mathbf{P}(S_n > z)$, $\overline{F}(z) = \overline{F}_1(z)$, $S_n = \sum_{j=1}^n Y_j$ and $(Y_j)_{j\geq 1}$ are i.i.d. random variables of the same distribution as Y.

Proof. We set

$$\alpha_n = \sup_{z\geq 0} \frac{\overline{F}_n(z)}{\overline{F}(z)}\,,$$

then we get

$$\begin{aligned}
\frac{\overline{F}_{n+1}(z)}{\overline{F}(z)} &= 1 + \frac{\int_0^z \mathbf{P}(S_n > z-t)\,\mathrm{d}F(t)}{\overline{F}(z)} \\
&\leq 1 + \alpha_n \frac{\mathbf{P}(Y_1 + Y_2 > z\,,\, Y_2 \leq z)}{\overline{F}(z)} \\
&= 1 + \alpha_n \left(\frac{\mathbf{P}(S_2 > z)}{\overline{F}(z)} - 1 \right).
\end{aligned}$$

Note here that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that

$$\sup_{z\geq T} \left(\frac{\mathbf{P}(S_2 > z)}{\overline{F}(z)} - 1 \right) \leq (1+\varepsilon)\,.$$

Therefore,

$$\alpha_{n+1} \le \sup_{0\le z\le T} \frac{\overline{F}_n(z)}{\overline{F}(z)} + \sup_{z\ge T} \frac{\overline{F}_n(z)}{\overline{F}(z)}$$
$$\le \alpha_n (1+\varepsilon) + K_0 \,,$$

where

$$K_0 = 1 + \frac{1}{\mathbf{P}(Y > T)} \,.$$

This inequality means that for $n \ge 1$

$$\alpha_{n+1} = (1+\varepsilon)\,\alpha_n + \beta_{n+1} \tag{3.39}$$

with $\alpha_1 = 1$ and $\beta_{n+1} = \alpha_{n+1} - (1+\varepsilon)\alpha_n \le K_0$. We can resolve this equation and find that

$$\alpha_n = (1+\varepsilon)^{n-1}\,\alpha_1 + \sum_{j=2}^{n} (1+\varepsilon)^j\,\beta_{n-j}$$
$$\le (1+\varepsilon)^{n-1} + K_0 \sum_{j=2}^{n} (1+\varepsilon)^j$$
$$\le K\,(1+\varepsilon)^n \,,$$

where $K = 1 + K_0/\varepsilon$. From here we obtain (3.38). $\square$

In Section 3.5 we have shown that the probability of non-ruin

$\phi(u)$ satisfies the equation (3.26). Now one needs to resolve this equation.

Proposition 3.4. *We assume that in the Cramér-Lundberg model the distribution function* $F_Y(\cdot)$ *for the random amounts* $(Y_j)_{j\geq 1}$ *has a density* f_Y *and* $\mu = \mathbf{E}\,Y_1 < \infty$. *Moreover, we assume that this model satisfies the net profit condition* (3.18), *i.e.* $c = \lambda\mu(1+\rho)$ *with* $\rho > 0$. *Then the solution of the equation* (3.26) *has the following form*

$$\phi(u) = p \sum_{j=0}^{\infty} q^j \, \mathbf{P}(\tilde{S}_j \leq u)\,, \tag{3.40}$$

where $p = \rho/(1+\rho)$, $q = 1/(1+\rho)$, $\tilde{S}_0 = 0$, $\tilde{S}_j = \sum_{i=1}^{j} \tilde{Y}_i$ *and* $(\tilde{Y})_{j\geq 1}$ *are i.i.d. random variables with the distribution function* $F_{Y,I}(\cdot)$ *defined in* (3.26).

Proof. Let us denote the right part in equality (3.40) by g, i.e.

$$g(u) = p \sum_{j=0}^{\infty} q^j \, \mathbf{P}(\tilde{S}_j \leq u)\,.$$

It is clear that this function is bounded, i.e.

$$g(u) \leq p \sum_{j=0}^{\infty} q^j = \frac{p}{1-q} = 1\,.$$

Moreover, one can see that this function satisfies the equation (3.26).

Indeed,

$$
\begin{aligned}
g(u) &= p + pq\,\mathbf{P}(\tilde{Y}_1 \le u) + p \sum_{j=2}^{\infty} q^j\, \mathbf{P}\left(\sum_{l=2}^{j} \tilde{Y}_l \le u - \tilde{Y}_1\right) \\
&= p + pq\,\mathbf{P}(\tilde{Y}_1 \le u) \\
&+ p \sum_{j=2}^{\infty} q^j \int_0^u \mathbf{P}\left(\sum_{l=2}^{j} \tilde{Y}_l \le u - t\right) \mathrm{d}F_{Y,I}(t) \\
&= p + q \int_0^u g(u-t)\,\mathrm{d}F_{Y,I}(t)\,.
\end{aligned}
$$

We show now that $g(u) = \phi(u)$. To this end we set $\delta(u) = g(u) - \phi(u)$. We have already seen that $|\delta(u)| \le 2$ for all $u \ge 0$. Let now u be a fixed positive number. Denote by $M_u = \sup_{0 \le t \le u} |\delta(t)|$. Then there exists $0 \le t_0 \le u$ such that $M_u = |\delta(u_0)|$ because the function $\delta(\cdot)$ is continuous on the interval $[0, u]$. Therefore,

$$
\begin{aligned}
M_u &= |\delta(t_0)| = q \left| \int_0^u \delta(u_0 - t)\,\mathrm{d}F_{Y,I}(t) \right| \\
&\le q \int_0^u |\delta(u_0 - t)|\,\mathrm{d}F_{Y,I}(t) \\
&\le q\, M_u\, \mathbf{P}(\tilde{Y}_1 \le u_0) \le q M_u\,.
\end{aligned}
$$

Taking into account that $q < 1$, we get that $M_u = 0$ for all $u > 0$. Therefore, $\phi(u) = g(u)$ for any $u \ge 0$. $\square$

Theorem 3.5. *We assume that all the conditions in Proposition 3.4 hold and the distribution function $F_{Y,I}(\cdot)$ is subexponential. Then*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{Y,I}(u)} = \rho^{-1} . \tag{3.41}$$

Proof. The equality (3.41) implies directly that

$$\psi(u) = 1 - \phi(u) = p \sum_{j=1}^{\infty} q^j \, \mathbf{P}(\tilde{S}_j \ge u) .$$

Thus, taking into account that

$$p \sum_{j=1}^{\infty} j \, q^j = \rho^{-1} ,$$

we have

$$\Delta(u) = \frac{\psi(u)}{\overline{F}_{Y,I}(u)} - \rho^{-1} = p \sum_{j=1}^{\infty} q^j \, \sigma_j(u) ,$$

where

$$\sigma_j(u) = \frac{\mathbf{P}(\tilde{S}_j \ge u)}{\overline{F}_{Y,I}(u)} - j .$$

Now we fixe $\varepsilon > 0$ such that $\theta = q(1+\varepsilon) < 1$. Then, by Proposition 3.3, we obtain that there is a positive constant K such that for any $j \ge 1$ and for all $u > 0$

$$q^j \, |\sigma_j(u)| \, \le \, K \, \theta^j \, + \, q^j \, j \, .$$

Moreover, for any $j \ge 1$

$$\lim_{u \to \infty} \sigma_j(u) \, = \, 0 \, .$$

Then the dominated convergence theorem directly implies that

$$\lim_{u \to \infty} \Delta(u) \, = \, p \sum_{j=1}^{\infty} q^j \lim_{u \to \infty} \sigma_j(u) \, = \, 0 \, .$$

Hence Theorem 3.5. $\square$

Example 3.4. *We consider the Cramér-Lundberg model in which the positive random variables $(Y_j)_{j \ge 1}$ are distributed according to the Pareto distribution function $F(\cdot)$ defined as*

$$F(z) \, = \, \mathbf{P}(Y \, \le \, z) \, = \, 1 \, - \, \frac{1}{(1+z)^{1+\alpha}}$$

for $z \ge 0$. In this case $\mu = 1/\alpha$ and, therefore, for all $z \ge 0$

$$F_{Y,I}(z) \, = \, 1 \, - \, \frac{1}{(1+z)^{\alpha}} \, .$$

We have seen already that this distribution function is subexponential. Therefore, in this case, in view of Theorem 3.5, we obtain that $u^\alpha \psi(u) \to \rho^{-1}$ *as* $u \to \infty$.

3.10 Exercises VII

Let F be some distribution function for the claim $Y > 0$. We denote $\overline{F}(x) = 1 - F(x)$. We say that F is light tailed if there exist a and $b > 0$ such that

$$\overline{F}(x) \leq a e^{-bx}$$

for all x (or that F is heavy-tailed). Let

$$x_l = \inf\{x | F(x) > 0\} \quad \text{and} \quad x_r = \sup\{x | F(x) < 1\}\,.$$

We set

$$e_F(u) = \mathbf{E}(Y - u | Y > u) \quad \text{for} \quad u \in (x_l, x_r)\,.$$

1. Show that $e_F(u)$ can be written as

$$e_F(u) = \frac{1}{\overline{F}(u)} \int_u^{+\infty} \overline{F}(x) \mathrm{d}x\,.$$

2. Show that if $F(x) > 0$ for all $x > 0$ and that F is continuous,

then we have for any $x > 0$

$$\overline{F}(x) = \frac{e_F(0)}{e_F(x)} \exp\left\{-\int_0^x \frac{1}{e_F(y)} dy\right\}.$$

3. Show that if $\lim_{u\to+\infty} e_F(u) = +\infty$, then F is heavy tailed.

4. Show that if F is heavy tailed, then the generator function of Y is infinite for any $z > 0$.

5. Calculate $e_F(u)$ when Y is an exponential random variable of a parameter $\lambda > 0$.

6. Calculate $e_F(u)$ when Y has the Pareto distribution

$$\overline{F}(x) = \left(\frac{\kappa}{\kappa + x}\right)^{\alpha}$$

of parameters $\kappa > 0$ and $\alpha > 1$.

7. Show that if Y has the Gamma distribution of order $m \geq 1$ and a parameter $\gamma > 0$, i.e.

$$\overline{F}(x) = \frac{\gamma^m x^{m-1}}{m!} e^{-\gamma x},$$

then F is light-tailed.

8. Show that if Y has the Weibull distribution with parameters

$c > 0$ and $\tau > 0$, i.e.

$$\overline{F}(x) = \exp\{-cx^{\tau}\}\,,$$

then F is light or heavy tailed depending on the value of τ.

3.11 Ruin problem with investment

In this section we consider an insurance company that invests its capital in a Black-Scholes market with the two assets $B = (B_t)_{t\geq 0}$ and $S = (S_t)_{t\geq 0}$ defined as

$$\begin{cases} \mathrm{d}B_t & = r\,B_t\,\mathrm{d}t\,, \quad B_0 = 1\,; \\ \mathrm{d}S_t & = aS_t\mathrm{d}t + \sigma S_t\,\mathrm{d}w_t\,, \quad S_0 > 0\,, \end{cases} \tag{3.42}$$

where $(w_t)_{t\geq 0}$ is a Brownian motion, r, a and σ are non-negative constants. Let $(\mathcal{F}_t)_{t\geq 0}$ be filtration on this model defined as

$$\mathcal{F}_t = \sigma\{w_s\,,\, X_s\,,\, s \leq t\}\,, \tag{3.43}$$

where $(X_t)_{t\geq 0}$ is the total claim amount process defined in (3.3). We assume that in each time moment $t \geq 0$ the insurance company has β_t of the assets (B) and γ_t of the assets (S). So, the wealth (the risk process) at the instant $t > 0$ is equal to

$$U_t = \beta_t B_t + \gamma_t S_t \,. \tag{3.44}$$

We denote by $\pi_t = (\beta_t, \gamma_t)$ and assume that the process $\pi = (\pi_t)_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. In this case π is said a financial strategy.

Definition 3.5. *Financial strategy $\pi = (\pi_t)_{t\geq 0}$ with $\pi_t = (\beta_t, \gamma_t)$ is said to be admissible if for any $t \geq 0$*

$$\int_0^t \left(|\beta_s| + \gamma_s^2\right) \mathrm{d}s < \infty \quad a.s.$$

and, for anyy $t \geq 0$,

$$U_t = \beta_t B_t + \gamma_t S_t = u + \int_0^t \beta_s \, \mathrm{d}B_s + \int_0^t \gamma_v \, \mathrm{d}S_v + Z_t \,, \tag{3.45}$$

where $u > 0$ is an initial endowment and $Z_t = ct - X_t$.

Proposition 3.5. *Let $u > 0$ and $\gamma = (\gamma_t)_{t\geq 0}$ be a square integrated process, i.e. for all $t > 0$*

$$\int_0^t \gamma_v^2 \, \mathrm{d}v < \infty \quad a.s.$$

We set

$$\beta_t = u + \int_0^t \gamma_v \, \mathrm{d}\widetilde{S}_v - \gamma_t \widetilde{S}_t + \widetilde{Z}_t \,, \tag{3.46}$$

where $\widetilde{S}_t = S_t/B_t$ *and* $\widetilde{Z}_t = \int_0^t B_v^{-1} \mathrm{d}Z_v$. *Then the financial strategy* $(\beta_t \,, \gamma_t)_{t\ge 0}$ *is admissible.*

Proof. First of all note that the process (3.46) is integrable, i.e. for any $t > 0$

$$\int_0^t |\beta_v| \, \mathrm{d}v < \infty \quad \text{a.s.}$$

The definition (3.46) implies that the discounted wealth process

$$\widetilde{U}_t = \frac{U_t}{B_t} = \beta_t + \gamma_t \widetilde{S}_t$$

admits the following stochastic differential:

$$\mathrm{d}\widetilde{U}_t = \gamma_t \, \mathrm{d}\widetilde{S}_t + \mathrm{d}\widetilde{Z}_t \,.$$

Therefore, Ito formula implies that

$$\begin{aligned} \mathrm{d}U_t &= \mathrm{d}B_t \widetilde{U}_t = B_t \mathrm{d}\widetilde{U}_t + \widetilde{U}_t \mathrm{d}B_t \\ &= \beta_t \, \mathrm{d}B_t + \gamma_t \left(\widetilde{S}_t \mathrm{d}B_t + B_t \mathrm{d}\widetilde{S}_t \right) + B_t \mathrm{d}\widetilde{Z}_t \,. \end{aligned}$$

Taking into account here that

$$\mathrm{d}Z_t = B_t \mathrm{d}\widetilde{Z}_t \quad \text{and} \quad \mathrm{d}S_t = \mathrm{d}\widetilde{S}_t B_t = \widetilde{S}_t \mathrm{d}B_t + B_t \mathrm{d}\widetilde{S}_t \,,$$

we get the equality (3.45). $\square$

We denote by $\varsigma = (\varsigma_t)_{t\ge 0}$ proportional strategy, i.e.

$$\varsigma_t = \frac{\gamma_t\, S_t}{\beta_t B_t + \gamma_t\, S_t} = \frac{\gamma_t \widetilde{S}_t}{\beta_t + \gamma_t \widetilde{S}_t} = \frac{\gamma_t \widetilde{S}_t}{u + \int_0^t \gamma_v \,\mathrm{d}\widetilde{S}_v + \widetilde{Z}_t} \,.$$

For this strategy we can rewrite the equation (3.45) as

$$U_t = u + \int_0^t (r + (a-r)\varsigma_s)\, U_s \,\mathrm{d}s \, + \, \sigma \int_0^t \varsigma_s \, U_s \mathrm{d}w_s \, + \, ct \, - \, X_t \,.$$

In this section we assume that

$$\varsigma_t \, \equiv \, \delta \,, \tag{3.47}$$

where $\delta \ge 0$ is a fixed nonrandom constant. In view of

$$\mathrm{d}\widetilde{S}_t = (a-r)\widetilde{S}_t \mathrm{d}t + \sigma \widetilde{S}_t \mathrm{d}w_t \,,$$

we obtain for $V_t = \gamma_t \, S_t$ the following stochastic differential equation

$$\mathrm{d}V_t = \delta(a-r)V_t \mathrm{d}t + \delta\sigma V_t \mathrm{d}w_t + \delta \mathrm{d}\widetilde{Z}_t \,, \quad V_0 = \delta u \,.$$

Through Ito formula we can represent the process $\mathbf{V}_t$ in the following form

$$\mathbf{V}_t = e^{\xi_t} \left(\delta\, u + \delta\, c \int_0^t e^{-\xi_s} B_s^{-1} \mathrm{d} Z_s \right) ,$$

where $\xi_t = a_1\, t + \sigma_\delta\, w_t$, $a_1 = \delta(a - r) - \sigma_\delta^2/2$ and $\sigma_\delta = \delta\sigma$. So, to get the property (3.47) we set

$$\gamma_t = \frac{e^{\xi_t} \left(\delta\, u + \delta\, c \int_0^t e^{-\xi_s} B_s^{-1} \mathrm{d} Z_s \right)}{\widetilde{S}_t} , \quad 0 \le t \le T .$$

For this strategy the risk process can be written as

$$U_t = u + a_\delta \int_0^t U_s \, \mathrm{d}s + \sigma_\delta \int_0^t U_s \mathrm{d} w_s + Z_t , \tag{3.48}$$

where $u > 0$ is the initial capital, $a_\delta = r + \delta(a - r)$ and $\sigma_\delta = \delta\sigma$. The ruin probability is

$$\psi(u) = \mathbf{P}(\inf_{t \ge 0} U_t < 0) . \tag{3.49}$$

We start to study this function for any $u \ge 0$.

Proposition 3.6. *The function $\phi(u) = 1 - \psi(u)$ satisfies for all*

$u \geq 0$ the following differential equation

$$\frac{\sigma_\delta^2 u^2 \phi^{''}(u)}{2} + \phi^{'}(u)(a_\delta u + c) - \lambda\phi(u) + \lambda \int_0^u \phi(u-y)\mathrm{d}F(y) = 0 \tag{3.50}$$

with the boundary conditions

$$c\phi^{'}(0) = \lambda\phi(0) \quad \textit{and} \quad \phi(+\infty) = 1\,. \tag{3.51}$$

Proof. We denote by $(\eta_t)_{t\geq 0}$ the solution of the following stochastic equation

$$\mathrm{d}\eta_t = (a_\delta \eta_t + c)\,\mathrm{d}t + \sigma_\delta \eta_t \mathrm{d}w_t\,, \quad \eta_0 = u\,.$$

By Ito formula, we can resolve it, i.e.

$$\eta_t \;=\; u\, e^{\zeta_t} \;+\; c\int_0^t e^{\zeta_t - \zeta_s}\,\mathrm{d}s \quad \text{and} \quad \zeta_t = \widetilde{a}\, t \;+\; \sigma_\delta\, w_t\,, \tag{3.52}$$

where $\widetilde{a} = a_\delta - \sigma_\delta^2/2$. Now we fixe some $h > 0$ and then, in view of the definition of ϕ, we can represent this function as

$$\begin{aligned}\phi(u) &= \mathbf{E}\left(\mathbf{1}_{\{\inf_{t\geq 0} U_t \geq 0\}}\right)\\ &= \mathbf{E}\left(\mathbf{1}_{\{\inf_{0\leq t\leq h} U_t \geq 0\}}\,\mathbf{E}\left(\mathbf{1}_{\{\inf_{t\geq h} U_t\geq 0\}}\,|\,\mathcal{F}_h\right)\right)\,.\end{aligned}$$

It should be noted that the process (3.48) is Markovian, i.e.

$$\mathbf{E}\left(\mathbf{1}_{\{\inf_{t\geq h}\, U_t\geq 0\}}\,|\,\mathcal{F}_h\right) = \phi(U_h)$$

and, therefore,

$$\phi(u) = \mathbf{E}\left(\mathbf{1}_{\{\inf_{0\leq t\leq h}\, U_t\geq 0\}}\,\phi(U_h)\right)\,.$$

Moreover, we can represent this function in the form

$$\phi(u) = A_1(u) + A_2(u) + A_3(u) \tag{3.53}$$

where $A_1(u) = \mathbf{E}\left(\mathbf{1}_{\{N_h=0\}}\,\phi(\eta_h)\right)$,

$$A_2(u) = \mathbf{E}\left(\mathbf{1}_{\{\inf_{0\leq t\leq h}\, U_t\geq 0\}}\,\mathbf{1}_{\{N_h=1\}}\phi(U_h)\right)$$

and

$$A_3(u) = \mathbf{E}\left(\mathbf{1}_{\{\inf_{0\leq t\leq h}\, U_t\geq 0\}}\,\mathbf{1}_{\{N_h\geq 2\}}\phi(U_h)\right)\,.$$

We rewrite the equation (3.53) as

$$\frac{A_1(u)-\phi(u)}{h} + \frac{A_2(u)}{h} + \frac{A_3(u)}{h} = 0\,. \tag{3.54}$$

It is clear that

$$A_1(u) - \phi(u) = e^{-\lambda h}\, \mathbf{E}\left(\phi(\eta_h) - \phi(u)\right).$$

In addition, it is well known (see, for example, in [8]) that the function ϕ is two times continuously differentiable. So, by the Ito formula

$$A_1(u) - \phi(u) = (e^{-\lambda h} - 1)\,\phi(u) + \int_0^h \mathbf{E}\left((a_\delta \eta_v + c)\,\phi^{'}(\eta_v) + \frac{\sigma_\delta^2\,\eta_v^2}{2}\,\phi^{''}(\eta_v)\right) \mathrm{d}v$$

and, therefore,

$$\lim_{h\to 0} \frac{A_1(u) - \phi(u)}{h} = (a_\delta u + c)\,\phi^{'}(u) + \frac{\sigma_\delta^2\, u^2}{2}\,\phi^{''}(u) - \lambda\phi(u).$$

Then we can represent the term $A_2(u)$ as

$$A_2(u) = \mathbf{E}\left(\mathbf{1}_{\{\eta_{T_1} \ge Y_1\}}\, \mathbf{1}_{\{N_h = 1\}} \phi(U_h)\right).$$

Note here that on the set $\{N_h = 1\}$

$$U_h = \left(\eta_{T_1} - Y_1\right) e^{\zeta_h - \zeta_{T_1}} + c \int_{T_1}^h e^{\zeta_h - \zeta_s}\, \mathrm{d}s,$$

where the process $(\zeta_t)_{t\geq 0}$ is given in (3.52). So, on the set

$$\Big\{N_h = 1\} \cap \{Y_1 \leq \eta_{T_1}\Big\} \tag{3.55}$$

we get

$$|U_h - u + Y_1| \leq (u + \eta_h^*)\left|e^{2\zeta_h^*} - 1\right| + \, c\,h\,e^{2\zeta_h^*} + \eta_h^*\,,$$

where

$$\eta_h^* = \sup_{0\leq s\leq h} |\eta_s - u| \quad \text{and} \quad \zeta_h^* = \sup_{0\leq s\leq h} |\zeta_s|\,.$$

From the definition of $(\eta_t)_{t\geq 0}$ it follows that

$$\eta_h^* \leq u|e^{\zeta_h^*} - 1| + c\,h\,e^{2\zeta_h^*}\,.$$

Therefore, on the set (3.55)

$$|U_h - u + Y_1| \leq B_h^*(\zeta_h^*)\,,$$

where

$$B_h^*(x) = \big(u + u|e^x - 1| + c\,h\,e^{2x}\big)\;|e^{2x} - 1| + 2c\,h\,e^{2x} + u|e^x - 1|\,.$$

Taking into account that the process $(\zeta_t)_{t\geq 0}$ is continuous, we obtain that

$$\lim_{h\to 0} \zeta_h^* = 0 \qquad \text{a.s.}$$

Moreover, using the properties of a Brownian motion, one can check directly that for all $\gamma > 0$ and $0 < t < \infty$

$$\mathbf{E}\, e^{\gamma \zeta_t^*} < \infty\,.$$

So, by the dominated convergence theorem

$$\lim_{h\to 0} \mathbf{E} B_h^*(\zeta_h^*) = 0\,.$$

In view of the independence of the processes $(\zeta_t)_{t\ge 0}$ and $(N_t)_{t\ge 0}$, we obtain that

$$\lim_{h\to 0} \frac{A_2(u)}{h} = \lambda \int_0^u \phi(u-y)\,\mathrm{d}F(y)\,. \tag{3.56}$$

Finally, for the last term in (3.54) it's easy to see that

$$\frac{A_3(u)}{h} \le \frac{\mathbf{P}\,(N_h \ge 2)}{h} \to 0\,, \quad \text{as} \quad h \to 0\,.$$

So, taking the limit in (3.54) as $h \to 0$ we get the equation (3.50). Hence Proposition 3.6. □

To study the asymptotic properties of the ruin probability (3.49) as $u \to \infty$, we set

$$\nu = \frac{2a_\delta}{\sigma_\delta^2} - 1 \,. \tag{3.57}$$

The following theorem gives us the asymptotic behavior of the ruin probability in the depending of this parameter (see, for example, [4] and [9]).

Theorem 3.6. *For the proportional strategy* (3.47) *the ruin probability* (3.49) *has the following asymptotic (as $u \to \infty$) properties*

1. *if $\nu \le 0$, then $\psi(u) = 1$ for all $u \ge 0$.*
2. *if $\nu > 0$ and $\mathbf{E}Y_1^\nu < \infty$, then there is a constant $0 < \psi_* < \infty$ such that*

$$\lim_{u\to\infty} u^\nu \, \psi(u) \, = \, \psi_* \,.$$

Now one needs to choose the proportional investment coefficient $\delta > 0$ in the strategy (3.47) such that the power parameter ν defined in (3.57) will be positive, i.e. $0 < \delta < \delta^*$, where

$$\delta^* \, = \, \frac{\sqrt{(a-r)^2 + 2r} - r + a}{\sigma^2} \,.$$

Therefore, if there exists $0 < \delta < \delta^*$ for which $\mathbf{E}\,Y_1^\nu < \infty$, then according to Theorem 3.6 for some constant $0 < \psi_* < \infty$

$$\lim_{u \to \infty} u^\nu \, \psi(u) = \psi_* \, .$$

Remark 3.4. *This result means that in the case where the net profit condition* (3.18) *does not hold true, i.e.* $c \le \lambda \mu$, *to avoid being bankrupt almost sure, the company is obliged to invest its capital in a Black-Scholes market through the strategy* (3.47) *with* $0 < \delta < \delta^*$ *and* $\mathbf{E}\, Y_1^\nu < \infty$.

3.12 Exercises VIII

1. Give the definition of an admissible strategy. Show that the set of admissible strategies is not empty.

2. Is there an admissible strategy with initial endowment $u = 20$ euros and $\gamma_t = (1 + S_t)^{-2}$? Clarify the answer.

3. Is there an admissible strategy with initial endowment $u = 2S_0$ and
$\gamma_t = 1/\sqrt{\widehat{S}_t}$ with $\widehat{S}_t = S_t / B_t$? Clarify the answer.

A Appendix

In this section we announce main limit results of the probability theory which can be found, for example, in [10].

A.1 Strong large numbers law

Theorem A.1. *Let* $(\xi_j)_{j\geq 1}$ *be i.i.d. random variables with* $\mathbf{E}|\xi_1| < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \xi_j = \mathbf{E}\xi_1 \qquad a.s.$$

A.2 Kolmogorov zero-one law

Theorem A.2. *Let* $(\xi_j)_{j\geq 1}$ *be a sequence of independent random variables and*

$$\mathcal{X} = \cap_{n\geq 1} \sigma\{(\xi_j)_{j\geq n}\}\,.$$

Then for any $A \in \mathcal{X}$ *the probability* $\mathbf{P}(A) = 0$ *or* $\mathbf{P}(A) = 1$.

A.3 Three series theorem

Theorem A.3. *Let* $(\xi_j)_{j\geq 1}$ *be a sequence of independent random variables. For almost sure convergence of the series* $\sum_{n\geq 1} \xi_j$ *neces-*

sary that for any $c > 0$ the following series are convergent

$$\sum_{n\geq 1} \mathbf{E}\,\xi_j^c\,, \quad \sum_{n\geq 1} \mathbf{E}\,(\xi_j^c - \mathbf{E}\,\xi_j^c)^2\,, \quad \sum_{n\geq 1} \mathbf{P}\left(|\xi_j| > c\right)\,,$$

and sufficiently that these series are convergent for some fixed $c > 0$, where $\xi^c = \xi \mathbf{1}_{\{|\xi_j|\leq c\}}$.

A.4 Central limit theorem

First, we recall the weak convergence for random variables.

Definition A.1. *The sequence of random variable is called convergent weakly to a random variable ξ, i.e. $\xi_n \Longrightarrow \xi$ as $n \to \infty$, if for any bounded continuous $\mathbb{R} \to \mathbb{R}$ function g*

$$\lim_{n\to\infty} \mathbf{E}\,g(\xi_n) = \mathbf{E}\,g(\xi)\,.$$

Theorem A.4. *Let $(\xi_j)_{j\geq 1}$ be i.i.d. random variables with $\mathbf{E}\xi_1 = 0$ and $\mathbf{E}\xi_1^2 = \sigma^2$. Then*

$$\frac{\sum_{j=1}^n \xi_j}{\sqrt{n}} \Longrightarrow \xi \quad as \quad n \to \infty\,,$$

where ξ is a Gaussian random variable with the parameters $(0, \sigma^2)$.

A.5 Iterated logarithm law

Theorem A.5. *Let* $(\xi_j)_{j\geq 1}$ *be i.i.d. random variables with* $\mathbf{E}\xi_1 = 0$ *and* $\mathbf{E}\xi_1^2 = \sigma^2$. *Then*

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \xi_j}{\sqrt{n \ln(\ln n)}} = \sigma\sqrt{2} \qquad a.s.$$

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