TERM STRUCTURE ESTIMATION, LIQUIDITY-INDUCED HETEROSKEDASTICITY AND THE PRICE OF LIQUIDITY RISK

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Abstract

Since the seminal paper of Vasicek and Fong (1982), the term structures of interest rates have been fitted assuming that yields are cross-sectionally homoskedastic. We show that this assumption does not hold when there are differences in liquidity, even for bonds of the same issuer. Lower turnover implies higher volatility. In addition, a minimum tick size for bond price negotiation will produce higher volatility for bonds approaching their maturity dates. To show these effects, we use data for Spanish sovereign bonds from 1988 to 2010, covering more than 700 bonds and 5000 trading days. We estimate the out-of-sample error for each bond and day. The variance of these errors is found to be negatively correlated with each bond's turnover and duration, while the mean of the errors is found to be directly correlated with the estimated variance. As a result, we propose a modified Svensson (1994) yield curve model to fit the term structure, adding a liquidity term and estimating parameters by weighted least-squared errors to take into account the liquidity-induced heteroskedasticity.

Resumen

Desde la publicación del trabajo de Vasicek y Fong (1982) se ha generalizado el ajuste de la estructura temporal de tipos de interés asumiendo que los rendimientos son homocedásticos. En este trabajo se muestra que dicha hipótesis no se mantiene cuando los activos presentan diferencias en liquidez, incluso cuando se trata de bonos del mismo emisor. Un bajo volumen negociado implica una mayor volatilidad. Además, la existencia de discontinuidades en la negociación del precio de los bonos producirá mayor volatilidad en aquellos que se encuentren más próximos a vencimiento. Para mostrar estos efectos, se ha trabajado con bonos del Estado español desde 1988 hasta 2010, considerando más de 700 títulos y 5.000 días observados. Con estos datos se han estimado los errores fuera de la muestra para cada día y título observados. La varianza de estos errores está negativamente correlacionada con la rotación y la duración de cada título, mientras que la media del error está directamente correlacionada con la varianza estimada. Por todo ello, en este trabajo se propone un modelo modificado de Svensson (1994) para estimar la curva de tipos de interés añadiendo un término por liquidez y estimando los parámetros por mínimos cuadrados ponderados, teniendo en cuenta la heterocedasticidad inducida por liquidez.

Palabras clave: heterocedasticidad, primas de liquidez, ajuste de la curva de tipos de interés, bonos del Estado español.

Códigos JEL: G12, C58, E43.

1   Introduction

Since the paper of Vasicek and Fong (1982), the term structure of interest rates has been estimated assuming that bond yields are homoskedastic, regardless of the methodology used (e.g. Nelson and Siegel, 1987; Svensson, 1994; Bliss, 1997; Jordan and Mansi, 2003). A vast majority of central banks use this assumption in their yield curve estimations (Bank for International Settlements, 2005). Nevertheless, the evidence presented in this paper shows that Vasicek and Fong's (1982) assumption of homoskedastic yield errors no longer holds when the bonds considered have different levels of liquidity.

Since differences in the level of liquidity would cause wider movements for less liquid bonds, on both the upside and the downside, liquidity considerations should imply differences in the variance of yields, even for bonds of the same issuer. The main direct implication of this heteroskedasticity is that parameter estimation of the yield curve would be inefficient.

There is little disagreement in the financial literature that liquidity is the second most important risk factor after credit risk in bond yields. However, it is one of the least understood areas of finance. Amihud and Mendelson (1986) stated that the first consequence of liquidity is the higher or lower return required by investors. These differences are known as liquidity premia and are deviations from the yields of different assets to compensate for differences in liquidity. These liquidity premia have been frequently examined in the literature (e.g. Amihud and Mendelson, 1986; Elton and Green, 1998; Alonso *et al*., 2004; Diaz *et al*., 2006).

In the recent financial crisis, where spreads between sovereign bonds (SBs) issued by different countries were used as a measure of credit risk differences, preferences for more liquid bonds may have been distorting the interpretation of these spreads, imposing an upward bias.[1] Liquidity considerations may have also distorted the differences between nominal bonds and inflation-linked bonds due to inflation compensation, since the higher liquidity of nominal bonds produces a downward bias in supposed market inflation expectations. Lately, the proposal for Eurobonds[2] has been the subject of intense political debate due to the threat of default on Greek SBs. The argument in favor of these assets points to the liquidity improvement over individual SBs, which could reduce the cost of issuance for all SBs (Delpla and von Weizsäcker, 2010). Therefore, while liquidity spreads have received increasing attention in the literature, the effect of liquidity factors on yield variance has been sidelined.

Alonso *et al*. (2004) define bond liquidity as the ease of the bond's conversion into money, whereas Díaz *et al*. (2006) define it as the ease with which a security can be traded within a short period without a significant impact on prices. The main consequence of the lack of liquidity is that, in the case of a trader willing to either buy or sell a given asset, the direction of the trade will have a sizable effect on the price, this movement being upward in the case of a buy order and downward for a sell order. Therefore, price changes will be higher for illiquid assets, implying differences in the associated variance of the bonds due to their different degrees of liquidity.

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1 Bond spreads increase during recessions due to phenomena called flight-to-liquidity and flight-to-quality (Goyenko *et al.*, 2011).
2 Defined as "pooled" sovereign debt instruments of the Member States.

Another issue raised in the literature is the way liquidity is measured. Some studies make inferences about bond liquidity or about the valuation implications of bond liquidity using proxies for liquidity such as security age (Sarig and Warga, 1989), security type (Amihud and Mendelson, 1991; Kamara, 1994), on-the-run/off-the-run status (Warga, 1992), trading volume (Elton and Green, 1998), and term to maturity (Shen and Starr, 1998).

For SBs, Alonso *et al*. (2004) stated that liquidity should be closely linked to the market makers' inventory risk and order-processing costs, which ultimately depend on the level of risk of the asset (duration) and the frequency with which a transaction will be executed (turnover). *On-the-run* issues are those more recently auctioned and tend to be more liquid than previously issued bonds (*off-the-run*) maturing on similar dates (Pasquariello and Vega, 2009). Nevertheless, even among these bonds, there can be differences in liquidity and, therefore, heteroskedasticity among yields.

By contrast, Amihud and Mendelson (1991) proposed a model for short-term interest rates where yields had a variance that was conditional on the time to maturity. Although this approach deals with the duration component of liquidity, it does not consider the trading volume factor. Elton and Green (1998) proposed a term structure model estimated by minimizing the mean root squared error, and then using the errors in a liquidity model where the parameters are determined using heteroskedasticity and autocorrelation-consistent estimation of the variance (HAC estimator). Díaz *et al*. (2011) considered the heteroskedasticity of the interest rates but used a time series approach rather than a cross-sectional liquidity-related approach.

Elton and Green (1998) proposed the trading volume as the main variable producing liquidity differences among bonds of the same issuers. However, since this variable is not always available, some authors have used proxies for it. The most common proxy is the classification of bonds as on-the-run (the most recently auctioned issue), off-the-run (the next most recently auctioned issue), and off-off-the-run (older bonds). That is the case for Alonso *et al*. (2004) and Diaz *et al*. (2006), both of whom also include a pre-benchmark category that includes the first days of a new issue, when there is still insufficient trading volume.

Alonso *et al*. (2004) have verified the existence of liquidity premia in negotiated prices in the Spanish SB market. The methodology used is that of Elton and Green (1998), based on the estimation of the term structure of interest rates. These authors incorporate the effect of liquidity in the estimation, introducing dummy variables for the different categories (e.g. *on-the-run/off-the-run)* of the bonds. The instantaneous forward rates are defined according to Svensson's (1994) method.

Díaz *et al*. (2006) analyze the liquidity structure of the Spanish Treasury bond market using trading volume, market share and auction status as proxies for liquidity to determine if the entry of Spain into Stage Three of the Economic and Monetary Union (EMU) has had repercussions on liquidity. The authors also analyze the impact of EMU on the volatility of yields in the Spanish Treasury market,[3] finding a dramatic decline since the market began pricing in EMU.

The main goal of this paper is to present a model capable of explaining the role played by liquidity in the departure of SB yields from a theoretical liquidity-free interest rate term

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3 Díaz *et al*. (2006) used Nelson and Siegel's (1987) exponential model to fit the daily term structures. They did not incorporate any specific liquidity effects.

structure. To do so, we propose a heteroskedastic model for the yields, where the variance is a function of the trading volume and the duration, allowing for consistent estimators of the yield curve, in the sense of White (1980). Additionally, following both Elton and Green (1998) and Alonso *et al*. (2004), we add a liquidity term to the estimation of Svensson's (1994) term structure model. However, we depart from both papers insofar as we use the heteroskedasticity variable estimated for the variance equation for the term structure instead of the trading volume, as in Elton and Green (1998), or the *on-the-run/off-the-run* status of bonds, as in Alonso et al. (2004). We thus use a model similar to a GARCH-in-mean model.

The rest of the paper is organized as follows. In Section 2 we introduce the problem of efficiency in term structure estimation. Section 3 describes the data. In Section 4 we show that liquidity factors produce heteroskedasticity in SB yields. In Section 5 we modify Svensson's (1994) model to incorporate liquidity when estimating the term structure, in both the mean and variance equations, and estimate it for Spanish SBs. In Section 6 we summarize the main conclusions of the paper.

2   Term structure estimation

The price (P_i) of a coupon-bearing SBs i, as is typical of long-term SBs, is equal to

$$P_i = \sum_{j=1}^{m} C_{ij} \cdot e^{-s(t_j|\Phi)\cdot t_j} + N_i \cdot e^{-s(t_m|\Phi)\cdot t_m} \tag{1}$$

where C_{ij} are the coupons paid in t_j ($j=1,\dots,m$), and N_i is the nominal paid at the expiration term (m). Each payment is discounted using an interest rate (s) that is a function of time (t) and a set of parameters (Φ). This function $s(t|\Phi)$ is also known as the estimated term structure. Alternatively, we could describe the bond pricing equation using yield to maturity (yield, y_i), defined as the constant interest rate that solves equation (1):

$$P_i = \sum_{j=1}^{n} C_{ij} \cdot e^{-y_i \cdot t_j} + N \cdot e^{-y_i \cdot t_m} \tag{2}$$

For the estimation of the parameter set Φ in the function $s(t|\Phi)$, we have two options:

1. Minimize the error between the observed price (P_i) and that derived from the estimated term structure $\hat{s}(t|\Phi)$, ($\hat{P}_i(\Phi)$),

$$P_i = \hat{P}_i(\Phi) + \varepsilon_i^p \tag{3}$$

2. Minimize the error between observed yields (i.e., those obtained in equation (2) using the observed price) and those derived from the estimated term structure ($\hat{y}_i(\Phi)$),

$$y_i = \hat{y}_i(\Phi) + \varepsilon_i^y \tag{4}$$

Vasicek and Fong (1982) assumed that the error term in equation (4) is homoskedastic ($E\left[(y_i - \hat{y}_i(\Phi))^2\right] = \sigma^2$). Under that assumption, price errors (ε_i^p) would have a variance given by

$$E\left[(P_i - \hat{P}_i(\Phi))^2\right] = \sigma^2 \cdot \left(\frac{dP_i}{dy}\right)^2 \tag{5}$$

Expression (5) is easy to interpret because the derivative of the bond price with respect to yield is equal to the bond modified duration (D_i) when price is normalized to one. Therefore, the term structure can be estimated by minimizing either the sum of squared errors in yields,

$$\widehat{\Phi} = \min_{\Phi} \sum_i (y_i - \hat{y}_i(\Phi))^2 \tag{6}$$

or the sum of squared weighted (by the inverse of duration) errors in prices,

$$\widehat{\Phi} = \min_{\Phi} \sum_i \left(P_i - \hat{P}_i(\Phi)\right)^2 \frac{1}{D_i^2} \tag{7}$$

to correct for heteroskedasticity in prices. Both solutions are considered equivalent. For instance, Gurkaynak *et al*. (2007) used the inverse of the squared duration to weight the errors in price in their estimation of the US Treasury yield curve. In addition, the Bank for International Settlements (2005) reported that, regardless of the type of term structure estimated, five out of eleven central banks estimate term structure by minimizing error in yields (Germany, Norway, Sweden, Switzerland, and the UK), whereas the other six central banks use the weighted errors in prices to estimate term structure (Belgium, Canada, Finland, France, Italy, Spain, and the US). Nevertheless, if Vasicek and Fong's (1982) assumption is rejected and the errors in yield are heteroskedastic, all those estimation methods would be inefficient.

3   Data

To check the validity of the assumption of the homoskedasticity of the errors in yields (ε_i^y), we estimate the term structure of interest rates for a long set of data. We need to use bonds from the same issuer with similar levels of credit risk to avoid distortions in the curve that are not related to liquidity: SBs are ideal for this purpose, since the number of securities traded at the same time is high enough to estimate the parameters in a term structure model. Therefore, we use Spanish SBs. These securities are classified, depending on their maturity, as *Letras del Tesoro* (equivalent to US Treasury bills, short-term zero-coupon bonds, with maturities of up to 18 months), *Bonos del Estado* (coupon-bearing bonds equivalent to US Treasury notes, with maturities ranging from three to five years, that earn a fixed coupon every year until maturity), and *Obligaciones del Estado* (similar to *Bonos del Estado* but issued withmaturities of 10, 15, and 30 years). In all cases, the nominal value is 1000 euros.

Spanish SBs are generally sold via auctions, where privileged dealers (market makers) usually buy a big share of these securities to sell later in the secondary market. Since January 1987, the Treasury uses a unique pricing mechanism (called the Spanish auction) that is a hybrid between uniform and discriminatory auctions (Abbink et al.,2006). The Spanish Treasury performs several auctions every month. Long-term securities (*Bonos* and *Obligaciones*) are auctioned by tranche, whereby the issue of a security is kept open over several consecutive auctions (three at least) to improve its liquidity. The securities allocated at such auctions are fungible because they share the same nominal coupons, interest payments, and redemption dates. When the total nominal amount issued reaches a sufficiently high outstanding volume, the corresponding security issuance is closed and a new one opened. In the case of *Letras*, liquidity is enhanced by a mechanism such that once a 12-month bill is close to the six-month term, it is auctioned again as a six-month bill, and again when it is close to the three-month term to redemption when it is auctioned as a three-month bill. Despite their different denominations, hereafter we refer to all *Letras, Bonos,* and *Obligaciones* as SBs.

The database is provided by Banco de España's public website and contains all recorded transactions in the secondary market, called the Spanish Public Debt Market.[4] As explained by Diaz *et al*. (2006), this database reports the numbers of transactions and the nominal and effective volumes for each issue, as well as the maximum, minimum, and average prices on a daily base. The database provides data from 1988 until 2010, for a total of almost 5000 trading days, 700 issues, and 121,758 observations. From a separate database from the same website, we gather information for each issue (coupons and dates of first issuance and redemption) for the pricing function. In other databases, such as Reuters or Bloomberg, dealers report their bid and ask offers for each bond. Nevertheless, these databases have no information on actual transactions (these must be reported to the Spanish Public Debt Market, and gathered by Banco de España's database) and the offers reported are not binding, so the information has less quality.

As Gurkaynak et al. (2007), from these data we produce daily estimations of the yield curve using Svensson's (1994) model, implementing the genetic algorithm (GA) proposed by Gimeno and Nave (2009), to ensure the stability of the nonlinear optimization. Svensson's (1994)

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4 Since 1988 Banco de España has been building a database of closing prices:
http://www.bde.es/webbde/es/secciones/informes/banota/series.html.

model (NSS) modifies the original work of Nelson and Siegel (1987) by using a second term—the one that Nelson and Siegel (1987) abandoned in their work—and adding two additional parameters (φ_5 and φ_6). The equation for the instantaneous *forward* rate is

$$f(m) = \varphi_1 + \varphi_2 \cdot e^{-\frac{m}{\varphi_5}} + \varphi_3 \cdot \frac{m}{\varphi_5} e^{-\frac{m}{\varphi_5}} + \varphi_4 \cdot \frac{m}{\varphi_6} e^{-\frac{m}{\varphi_6}} \tag{8}$$

Equation (8) generates a complete family of *forward* curves that reflects a great variety of term structure shapes (Gurkaynak *et al*., 2007; Gimeno and Nave, 2009). Integrating equation (8) between [0, *m*] and dividing by *m*, results in an equation that relates spot interest rates to time to maturity:

$$s(m) = \phi_1 + \phi_2 \frac{1 - e^{-m/\phi_5}}{\left(m/\phi_5\right)} + \phi_3 \left[\frac{1 - e^{-m/\phi_5}}{\left(m/\phi_5\right)} - e^{-m/\phi_5} \right] + \phi_4 \left[\frac{1 - e^{-m/\phi_6}}{\left(m/\phi_6\right)} - e^{-m/\phi_6} \right] \tag{9}$$

We chose this model based on its appearance in a considerable number of studies, the great number of central banks that use it (Bank for International Settlements, 2005), including Banco de España, and the model's performance for the Spanish government debt market compared to that of other parametric models (Núñez, 1995).

Following Bliss (1997), for each of the 5000 days in the sample, we compute the *out-of-sample* errors in the yields of each bond traded. Since these are *out-of-sample* errors (each yield is obtained from a term structure estimated with the rest of the bonds traded that day), we have to estimate 121,758 term structures (one for each excluded bond and trading day). Term structure estimations using coupon-bearing bonds are extremely non-linear (Gimeno and Nave, 2009), so for each of the term structures we run 30 GAs to ensure that the estimated parameters do not correspond to a local minimum, raising the total number of term structure estimations above 3.5 million GAs. For these estimations, we use as a target function the minimum squared weighted errors in the prices of equation (7), equivalent to the minimum squared errors in yields but faster to compute.

4 Liquidity Analysis

We have estimated the deviations (ε_{it}^{y}) of observed yields from those implied by the estimated term structure of interest rates: the difference between the quoted yield of a bond (*l*) and its yield implied by the *out-of-sample* NSS term structure model for a given day (*t*). If these deviations are affected by liquidity considerations, they should be strongly influenced by two factors, one related to the ease of closing trades and the other to market microstructure. The first factor is the turnover or trading volume (T_{it}). When a bond is rarely traded, opposite offers are difficult to match and a willing seller (buyer) would have to accept a lower (higher) price to fulfill the transaction. Warga (1992) and Alonso et al. (2004), among others, classify securities as *on-the-run/off-the-run* as a proxy for trading volume when turnover data are unavailable. Nevertheless, Elton and Green (1998) signaled that trading volume was a more robust measure of asset liquidity than these other proxies. In our sample, as can be seen in Figure 1, error dispersion is higher for those days when the bond's trading volume is lower. Figure 1 shows the funnel shape characteristic of heteroskedasticity.

Another alternative is the bid–ask spread. However, this measure has two main drawbacks: First, there is no single bid–ask spread for each trading day, since it changes during the day. Although we could compromise with the spread at closing time, the second and more relevant issue is the quality on the bid–ask data. As mentioned in the previous section, information on bid–ask spreads includes non-binding quotes, which can distort the real bid–ask spread. Other problem is the depth of both bid and asks: how many bonds are available to buy or sell at those prices. Sometimes, it may be possible to discern binding from non-binding quotes but, given the scope of our study, we are unable to recover them for each bond and each day.

The tick size is a market microstructure characteristic that produces a second relevant factor from a liquidity point of view. Although all bond prices have the same tick size, changes in price have completely different effects in terms of yield. To illustrate this effect, suppose we have a zero-coupon bond where prices change at a minimum Δ:

$$P = e^{-r \cdot m}$$
$$P + \Delta = e^{-r_\Delta \cdot m}$$

Therefore, a change in prices implies a change in yield equal to

$$r_\Delta - r = \frac{1}{m} \cdot \ln \frac{P+\Delta}{P} \tag{10}$$

Although the logarithm of the price change will be similar for all bonds, changes in yields are inversely related with the bond's term to maturity (*m*). In the case of coupon-bearing bonds, equation (10) is much more complex, but the effect can be approximated by the time to maturity (D_{it}), so those close to maturity will experience, for the same price change, higher return swings than the rest. In this sense, Amihud and Mendelson (1991) found evidence of a liquidity premium that was a decreasing convex function of time to maturity.

Even when prices are allowed to change in a continuum rather than as discrete values, the effect will still be present. Dealers will only trade on a bond if the expected profit compensates them from the trading costs. This profit will be related with the difference between the purchase and sale prices. So, when a bond is close to maturity, dealers will

accept higher deviations from their bond valuation before it compensates them for the cost of taking a position on that bond.

Figure 2 shows the funnel pattern when comparing errors in yield with bond duration. Although this is a further signal of yield heteroskedasticity, this outcome may not be due to a second factor but, rather, due to the differences in turnover previously mentioned: Bonds close to maturity also tend to be the least traded. We need a multivariate model to consider both factors jointly.

As stated above, liquidity constraints would produce wider movements for less liquid bonds, in both the upside and the downside. Therefore, liquidity considerations should produce differences in the variance of ε_{it}^{y} (heteroskedasticity). In this sense, Amihud and Mendelson (1991) proposed a model where yields had a variance conditional on time to maturity. The unobserved variance variable (h_{it}) would depend on both turnover (there would be a negative relation between turnover and variance) and duration (there would be a positive relation between the inverse of the duration and the variance). Thus, a heteroskedastic model for the yield errors (ε_{it}^{y}) would be equal to that in equation (11) (model 1):

$$\varepsilon_{it}^{y} = \beta_0 + \sqrt{h_{it}} \cdot u_{it} \qquad\qquad u_{it} \sim N[0,1]$$

$$\log h_{it} = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \qquad\qquad (11)$$

In the variance equation, we would expect $\gamma_1 < 0$ and $\gamma_2 > 0$ if both the turnover and tick size explain the variance of ε_{it}^{y}. In Table 1, model 1 represents the estimations of the parameters of equation (11). As can be seen in Table 1, both the turnover and duration coefficients have the expected signs (negative for turnover and positive for the inverse of the duration). This result confirms that yields are not homoskedastic, a feature we cover in the models of next section.

Although these variables affect primarily the variance of ε_{it}^{y} (h_{it}), this is far from the only effect expected. Differences in the variance of the yields of different bonds imply different levels of risk for investors. Therefore, we would expect investors to ask for higher returns in the case of bonds susceptible to higher liquidity volatilities. Thus, the level of variance (h_{it}) would also affect the yield level (the level equation). This second model (model 2) is

$$\varepsilon_{it}^{y} = \beta_0 + \beta_1 \cdot \log h_{it} + \sqrt{h_{it}} \cdot u_{it} \qquad\qquad u_{it} \sim N[0,1]$$

$$\log h_{it} = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \qquad\qquad (12)$$

The variance equation remains invariant with respect to the specification of equation (11), where we expect $\gamma_1 < 0$ and $\gamma_2 > 0$ if both turnover and tick size explain the range of movements in ε_{it}^{y}. Nevertheless, in the level equation, the heteroskedastic behavior of ε_{it}^{y} would be derived from the first term ($\sqrt{h_{it}} \cdot u_{it}$), while these differences in volatility would be compensated for by a higher liquidity premium ($\beta_1 \cdot \log h_{it}$). Therefore, we can call the parameter β_1 the price of liquidity risk. We would expect $\beta_1 > 0$, implying that investors demand a premium for the risk they assume.[5]

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[5] Although it is reasonable to suppose that the price of risk change in time (increasing with the crisis and decreasing in normal times) for simplicity we have suppose in this section that the price of risk is constant. In next section, we present a model where the price of risk is estimated in a daily base.

The second model of Table 1 presents the parameter estimations of model (12). As can be seen in Table 1, the parameters estimated in the variance equation are similar to those obtained in model 1 and both the turnover and duration coefficients have the expected signs (negative for turnover and positive for the duration). The main difference is in the case of the level equation, where we also find the expected positive (and significant) price of risk.

5   A proposal for a modified Svensson model

In the previous section we show that Vasicek and Fong's (1982) assumption of homoskedastic yield errors no longer holds when the bonds have different levels of liquidity. Under these circumstances, the usual optimization functions (equations (6) and (7)) exacerbate the observed heteroskedasticity instead of correcting it. As a result, term structure estimations (such as those obtained in Section 3) are not efficient, tending to use the wrong weighting, since they rely too much on duration and avoid others sources of uncertainty, such as turnover.

In fact, the variance for the price error would be better described by equation (13), where the constant yield variance (σ^2) has been replaced by the heteroskedastic version (h_i) proposed in equations (11) and (12):

$$E\left[\left(P_i - \hat{P}_i(\Phi)\right)^2\right] = h_i \cdot \left(\frac{dP_i}{dy}\right)^2 \tag{13}$$

$$\log h_{it} = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \tag{14}$$

In this section we propose several ways to include the liquidity factor in the term structure estimation.

5.1   Liquidity weighting

A straightforward way to estimate the term structure is to jointly estimate the parameters of both the term structure (equation (9)) and the variance equation (equation (14)), using, as the function we want to minimize, the squared errors in prices weighted by the variance of yields times the squared bond duration:

$$\left(\hat{\Phi}, \hat{\gamma}\right) = \min_{\Phi, \gamma} \sum_i \left(P_i - \hat{P}_i(\Phi)\right)^2 \frac{1}{D_i^2 h_i(\gamma)} \tag{15}$$

Although we apply the optimization criteria of equation (15) to estimate Svensson's model, this modification can also be used for any other term structure estimation methodology, whether parametric—for example, Nelson and Siegel's (1987) model—or non-parametric—for example, Vasicek and Fong's (1982) splines.

5.2   Liquidity premium

The previous model specification does not take into account the presence of an eventual liquidity premium; it just takes into account the variability in prices derived from these liquidity considerations. To include this premium in the model, we should vary the pricing equation (equation (1)) to add an additional term:

$$\hat{P}_i = \sum_{j=1}^m C_{ij} \cdot e^{-(s(t_j,\varphi)\cdot t_j + \alpha \cdot h_i)} + N_i \cdot e^{-(s(t_m,\varphi)\cdot t_m + \alpha \cdot h_i)} \tag{16}$$

This modification is equivalent to multiplying the estimated price by $e^{-\alpha \cdot h}$ or to adding a premium ($\alpha h / t$) to the bond yield. The variance equation (equation (15)) would remain valid and the function we want to optimize now is

$$\left(\hat{\Phi}, \hat{\gamma}, \hat{\alpha}\right) = \min_{\Phi, \gamma, \alpha} \sum_i \left(P_i - \hat{P}_i(\Phi, \gamma, \alpha)\right)^2 \frac{1}{D_i^2 h_i(\gamma)} \tag{17}$$

The introduction of the liquidity model proposed in equation (16) is similar to that of Elton and Green (1998), where the logarithm of the trading volume was added for the pricing equation, although the authors did not consider the liquidity-induced heteroskedasticity to modify their weighting.

5.3 Term structure estimation

Tables 2a–2c present term structure estimations for three different days: April 20, 2010 (on the way to first euro area sovereign debt crisis); May 11, 2010 (right in the middle of the first Greek sovereign debt crisis); and July 7, 2010 (in the middle of the market easing after the publication of the first European banks' stress tests). For each day, we compute four different estimations:

1. A traditional Svensson model (equation (9)), minimizing equation (7), that is, traditional weighting error prices using the squared modified bond duration.

2. A Svensson model (equation (9)), but minimizing equation (17). To obtain the estimated variances $(\hat{h}_{it})$, we need to compute the weights, which we carry out in a two-step process. First, we compute the squared differences between the observed and estimated yields (y) from the previous model and then estimate a regression similar to the one we would use in a White heteroskedasticity test,

$$\log(y_{it} - \hat{y}_{it})^2 = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \qquad (18)$$

Finally, we use this modeled variance to estimate the term structure, minimizing equation(17).

3. Instead of two steps, a joint estimation of the mean (Svensson's equation (9)) and variance (equation (14)) estimation (minimizing equation (17)). In this case, we do not need to rely on obtaining the traditional duration-weighted estimators in the first step of the variance equation.

4. A model with a liquidity premium (equation (16))—where we include compensation for liquidity risk (α) in the price equation as well as a variance equation (equation (14))—that we jointly estimate using the optimization approach of equation (17).

As can be seen in Figures 3-5, yield curves can vary greatly, depending on the specification. Furthermore, the liquidity premium can change from day to day: It was higher in the middle of the Greek debt crisis (May 11, 2010) than before or after.

6   Conclusions

In this paper we show how differences in the liquidity of bonds with the same credit risk produce yields that differ from those expected in a theoretical liquidity-free term structure of interest rates. We observe that differences in bonds' liquidity affect their volatility. Bond yields have an expected value equal to that associated with the term structure of interest rates and a variance that depends on liquidity factors that we identify with the bond's trading volume and duration. We find that the variance is negatively related to turnover and positively related to duration. This finding suggests that liquidity differences between bonds of the same issuer can produce heteroskedasticity.

The main direct implication of this heteroskedasticity is for the estimation of the yield curve. Vasicek and Fong (1982) estimated the term structure of interest rates by assuming that bond returns are homoskedastic and they proposed an error correction that consisted in weighting the squared price errors by the inverse of duration. Our findings imply that this hypothesis does not hold for bonds with differences in liquidity levels, even when they are of the same issuer. Therefore, estimations of cross-sectional models of term structure should be corrected to take into account the liquidity factor.

Finally, as a solution to the rejection of Vasicek and Fong's (1982) hypothesis, we propose the use of a new optimization program that takes liquidity factors into account. Additionally, we propose a Svensson model modified by adding a liquidity risk premium, drawing on the findings of this paper.

TABLE 1: Heteroskedastic models for out-of-sample yield errors.

	Model 1		Model 2	
Level Equation				
Intercept	-0.989		-0.240	
$\log(h_{it})$			0.184	***
Variance Equation				
Intercept	3.373		3.290	
$\log(Turnover_{it})$	-0.133	***	-0.129	***
$1/Duration_{it}$	0.139	***	0.142	***
# of observations	121,758		121,758	
# of days	4,996		4,996	
# of bonds	662		662	

In both models 1 and 2, the dependent variable for each ith bond and day is the difference between the quoted yield and that derived from an NSS term structure model estimated each day for all Spanish SBs traded that day, but excluding the ith bond. Turnover is measured in thousands of euros and duration is measured in years. Model 1 corresponds to the model specified in equation (11), whereas model 2 corresponds to the model specified in equation (12).Individual likelihood ratio tests were computed for each parameter (outside the intercepts) under the null hypothesis of a non-significant variable. The superscripts ***, **, and * denote rejection of the null at the 1%, 5%, and 10% levels, respectively.

TABLE 2a: Term structure of Spanish government bonds, April 20, 2010.

	Duration Weighted	Liquidity Weighted	Mean–Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.023	0.023	0.022	0.022
φ_2	-0.020	-0.020	-0.020	-0.020
φ_3	0.062	0.060	0.067	0.067
φ_4	0.069	0.068	0.081	0.081
φ_5	6.983	6.965	6.593	6.776
φ_6	25.991	25.260	38.915	38.915
Variance equation				
γ_0	-	-13.392	12.971	12.971
γ_1	-	-0.191	-0.235	-0.235
γ_2	-	0.667	1.458	1.458
Price of risk				
α	-	-	-	0.000121

The duration-weighted model corresponds to the estimation of model (9) using optimization program (7). The liquidity-weighted model corresponds to the estimation of model (9) using optimization program(15), where liquidity variance was previously estimated from equation (14). The mean–variance joint estimation model is equal to the liquidity-weighted model, but the variance and mean equations (14) and (9), respectively, are estimated simultaneously. The liquidity premium model includes mean equation (16), variance equation (14), and optimization program (17).

TABLE 2b: Term structure of Spanish government bonds, May 11, 2010.

	Duration Weighted	Liquidity Weighted	Mean–Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.050	0.053	0.058	0.058
φ_2	-0.045	-0.053	-0.058	-0.058
φ_3	0.000	0.002	0.036	0.044
φ_4	0.107	0.075	0.039	0.029
φ_5	2.668	9.092	8.881	7.414
φ_6	138.866	6.987	2.245	0.806
Variance equation				
γ_0	-	-14.743	-24.633	-0.003
γ_1	-	-0.010	-0.057	-0.149
γ_2	-	0.918	0.916	1.697
Price of risk				
α	-	-	-	0.000

The duration-weighted model corresponds to the estimation of model (9) using optimization program (7). The liquidity-weighted model corresponds to the estimation of model (9) using optimization program (15), where liquidity variance was previously estimated from equation (14). The mean–variance joint estimation model is equal to the liquidity-weighted model, but the variance and mean equations (14) and (9), respectively, are estimated simultaneously. The liquidity premium model includes mean equation (16), variance equation (14), and optimization program (17).

TABLE 2c: Term structure of Spanish government bonds, July 7, 2010.

	Duration Weighted	Liquidity Weighted	Mean–Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.035	0.036	0.030	0.027
φ_2	-0.032	-0.036	-0.030	-0.027
φ_3	0.000	0.009	0.001	0.000
φ_4	0.082	0.078	0.097	0.104
φ_5	0.612	0.935	0.394	0.300
φ_6	20.371	16.864	16.221	15.140
Variance equation				
γ_0	-	-12.637	-0.111	0.159
γ_1	-	1.106	-0.119	0.830
γ_2	-	22.858	-0.128	0.843
Price of risk				
α	-	-	-	0.000000

The duration-weighted model corresponds to the estimation of model (9) using optimization program (7). The liquidity-weighted model corresponds to the estimation of model (9) using optimization program (15), where liquidity variance was previously estimated from equation (14). The mean–variance joint estimation model is equal to the liquidity-weighted model, but variance and mean equations (14) and (9), respectively, are estimated simultaneously. The liquidity premium model includes mean equation (16), variance equation (14), and optimization program (17).

FIGURE 1. Errors in yields of Spanish SBs versus turnover
Out-of-sample errors were estimated using Svensson's (1994) model, 1988 to 2010.
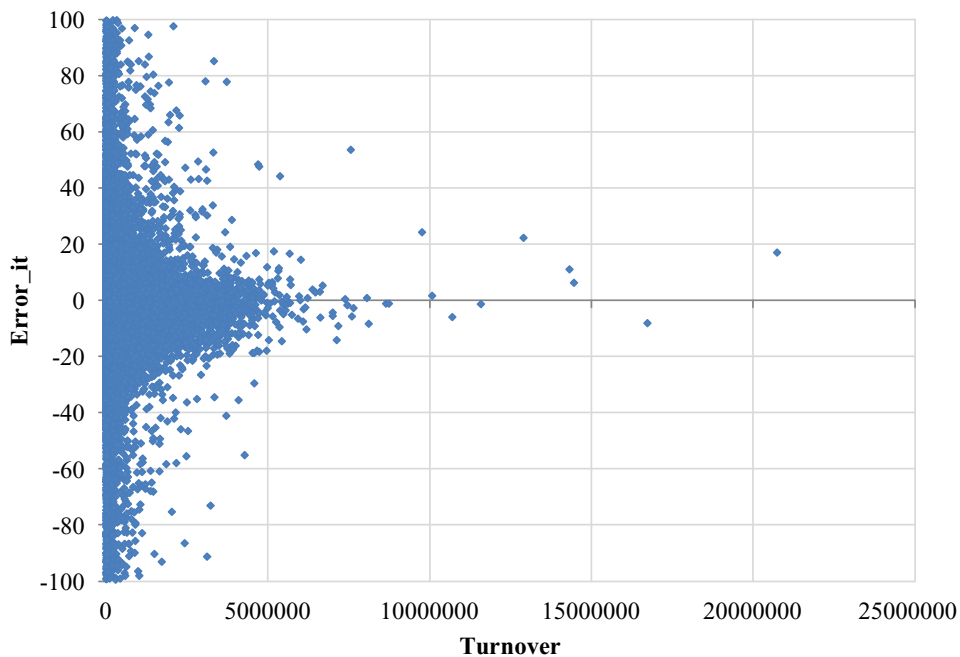
FIGURE 2. Errors in yields of Spanish SBs versus duration
Out-of-sample errors were estimated using Svensson's (1994) model, 1988 to 2010.
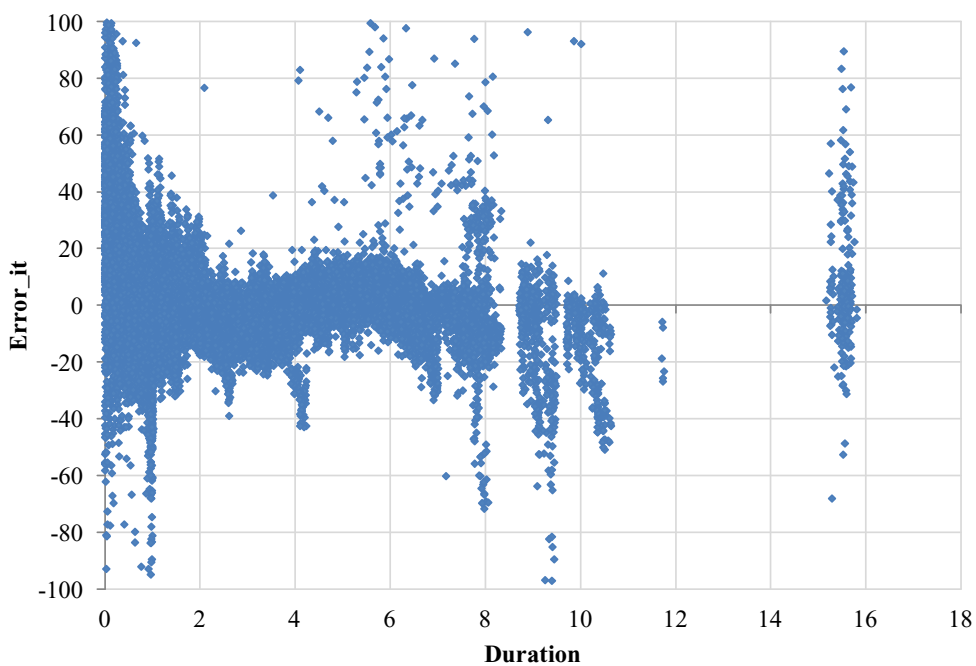
Figure 3. Term structure of Spanish government bonds, April 20, 2010
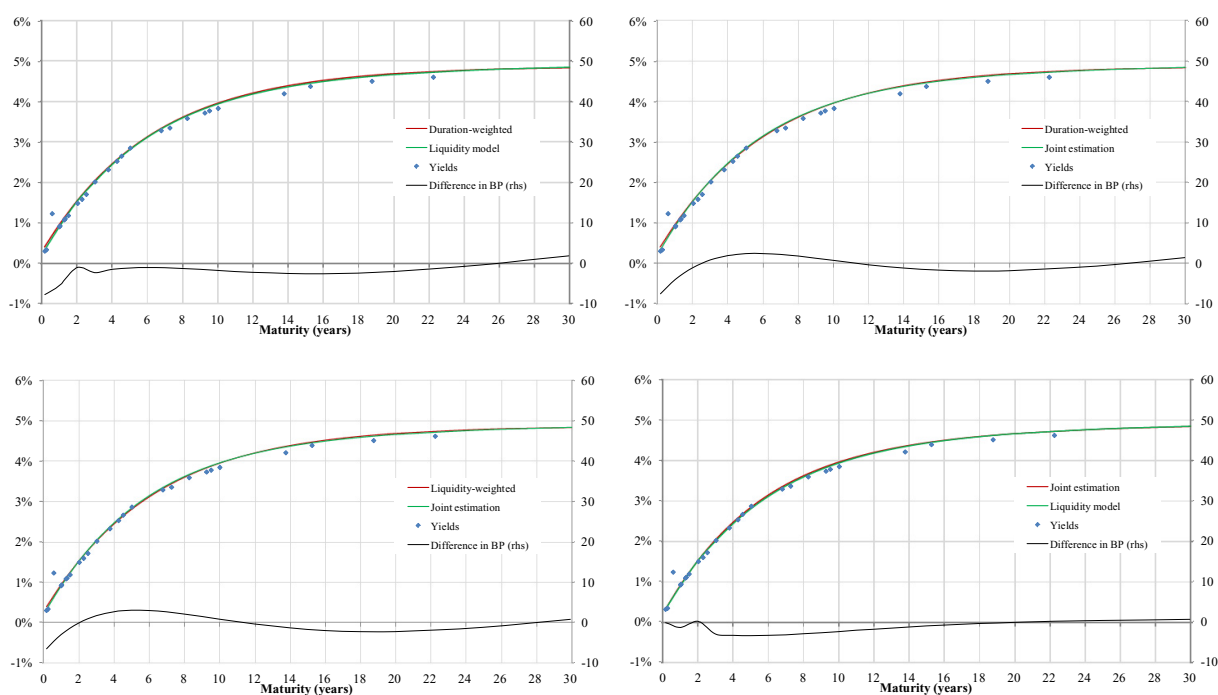
This figure shows the par yield curves, where the dots are yields versus time to maturity. The chart in the upper left represents the traditional estimation curve based on weighting observations by their squared duration (red) and the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond. Differences between both curves are represented in the right-hand axis in basis points (black). The chart in the upper right compares the traditional squared duration weighting (red) and liquidity weighting without liquidity premia (green). The chart in the lower left compares the liquidity weighting estimation in a two-step approach (red) and a joint estimation of level and variance equations (green). The chart in the lower right compares liquidity weighting (red) with the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond.

Figure 4. Term structure of Spanish government bonds, May 11, 2010

This figure shows the par yield curves, where the dots are yields versus time to maturity. The chart in the upper left shows the traditional estimation curve based on weighting observations by their squared duration (red) and the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond. The differences between both curves are shown in the right-hand axis in basis points (black). The chart in the upper right compares the traditional squared duration weighting (red) and liquidity weighting without liquidity premia (green). The chart in the lower left compares the liquidity weighting estimation in a two-step approach (red) and a joint estimation of level and variance equations (green). The chart in the lower right compares liquidity weighting (red) with the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond.

Figure 5. Term structure of Spanish government bonds, July 7, 2010
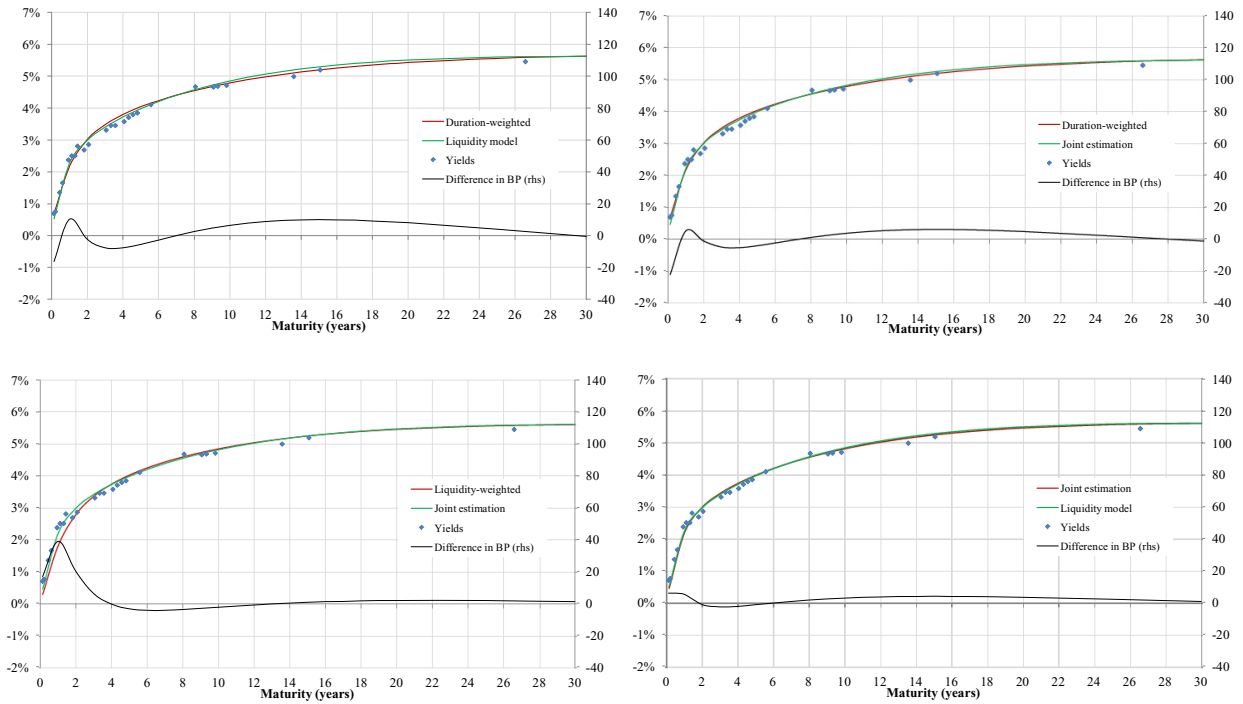
This figures show the par yield curves, where the dots are yields versus time to maturity. The chart in the upper left shows the traditional estimation curve based on weighting observations by their squared duration (red) and the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond. The differences between both curves are shown in the right-hand axis in basis points (black). The chart in the upper right compares the traditional squared duration weighting (red) and liquidity weighting without liquidity premia (green). The chart in the lower left compares the liquidity weighting estimation in a two-step approach (red) and a joint estimation of level and variance equations (green). The chart in the lower right compares liquidity weighting (red) with the modified Svensson model, where the curve is estimated using liquidity weighting and adding a liquidity premium (green), although the plotted curve does not include this premium, which would be different for each bond.

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