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Abstract

We establish the asymptotic normality of marginal sample quantiles for S-mixing vector stationary processes. *S*-mixing is a recently introduced and widely applicable notion of dependence. Results of some Monte Carlo simulations are given.

Keywords: Quantiles, *S*-mixing.

JEL classification: C01.

Resumen

Establecemos la normalidad asintótica de cuantiles muestrales marginales para un proceso vectorial estacionario *S*-mixing, una noción de dependencia recientemente propuesta y que cubre un amplio rango de procesos temporales. También mostramos resultados de una simulación de Monte Carlo.

Palabras clave: cuantiles, S-mixing.

Código JEL: C01

1 Introduction

Let us assume that $\{\boldsymbol{X}_t, t \in \mathbb{Z}\}$ is a strictly stationary and ergodic p–variate process. We denote by $X_t^{(i)}$ the i–th component of $\boldsymbol{X}_t$ with marginal distribution function $F^{(i)}(x) = P(X_t^{(i)} \leq x)$, and τ–quantile $q_\tau^{(i)} = \inf\{x|F^{(i)}(x) \geq \tau\}$. Analogously, let

$$F_n^{(i)}(x) = \frac{1}{n}\sum_{k=1}^{n} I\{X_k^{(i)} \leq x\}$$

be the i–th marginal empirical distribution function and denote the empirical τ–quantile by

$$q_{\tau,n}^{(i)} = \inf\{x|F_n^{(i)}(x) \geq \tau\}.$$

The aim of this note is to investigate the asymptotic behaviour of the marginal sample quantiles for p–dimensional stationary processes $\{\boldsymbol{X}_t\}$ and obtain the asymptotic normality of the empirical quantile vector $(q_{\tau_1,n}^{(1)}, \ldots, q_{\tau_p,n}^{(p)})'$ for given $\tau_1, \ldots, \tau_p \in [0, 1]$.

Asymptotic properties of sample quantiles have been of great interest since Cramér (1946) [5] gave the joint asymptotic normality of sample quantiles coming from an independent and identically distributed ($i.i.d.$) univariate population. Babu and Rao (1988) [1] derived the joint asymptotic normality of marginal sample quantiles coming from an $i.i.d.$ multivariate population. In the case of dependent processes, Sen (1968) [14] extended the results of Bahadur (1966) [2] to m–dependent stochastic processes and showed the asymptotic normality of sample quantiles. Duta and Sen (1971) [8] extended those results to show the asymptotic normality of sample quantiles for multivariate autoregressive processes. Sen (1972) [15] investigated the asymptotic almost sure representation of a sample quantile for a stationary process of ϕ–mixing random variables. He showed that Bahadur's (1966) [2] asymptotic almost sure representation of sample quantiles also holds under the ϕ–mixing condition. One of the most recent papers in this context is Wu (2005) [17] who obtained Bahadur's representation for a class of linear and non-linear (scalar) processes. Oberhofer and Haupt (2005) [11] showed the asymptotic distribution of the unconditional quantile estimator and the joint asymptotic normality of several quantiles by using results on convex stochastic optimization and mixing properties of an indicator process.

In this note, we investigate the asymptotic behaviour of sample quantiles for a vector stationary process. The use of quantiles in statistical inference abound, in particular in the context of heavy tails where moments may not exist. They have been used for the estimation of parameters (Dominicy and Veredas (2012) [7] and Dominicy et al. (2012) [6]). Multivariate dependencies between financial products have often been computed with copulas. Fermanian and Scaillet (2003) [9] use sample marginal quantiles from time series data as a non–parametric method to estimate copulas. The 2007-2010 financial and the 2009-2012 European sovereign debt crises have highlighted the importance of tail –or rare– events. When they occur, their effect is spread over the system, creating tail correlation. Ricci and Veredas (2012) [12] introduce TailCoR, a new measure of tail correlation for financial time series that is based on sample marginal quantiles.

The classical approach to obtain limiting distributions for statistics of weakly dependent processes is to impose mixing conditions (i.e. α–, ϕ–, ψ–, and β–mixing). Those classical mixing conditions are interesting and lead to acute results. However, often they are not only difficult to be verified but require as well strong smoothness of the process, and so their range of applications in time series context is somewhat limited. A drastic example is an AR(1) process with iid Bernoulli innovations (ε_t) (i.e. $\varepsilon_t = \pm 1$ with probability 1/2). When $X_t = \frac{1}{2}X_{t-1} + \varepsilon_t$ the dependence amoung the variables X_t decays obviously very fast but surprisingly this process is not mixing (see Rosenblatt (1985) [13]).

The inadequacy of classical mixing conditions in the time series context has lead to a number of new approaches for dealing with weak dependence in the last couple of years. One of these approaches is the so called S–mixing introduced by Berkes et al. (2009) [3]. S–mixing is attractive since its verification is almost immediate (it is a trivial exercise to verify it for the AR-processes) and it nests a large number of well–known and well–used econometric models such as linear processes (especially ARMA models), GARCH models and its extensions, and stochastic volatility models, among others. A remarkable property of S–mixing is the fact that it doesn't require any higher order moment assumptions to be verified. Since we are interested in quantiles and processes that are probably heavy-tailed, this is of particular interest. Furthermore, using the results of Berkes et al. (2009) [3] we avoid establishing Bahadur's (1966) [2] representation, which is a theoretically interesting result, but not necessary for the asymptotic normality of quantiles. We refer to Berkes et al. (2009) [3] for a number of examples within this framework.

The remaining sections are laid out as follows. In Section 2 we review briefly the S–mixing property and state the joint asymptotic distribution for marginal quantiles. Section 3 covers the Monte Carlo study and Section 4 concludes. The proof of the theorem, other technical results, and figures are relegated to the Appendix.

2 Setup and main result

Throughout this note, we assume that the p–dimensional process $\{\boldsymbol{X}_t, t \in \mathbb{Z}\}$ has the following form

$$\boldsymbol{X}_t = f(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t-1}, \ldots), \tag{1}$$

where $\{\boldsymbol{\varepsilon}_t\}$ is an *i.i.d.* sequence taking values in a measurable space $\mathbb{S}$ and $f : \mathbb{S}^\infty \to \mathbb{R}^p$ is a measurable function. Representation (1) is quite natural in time series and it implies that the process $\{\boldsymbol{X}_t\}$ is strictly stationary and ergodic. As the aim is to show the asymptotic normality of the empirical quantile vector

$$(q^{(1)}_{\tau_1,n}, \ldots, q^{(p)}_{\tau_p,n})'$$

for given $\tau_1, \ldots, \tau_p \in [0, 1]$, we need to impose stronger conditions on the structure of $\{\boldsymbol{X}_t\}$. We use the S–mixing condition introduced in Berkes et al. (2009) [3].[1]

Definition 1 A random process $\{\boldsymbol{X}_t, t \in \mathbb{Z}\}$ is called *S–mixing* if it satisfies the following two conditions.

i) For any $t \in \mathbb{Z}$ and $m \in \mathbb{N}$, there exists some approximating random vectors $\boldsymbol{X}_{tm}$ such that $P(|\boldsymbol{X}_t - \boldsymbol{X}_{tm}| \geq \gamma_m) \leq \delta_m$, for some numerical sequences $\gamma_m \to 0$ and $\delta_m \to 0$.

ii) For any disjoint intervals $I_1, \ldots, I_r$ of integers and any positive integers $m_1, \ldots, m_r$, the vectors $\{\boldsymbol{X}_{tm_1}, t \in I_1\}, \ldots, \{\boldsymbol{X}_{tm_r}, t \in I_r\}$ are independent provided the separation between I_k and I_l is greater than $m_k + m_l$.

[1] Originally introduced for scalar processes, this notion of mixing is straightforwardly generalized to vector–valued series.

S–mixing requires approximating random vectors $\boldsymbol{X}_{tm}$ that can be constructed in various ways. Among them, we will use a coupling method. Let $\{\boldsymbol{\varepsilon}_t^{(k)},\ t \in \mathbb{Z},\ k \in \mathbb{Z}\}$ be an *i.i.d.* array of random elements all having the same law as $\boldsymbol{\varepsilon}_0$. Now we set the approximating random vectors as

$$\{\boldsymbol{X}_{tm},\ t \in \mathbb{Z}\} = \{f(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t-1}, \ldots, \boldsymbol{\varepsilon}_{t-m}, \boldsymbol{\varepsilon}_{t-m-1}^{(t)}, \ldots),\ t \in \mathbb{Z}\}. \tag{2}$$

This way of construction leads $\boldsymbol{X}_{tm}$'s to have same marginal distributions as those of $\boldsymbol{X}_t$'s, while being now m–dependent. If the process is weakly dependent, then with increasing m the approximations $\boldsymbol{X}_{tm}$ are expected to converge (in some sense to be specified) to $\boldsymbol{X}_t$.

Before stating the theorem, we need two assumptions. The first relates to the marginal distribution functions $F^{(i)}$, while the second relates to the dependence structure of $\boldsymbol{X}_t$.

Assumption 1 For any $i = 1, \ldots, p$, the distribution $F^{(i)}(x)$ has a density $f^{(i)}(x)$ that is positive and continuous in a neighbourhood of $q_{\tau_i}^{(i)}$ and $f^{(i)}(x)$ is uniformly bounded by some constant B.

Assumption 2 The process $\{\boldsymbol{X}_t, t \in \mathbb{Z}\}$ has representation (1) and with approximations (2) it is S–mixing with coefficients $\gamma_m = \delta_m = O(m^{-A})$, $A > 4$.

Equipped with the definition of S–mixing and the assumptions, we present the theorem.

Theorem 1 *Let $\{\boldsymbol{X}_t\}$ be a stationary process satisfying Assumptions 1 and 2. Let us define* $\boldsymbol{V} = \mathrm{diag}\left(f^{(1)}(q_{\tau_1}^{(1)}), \ldots, f^{(p)}(q_{\tau_p}^{(p)})\right)$ *and* $\boldsymbol{Q} = \sum_{h \in \mathbb{Z}} E\boldsymbol{T}_0\boldsymbol{T}_h'$ *with*

$$\boldsymbol{T}_k = \left(I\{X_k^{(1)} \le q_{\tau_1}^{(1)}\} - \tau_1, \ldots, I\{X_k^{(p)} \le q_{\tau_p}^{(p)}\} - \tau_p\right)'.$$

Then

$$\sqrt{n}\left(q_{\tau_1,n}^{(1)} - q_{\tau_1}^{(1)}, \ldots, q_{\tau_p,n}^{(p)} - q_{\tau_p}^{(p)}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Psi}),$$

where $\boldsymbol{\Psi} = \boldsymbol{V}^{-1}\boldsymbol{Q}\boldsymbol{V}^{-1}$. The element in the i–th row and j–th column of the matrix $\boldsymbol{\Psi}$ is given by

$$\psi_{ij} = \frac{\sum_{h \in \mathbb{Z}}\left(P\big(\{X_0^{(i)} \le q_{\tau_i}^{(i)}\} \cap \{X_h^{(j)} \le q_{\tau_j}^{(j)}\}\big) - \tau_i\tau_j\right)}{f^{(i)}(q_{\tau_i}^{(i)})f^{(j)}(q_{\tau_j}^{(j)})}.$$

3 Monte Carlo study

We carry out a Monte Carlo experiment to assess the finite sample performance of the sample quantiles in a bivariate setup. Extensions of this experiment to higher dimensions are straightforward but they do not add value added since we are interested in marginal quantiles. We generate 500 draws of 100 (small sample) and 1000 (large sample) observations from two bivariate dynamic models, one for the location (a VAR(1)) and another for the scale (a CCC-GARCH(1,1)).[2] In both settings we assume that the vector of innovations $\boldsymbol{\epsilon}_t$ follows a multivariate Student–t distribution with zero location, identity dispersion matrix,

[2]The bivariate VAR(1) model is defined as $\boldsymbol{x}_t = \boldsymbol{c} + \boldsymbol{A}\boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}_t$, where $\boldsymbol{c}$ is a 2×1 vector of intercepts, $\boldsymbol{A}$ is a 2×2 matrix and $\boldsymbol{\epsilon}_t$ is a 2×1 vector of innovations.

and the degrees of freedom being either $\nu = 10$ or $\nu = 20$. Two remarks to these choices. First, $\nu = 20$ produce moderate tails in the distribution of the innovations, while those for $\nu = 10$ are fairly heavy. Second, the unconditional distribution of $\boldsymbol{x}_t$ has heavier tails in the $CCC - GARCH(1,1)$ model than those of the innovations, and hence for $\nu = 10$ the tail of the observations are heavier than those of the innovations.[3]

For the VAR(1) model, we set the vector of constants $\boldsymbol{c}$ equal to zero and $\boldsymbol{A}$ to

$$\boldsymbol{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}.$$

As for the CCC–GARCH(1,1), the correlation is 0.5 and the parameters for the conditional dispersions are given by $\boldsymbol{\theta} = [\omega_1, \alpha_1, \beta_1, \omega_2, \alpha_2, \beta_2] = [0.05, 0.05, 0.7, 0.05, 0.05, 0.7]$. Thus, we consider 8 scenarios and for each we estimate the quantiles for $\tau = \{0.01, 0.10, 0.25, 0.5\}$. Results are displayed in the form of Q–Q plots (sample distribution of the 500 estimated quantiles against the Gaussian distribution) in Figures 1–4 in the Appendix. Each figure has two panels, for 100 and 1000 observations respectively. The first two figures are for the VAR(1) model, while the other two are for the CCC–GARCH(1,1).[4]

Results are in line with the intuition. For the VAR, the sample distribution of the estimated quantiles is well approximated by the asymptotic distribution of 0.25 and 0.50 quantiles for any tail thickness and sample size. The sample distribution of extreme quantiles show slight departures from Gaussianity for $n = 100$, in particular for $\nu = 10$, but it improves substantially for $n = 1000$. Similar conclusions are drawn for the CCC–GARCH(1,1), except that, due to the volatility clustering, more observations are needed in the case of $\nu = 10$ for the sample distribution to be closely represented by the asymptotic counterpart.

4 Conclusions

In this note we show the asymptotic normality of marginal sample quantiles for vector stationary processes under the S–mixing condition. The results obtained via Monte Carlo simulations confirm the theoretical result.

The bivariate CCC-GARCH(1,1) model is defined as $\boldsymbol{x}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\epsilon}_t$ where $\boldsymbol{\epsilon}_t$ is an *i.i.d.* vector standardized error process, and $\boldsymbol{H}_t = [h_{ij,t}]$ is the 2×2 conditional dispersion matrix of $\boldsymbol{x}_t$, expressed as $\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{P} \boldsymbol{D}_t$ where $\boldsymbol{D}_t = diag(\boldsymbol{H}_t^{1/2})$ and $\boldsymbol{P} = [\rho_{ij}]$ is positive definite with $\rho_{ii} = 1$ for $i = 1, 2$. The diagonal elements follow a GARCH(1,1) model $h_{ii,t} = \omega_i + \alpha_i x_{ii,t-1}^2 + \beta_i h_{ii,t-1}$, and the off–diagonal are given by $h_{ij,t} = h_{ii,t}^{1/2} h_{jj,t}^{1/2} \rho_{ij}$, for $1 \leq i \neq j \leq 2$.

[3] This reasoning follows the same lines as the well known fact that the unconditional distribution entailed by a Gaussian GARCH(1,1) has standardized kurtosis larger than 3.

[4] We did further simulations with different specifications of the conditional mean and variance. Results, available under request, do not change qualitatively the conclusions.

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Appendix: proofs

For simplicity, we will henceforth use the following notation $X_t = X_t^{(i)}$, $F = F^{(i)}$ and $q_\tau = q_\tau^{(i)}$ for some generic component, its distribution and its τ–quantile, respectively. Let us further note that, since all marginal distributions are continuous, the event $\{X_t = X_s\}$ has probability zero if $t \neq s$. This means that the sample points $X_1, \ldots, X_n$ are all distinct and thus the jumps of the empirical distribution function have size $1/n$. Consequently

$$|F_n(q_{\tau,n}) - \tau| \leq n^{-1} \quad \text{a.s.} \tag{3}$$

Before giving the proof of our main theorem, we need to state and prove some lemmas.

Lemma 1 *Under the Assumptions 1 and 2 we have*

$$(q_{\tau_1,n}^{(1)}, \ldots, q_{\tau_p,n}^{(p)})' \to (q_{\tau_1}^{(1)}, \ldots, q_{\tau_p}^{(p)})' \quad \text{a.s.}$$

Proof. It is sufficient to give the proof component–wise. Since $F(x)$ is continuous, monotonously increasing and strictly monotone in a neighbourhood of q_τ we have $q_{\tau,n} \to q_\tau$ if and only if $F(q_{\tau,n}) \to F(q_\tau)$. Thus we evaluate

$$\begin{aligned} |F(q_{\tau,n}) - F(q_\tau)| &\leq |F(q_{\tau,n}) - F_n(q_{\tau,n})| + |F_n(q_{\tau,n}) - F(q_\tau)| \\ &\leq \sup_{x\in\mathbb{R}} |F(x) - F_n(x)| + |F_n(q_{\tau,n}) - \tau|. \end{aligned} \tag{4}$$

Now the result follows from the Glivenco–Cantelli theorem for stationary processes (see e.g. Stute and Schumann (1980) [16]) and (3). □

Our next lemma is a special case of the main result in Berkes et al. (2009) [3]. To state it we introduce

$$R(x,n) = \sum_{k=1}^{n} Y_k(x) \quad \text{where} \quad Y_k(x) = I\{X_k \leq x\} - F(x), \quad x \in \mathbb{R}.$$

Lemma 2 *Under Assumptions 1 and 2 the series*

$$\Gamma(x,x') = \sum_{-\infty<k<\infty} EY_0(x)Y_k(x') \tag{5}$$

converges absolutely for every choice of parameters $(x,x') \in \mathbb{R}^2$. Moreover, there exists a two-parameter Gaussian process $K(x,n)$ such that $EK(x,n) = 0$ and $EK(x,n)K(x',n') = (n \wedge n')\,\Gamma(x,x')$ and for some $\varepsilon > 0$

$$\sup_{0\leq n\leq N} \sup_{x\in\mathbb{R}} |R(x,n) - K(x,n)| = o\big(N^{1/2}(\log N)^{-\varepsilon}\big) \quad \text{a.s.} \tag{6}$$

The next lemma will allow us to deduce asymptotic normality for the quantiles from the empirical process.

Lemma 3 *There exists a $q_{\tau,n}^*$ in the interval bounded by q_τ and $q_{\tau,n}$ such that we have the following representation:*

$$\sqrt{n}\,(F(q_\tau) - F_n(q_\tau)) = \sqrt{n} f(q_{\tau,n}^*)(q_{\tau,n} - q_\tau) + o_P(1).$$

Proof. By the mean value theorem we have

$$F(q_{\tau,n}) - F(q_\tau) = f(q_{\tau,n}^*)(q_{\tau,n} - q_\tau).$$

Noting that $F(q_\tau) = \tau$ we get via (3) that the left hand side above is equal to $F(q_{\tau,n}) - F_n(q_{\tau,n}) + O_p(1/n)$. It remains to show that

$$\sqrt{n}\left([F(q_{\tau,n})-F_n(q_{\tau,n})]-[F(q_\tau)-F_n(q_\tau)]\right)=o_p(1). \tag{7}$$

By Lemma 2 we infer that there exists an $\varepsilon>0$ and a Gaussian process $K(x,n)$ such that

$$\sup_{x\in\mathbb{R}}\left|\sqrt{n}(F_n(x)-F(x))-\frac{1}{\sqrt{n}}K(x,n)\right|=O_p\left((\log n)^{-\varepsilon}\right).$$

Hence (7) follows if $\left|\frac{1}{\sqrt{n}}K(q_\tau,n)-\frac{1}{\sqrt{n}}K(q_{\tau_n},n)\right|\xrightarrow{\mathcal{P}}0$. In the proof of Lemma 6 of Berkes et al. (2009) [3] it is shown that there is a $\tau>0$ such that

$$E\left|\frac{1}{\sqrt{n}}K(x,n)-\frac{1}{\sqrt{n}}K(x',n)\right|^2\le C|x-x'|^\tau.$$

This implies (see e.g. Lemma 2 in Lai (1974) [10]) that the processes $\{n^{-1/2}K(x,n),\,x\in\mathbb{R}\}$ have continuous sample paths. Since the law of $\{n^{-1/2}K(x,n),\,x\in\mathbb{R}\}$ is independent of n, we get in view of Lemma 1 the proof by routine arguments. □

Via the Cramér–Wold device, the next lemma gives asymptotic normality of the marginal empirical distribution functions evaluated at fixed arguments.

Lemma 4 *Under Assumptions 1 and 2 we have that for any p–vectors $\boldsymbol{v}=(v_1,\dots,v_p)'$ with $|\boldsymbol{v}|=1$ and $\boldsymbol{x}=(x_1,\dots,x_p)'$ that*

$$Z_n:=\sqrt{n}\sum_{i=1}^p v_i\left(F_n^{(i)}(x_i)-F^{(i)}(x_i)\right)\xrightarrow{\mathcal{D}}N(0,\boldsymbol{v}'\boldsymbol{Q}\boldsymbol{v}),$$

where $\boldsymbol{Q}$ is defined in Theorem 1.

Proof. Define

$$\eta_k=\sum_{i=1}^p v_i\left(I\{X_k^{(i)}\le x_i\}-F^{(i)}(x_i)\right)$$

and

$$\eta_{km}=\sum_{i=1}^p v_i\left(I\{X_{km}^{(i)}\le x_i\}-F^{(i)}(x_i)\right),$$

where $X_{km}^{(i)}$ are defined as in (2). Then $Z_n=n^{-1/2}\sum_{k=1}^n\eta_k$. By the Cauchy–Schwartz inequality we have

$$\begin{aligned}
E|\eta_k-\eta_{km}|^2&\le\sum_{i=1}^p E\left|I\{X_k^{(i)}\le x_i\}-I\{X_{km}^{(i)}\le x_i\}\right|^2\\
&\le\sum_{i=1}^p E\left(I\{X_k^{(i)}\in[x_i-\delta,x_i+\delta]\}+I\{|X_k^{(i)}-X_{km}^{(i)}|>\delta\}\right)\\
&\le p(2B\delta+P(|\boldsymbol{X}_k-\boldsymbol{X}_{km}|>\delta).
\end{aligned}$$

Choosing $\delta=\delta_m=m^{-4}$ we obtain that $\sum_{m\ge1}(E|\eta_k-\eta_{km}|^2)^{1/2}<\infty$. By a slightly adapted version of Theorem 19.3. in Billingsley (1999) [4] this implies that the sequence $\{\eta_k\}$ satisfies the central limit theorem:

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_k \xrightarrow{\mathcal{D}} N(0, \sigma^2),$$

where $\sigma^2 = \sum_{h \in \mathbb{Z}} \mathrm{Cov}(\eta_0, \eta_h)$. The claim is now immediate. $\square$

This leads to the proof of the main theorem.

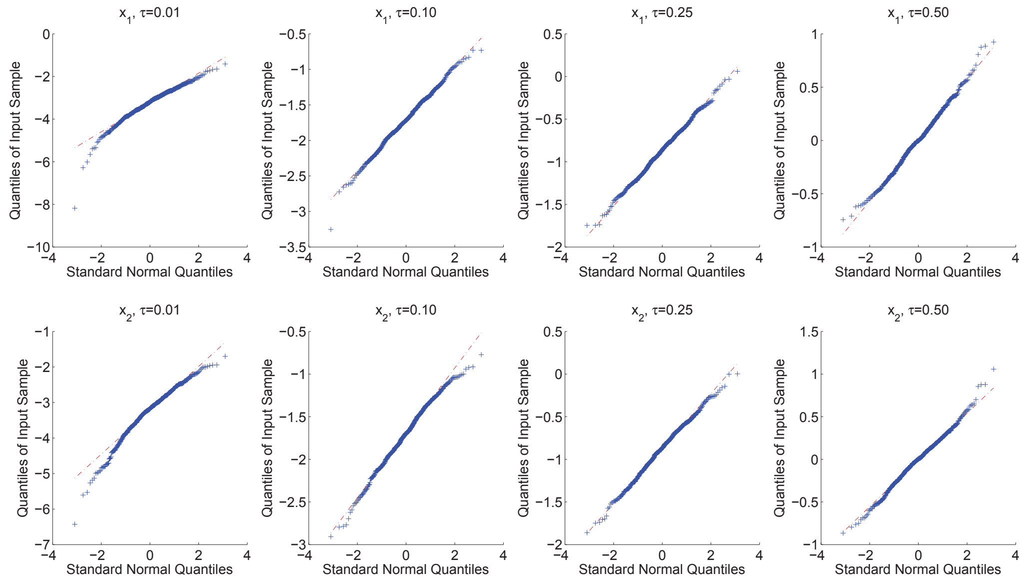
Proof of Theorem 1. Lemma 1 shows that under our assumptions

$$f(q^*_{\tau,n}) \xrightarrow{\mathcal{P}} f(q_\tau) > 0.$$

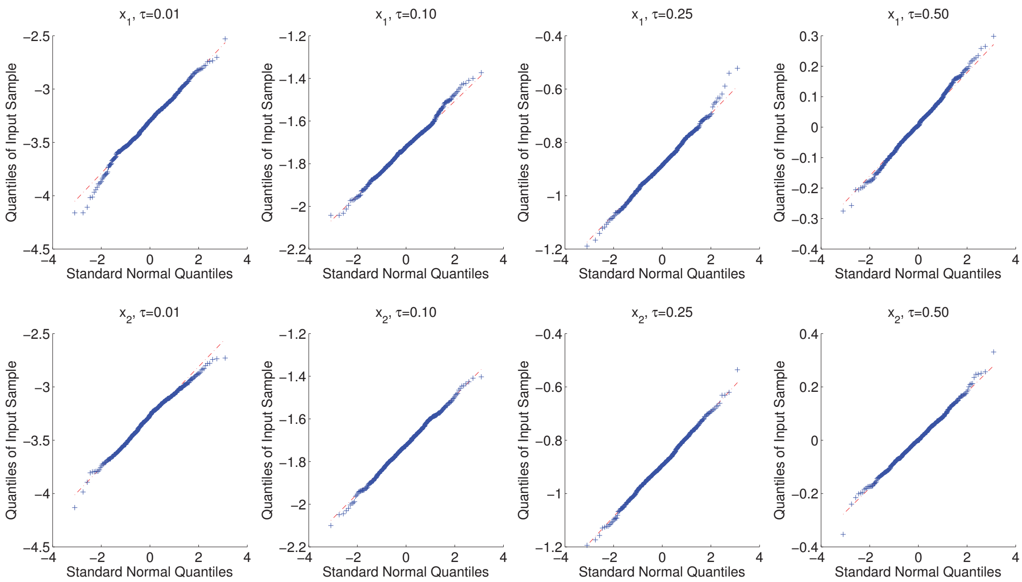
A routine application of the Cramér–Wold device and Slutzky's theorem to Lemma 3 and Lemma 4 yields the desired asymptotic normality of the empirical quantiles. $\square$

Appendix: figures

Figure 1: VAR(1) WITH $\nu = 10$

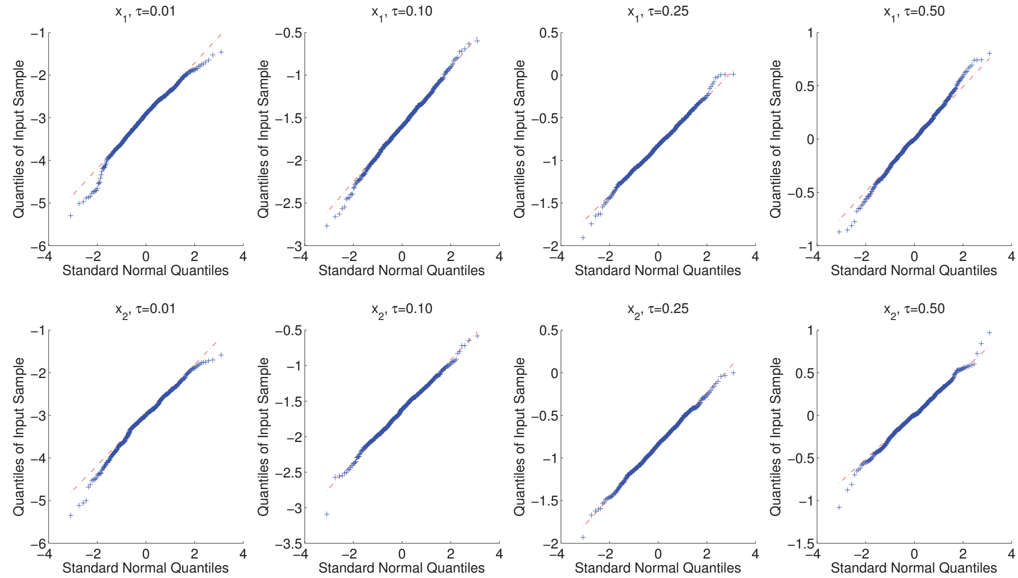


(a) n=100

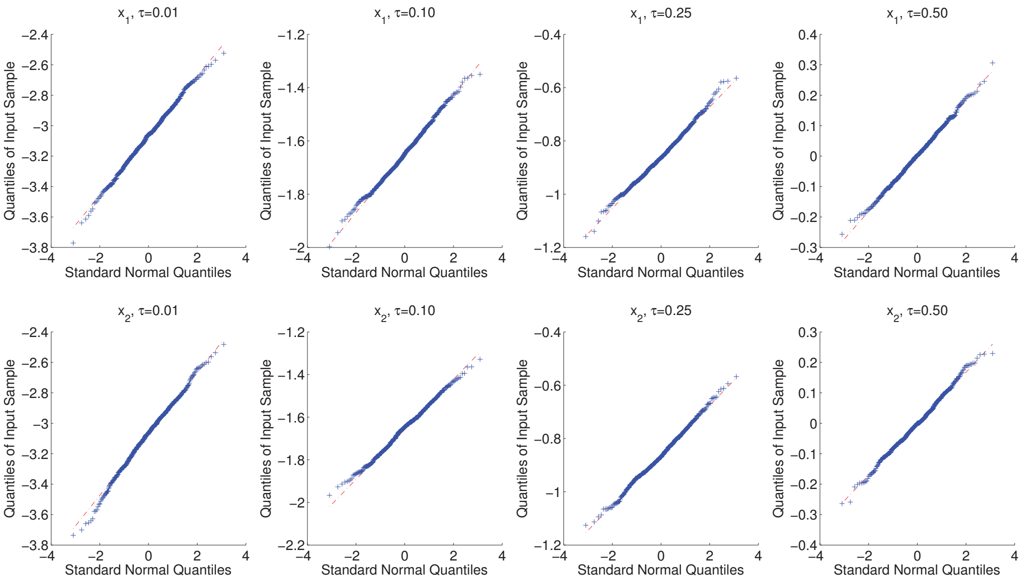


(b) n=1000

Figure 2: VAR(1) WITH $\nu = 20$

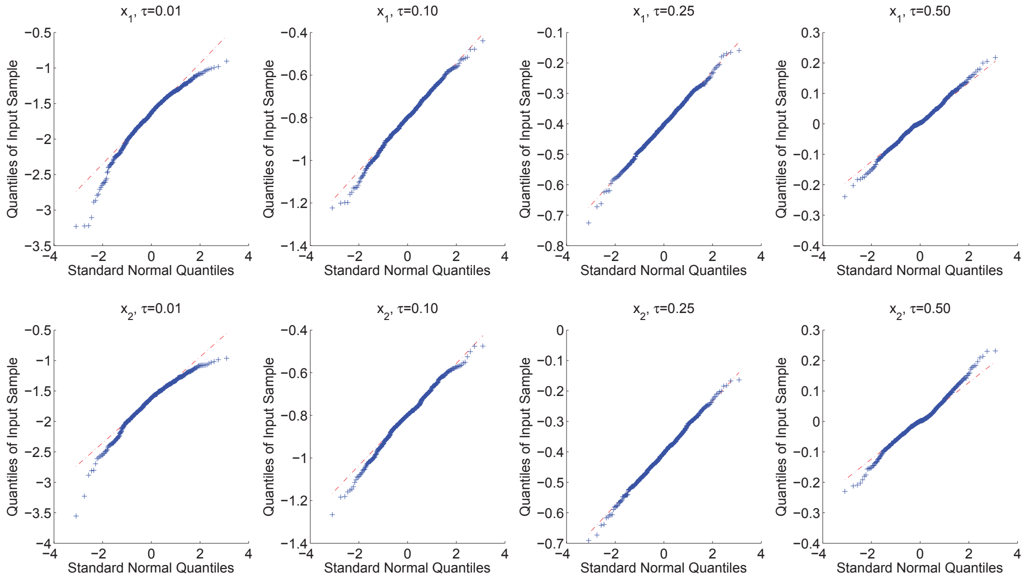


(a) n=100

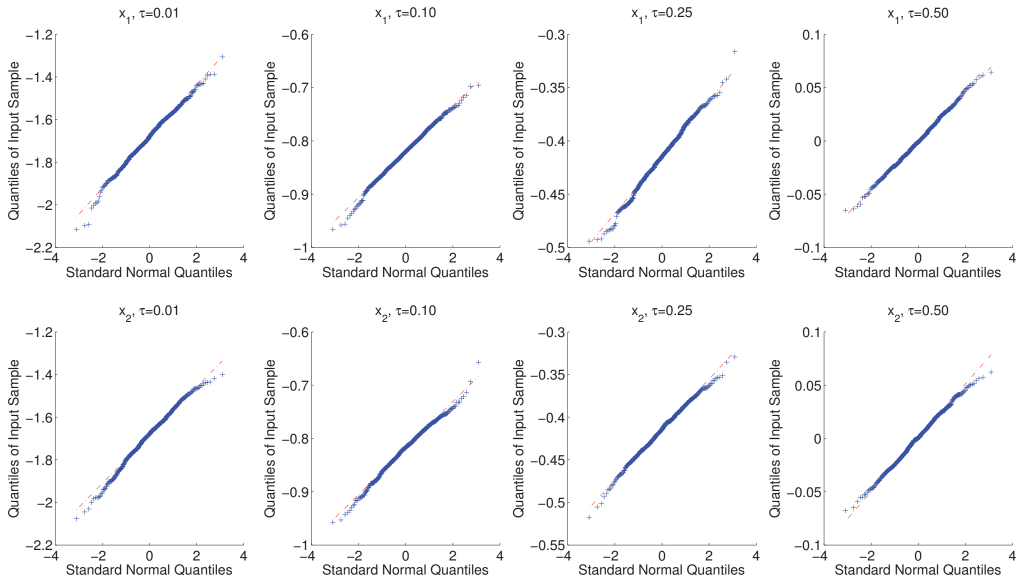


(b) n=1000

Figure 3: CCC-GARCH(1,1) WITH $\nu = 10$

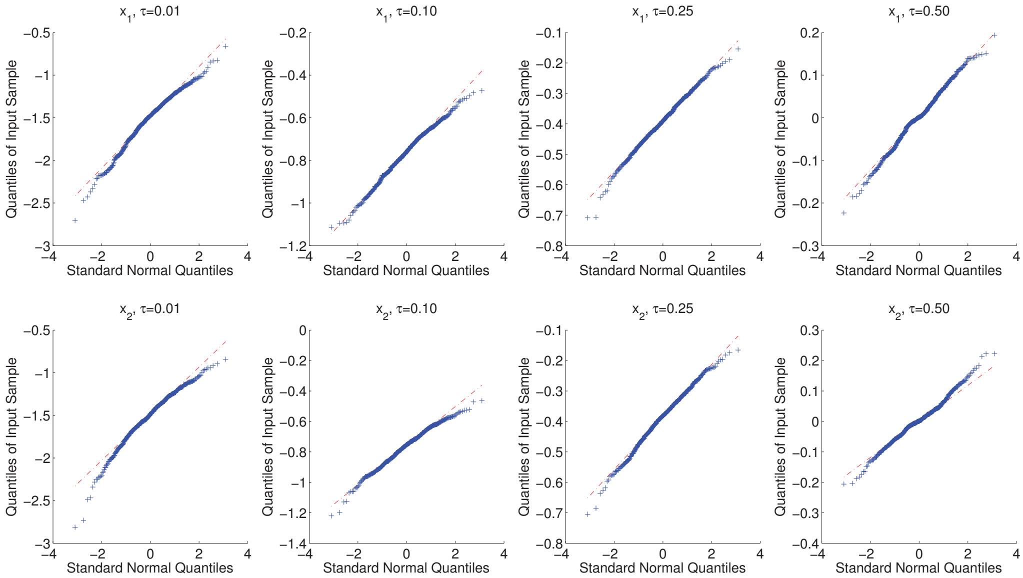


(a) n=100

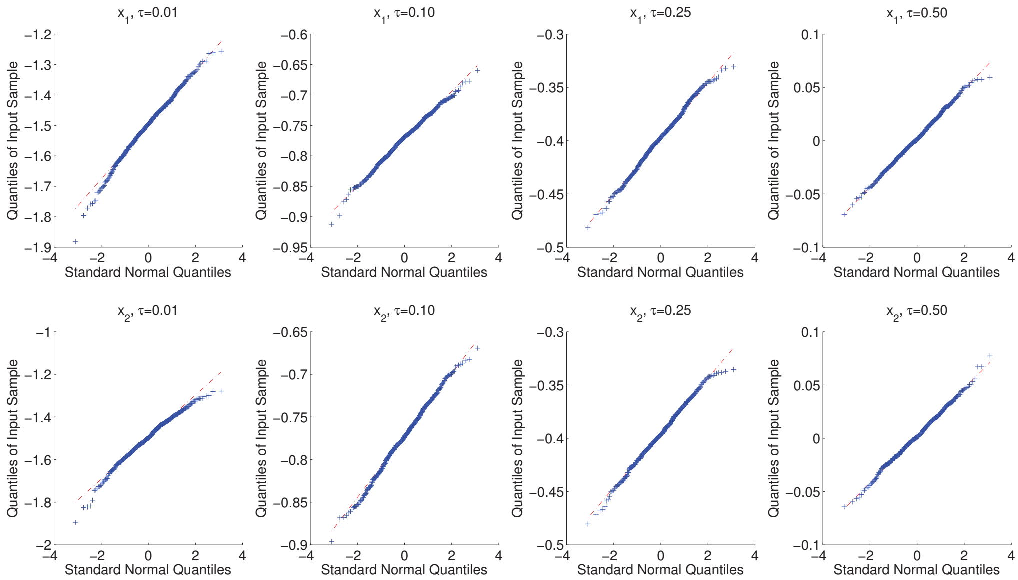


(b) n=1000

Figure 4: CCC-GARCH(1,1) WITH $\nu = 20$



(a) n=100



(b) n=1000

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