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Abstract

The time aggregation properties of the Hodrick-Prescott (HP) filter to decompose a time series into trend and cycle are analized for the case of annual, quarterly, and monthly data. It is seen that aggregation of the disagreggate component estimators cannot be obtained as the exact result from applying an HP filter to the aggregate series (and viceversa). Nevertheless, using several criteria, one can find HP decompositions for different levels of aggregation that provide similar results. The approximation works better for the case of temporal aggregation than for systematic sampling. The criterion finally proposed to find "close to equivalent" HP filters for different frequencies of observation is trivial to apply, and does not depend on the particular series at hand, nor on the series model.

1. Introduction

The subjectiveness in the concept of business cycle has derived in multiple methodologies for its identification (see, for example, Canova (1998)). Be that as it may, and despite substantial academic criticism (see, for example, Cogley and Nason (1995), Harvey (1997), or King and Rebelo (1993)), the Hodrick-Prescott (HP) filter has become the core of the paradigm for business-cycle estimation in short-term economic analysis at policy making institutions. The HP filter decomposes a time series into two components: a long-term trend component and a stationary cycle (see Hodrick and Prescott (1980), Kydland and Prescott (1990), and Prescott (1986)); it is a linear filter that requires previous specification of a parameter known as lambda, λ. This parameter tunes the smoothness of the trend, and depends on the periodicity of the data and on the main period of the cycle that is of interest to the analyst. Nevertheless, as pointed out by Wyne and Koo (1997), the parameter does not have an intuitive interpretation for the user, and its choice is considered perhaps the main weakness of the HP method (Dolado et al. (1993)).

For quarterly data (the frequency most often used for business-cycle analysis), there is an implicit consensus in employing the value of λ=1600, originally proposed by Hodrick and Prescott (1980) based on a somewhat mystifying reasoning ("*…a 5% cyclical component is moderately large, as is a 1/8 of 1% change in the growth rate in a quarter…*"). Still, the consensus around this value undoubtedly reflects the fact that analysts have found it useful. The consensus, however, disappears when other frequencies of observation are used. For example, for annual data, Baxter and King (1999) recommend the value λ=*10* because it approximates a band pass filter that removes from the cycle periodicities larger than 8 years; Dolado et al. (1993) employ λ=400 as a result of dividing 1600 by *4*, the ratio of the number of observations per year for quarterly and annual data (this value of λ is also considered, together with λ=100, by Apel et al. (1996)); while Backus and Kehoe (1992), Giorno et al. (1995) or European Central Bank (2000) use the value λ=100. Concerning monthly data (a frequency seldom used), the popular econometrics program E-views uses the default value 14400, while the Dolado et al. reasoning would lead to λ=4800. This paper deals with the problem of finding appropriate values of λ that yield consistent results under aggregation (or disaggregation). Although the solution we shall eventually propose extends trivially to any other frequency, we shall concentrate the discussion on annual, quarterly, and monthly observations.

Let f denote the number of observations per year (hence f=12 for monthly data, f=4 for quarterly data, and f=1 for annual data), and denote by λ_f the HP parameter used for that frequency of observation. In order to compute values of λ that are consistent under aggregation (we shall refer to them as "equivalent" values), the basic comparison will be the following. Let h and l (h>l) denote a higher and a lower frequency of observation. Firstly, for a fixed λ_h for the higher frequency, we shall find a value λ_l, for the lower frequency, such that the components obtained with direct and indirect estimation are as close as possible. Direct estimation denotes the decomposition obtained by applying λ_l to the low frequency data; indirect estimation refers to the decomposition obtained by aggregating the components obtained by applying λ_h to the high frequency data. Secondly, for a given λ_l for the low frequency data, we shall try to find λ_h, for the high frequency data such that the difference between direct and indirect adjustment is as small as possible. Although, as shall be seen, the extension to other values is immediate, in the discussion that follows we shall use the widely accepted value of λ_Q=1600 for quarterly data as the pivotal value for the comparisons.

2. The Hodrick-Prescott Filter

We shall adopt the standard Box and Jenkins (1970) terminology, whereby an $ARIMA(p,d,q)$ model denotes the model

$$(1+f_1 B+\ldots+f_p B^p)\nabla^d x_t = (1+q_1 B+\ldots+q_q B^q)a_t \qquad ,$$

where B is the lag operator, such that $B^j x_t = x_{t-j}$, $\nabla = 1-B$ is the regular difference, and a_t is a white-noise innovation. When $p=0$, the model name will be abbreviated to $IMA(d,q)$. Further, if x_t requires d differences to become stationary, x_t will be called "integrated of order d ", or $I(d)$.

Assume we are interested in decomposing a time series (x_1 … x_T) into a long-term trend m_t and a residual, c_t, to be called "cycle". The HP filter provides the sequences (m_1 … m_T) and (c_1 … c_T) such that

$$x_t = m_t + c_t \qquad t=1,\ldots,T, \tag{2.1}$$

and the loss function

$$\sum_{t=1}^{T} c_t^2 + l \sum_{t=3}^{T} \left(\nabla^2 m_t\right)^2 \tag{2.2}$$

is minimized. The first term in (2.2) penalizes large residuals (i.e., poor fit), while the second term penalizes lack of smoothness in the trend. The parameter λ regulates the trade-off between the two criteria. The solution is given by the expression $\hat{m} = A^{-1}x$, with $A = I + l KK'$, where $\hat{m}$ is the estimated value of the vector (m_1 … m_T)', x is the vector (x_1 … x_T)', K is a $(T\text{-}2)xT$ matrix with elements $K_{ij}=1$ if $i=j$ or $i=j+2$, $K_{ij}=-2$ if $i=j+1$, and $K_{ij}=0$

otherwise (see Danthine and Girardin (1989)). King and Rebelo (1993) showed that the filter could be given a model-based interpretation whereby (x_1 … x_T) is the realization of a stochastic process consisting of the sum of an *IMA(2,0)* stochastic trend, plus an orthogonal white-noise residual. Under these assumptions, the HP filter solution is equivalent to finding the minimum mean square error (MMSE) estimator of m_t and c_t. As shown in Harvey and Jaegger (1993), the Kalman filter can be used to obtain those estimators. Kaiser and Maravall (1999) show that the HP estimators can also be given a slightly different model-based interpretation, that implies an alternative computational algorithm. We summarize this approach.

Letting "*w.n. (0,n)*" denote a white noise (niid) variable with zero mean and variance n, the models for m_t and c_t can be expressed as

$$\nabla^2 m_t = a_{mt} \quad , \quad a_{mt} \sim w.n.\ (0\ ,n_m) \quad ; \tag{2.3a}$$

$$c_t \sim w.n.\ (0\ ,n_c) \quad ; \tag{2.3b}$$

with a_{mt} orthogonal to c_t. From (2.1) it follows that

$$\nabla^2 x_t = a_{mt} + \nabla^2 c_t \tag{2.4}$$

and hence the reduced form for x_t is an *IMA(2,2)* process, say

$$\nabla^2 x_t = q_{HP}(B) a_t = (1 + q_1 B + q_2 B^2) a_t \ , \quad a_t \sim w.n.\ (0\ ,n_a) \quad , \tag{2.5}$$

where the identity

$$q_{HP}(B) a_t = a_{mt} + \nabla^2 c_t \tag{2.6}$$

determines the parameters q_1, q_2, and n_a. The fact that the r.h.s. of (2.6) is the sum of the two orthogonal components, one of them white noise, implies that $\nabla^2 x_t$ has a strictly positive spectral minimum, or, in other words, that $q_{HP}(B) a_t$ is an invertible process; therefore, $q_{HP}(B)^{-1}$ will always converge. The MMSE estimator of m_t and c_t can now be obtained with the Wiener-Kolmogorov (WK) filter. When

$$\lambda = n_c / n_m \tag{2.7}$$

these estimators are the ones obtained with the HP filter.

In general for a stationary component n_t with moving average (*MA*) expression

$$n_t = y_n(B) b_t \quad , \qquad\qquad b_t \sim w.n.\ (0, n_b) \qquad , \tag{2.8}$$

contained in a stationary series with MA expression

$$x_t = y_x(B) a_t \quad , \qquad\qquad a_t \sim w.n.\ (0, n_a) \qquad ; \tag{2.9}$$

the WK filter that provides the MMSE estimator of n_t (when a doubly infinite realization of x_t is available) is given by

$$J_n(B,F) = \frac{v_b}{v_a} \frac{y_n(B) y_n(F)}{y_x(B) y_x(F)} \qquad , \tag{2.10}$$

where $F(=B^{-1})$ denotes the forward operator, such that $F^j x_t = x_{t+j}$. When the series - and perhaps the component- is nonstationary (NS), the MA representation (2.9) - and perhaps (2.8) - will not converge. But if (2.10) converges in B and F, (2.10) can be seen as the WK filter extended to NS series, and still provides the MMSE of the component (Bell (1984) and Maravall (1988)). From (2.3) and (2.5), after simplification, it is obtained that, for the HP trend and cycle, the filters are given by

$$J_m(B,F) = \frac{v_m}{v_a} \frac{1}{q_{HP}(B) q_{HP}(F)} \qquad ; \tag{2.11}$$

$$J_c(B,F) = \frac{v_c}{v_a} \frac{(1-B)^2 (1-F)^2}{q_{HP}(B) q_{HP}(F)} \qquad . \tag{2.12}$$

It is immediately seen that $J_m(B,F) + J_c(B,F) = 1$ (which implies $x_t = \hat{m}_t + \hat{c}_t$). Therefore, we shall concentrate attention in $J_m(B,F)$, and $J_c(B,F)$ can be trivially obtained as $J_c(B,F) = 1 - J_m(B,F)$.

The filters (2.11) and (2.12) are symmetric and centered and, because of invertibility of $q_{HP}(B)$, convergent. Since (2.6) implies

$$q_{HP}(B) q_{HP}(F) v_a = v_m + (1-B)^2 (1-F)^2 v_c \quad , \tag{2.14}$$

and considering (2.7), the filter (2.11) can alternatively be expressed in terms of the HP parameter λ as:

$$J_m(B,F) = \frac{1}{1 + l(1-B)^2 (1-F)^2} \qquad . \tag{2.15}$$

The estimator of m_t will thus be obtained through

$$\hat{m}_t = J_m(B,F)x_t \qquad . \tag{2.16}$$

It will prove useful to look at the frequency domain representation of the filter (2.15), given by its Fourier transform (FT). If $w \in [0, p]$ denotes the frequency measured in radians, replacing B by the complex number e^{-iw}, and using the identity $2\cos(jw) = e^{-ijw} + e^{ijw}$, the FT of (2.15) is given by

$$G_m(w, l) = \frac{1}{1 + 4l(1 - \cos(w))^2} \tag{2.17}$$

and $G_c(w, l) = 1 - G_m(w, l)$. Expression (2.17) is the gain function of the filter. Equating the pseudo-autocovariance functions (ACF) of the two sides of (2.16), and taking the FT yields

$$S_{\hat{m}}(w, l) = [G_m(w, l)]^2 S_x(w) \tag{2.18}$$

where $S_{\hat{m}}(w, l)$ and $S_x(w)$ are the pseudo-spectra of $\hat{m}_t$ and x_t. Given that the pseudo-spectrum provides a description of how different frequencies contribute to the variation of x_t, expression (2.18) shows that the squared gain of the filter indicates which frequencies are selected from x_t to provide the estimator $\hat{m}_t$.

Figure 1a shows the shape of the squared gain of (2.15) for different values of λ. The larger the value of λ, the smaller is the contribution of high frequencies, and hence smoother trends will be obtained. Figure 1b displays a realization of an *IMA(2,2)* process, and Figures 1c and 1d compare the trends and cycles, respectively, obtained for the different values of λ. Naturally, smoother trends imply cycles with larger amplitude and longer periods.

The WK filter (2.15) extends from $-\infty$ to ∞. Its convergence, however, would allow us to use a finite truncation. But, as characterizes all 2-sided filters, estimation of the component at both ends of a finite series requires future observations, still unknown, and observations prior to the first one available. As shown by Cleveland and Tiao (1976) the optimal (MMSE) estimator for end points can be obtained by extending the series at both ends with forecasts and backcasts, so that expression (2.16) remains valid with x_t replaced by the extended series. There is no need however to truncate the filter: using the Burman-Wilson algorithm in Burman (1980), Kaiser and Maravall (2001) present the algorithm for the HP filter case, and show how the effect of the infinite extensions can be exactly captured with only 4 forecasts and backcasts. The WK application of the HP filter is computationally efficient and analytically convenient.

Figure 1a: Gain of the HP filter for different values of λ

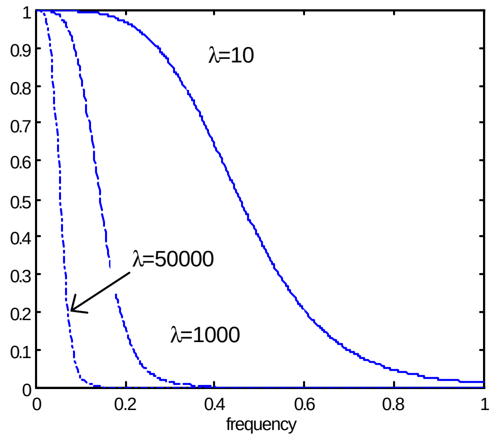


Figure 1b: Realization of an *IMA(2,2)* process

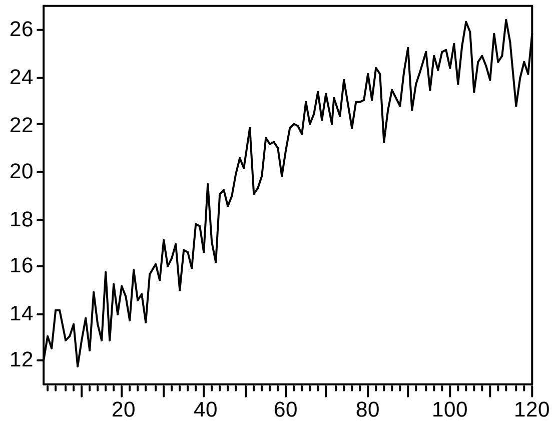


Figure 1c: Estimated trends for different values of λ

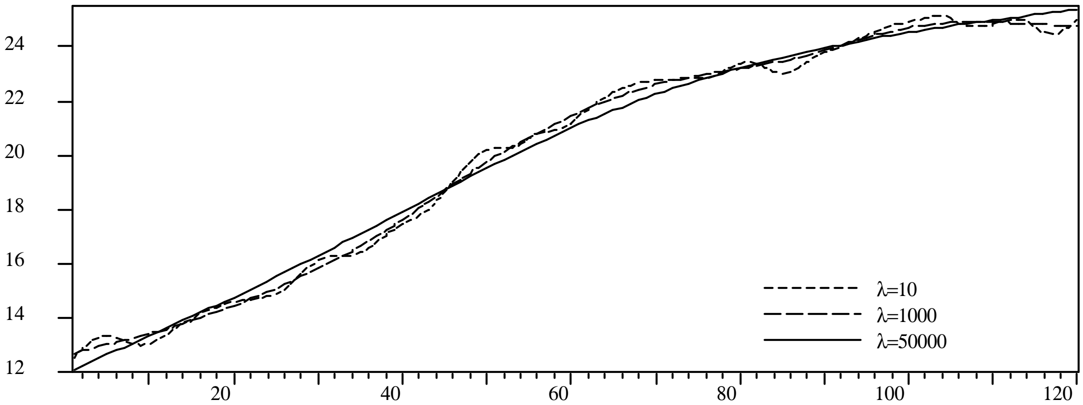
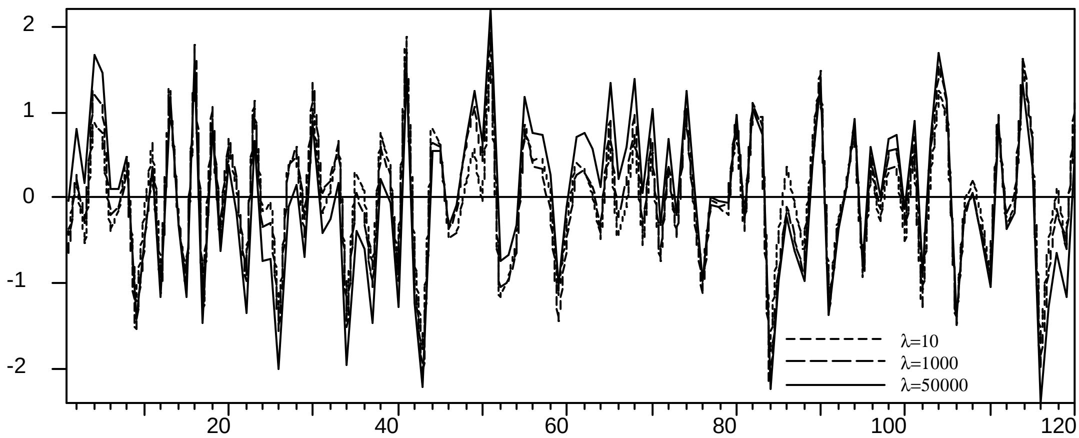


Figure 1d: Estimated cycles for different values of λ



We shall consider two types of aggregation. In the first one the aggregate variable is the sum (or average) of the disaggregate variable; in the second one, the aggregate variable is obtained by periodically sampling one observation from the disaggregate variable. The first case will be denoted "temporal aggregation", and the second case, "systematic sampling". Although temporal aggregation is the most frequently used method, systematic sampling affects some important variables such as, for example, European Monetary Union monetary aggregates.

Two important remarks should be made:

a) Given that seasonal variation should not contaminate the cyclical signal, the HP filter should be applied to seasonally adjusted series. As shown in Kaiser and Maravall (2001), the presence of higher transitory noise in the seasonally adjusted series can also contaminate the cyclical signal and its removal may be appropriate. In this paper, the issue of seasonal adjustment or noise removal will not be addressed, and we shall assume that the series does not contain seasonality, or that it has been appropriately removed.

b) It is well known (see Baxter and King (1999)) that the HP filter displays unstable behavior at the end of the series. As shown in Kaiser and Maravall (2001), end-point estimation is considerably improved if the extension of the series needed to apply the filter is made with proper forecasts, obtained not with model (2.5), but with the correct $ARIMA$ model for the series in question. We shall always apply the filter in this manner.

3. Temporal Aggregation of the Hodrick-Prescott Filter

In order to analyze the aggregation properties of the HP filter, it will prove helpful to use the WK representation. We have mentioned that the HP filter can be seen as the filter that provides the MMSE estimator of an unobserved component m_t in the full model consisting of equations (2.1), (2.3), and (2.5), with c_t white noise and λ given by (2.7). Let the aggregate series consist of non-overlapping sums of k consecutive values of the disaggregate series. We adopt the following notation:

- Aggregate series will be represented by capital letters; disaggregate series by small letters.
- T and t denote the time sub-index for the aggregate and disaggregate series respectively. Thus, if T and t refer to the same date $X_T = x_t + x_{t-1} + ... + x_{t-k+1}$, and in general,

$$X_{T-j} = x_{t-jk} + x_{t-jk-1} + ... + x_{t-(j+1)k+1} \quad , \quad j=1,2,.... \tag{3.1}$$

- b denotes the backward operator and ∇ the difference operator for the disaggregate series. Thus $\nabla_j x_t = (1-b^j)x_t = x_t - x_{t-j}$.

- S_k is the aggregation operator for the disaggregate series: $S_k = 1 + b + ... + b^{k-1}$. Thus

$$X_{T-j} = S_k x_{t-jk} \quad ; \tag{3.2}$$

$$\nabla_k = \nabla S_k \quad . \tag{3.3}$$

- B denotes the backward operator and D the difference operator for the aggregate series. Thus $DX_T = (1-B)X_T = X_T - X_{T-1}$.

- The same convention applies to the disaggregate and aggregate components, m_t, and M_T, and c_t and C_T.

- We shall denote an "HP-type decomposition" the decomposition of an $IMA(2,2)$ model into the sum of an $IMA(2,0)$ model plus orthogonal white noise.

Starting with a disaggregate HP-type decomposition, in order to derive the aggregate filter, we aggregate series and components, and see if they are consistent with an HP-type decomposition. As shown by Stram and Wei (1986), aggregation preserves the order of integration of the series, thus, from (2.5), we consider D^2X_T. From the identity $D^2X_T = X_T - 2X_{T-1} + X_{T-2}$ it is easily found using (3.2) that $D^2X_T = \nabla_k^2 S_k x_t$, or using (2.5) and (3.3),

$$D^2X_T = S_k^3 q_{HP}(b)a_t = \alpha(b)a_t \quad , \tag{3.4}$$

where $\alpha(b)$ is a polynomial in b of order $3k$-1. From

$$D^2X_{T-j} = a_{t-jk} + \alpha_1 a_{t-jk-1} + ... + \alpha_{3k-1} a_{t-(3+j)k+1} \qquad j=0,1,2... \quad ,$$

it is seen that the ACF of D^2X_T is such that r_1 and r_2 are nonzero, while $r_j = 0$ for $|j|>2$. Therefore, D^2X_T follows an $IMA(2,2)$ process.

Proceeding in a similar way with the aggregate trend component, it is found that

$$D^2M_T = \nabla_k^2 S_k m_t = S_k^3 a_{mt} \quad , \tag{3.5}$$

where $\alpha_m(b) = S_k^3$ is a polynomial in b of order $3k$-3. From

$$D^2M_{T-j} = a_{m,t-kj} + \alpha_{m,1} a_{m,t-kj-1} + ... + \alpha_{m,3k-3} a_{m,t-(3+j)k+3} \qquad j=0,1,2... \quad ,$$

it follows that $D^2 M_T$ is also an $IMA(2,2)$ model. As for the cycle, it is immediately seen that C_T is a white-noise variable, with variance $v_C = k\, v_c$.

In summary, aggregation of the HP filter yields an $IMA(2,2)$ series, an $IMA(2,2)$ trend, and a white-noise cycle. The trend specification implies that the aggregate components cannot be seen as the result of a direct HP-type decomposition of the aggregate series: the HP-filter does not preserve itself under temporal aggregation.

Be that as it may, from an applied point of view it is important to know whether the departure of the aggregate components from an HP-type decomposition is relatively large or, on the contrary, close to negligible, in which case an HP-type decomposition of the aggregate series could approximate well the components obtained with the indirect procedure. We shall consider two separate problems:

a) Given the disaggregate HP decomposition, and the components for the aggregate series obtained by aggregating the disaggregate components, can we obtain a value of λ that provides a direct HP decomposition of the aggregate series with components that are close to the former ones?

b) Given a direct HP-decomposition of the aggregate series, can we obtain a value of λ that provides an HP-decomposition of the disaggregate series with components that, when aggregated, are close to the components of the direct decomposition?

The two problems could be discussed following the general approach of Stram and Wei (1986). For our purposes, however, we can follow an alternative, much simpler, approach. Assuming that the value λ_Q=1600 for quarterly data provides the reference value, question a) is relevant to determine the equivalent λ_A for annual data; question b) is relevant to determine the equivalent value of λ_M for monthly data.

From (2.6), (3.3) and (3.4),

$$D^2 X_T = S_k^3 (a_{mt} + \nabla^2 c_t) = S_k^3 a_{mt} + S_k \nabla_k^2 c_t \quad , \tag{3.6}$$

where T and t always are assumed to refer to the same date. The ACF of the r.h.s. of (3.6) is given by

$$S_k^3 \overline{S}_k^3 v_m + S_k \nabla^2 \overline{S}_k \nabla^2 v_c \qquad , \tag{3.7}$$

where an upper bar denotes the corresponding polynomial with B replaced by F (i.e. $\overline{\nabla}_k = 1 - F^k$). Given that, from (3.4),

$$D^2 X_{T-j} = S_k^3 q_{HP}(b) a_{t-jk} \qquad j=0,1,2 \quad ,$$

the variance and lags 1 and 2 autocovariances of $D^2 X_{T-j}$, to be denoted G_0, G_1, and G_2, measured in the aggregate time units, are equal to the variance and lags k and $2k$ autocovariances of (3.7), to be denoted g_0, g_{1k}, and g_{2k}, measured in the disaggregate time units. If the direct HP decomposition of the aggregate series X_T is given by

$$\begin{aligned} X_T &= M_T + C_T \\ D^2 M_T &= A_{MT} \ , \qquad A_{MT} \sim w.n.\ (0, n_M) \qquad , \end{aligned} \tag{3.8}$$

with C_T white noise $(0, n_C)$ orthogonal to A_{MT}, setting $G_j = g_{jk}$ ($j=0,1,2$), a system of three equations is obtained, of the type:

$$\begin{aligned} v_M + 6 v_C &= a_{10} v_m + a_{20} v_c \\ -4 v_C &= a_{11} v_m + a_{21} v_c \\ v_C &= a_{12} v_m + a_{22} v_c \end{aligned} \quad . \tag{3.9}$$

The system can be standardized by dividing by n_m. Given n_m and n_c or, considering (2.7), the HP parameter λ for the disaggregated decomposition, the system of three equations with two unknowns will not have a solution for n_M and n_C, i.e., for the HP-parameter of the aggregate decomposition. Inversely, given n_M and n_C, the system will not have a solution for n_m and n_c.

Focusing on the relationship between monthly, quarterly, and annual aggregation the coefficients a_{ij} in (3.9) can be easily obtained from (3.7), yielding the matrices

$$A_3 = \begin{bmatrix} 141 & 18 \\ 50 & -12 \\ 1 & 3 \end{bmatrix} \ , \qquad A_4 = \begin{bmatrix} 580 & 24 \\ 216 & -16 \\ 6 & 4 \end{bmatrix} \ , \qquad A_{12} = \begin{bmatrix} 137292 & 72 \\ 53768 & -48 \\ 2002 & 12 \end{bmatrix} \ ,$$

where the subindices correspond to k=3, k=4, and k=12. For the quarterly value λ=1600, setting n_m=1 and n_c=1600, it is easily seen that no equivalent annual value for λ can be found because the system has no solution for n_M and n_C. Likewise, setting n_M =1 and n_C =1600 no equivalent monthly value for λ can be obtained because the system has no solution for n_m and n_c. These results are obvious from the structure of (3.9): the ratio of the l.h.s. of the last two equations is equal to *-4*, while this needs not be true for the ratio of the two r.h.s.

Letting λ_M, λ_Q, and λ_A denote the value of the HP-parameter for the decomposition of the monthly, quarterly and annual series, respectively, we can try to find values of λ_M and λ_A consistent with λ_Q=1600 by solving the system consisting of the first two equations of (3.9). This way of proceeding yields eventually

$$\lambda_M = 115204 \quad ; \qquad \lambda_A = 7.02 \quad . \tag{3.10}$$

For these values, the ratio of the r.h.s. of the last two equations of (3.9) is approximately *-3.99* and *-3.96*, respectively, and hence very close to *-4*. Thus, although there are no values of λ_M and λ_A that provide an exact HP decomposition of the monthly and annual series consistent with the quarterly one, the previous approximation provides values that are surprisingly close to an exact solution. When, instead of the first and second equation of the system (3.9), the first and third equation are chosen, the value of λ_M decreases while that of λ_A increases by relatively small amounts, and the approximation is also close to the exact solution. More generally, we can solve the system (3.9) in a least-square sense, by minimizing

$$\sum_{j=0}^{2} \left[\Gamma_j (v_M, v_C) - g_{jk} (v_m, v_c) \right]^2 . \tag{3.11}$$

To find λ_A, we set n_m=1 and n_c=1600, and minimize (3.11) with respect to n_M and n_C.; on the other hand, to obtain λ_M, we set n_M =1 and n_C =1600 and solve (3.11) with respect to n_m and n_c. This yields

$$\lambda_M = 114013 \quad ; \qquad \lambda_A = 7.19 \tag{3.12}$$

values that are very close to those in (3.10). Thus, under temporal aggregation, there seems to be a range of values for λ_M and λ_A that can be seen as close to being equivalent.

4. Aggregation of the Hodrick-Prescott Filter under Systematic Sampling

The previous section considered the case of temporal aggregation, namely, when the aggregate series is the sum of non-overlapping sequences of consecutive values. We consider now the case in which the aggregate variable is obtained by systematic sampling every k periods. Following the same notation as before, if T and t refer to the same date, the aggregate series is obtained as

$$X_{T-j} = x_{t-jk} \qquad j=0,1,2...$$

Proceeding as in the previous section, in order to derive the aggregate filter, our first interest is to obtain the models for the aggregate series and components. From

$$D^2 X_T = X_T - 2X_{T-1} + X_{T-2} = (1-2b^k + b^{2k})x_t = \nabla_k^2 x_t \qquad ,$$

using (2.5) and (3.3) it is found that

$$D^2 X_T = S_k^2 q_{HP}(b)a_t = a(b)a_t \tag{4.1}$$

where $a(b)$ is of order $2k$. Expression (4.1) remains valid for X_{T-j} if a_t is replaced by a_{t-jk}, $j=1,2...$ from which it is immediately found that D^2X_T has nonzero covariances only for lags 0, ±1 and ±2. Therefore, similarly to the temporal aggregation case, the aggregate variable X_T follows an $IMA(2,2)$ model.

Using a similar derivation for the (systematically sampled) trend aggregate component, it is obtained that

$$D^2 M_T = M_T - 2M_{T-1} + M_{T-2} = (1-2b^k + b^{2k})m_t = \nabla_k^2 m_t$$

or using (2.3) and (3.3),

$$D^2 M_T = S_k^2 a_{mt} \tag{4.2}$$

where S_k^2 is a polynomial in b of order $2(k-1)$. The autocovariances for lag 0 and lag jk of the r.h.s. of (4.2) provide the autocovariance for lag 0 and j of D^2M_T. It is easily checked that this implies that M_T is an $IMA(2,1)$ process. As for the aggregate cycle, C_T $(=c_t)$ is a white noise variable, with variance $n_C = n_c$.

Summarizing, although aggregation under systematic sampling implies a model for the aggregate trend different than the one implied by temporal aggregation (an *IMA(2,1)* versus an *IMA(2,2)* model), it still remains true that the decomposition for the aggregate obtained by aggregating the disaggregate components is not of the HP type. The HP decomposition does not preserve itself under systematic sampling.

Proceeding as before, plugging (2.6) in (4.1) yields

$$D^2 X_T = S_k^2 (a_{mt} + \nabla^2 c_t) = S_k^2 a_{mt} + \nabla_k^2 c_t \tag{4.3}$$

and the ACF of the r.h.s. of (4.3) is equal to

$$S_k^2 \bar{S}_k^2 v_m + \nabla_k^2 \bar{\nabla}_k^2 v_c \quad . \tag{4.4}$$

Therefore, as in the case of temporal aggregation, equating the lag-0, lag-1 and lag-2 covariances of $D^2 X_T$ to the lag-0, lag-k and lag-$2k$ covariances in (4.4) yields a system of three equation of the type (3.9), where the a_{ij}-coefficients are determined from (4.4). For k=3,4 and 12, the matrices of those coefficients are now given by

$$A_3 = \begin{bmatrix} 19 & 6 \\ 4 & -4 \\ 0 & 1 \end{bmatrix} \quad ; \quad A_4 = \begin{bmatrix} 44 & 6 \\ 10 & -4 \\ 0 & 1 \end{bmatrix} \quad ; \quad A_{12} = \begin{bmatrix} 1156 & 6 \\ 286 & -4 \\ 0 & 1 \end{bmatrix} \quad ,$$

and it is easily seen that, given n_m and n_c, in none of the three cases the system has a solution for n_M and n_C; similarly, given n_M and n_C, the system has no solution for n_m and n_c.

Although no exact solution exists, our interest as before is to see if we can find values for λ_M and λ_A that produce HP decompositions for the monthly and annual series that are approximately consistent with the HP decomposition obtained for the quarterly series with λ_Q=1600. First, we proceed to solve the system of the first two equations for the case of annual aggregation: setting n_m=1 and n_c=1600, we obtain the solution for n_M and n_C. Second, we solve the system of the first two equations for the case of monthly to quarterly aggregation: setting n_M =1 and n_C =1600, we obtain the solution for n_m and n_c. The values for the HP-parameter are

$$\lambda_M = 40001 \quad ; \quad \lambda_A = 27.08 \quad . \tag{4.5}$$

If the variances obtained are inserted in the third equation, the ratio of the second to third equation, which should be *-4* for the exact solution, turns out to be approximately *-3.99* in both cases. The conclusion is the same as for the temporal aggregation case: the approximate solution is very close to satisfying exactly the full system of three equations.

To find the least squares solution to the system of three equations, for the quarterly to annual aggregation case, from (3.11) and λ_Q=1600 it is seen that the expression

$$SS = \left(v_M + 6v_C - 9644\right)^2 + \left(-4v_C + 6390\right)^2 + \left(v_C - 1600\right)^2$$

has to be minimized with respect to n_M and n_C. Analogously, for the quarterly to monthly disaggregation case, we minimize the expression

$$SS = \left(19v_m + 6v_c - 9601\right)^2 + 16\left(v_m - v_c + 1600\right)^2 + \left(v_c - 1600\right)^2$$

with respect to n_m and n_c. Using (2.7) this yields the values:

$$\lambda_M=39627 \quad ; \quad \lambda_A=27.49 \quad . \tag{4.6}$$

Again the two sets of values (4.5) and (4.6) are very close.

In summary, the lack of aggregation property of the HP filter seems more apparent that real. Although there will be no set of values λ_M, λ_Q and λ_A that will exactly satisfy the aggregation constraints, we can find values that provide close approximations. The procedure we have used to obtain these approximation is rather simple, but requires however some prior work by the analyst. Moreover, we have looked at straightforward aggregation of the filter, but we do not know what this aggregation implies in terms of the cycle features: will the properties of the filter (in particular, the gain) be affected? Will the dominant period in the cycle be different? Ultimately, it would be convenient to obtain a rule, trivial to apply, that would permit the analyst to find equivalent values of λ to use for any type of observation frequency. To this issue we turn next.

5. Aggregation by Fixing the Period for which the Gain is One Half (the Cycle of Reference).

In the engineering literature, a well-known family of filters designed to remove (or estimate) the low-frequency component of a series is the Butterworth family (see, for example Gomez (1999)). The filter is described by its gain function which, for the two-sided expression and the sine-type subfamily of filters, can be expressed as

$$G_m(w) = \left[1+\left(\frac{sin(w/2)}{sin(w_0/2)}\right)^{2d}\right]^{-1} \quad , \quad 0 \le w \le p \quad , \tag{5.1}$$

and depends on two parameters, d and w_0. From (5.1), $G(w_0)$=1/2, and hence the parameter w_0 can be seen as the frequency for which 50% of the filter gain has been achieved. We shall refer to the cycle associated with that frequency as the "cycle of reference". Setting d=2 and defining $b = \left[sin^4(w_0/2)\right]^{-1}$, the gain can also be expressed as

$$G_m(w) = \left[1 + b sin^4(w/2)\right]^{-1} . \tag{5.2}$$

From the identity $2sin^2(w/2) = 1 - \cos(w)$, (5.2) can be rewritten as

$$G_m(w) = \left[1 + (b/4)(1-\cos(w))^2\right]^{-1} ,$$

which, considering (2.17), shows that the filter is precisely the HP filter, with $\lambda = b/16$, or

$$l = \left[4(1-\cos(w_0))^2\right]^{-1} \quad . \tag{5.3}$$

Therefore, knowing the parameter w_0, the HP filter parameter λ can be easily obtained, and viceversa. In this way, λ can be given an alternative interpretation in terms of the frequency associated with $G(w_0)$=1/2. If t denotes the period it takes for completion of the full cycle of $\cos(w_0)$, t is related to the frequency (in radians) through

$$t = 2p/w \quad . \tag{5.4}$$

Using (5.3) and (5.4), we can express the period t directly as a function of λ, as

$$t = 2p \,/\, a\cos(1 - \frac{1}{2\sqrt{l}}) \qquad . \tag{5.5}$$

Equations (5.3)-(5.5) allow us to move from period to frequency, and then to λ (and viceversa) in a simple way. For example, with λ=1600 applied to quarterly data, it is found that t =*39.7* quarters, or close to a 10 year period. A possible alternative criterion for finding values of the HP parameter λ that provide relatively consistent and convenient results under aggregation or disaggregation is to preserve the period of the cycle of reference.

Assume we accept λ_Q=1600 as the appropriate value of λ for quarterly series. We have seen that λ_Q=1600 implies a period of 39.7 quarters. Thus, for annual data, the period of the cycle of reference is, according to (5.5), t =9.9 years. From (5.4), $\omega_A=2p/9.9$, and (5.3) yields λ_A=6.65. On the other hand, the period of 39.7 quarters is, in terms of monthly observations, equal to 119.1 months. Using (5.4) and (5.5), it is found that the equivalent value of λ for monthly data is λ_M=129119. Thus, using this criterion, the values of λ that are consistent for the monthly and annual frequencies of observation, in the sense that they will preserve the period of the cycle of reference implied by λ_Q=1600, are

$$\lambda_M=129119 \quad ; \quad \lambda_A=6.65 \quad , \tag{5.6}$$

close to the values in (3.10) and (3.12), derived previously for the case of temporal aggregation. They are considerably different, however, from the values obtained for the case of systematic sampling.

The procedure of finding λ's that are approximately consistent under aggregation by means of fixing the period of the cycle of reference (i.e., the cycle for which the gain of the filter is ½), is simple to apply, and can be used for aggregating or disaggregating series with any frequency of observation.

An example can illustrate the types of approximation implied. In Giorno et al. (1995) the method used by the OECD for the estimation of the output gap is described: it uses the HP filter with λ_Q=1600 and λ_A=100. These values are referred to as "de facto industry standards" (the European Central Bank (2000) has also used similar values); the two values however are inconsistent. Using λ_A=100 for annual data, from (5.5), the period of the cycle of reference is t_A=19.8 years or, very approximately, 20 years, which in terms of quarterly data becomes t_Q=79.2 quarters. From (5.3) and (5.4), this implies the use of λ_Q=25199, considerably higher that the "de facto" value of 1600. For the same example of Figure 1, Figure 2 compares the cycles (for annual data) obtained directly with λ_A=100 and, indirectly with λ_Q=1600, the two "de facto" values. The cycles display important differences. Which of the two estimators of the cycle should be chosen? It would be clearly desirable to achieve a higher degree of consistency, and perhaps the use of a 10-year period for the cycle of reference is more appropriate for short-term policy than the use of a 20-year one. Figure 3 compares the direct and indirect cycle estimators for annual data with λ_A=6.7 and λ_Q=1600, the two values associated with the 10-year-period cycle of reference. They are certainly much closer.

A comment should be made on the interpretation of the cycle of reference. On occasion, the HP filter is described as a filter that removes (or estimates) cycles with periods longer than 6 years (see Baxter and King (1999) or Canova (1998)). This description is somewhat misleading. The slope of the gain function is not vertical, and hence there is a range of cyclical periods for which the removal is only partial. For example, as we have just seen with λ_Q=1600, the gain will be 50% for a 10-year cycle. From (2.17) it is found that, roughly, the percentage decreases to 30% for an 8-year cycle and to about 10% for a 6-year cycle; it increases to 70% for a 12-year cycle, and to 90% for a 16-year one. Thus one can say that, approximately, cycles with period below 6 years will be excluded from the trend, while cycles with periods above 16 years will be fully included. For cycles with periods within these two limits, the participation will gradually increase with the length of the period.

Figure 2: Cycles in annual data when λ_Q=1600 and λ_A=100.

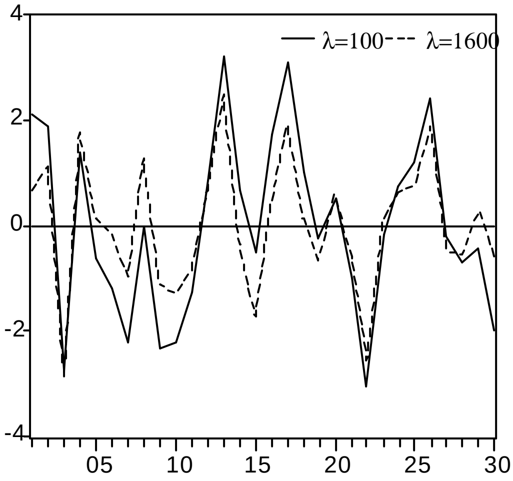
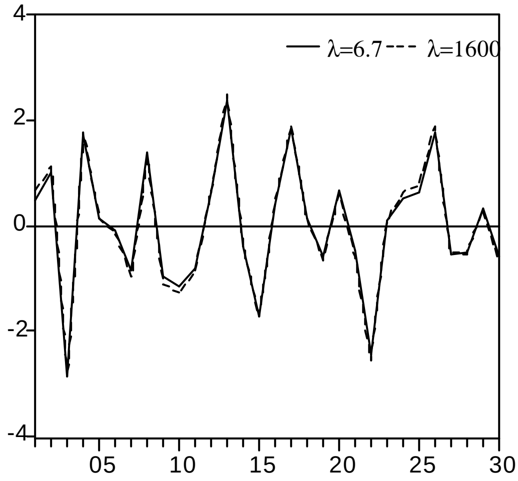


Figure 3: Cycles in annual data when λ_Q=1600 and λ_A=6.7



6. Aggregation by Fixing the Period Associated with the Maximum of the Cycle Spectrum

Ultimately, the properties of the cycle obtained are a combination of two factors: on the one hand, the characteristics of the filter employed; on the other hand, the stochastic properties of the series in question. In the previous two sections we have looked at the first of these factors. We consider now their interaction. To describe the characteristics of the estimated cycle the most appropriate tool is its spectrum. Similarly to (2.16), we can write the estimator of the cycle as

$$\hat{c}_t = J_c(B,F)x_t \quad , \tag{6.1}$$

where $J_c(B,F)$ is given by (2.12). Inspection of (2.12) and (6.1) shows that the filter will cancel up to four unit AR roots in the series. Given that the number of regular differences needed to render a series stationary is, in practice, always smaller than four, it follows that $\hat{c}_t$ will be a stationary process. Its spectrum can be expressed as

$$S_{\hat{c}}(w,l) = [G_c(w,l)]^2 S_x(w) \quad , \tag{6.2}$$

where $G_c(w,l)$ is the gain of the filter (2.12), and $S_x(w)$ is the (pseudo) spectrum of x_t; therefore the spectrum of the cycle estimator is straightforward to obtain. Clearly, series with different stochastic structures will imply different spectra for the cycle even when the same HP filter is used.

As an example, consider the spectra of the cycles for two series that follow a standard and a second-order random-walk model, namely

$$\nabla x_{1t} = a_t \quad , \tag{6.3a}$$

$$\nabla^2 x_{2t} = a_t \quad . \tag{6.3b}$$

From (2.12) and (6.1), the estimators of the cycle can be expressed in terms of the innovations a_t as

$$\hat{c}_{1t} = k_c \frac{(1-B)(1-F)^2}{q_{HP}(B)q_{HP}(F)} a_t \quad , \tag{6.4a}$$

$$\hat{c}_{2t} = k_c \frac{(1-F)^2}{q_{HP}(B)q_{HP}(F)} a_t \quad , \tag{6.4b}$$

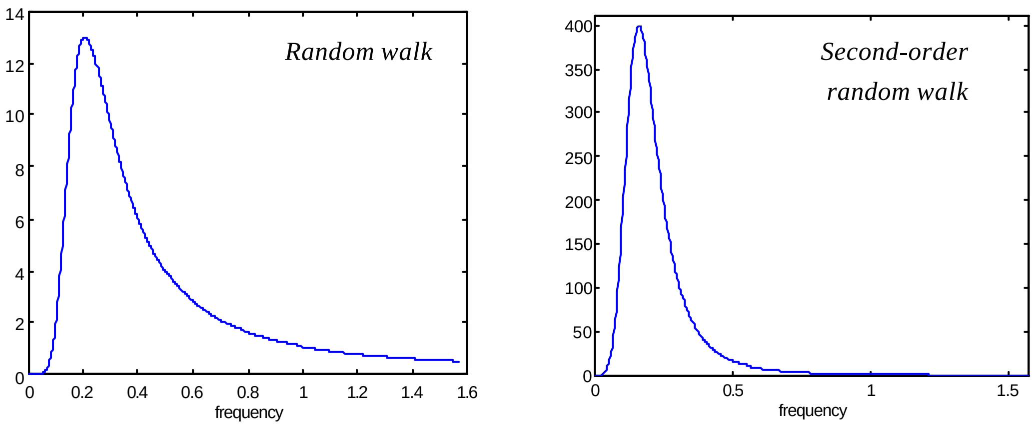
where $k_c = n_c / n_a$. The FT of the ACF of (6.4a) and (6.4b) yield the two spectra, namely,

$$S_{c1}(w) = \frac{8\lambda^2(1-\cos w)^3}{\left[1+4\lambda(1-\cos w)^2\right]^2} V_a \quad , \tag{6.5a}$$

$$S_{c2}(w) = \frac{4\lambda^2(1-\cos w)^2}{\left[1+4\lambda(1-\cos w)^2\right]^2} V_a \quad . \tag{6.5b}$$

The two spectra are displayed in Figure 4 for λ=1600.

Figure 4: Spectra of the cycle component of first and second-order random walk (λ=1600)



Although different, both have the well-defined shape of the spectrum of a stochastic cyclical component, with the variance concentrated around the spectral peak. The cycle associated with the frequency for which the spectrum reaches a maximum will be denoted the "cycle of dominance", and possibly represents the most relevant single descriptive feature of the HP cycle. Thus a natural criterion for aggregation could also be preservation of the cycle of dominance. (This is similar to approximating spectral densities by preserving the mode, see for example Durbin and Koopman (2000)).

Thus an alternative procedure to find HP filters that are approximately consistent under time aggregation is the following. First, given λ_Q, obtain $S_c(w)$ for the quarterly cycle. Second, compute the period for which the maximum of $S_c(w)$ is achieved. Preserving this period implies, for annual data, to obtain the value of λ_A associated with the period $t_Q/4$; for monthly data, to derive the value of λ_M associated with the period $3t_Q$. An advantage of this approach is that it combines the characteristics of the filter with the specific features of the series of interest. On the other hand, it has the disadvantage that no general equivalence between λ's for different frequencies of observation can be obtained, since the equivalence depends on the *ARIMA* model for the series. As a consequence, two issues are of interest:

a) what is the equivalence for some of the most relevant *ARIMA* models;

b) if the much simpler criterion of fixing the period associated with the cycle of reference of the previous section (that does not depend on the series model) is used, will the results be much different from the ones obtained with the criterion of fixing the period associated with the cycle of dominance?

In order to apply the criterion we need to obtain the spectrum of the disaggregate and aggregate cycle, which depends on the model for the series. Two cases will be distinguished, namely, an *I(1)* and an *I(2)* series. Further, since, in both cases, our starting point will be the value of λ_Q for the quarterly series, and our aim is to find the monthly λ_M and annual λ_A, we need to consider models that are consistent under aggregation. For the *I(1)* case, a model consistent under both temporal aggregation and systematic sampling is the *I(1,1)* model

$$\nabla z_t = (1+mB)b_t \qquad b_t \sim w.n.\ (0,V_b) \qquad ; \tag{6.6a}$$

and, for the *I(2)* case, the same is true for the *IMA(2,2)* model

$$\nabla^2 z_t = (1+m_1B+m_2B^2)b_t \qquad b_t \sim w.n.\ (0,V_b) \qquad ; \tag{6.6b}$$

(treating the unit *AR* roots as *AR* polynomials, the results are in Brewer (1973)). Following the previous notation, when the model refers to the disaggregate series, we have $z_t=x_t$, $m=q$, $b_t=a_t$, and $V_b=V_a$; while for the aggregate series, $z_t=X_T$, $m=Q$, $b_t=A_t$, and $V_b=V_A$. It is worth noticing that the *IMA(d,d)* formulation is attractive because it is the limiting model for time aggregates of *ARIMA(p,d,q)* models (Tiao (1972)).

Let, in general, $q_Q=(q_1,q_2)$ denote the *MA* parameters of the quarterly model ($q_2=0$ for the *IMA(1,1)* case), and q_T denote the vector with the *MA* parameters of the transformed model (annual or monthly). Likewise, let $S_Q(w,q,l)$ and $S_T(w,q,l)$ denote the spectra of the quarterly cycle and of the cycle for the transformed series, respectively. In all cases (temporal aggregation or systematic sampling, aggregation of quarterly to annual data or disaggregation of quarterly to monthly data, *IMA(1,1)* or *IMA(2,2)* models) the procedure to obtain the equivalent values of λ for the transformed series can be summarized as follows.

1. Given q_Q and λ_Q, obtain the frequency w_Q such that $S_Q(w,q_Q,l_Q)$ is maximized in the interval $w \in [0,p]$, and the associated period t_Q.
2. Transform t_Q into t_T and obtain the associated frequency w_T.
3. Use the relationship between the variance and covariances of the disaggregate and aggregate series to find q_T given q_Q.
4. Find l_T such that $S_T(w_T,q_T,l)$ is maximized. This l_T is the equivalent value of λ_Q.

Although the procedure is general, in our application we fix λ_Q=1600 for the quarterly data.

6.1. *IMA(1,1)* Model

When z_t follows model (6.6a), it is straightforward to find that the spectrum of the cycle in the HP decomposition of z_t is given by

$$S_c(w,m,l) = \frac{8l^2(1-\cos w)^3}{\left[1+4l(1-\cos w)^2\right]^2}(1+m^2+2m\cos w)V_b \quad , \tag{6.7}$$

and maximizing $S_c(w,m,l)$ with respect to w yields

$$\tilde{w} = a\cos\left[1+\frac{m}{l(1+m)^2} - \sqrt{\frac{3}{4l}+\frac{m^2}{l^2(1+m)^4}}\right] \quad . \tag{6.8}$$

Alternatively, solving for λ, it is obtained that

$$\tilde{l} = \frac{3}{4\left[1-\cos\tilde{w}\right]} - \frac{2m}{(1+m)^2(1-\cos\tilde{w})} \quad . \tag{6.9}$$

Figure 5 presents the function $\tilde{t} = f(l)$ derived from (5.4) and (6.8) for fixed values of the *MA* parameter m. Three values of m are considered: m=0, -0.4, -0.8. The figure illustrates the important influence of l on the cycle period although, when λ is large, relatively large variations in its value hardly affect the length of the period. The figure also reveals how little effect m has on the period of the cycle of dominance since the three values of m provide very similar lines. This fact is confirmed by Figure 6, which displays the spectra of the cycle for l=1600 and three values of m. Although m strongly affects the stochastic variance of the cycle, the frequency for which the maximum is achieved is seen to be practically constant.

Figure 5. *IMA(1,1)*: Period of the cycle of dominance as a function of λ

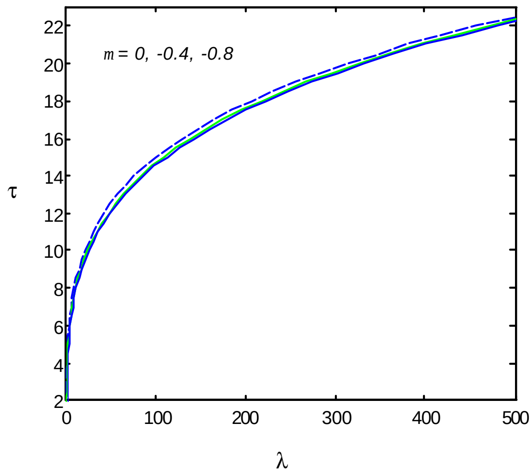
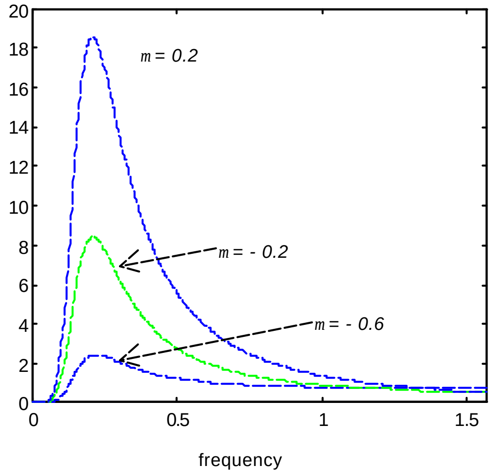


Figure 6: *IMA(1,1): S*pectrum of the cycle for various μ



(a) From Quarterly to Annual Data

Steps (1) and (2) above are common to the case of temporal aggregation and systematic sampling: setting λ_Q=1600 and $m=q_Q$, expression (6.8) yields w_Q, or $t_Q = 2p/w_Q$. For annual series, preserving the period of the cycle of dominance implies setting $t_A = t_Q/4$ and $w_A = 2p/t_A$. In order to obtain λ_A form (6.9) we need to obtain the model for the annual series; this model is different for the two types of aggregation.

For the temporal aggregation case, proceeding as in Section 3 and using the same notation, it is found that

$$DX_T = X_T - X_{T-1} = S_4^2(1+q_Q b)a_t \quad , \tag{6.10}$$

where, as before, T and t refer to the same date, expressed in annual and quarterly time units. Similarly, $DX_{T-1} = S_4^2(1+q_Q b)a_{t-4}$, from which it follows that X_T is an *IMA(1,1)* model,

$$DX_T = (1+q_A B)A_t \quad . \tag{6.11}$$

Therefore the variance and lag-*1* autocovariance of the r.h.s. of (6.11),

$$\Gamma_0 = (1+q_A^2)V_A \quad , \tag{6.12a}$$

$$\Gamma_1 = q_A V_A \quad , \tag{6.12b}$$

have to be equal to the variance and lag-4 autocovariance of the r.h.s. of (6.10), which, after simplification, are equal to

$$g_0 = (44 + 80q_Q + 44q_Q^2)V_a \quad , \tag{6.13a}$$

$$g_4 = (10 + 24q_Q + 10q_Q^2)V_a \quad . \tag{6.13b}$$

Equating the r.h.s. of (6.12a, b) with the r.h.s. of (6.13a, b) yields

$$\frac{1+q_A^2}{q_A} = \frac{44 + 80q_Q + 44q_Q^2}{10 + 24q_Q + 10q_Q^2} \quad .$$

Letting $c = (44 + 80q_Q + 44q_Q^2)/(10 + 24q_Q + 10q_Q^2)$, the MA parameter of the annual $IMA(1,1)$ model is given by the invertible solution of equation

$$q_A^2 - cq_A + 1 = 0 \quad . \tag{6.14}$$

For the case of systematic sampling, equation (6.10) is replaced by

$$DX_T = x_t - x_{t-4} = (1-b^4)x_t = S_4(1+q_Q b)a_t \quad , \tag{6.15}$$

so that, after simplification, the system (6.13) is replaced by

$$g_0 = (4 + 6q_Q + 4q_Q^2)V_a \quad , \tag{6.16a}$$

$$g_4 = q_Q V_a \quad . \tag{6.16b}$$

Defining $c = (4 + 6q_Q + 4q_Q^2)/q_Q$, the MA parameter for the $IMA(1,1)$ annual model is again the invertible solution of (6.14).

Having obtained q_A, setting m=q_A, and $\tilde{w} = w_A$ in (6.9), the equivalent value of λ for annual series, λ_A, is obtained. The period associated with the cycle spectral maximum will be identical for the quarterly and annual series.

(b) From Quarterly to Monthly Data

Step (1) is as in the previous case. For λ_Q=1600 and $m=q_Q$, (6.8) yields w_Q and hence the period t_Q associated with the cycle spectral peak. Preserving this period implies for monthly data, setting $t_M = 3t_Q$ and $w_M = 2p/t_M$. In order to obtain the equivalent value for λ_M through (6.9) we need to derive the model for the disaggregate monthly series. Given that the quarterly model is an *IMA(1,1)* model, the monthly series will also follow an *IMA(1,1)* model. As before, to obtain the *MA* parameter we need to distinguish between temporal aggregation and systematic sampling.

Under temporal aggregation, expression (6.10) remains unchanged, with S_4 replace by $S_3=1+B+B^2$. The relationship between the variance and lag-*1* autocovariance for the quarterly series and the variance and lag-3 autocovariance for the monthly series is found to be

$$(1+q_Q^2)V_A = (19+32q_M + 19q_M^2)V_a \tag{6.17a}$$

$$q_Q V_A = (4+11q_M + 4q_M^2)V_a \tag{6.17b}$$

Letting $c_1=(1+q_Q^2)/q_Q$, and solving (6.17) for q_M, it is obtained that q_M, the *MA* parameter of the disaggregate monthly model, is the invertible solution of the equation $x^2+c_2x+1=0$, where $c_2=(32-11c_1)/(19-4c_1)$. The equation has complex solutions when $q_Q{}^3$ 0.3 so that *IMA(1,1)* monthly models aggregate into *IMA(1,1)* quarterly models with the *MA* parameter restricted to the range $-1<q_Q<0.3$. In practice, it is unlikely that the estimated value of q_Q falls outside this range.

For the case of systematic sampling, equation (6.15) remains valid, with S_4 replaced by S_3. The system of covariance equations (6.1) is then replaced by:

$$(1+q_Q^2)V_A = (3+4q_M + 3q_M^2)V_a$$

$$q_Q V_A = q_M V_a \quad ,$$

so that, if $c_1=(1+q_Q^2)/q_Q$ and $c_2=(4-c_1)/3$, the value of q_M is the invertible solution of the equation $x^2+c_2x+1=0$. The system yields complex solutions when $q_Q>0.33$ and hence systematic sampling of monthly *IMA(1,1)* models yields quarterly *IMA(1,1)* models with the *MA* parameter restricted to the range $-1<q_Q<0.33$, very similar to the temporal aggregation case.

Table 1 displays the equivalent monthly and annual values of l, with the quarterly value set at l_Q=1600, obtained with the criterion of preserving the period associated with the cycle spectral peak, when the series follows an *IMA(1,1)* process, and for different values of the *MA* parameter q_Q. It is seen, first, that when q_Q is not close to -1, the period associated with the cycle spectral peak (i.e., the cycle of dominance) takes a value between roughly 7 and 7.5 years.

Table 1: *IMA(1,1)*: monthly and annual λ values that preserve the period of the cycle spectral peak for λ_Q=1600.

θ_Q	period of the cycle of dominance (in years)	equivalent values of λ			
		temporal aggregation		systematic sampling	
		annual (λ_A)	monthly (λ_M in 10^3)	annual (λ_A)	monthly (λ_M in 10^3)
-0.9	3.22	8.38	127.5	33.24	45.3
-0.8	5.72	6.53	129.3	20.86	71.4
-0.7	6.74	6.22	129.6	14.21	97.8
-0.6	7.14	6.12	129.8	10.85	112.0
-0.5	7.32	6.07	129.8	9.14	119.4
-0.4	7.41	6.05	129.8	8.21	123.4
-0.3	7.47	6.04	129.9	7.66	125.8
-0.2	7.50	6.03	129.9	7.33	127.2
-0.1	7.52	6.02	129.9	7.11	128.2
0.0	7.53	6.02	129.9	6.97	128.8
0.1	7.54	6.02	129.9	6.88	129.2
0.2	7.55	6.02	129.9	6.81	129.8
0.3	7.55	6.01	129.9 (*)	6.77	129.7
0.4	7.56	6.01	-	6.74	-
0.5	7.56	6.01	-	6.72	-
0.6	7.56	6.01	-	6.70	-
0.7	7.56	6.01	-	6.69	-
0.8	7.56	6.01	-	6.69	-
0.9	7.56	6.01	-	6.68	-

(*) Computed for θ_Q = 0.295. Values of θ_Q below this line cannot be obtained by aggregation of monthly IMA(1,1) models.

When aggregation of the series is made through temporal aggregation the results are seen to be very stable in all cases. The monthly equivalent values l_M are always close to 130000, and the annual equivalent value l_A, unless q_Q is close to -1, is slightly above 6. These values are close to the ones obtained with the criterion of preserving the period of the cycle of reference (i.e., the value of w for which the gain of the HP filter is ½), given by (5.6).

When aggregation is achieved through systematic sampling, the results are less stable, and the criterion of preserving the cycle of dominance provides a less satisfactory approximation in the neighborhood of q_Q=-1. Nevertheless for the range -0.7<q_Q<1, the results are considerable stable and not too distant to the ones obtained for the temporal aggregation case. (Notice that, if l_Q=1600 is chosen as the quarterly value, the annual value l_A=100 used by the OECD and the ECB is far from any of the values contained in the table).

Altogether, from an applied point of view, the differences between the equivalent λ values are moderate, and the approximation (discussed in Section 5) based on the criterion of preserving the cycle of reference, which is trivial to compute and does not depend on the particular series at hand nor on the series model, provides in most cases a reasonably close approximation. The only exception may be series with values of q_Q close to -1, in which case l_A might need to be increased and l_M decreased.

The following example illustrates the closeness of the approximations. "Monthly" series were generated for the *IMA(1,1)* model (6.6a) with q =-0.3 and $V_a=1$. The monthly series was aggregated into quarterly and annual series. Using the monthly values of λ_M derived with the criteria of preserving the cycle of reference and the cycle of dominance, the monthly HP cycle was obtained. These monthly cycles were then aggregated into quarterly and annual ones. Then, setting λ_Q=1600, the direct quarterly cycle was obtained, and aggregated into annual. Finally with the two annual values of λ_A implied by the two criteria, the direct annual cycle was obtained.

Figure 7 compares the monthly, quarterly and annual cycles obtained with the direct and indirect procedures, for the two cases of temporal aggregation and systematic sampling. Part a) compares the quarterly cycles obtained through aggregation of the monthly cycles using the two criteria of preserving the cycle of reference (*IR*) and of dominance (*ID*), with the one obtained through direct adjustment of the quarterly series with λ_Q=1600 (*D*). Part b) compares the annual cycles obtained by aggregation of the monthly cycles using the two criteria (*IMR* and *IMD*), with the ones obtained indirectly through aggregation of the direct estimation of the quarterly cycle (*IQ*), and with the ones obtained through direct estimation on the annual data using the two criteria (*DR* and *DD*). In all cases, the differences between the different cycle estimators are mild. This conclusion remains valid for other values of q_M, except for the case of systematic sampling with q_M <-0.8.

Figure 7a: Quarterly cycles

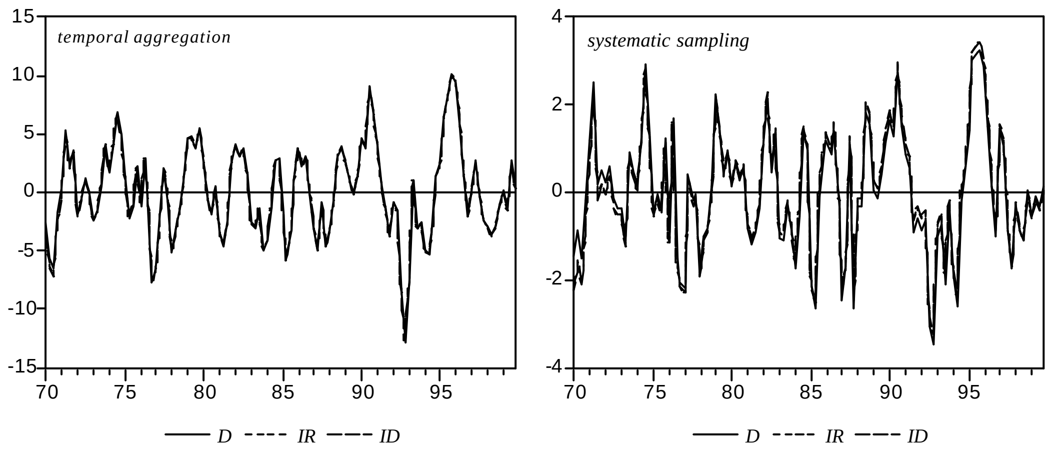
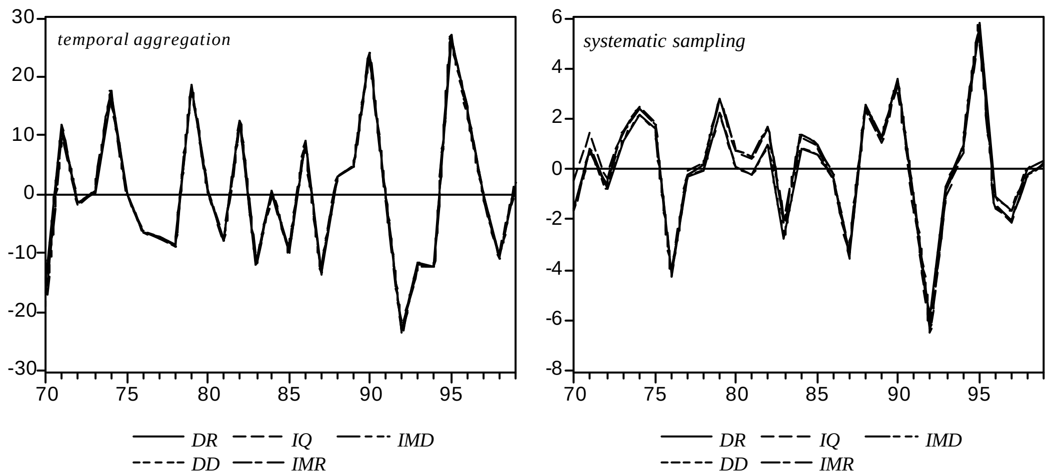


Figure 7b: Annual cycles



6.2. *IMA(2,2)* Model

When z_t follows the *IMA(2,2)* model given by (6.6b), from (2.12) and (6.1) it is found that the HP cycle follows the model

$$\hat{c}_t = \frac{l\,(1-F)^2\,(1+m_1 B + m_2 B^2)}{1+\,l\,(1-B)^2\,(1-F)^2}\,a_t \qquad ,$$

with spectrum

$$S_c(w,\,l\,,m_1\,,m_2) = \frac{4\,l^2\,(1-\cos w)^2\,(1+m_1^2+m_2^2+2m_1(1+m_2)\cos w+2m_2\cos 2w)}{\left[1+4\,l\,(1-\cos w)^2\right]^2}V_a \quad . \qquad (6.18)$$

The spectrum is maximized for

$$\tilde{l} = \frac{1}{4(1-\cos w)} - \frac{m_1 + m_1 m_2 + 4m_2 \cos w}{2(1-\cos w)\left[1 + m_1(m_1 + 1 + \cos w) + m_2(m_2 - 2) + (2 + m_1)\cos w\right]} \quad , \tag{6.19}$$

and the value of w for which (6.18) attains a maximum can be found numerically from the first-order conditions; we represent this value as

$$\tilde{w} = \tilde{w}(\,l\,, m_1, m_2) \qquad . \tag{6.20}$$

Proceeding as in the previous section, given m_1 and m_2 for the quarterly model and λ_Q, we use (6.20) to compute the frequency of the quarterly cycle of dominance, and the associated period. Expressing this period in terms of annual and monthly data, we obtain the annual and monthly associated frequencies. Once we know the MA parameters m_1 and m_2 of the annual and monthly model, (6.19) provides the values of l_A and λ_M equivalent to λ_Q. The monthly, quarterly, and annual series follow $IMA(2,2)$ models, but the procedure used in the previous section to derive the relationship between the MA parameters of the models becomes very cumbersome. We follow instead the Stram-Wei procedure described the Appendix A.

Let x_t, (q_1, q_2), and V_a denote the disaggregate series, the MA parameters of its model, and its innovation variance, respectively. Likewise, let X_T, (Q_1,Q_2), and V_A denote the aggregate series, the MA parameters of its model, and its innovation variance. If (g_0,g_1,g_2) and (G_0,G_1,G_2) represent the variance, lag-1, and lag-2 autocovariances of $Ñ^2 x_t$ and $Ñ^2 X_T$, respectively, we have

$$g_0 = (1 + q_1^2 + q_2^2)V_a \quad , \tag{6.21a}$$

$$g_1 = q_1(1 + q_2)V_a \quad , \tag{6.21b}$$

$$g_2 = q_2 V_a \quad , \tag{6.21c}$$

and, replacing (q_1,q_2), and V_a by (Q_1,Q_2), and V_A, similar expressions hold for G_0, G_1 and G_2. If g and G denote the vectors $g=(g_0,g_1,g_2)'$ and $G=(G_0,G_1,G_2)'$, the Stram-Wei procedure permits us to express the relationship between g and G as

$$G=Mg \ ,$$

where M is a $(3\text{x}3)$ matrix whose construction is detailed in Appendix A. Thus, given g, one can obtain G and, using the inverse relationship $g=M^{-1}G$, given G, one can obtain g. The aggregate/disaggregate *MA* parameters are found by factorizing the ACF obtained, as explained in the Appendix B. The M matrices relevant to our application are found to be

	Temporal Aggregation	Systematic Sampling
Quarterly to Annual Aggregation	$M=\begin{bmatrix} 580 & 1092 & 912 \\ 216 & 456 & 512 \\ 6 & 22 & 56 \end{bmatrix}$	$M=\begin{bmatrix} 44 & 80 & 62 \\ 10 & 24 & 32 \\ 0 & 0 & 1 \end{bmatrix}$
Monthly to Quarterly Aggregation	$M=\begin{bmatrix} 141 & 252 & 180 \\ 50 & 111 & 132 \\ 1 & 6 & 21 \end{bmatrix}$	$M=\begin{bmatrix} 19 & 32 & 20 \\ 4 & 11 & 16 \\ 0 & 0 & 1 \end{bmatrix}$

Table 2 is analogous to Table 1 for the *IMA(2,2)* case, and displays the equivalent monthly and annual l values when the quarterly value is l_Q=1600, using the criterion of preserving the period associated the cycle spectral peak (the quarterly *MA* values $q_{Q,1}$ and $q_{Q,2}$ are restricted to lie in the invertible region; see Box and Jenkins, 1970, p.73). Altogether, the values of the q parameters have a moderate effect on l_A and l_M, and the values equivalent to l_Q=1600 are similar to those obtained for the *IMA(1,1)* case. As before, they are also close to those obtained with the criterion of preserving the cycle of reference.

Table 2: *IMA(2,2)*: monthly and annual λ values that preserve the period of dominance for λ_Q=1600.

$\theta_{Q,1}$	$\theta_{Q,2}$	period of the cycle of dominance (years)	equivalent values of λ: temporal aggregation, annual (λ_A)	temporal aggregation, monthly (λ_M in 10^3)	systematic sampling, annual (λ_A)	systematic sampling, monthly (λ_M in 10^3)
0.2	0.0	9.9	6.02	131.8	6.24	129.6
0.0	0.0	9.9	6.03	128.6	6.24	129.6
0.0	0.2	10.0	6.01	131.2	6.23	131.8
-0.2	0.0	9.9	6.03	127.7	6.24	129.6
-0.2	0.2	10.0	6.01	131.1	6.23	131.4
-0.4	0.0	9.9	6.04	125.2	6.26	129.6
-0.4	0.2	10.0	6.01	130.8	6.23	130.9
-0.6	0.0	9.7	6.05	117.7	6.29	129.5
-0.6	0.2	9.9	6.02	129.7	6.24	129.8
-0.8	0.2	9.9	6.04	125.7	6.28	125.6
-0.8	0.4	10.0	5.98	133.5	6.22	133.9
-1.0	0.2	9.4	5.99	102.9	6.59	102.7
-1.0	0.4	10.0	5.98	132.9	6.24	133.0
-1.4	0.6	10.1	5.74	140.8	6.23	140.9

The following example illustrates the closeness of the approximations in the same way as we did for the *IMA(1,1)* model. "Monthly" series were generated for the *IMA(2,2)* model (6.6b) with q_1 =-0.6, q_2 =0.2 and V_a=1. The monthly series was aggregated into quarterly and annual series, and then direct and indirect cycles were estimated using the previous criterion. Figure 8 is the same as Figure 7 for the case of the *IMA(2,2)* model. As we can see the all estimated cycles are almost identical.

Figure 8a: *IMA(2,2)* model: quarterly cycles

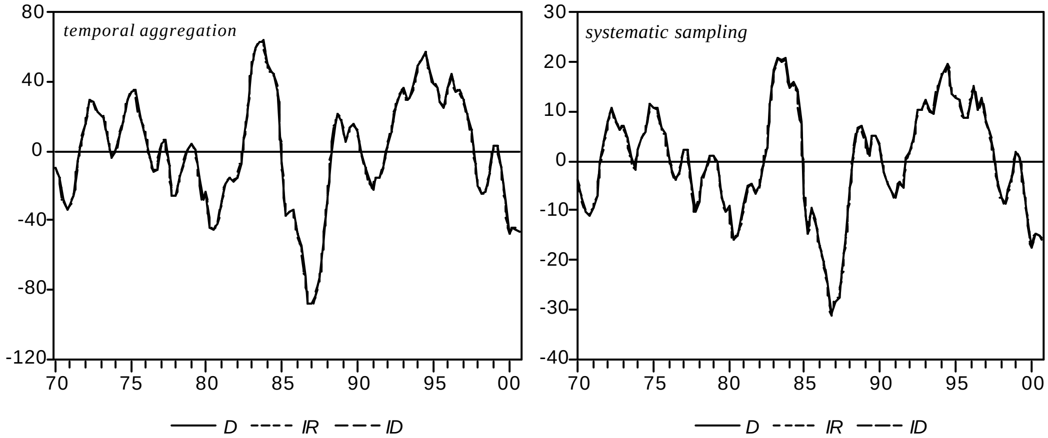
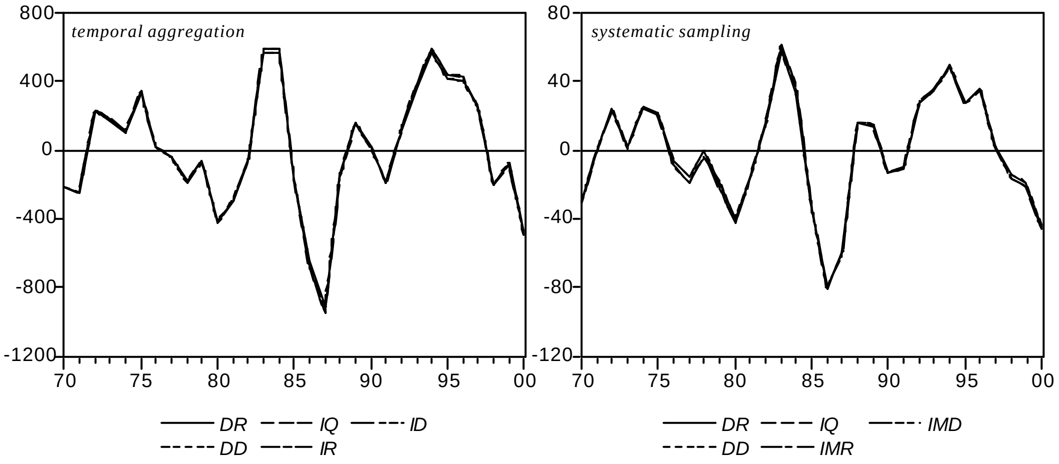


Figure 8b: *IMA(2,2)* model: annual cycles


temporal aggregation
systematic sampling
800
400
0
-400
-800
-1200
80
40
0
-40
-80
-120
70
75
80
85
90
95
00
DR
IQ
ID
DD
IR
DR
IQ
IMD
DD
IMR

7. Least squares minimization of the distance between direct and indirect cycle.

For a particular application, it is always possible to compute close-to-equivalent values of λ through least-squares minimization of the distance between the direct and indirect aggregate cycles. If the indirect cycle is respected, and λ_0 is the value of λ applied the disaggregate series, the value λ_d to use for direct adjustment is given by

$$\hat{l}_d = \arg min \sum_T \left[\hat{C}_{i,T}(l_0) - \hat{C}_{d,T}(l_d)\right]^2 \qquad (4.3)$$

where $\hat{C}_{i,T}(l_0)$ and $\hat{C}_{d,T}(l_d)$ denote the estimated indirect and direct aggregate cycle, respectively. This procedure is, of course, considerably more complex and cumbersome than the simple expressions (5.3) and (5.5), based on the criteria of preserving the cycle of reference. Further, being dependent on the particular realization, minimization of the distance between direct and indirect estimation, may produce variability in the values of λ, that could yield inconsistencies for the different levels of aggregation. It is nevertheless of interest to look at whether the (case-by-case) solution (4.3) is likely to yield values of λ that strongly depart from the values obtained with the previous criteria.

We only looked at the case of aggregating quarterly series into annual ones (using λ_Q=1600 for direct estimation of the quarterly cycle), under temporal aggregation and systematic sampling, and for the *IMA(1,1)* and *IMA(2,2)* models for different values of the parameters. For each one of the cases, only 100 simulations were made; the results seemed clearly stable given our level of precission (first decimal point in λ_A). For each simulation, expression (4.3) was solved and $\hat{l}_d$ estimated; then the mean and standard deviation of the $\hat{l}_d$ *'s* obtained were computed.

Again, except for the case of systematic sampling an *IMA(1,1)* model with a large and negative value of its *MA* parameter, the values of λ_A are considerable stable and relatively close to the ones obtained with the previous criteria. One remarkable feature is that the single value obtained with the criteria of preserving the cycle of reference (λ_A=6.65, see expression (5.6)) is, in none of the cases, significantly different from the values in Tables 3 and 4. Comparison of Tables 1 and 3, and of Tables 2 and 4, shows that the least-squares minimum distance procedure yields systematically slightly large values of λ. This can be seen to be a result of the asymmetry around its mode of the spectrum, used when the period of the cycle of dominance is preserved.

Table 3: Least square minimization: *IMA(1,1)* models.

θ_Q	temporal aggregation λ_A		systematic sampling λ_A	
	mean	std.dev	mean	std.dev
-0.9	7.3	1.0	(*)	(*)
-0.8	6.9	0.7	(*)	(*)
-0.7	6.8	0.5	(*)	(*)
-0.6	6.8	0.4	15.1	11.4
-0.5	6.7	0.3	14.0	16.7
-0.4	6.6	0.2	10.8	13.3
-0.3	6.7	0.2	8.8	3.7
-0.2	6.7	0.2	8.4	4.1
-0.1	6.6	0.2	7.5	2.0
0.0	6.6	0.2	7.4	1.7
0.1	6.7	0.2	7.6	1.3
0.2	6.7	0.2	7.1	1.2
0.3	6.6	0.2	7.2	1.3
0.4	6.7	0.2	7.1	1.2
0.5	6.6	0.1	7.2	1.1
0.6	6.6	0.1	7.1	1.2
0.7	6.6	0.1	7.1	1.1
0.8	6.6	0.1	7.0	1.2
0.9	6.6	0.1	7.0	1.2

(*) Numerical problems because of the flat surface of the objective function around the minimum.

Table 4: Least square minimization: *IMA(2,2)* models.

$\theta_{Q,1}$	$\theta_{Q,2}$	temporal aggregation λ_A		systematic sampling λ_A	
		mean	std.dev	mean	std.dev
-0.9	0	6.6	0.1	7.2	1.4
-0.8	0	6.5	0.1	6.7	0.9
-0.6	0	6.5	0.1	6.7	0.7
-0.4	0	6.5	0.1	6.7	0.6
-0.2	0	6.5	0.1	6.6	0.6
0.0	0	6.5	0.1	6.5	0.7
0.2	0	6.5	0.1	6.4	0.6
0.4	0	6.5	0.1	6.6	0.6
0.6	0	6.5	0.1	6.6	0.6
0.8	0	6.5	0.1	6.5	0.6
0.9	0	6.5	0.1	6.6	0.4
-0.6	-0.3	6.6	0.1	6.9	1.1
-0.6	0.3	6.5	0.1	6.5	0.7
-0.5	-0.3	6.5	0.1	6.6	0.9
-0.5	0.3	6.5	0.1	6.6	0.6
-0.4	-0.3	6.5	0.1	6.6	0.9
-0.4	0.3	6.5	0.1	6.6	0.6
0.4	-0.3	6.5	0.1	6.6	0.6
0.4	0.3	6.5	0.1	6.5	0.6
0.5	-0.3	6.5	0.1	6.6	0.5
0.5	0.3	6.5	0.1	6.5	0.5
0.6	-0.3	6.5	0.1	6.6	0.6
0.6	0.2	6.5	0.1	6.5	0.5

8. Conclusions

We have analyzed the time aggregation properties of the Hodrick-Prescott (HP) filter, focusing on monthly, quarterly, and annual observations. Two types of aggregation have been considered: Temporal Aggregation, whereby the aggregate series consists of (non-overlapping) sums (or averages) of disaggregate values, and Systematic Sampling, whereby the aggregate series is equal to a value of the disaggregate series sampled at periodic intervals. The main results can be summarized as follows.

For the two types of aggregation, the HP filter does not preserve itself under aggregation, in the following sense. If m_t and c_t are the trend and cycle obtained with the HP filter applied to the disaggregate data, the aggregates obtained with them cannot be seen as the exact result of an HP filter applied to the aggregate data or, alternatively, if M_t and C_t are obtained from applying the HP filter to the aggregate data, they cannot be interpreted as the exact result from aggregating the components obtained with an HP filter applied to the disaggregate data.

Be that as it may, for the temporal aggregation case (the case most encountered in practice), it is possible to find λ values for monthly, quarterly and annual data, that behave well under aggregation, in the sense that direct and indirect adjustment of the aggregate data is extremely close. These "equivalent" values of λ can be obtained through several criteria, namely:

a) Least squares minimization of the distance between the aggregate and disaggregate HP filters.

b) Preserving the period of the cycle for which the gain of the filter is ½ (the cycle of reference).

c) Preserving the period of the cycle associated with the cycle spectral peak (the cycle of dominance).

d) Least squares minimization of the distance between the direct and indirect estimates of the cycle.

Criterion d) is the least appealing because it depends on the particular realization of the time series at hand and provides the most computationally complicated solution. Criterion c) is relatively complex and depends on the particular model identified for the series. Criterion b) provides general results, obtained through a very simple rule, trivial to compute, that can be easily applied to aggregate/disaggregate data with any frequency of observation. For the case of temporal aggregation, all criteria lead to similar results for the equivalent values of λ. In particular, if the quarterly value λ_Q=1600 is used, the range of equivalent values for the monthly and annual HP parameters are, approximately,

$$115000 < \lambda_M < 130000 \quad , \quad 6 < \lambda_A < 7 \quad ,$$

and, for values within these ranges, differences between direct and indirect aggregation are, to all effects, negligible.

For the case of systematic sampling, the aggregation properties of the filter are less satisfactory, the results are less robust with respect to the stochastic properties of the series, and the equivalent monthly, quarterly, and annual values of λ are more volatile. As far as the trend-cycle decomposition of the series is concerned, aggregation by averaging is better behaved than aggregation through systematic sampling. Still, even in this latter case, unless the series has a narrow spectral peak for the zero frequency (i.e., an *MA* root not far from -1) the equivalent values of λ yield reasonably close results. If, as before, we set $\lambda_Q=1600$, the range of values for the monthly and annual HP parameters are, approximately,

$$100000 < \lambda_M < 140000 \qquad 6 < \lambda_A < 14 \quad ,$$

Although wider than the ranges for the temporal aggregation case, for values within those ranges, the difference between direct and indirect estimation is of moderate size. It is worth pointing out that the values obtained with criterion a) for the case of systematic sampling, given by (4.6), are not included in the previous range. This is due to the fact that WK enforcement of the HP filter implicitly assumes an *IMA(2,2)* model for the series with one of the *MA* roots very close to -1.

Bearing this exception in mind (systematic sampling of models with a unit root close to -1) it seems safe to conclude that the criterion of preserving the period of the cycle for which the HP filter gain is 50% provides the most appealing solution to the problem of providing HP cycles for different data frequency that are reasonably compatible. First, as seen in Section 7, the values of λ provided by this criterion are extremely close the ones that minimize the least-squares distance between direct and indirect estimation. Besides, the solution does not depend on the series realization, nor on the series model; it consists of the application of the simple expressions (5.3) and (5.5) to move among different frequencies. Table 5 exhibits several triplets (λ_A, λ_Q and λ_M) with equivalent values of λ for the annual, quarterly and monthly periodicities, under the criterion of preserving the cycle of reference; the table also includes the period of this cycle. This period represents the value above which most of the series variation is assigned to the trend. According to Table 5, a short-term interest would indicate a reference period of about 10 years. The choice of this period is, ultimately, arbitrary and should reflect the interest of the analyst, though consistency of the cycle for different time units would seem a desirable property.

The table shows how the values of λ used in practice for the different levels of aggregation are not quite equivalent. Given that larger values of λ imply longer periods of the cycle of reference, the values used in practice show that, for more aggregated data, cycles with longer periods have been seeked. This choice may well reflect the fact that annual data implicitly implies a larger perspective due to the time length between observations, so that, for example, policy based on monthly data obviously can be more reactive to the short-term than policy based on annual data. But when the disaggretate data is available, this reason for inconsistency with respect to the time units makes no sense.

Table 5: Annual, quarterly, and monthly values of λ that are (approximately) compatible.

λ_A	λ_Q	λ_M	cycle of reference (years)
1	179	***14,400***	5.7
5	1,190	95,972	9.2
6	1,437	115,975	9.7
7	***1,600***	129,119	9.9
10	2,433	196,474	11.0
15	3,684	297,715	12.2
20	4,940	399,339	13.2
25	6,199	501,208	13.9
30	7,460	603,250	14.6
35	8,723	705,424	15.2
40	9,986	807,702	15.7
70	17,585	1,422,774	18.1
100	25,199	2,039,248	19.8
200	50,633	4,098,632	23.6
400	101,599	8,225,728	28.0

The most often used values are indicated in bold

Appendix A: Construction of the Stram-Wei Aggregation Matrix

The Stram-Wei aggregation matrix, M, relates the second moments of the stationary transformation of the aggregate series with those of the corresponding disaggregate series. We consider $IMA(d,q)$ models, so that the stationary tranformation of the series x_t is Ñ$^d x_t$.

Let $k=h/j$ be the order of aggregation, where h and j are the number of observations per year for the disaggregated and aggregated series respectively, and let d be the order of integration of the disaggregate series. Define $n=(d+1)$ for temporal aggregation and $n=d$ for systematic sampling.

Let g_i and G_i be the autocovariance of order i for the stationary transformation of the disaggregate and aggregate series respectively. Stram and Wei (1986) prove the following relationship for the case of temporal aggregation:

$$\Gamma_i = (1+B+....+B^{k-1})^{2n} g_{(ki+n(k-1))} \qquad i=0,1,\dots \tag{B1}$$

The systematic sampling case is not consider by Stram and Wei (1986) but, proceeding in a similar manner it is straightforward to find that the relationship (B1) also holds.

If x_t follows an $IMA(d,q)$ model, then the aggregate series X_T follows an $IMA(d,Q)$ process with Q£$[k^{-1}(n(k-1)+q)]$, where [x] denotes the integer part of x. If g and G denote the vectors $g=(g_0,g_1,g_2)'$ and $G=(G_0,G_1,G_2)'$, the Stram-Wei procedure permits us to express the relationship between g and G as

$$G=Mg\ .$$

Let $S=(1+B+B^2+\ldots+B^{(k-1)})$ be the aggregation operator, and c be a $1x(2n(k-1)+1)$ vector with elements (c_i) the coefficients of B^i in the polynomial S^{2n}. Define the matrix A as the following $(Q+1)(kQ+2n(k-1)+1))$ matrix:

$$A = \begin{bmatrix} c & 0_k & 0_{(Q-1)k} \\ 0_k & c & 0_{(Q-1)k} \\ 0_{2k} & c & 0_{(Q-2)k} \\ \dots & \dots & \dots \\ 0_{(Q-1)k} & c & 0_k \\ 0_{(Q-1)k} & 0_k & c \end{bmatrix}$$

where 0_j is a $1xj$ vector of zeros.

Adding the column $(n(k\text{-}1)+1\text{-}j)$ of matrix A to the column $(n(k\text{-}1)+1+j)$ of the same matrix, for $j=1$ to $n(k\text{-}1)$, and then, deleting the first $n(k\text{-}1)$ columns, we obtain a new matrix A^*. Then, the matrix M consists of the first $q+1$ columns of A^*.

Consider as example a quarterly $IMA(1,1)$ model which is aggregated to annual frequency. In this case we have $k=4$, $d=1$, and $q=Q=1$.

For temporal aggregation $n=(d+1)=2$ and considering the coefficients of $S^{2n}=(1+B+B^2+B^3)^4$, given by $c=(1\ \ 4\ \ 10\ \ 20\ \ 31\ \ 40\ \ 44\ \ 40\ \ 31\ \ 20\ \ 10\ \ 4\ \ 1)$, it is found that A is the following $(2x17)$ matrix:

$$A=\begin{bmatrix} 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \end{bmatrix},$$

from which it is obtained that $A^*=\begin{bmatrix} 44 & 80 & 62 & 40 & 20 & 8 & 2 & 0 & 0 & 0 & 0 \\ 10 & 24 & 32 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \end{bmatrix}$. The matrix M is $\begin{bmatrix} 44 & 80 \\ 10 & 24 \end{bmatrix}$ and hence $\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}=\begin{bmatrix} 44 & 80 \\ 10 & 24 \end{bmatrix}\begin{bmatrix} g_0 \\ g_1 \end{bmatrix}$.

Given that $g_0=(1+q_Q^2)V_a$ and $g_1=q_Q V_a$ we have the same relationships as in (6.13a) and (6.13b) that allows us to obtain the parameters of the aggregate model.

For the case of systematic sampling $n=d=1$, the vector c contains the coefficients of $S^{2n}=(1+B+B^2+B^3)^2$, that is $c=(1\ \ 2\ \ 3\ \ 4\ \ 3\ \ 2\ \ 1)$, and A is the following $(2x11)$ matrix:

$$A=\begin{bmatrix} 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

Then, $A^*=\begin{bmatrix} 4 & 6 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix}$, and the matrix M is $\begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$. Therefore we obtain the same relationship as in (6.16a) and (6.16b), that allow us to obtain the parameters of the aggregate model.

Appendix B: Factorization of an *MA(2)* process

Given the variance, g_0, and the lag-1 and lag-2 autocovariances g_1 and g_2, of an $MA(2)$ model, the MA parameters can be obtained by solving the nonlinear system of equations (6.21). We provide a much simpler procedure, based on the general algorithm in Maravall and Mathis (1994, appendix A).

Compute $b=g_1/g_2$ and $c=g_0/g_2-2$, and let y_1 and y_2 be the two solutions of the equation $y^2+by+c=0$. Consider first the case in which the two roots are real. Solve the two equations $z^2-y_jz+1=0,\ j=1,2$, and select in each case the root z_j such that $|z_j|$ ³ 1. If the two roots selected are z_1 and z_2, the $MA(2)$ polynomial is given by $q(B)=(1-z_1B)(1-z_2B)$.

When the roots y_1 and y_2 are complex we proceed as follows. Let $y_1=a+bi$ and $y_2=a-bi$; define $k=a^2-b^2-4,\ m=2ab,\ h=+\left[\left(|k|+\left(k^2+m^2\right)^{1/2}\right)/2\right]^{1/2}$. If k ³ 0, let $c=h,\ d=m/2h$; if $k<0$, let $d=[sign(m)]h,\ c=m/2d$. Consider the two complex numbers

$$z_1 = z_1^r + z_1^i i \ ;\ z_2 = z_2^r + z_2^i i\ ,$$

where $z_1^r = (-a+c)/2$, $z_1^i = (-b+d)/2$, $z_2^r = (-a-c)/2$ and $z_2^i = (-b-d)/2$ and denote with z_j the one with the smallest modulus. Then

$$q_1 = 2z_j^r$$

$$q_2 = \left(z_j^r\right)^2 + \left(z_j^i\right)^2 ,$$

and the $MA(2)$ polynomial is given by $q(B)=1+q_1B+q_2B^2$. The innovation variance can be obtained through $V_a = g_0/(1+q_1^2+q_2^2)$.

References

Apel, M., Hansen, J., and Lindberg, H. (1996) “Potential Output and Output Gap”, *Quarterly Review of the Bank of Sweden* 3, 24-35.

Backus D.K. and Kehoe, P.J. (1992), "International Evidence on the Historical Properties of Business Cycles", *The American Economic Review*, 82, 864-888.

Baxter M. and King, R.G. (1999), "Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series", *Review of Economics and Statistics* 81, 575-593.

Bell, W.R. (1984), “Signal Extraction for Nonstationary Time Series”, *Annals of Statistics* 12, 644-664.

Box, G.E.P. and Jenkins, G.M. (1970), *Time Series Analysis: Forecasting and Control*, San Francisco: Holden-Day.

Brewer, K.R.W. (1973), “Some Consequences of Temporal Aggregation and Systematic Sampling for ARMA and ARMAX Models”, *Journal of Econometrics*, 1, 133-154.

Burman, J.P. (1980), "Seasonal Adjustment by Signal Extraction", *Journal of the Royal Society A* 143, 321-337.

Canova, F. (1998), "Detrending and Business Cycle Facts", *Journal of Monetary Economics* 41, 475-512.

Cleveland, W.P. and Tiao, G.C. (1976), “Decomposition of Seasonal Time Series: a Model for the X-11 Program”, *Journal of the American Statistical Association* 71, 581-587.

Cogley, T. and Nason, J.M. (1995), “Effects of the Hodrick-Prescott Filter on Trend and Difference Stationary Time Series: Implications for Business Cycle Research”, *Journal of Economic Dynamics and Control* 19, 253-278.

Danthine, J.P. and Girardin, M. (1989), “Business Cycles in Switzerland: a Comparative Study”, *European Economic Review* 33, 31-50.

Dolado, J.J., Sebastián, M., Vallés, J. (1993), "Cyclical Patterns of the Spanish Economy", *Investigaciones Económicas*, XVII, 445-473.

Durbin, J. and Koopman, S.J. (2000), “Time Series Analysis of Non-gaussian Observations Based on State Space Models form both Classical and Bayesian Perspectives”, *Journal of the Royal Statistical Society B,* 62, 3-56.

European Central Bank (2000), *Monthly Bulletin*, October.

Giorno, Cl., Richardson, P., Roseveare, D. and van den Nooord, P. (1995), “Estimating Potential Output, Output Gaps and Structural Budget Balances”, OECD Economics Department Working Papers, 152.

Gomez, V. (1999), “Three Equivalent Methods for Filtering Nonsationary Time Series”, *Journal of Business and Economic Statistics* 17, 109-116.

Harvey, A.C. (1997), “Trends, Cycles and Autoregressions”, *Economic Journal*, 107, 192-201.

Harvey, A.C. and Jaeger, A. (1993), “Detrending, Stylized Facts and the Business Cycle”, *Journal of Applied Econometrics* 8, 231-247.

Hodrick, R.J., Prescott E.C. (1980), "Postwar U.S. Business Cycles: an Empirical Investigation". mimeo, Carnegie-Mellon University, Pittsburgh, P.A.

Kaiser, R. and Maravall A. (1999), “Estimation of the Business-Cycle: A Modified Hodrick-Prescott filter”, *Spanish Economic Review* 1, 175-206.

Kaiser R., Maravall A. (2001), *Measuring Business Cycles in Economic Statistics*, Lecture Notes in Statistics 154, New York: Springer-Verlag.

King, R.G. and Rebelo S.T. (1993), "Low Frequency Filtering and Real Business Cycles", *Journal of Economic Dynamics and Control* 17, 207-233.

Kydland, F.E. and Prescott, E. C. (1990), "Business Cycles: Real Facts and Monetary Myth", *Federal Reserve Bank of Minneapolis Quarterly Review (Spring)*, 3-18.

Maravall, A. (1988), “A Note on Minimum Mean Squared Error Estimation of Signals with Unit Roots”, *Journal of Economic Dynamics and Control* 12, 589-593.

Maravall, A. and Mathis, A. (1994), “Encompassing Univariate Models in Multivariate Time Series”, *Journal of Econometrics* 61, 197-233.

Prescott, E. (1986), “Theory ahead of Business Cycle Measurement”, *Carnegie-Rochester Conference Series on Public Policy* 25, 11-66.

Tiao, G.C. (1972), “Asymptotic Behavior of Time Series Aggregates”, *Biometrika* 59, 525-531.

Wei, W.S.W. and Stram, D.O. (1986), "Temporal Aggregation in the Arima Process". *Journal of Time Series Analysis*, 7, 279-292.

Wynne M.A. and Koo J. (1997), "Business Cycles Under Monetary Union EU and US Business Cycles Compared", mimeo, Federal Reserve Bank of Dallas. 97-07.