SHORT-TERM AND LONG-TERM TRENDS, SEASONAL ADJUSTMENT, AND THE BUSINESS CYCLE

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ABSTRACT

In this monograph, first, we analyze in detail some of the major limitations of the standard procedure to estimate business cycles with the Hodrick-Prescott (HP) filter. By incorporating time series analysis techniques, it is seen how some intuitive and relatively simple modifications to the filter can improve significantly its performance, in particular in terms of cleanness of the signal, smaller revision, stability of end-period estimators, and detection of turning points.

Then, we show how the modified filter can be seen as the exact solution of a well-defined statistical problem, namely, optimal (minimum mean squared error) estimation of components in a standard unobserved-component model, where the observed series is decomposed into a trend, a cycle, a seasonal, and an irregular component. This problem is straightforward to solve with Kalman or Wiener-Kolmogorov filter techniques. The models for the components incorporate some-a-priori features, that reflect the ad-hoc nature of the HP filter, and some series-dependent features, that ensure that the aggregate ARIMA model implied by the components is exactly the parsimonious ARIMA model identified directly on the observed series. It is shown how the model-based interpretation greatly facilitates diagnostics and inference, thereby facilitating systematic analysis and improvement. The procedure is trivially implemented with already available free software.

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Preface

This monograph addresses the problem of measuring economic cycles (also called business cycles) in macroeconomic time series. In the decade that followed the Great Depression, economists developed an interest in the possible existence of (more or less systematic) cycles in the economy; see, for example, Haberler (1944) or Shumpeter (1939). It became apparent that in order to identify economic cycles, one had to remove from the series seasonal fluctuations, associated with short-term behavior, and the long-term secular trend, associated mostly with technological progress. Burns and Mitchell (1946) provided perhaps the first main reference point for much of the posterior research. Statistical measurement of the cycle was broadly seen as capturing the variation of the series within a range of frequencies, after the series has been seasonally adjusted and detrended. (Burns and Mitchell suggested a range of frequencies associated with cycles with a period between 6 and 32 quarters.)

Statistical methods were devised to estimate cyclical variation, and these gradually evolved towards methods fundamentally based on the application of moving average filters to the series; see, for example, Bry and Boschan (1971). The last 20 years have witnessed methodological research on two broad fronts. The first front dealt with further developments of the moving-average type of approach; the second front was the development of more complex statistical approaches. Examples of research in both directions can be found in Sims (1977), Lahiri and Moore (1991), Stock and Watson (1993) and Hamilton (1994). Although the first approach is known to present serious limitations, the new and more sophisticated methods developed in the second approach (most notably, multivariate and nonlinear extensions) are at an early stage, and have proved still unreliable, displaying poor behavior when moving away from the sample period. As Baxter and King (1995) point out, we still face at present the same basic question "as did Burns and Mitchell fifty years ago: how should one isolate the cyclical component of an economic time series? In particular, how should one separate business-cycle elements from slowly evolving secular trends, and rapidly varying seasonal or irregular components?"

Be that as it may, it is a fact that measuring (in some way) the business cycle is an actual pressing need of economists, in particular of those related to the functioning of policy-making agencies and institutions, and of applied macroeconomic research. Lacking a practical and reliable alternative, moving-average methods are the ones actually used, to the point that economic agencies (such as the OECD, the International Monetary Fund, or the European Central Bank) often have internal rules or recommendations to measure economic cycles that are MA-type methods. One can say that, very broadly, within the set of applied business-cycle analysts, there has been a convergence towards what could be called "Hodrick-Prescott" (HP) filtering, a methodology proposed by Hodrick and Prescott (1980); see also Kydland and Prescott (1982) and Prescott(1986). The emergence of the HP filter as a paradigm has, probably, been fostered by economic globalization and European integration, which has required a relatively high level of methodological homogeneity in order to compare countries. An unavoidable consequence is that, to some degree, the procedure eventually is often used as a black box.

Academic criticism of the HP filter has pointed out some serious drawbacks. But, beyond the criticism, not much effort has been spent on addressing these shortcomings. An important exception is Baxter and King (1995), where an alternative (related) filter is proposed, that improves smoothness of the estimators for the central years, but avoids estimation at both extremes of the series (including, of course, current and recent periods.) Systematic improvement of the filter performance has been clearly hampered by its ad-hoc nature and the lack of an underlying statistical model with a precise definition of the components (see, for example, Harvey, 1985, and Crafts, Leyburne and Mills, 1989).

In this monograph, first, we analyze in detail some of the major limitations of HP filtering. By incorporating time series analysis techniques, mostly developed over the last 20 years as an aftermath of the explosion in the use of ARIMA-type models (Box and Jenkins, 1970), it is seen how some intuitive and relatively simple modifications to the filter can improve significantly its performance, in particular in terms of cleanness of the signal, smaller revision, stability of end-period estimators, and detection of turning points.

Then, we show how the modified filter can be seen as the exact solution of a well-defined statistical problem, namely, optimal (minimum mean squared error) estimation of components in a standard unobserved-component model, where the observed series is decomposed into a trend, a cycle, a seasonal, and an irregular component. This problem is straightforward to solve with the well-known Kalman or Wiener-Kolmogorov filter techniques (see, for example, Harvey, 1989, or Maravall, 1995). The models for the components incorporate some a-priori features, that reflect the ad-hoc nature of the HP filter, and some series-dependant features, that ensure that the aggregate ARIMA model implied by the components is exactly the parsimonious ARIMA model identified directly on the observed series. Further, summing the trend and the cycle, the standard trend-cycle / seasonal / irregular decomposition of Burman (1980) or Hillmer and Tiao (1982) is obtained. An obvious advantage of the model-based interpretation is that it greatly facilitates diagnostics and inference, thereby facilitating systematic analysis and improvement. Finally, it is shown how the procedure is trivially implemented with already available free software.

1 Introduction

There are two different uses of trends in applied work. First, in short-term monitoring and seasonal adjustment, trends are equal to

$$p_t = x_t - (s_t + u_t)$$

where x_t is the observed series, s_t is the seasonal component, and u_t is the irregular component, that typically captures white (or close to it) noise behavior. These short-term trends are discussed in Maravall (1993), and examples are the trends produced by the Henderson filters in X11 or X12, or the ones obtained in the model-based decomposition of a series, as in programs STAMP or SEATS (see Findley et al, 1998, Koopman et al, 1996, and Gómez and Maravall, 1996). Since they only differ from the seasonally adjusted (SA) series by a highly erratic component, often they will contain variation of the series within the range of cyclical frequencies (which can be broadly defined as the one between the zero and the fundamental seasonal frequency). As a consequence, these trends will only be of interest as a short-term signal (for example, to monitor period-to-period growth). An example is provided in Figures 1.1a and 1.1b. The gain of the filter extends over a wide range of cyclical frequencies, and the trend is seen to contain short-term cyclical oscillations. Throughout the paper, these short-term trend will be referred to as trend-cycles, and denoted p_t; on occasion, they will also be called "noise free" SA series.

The second use of trends is in business cycle analysis, where the cycle is typically measured as what is left of the series, after detrending and seasonal adjustment. Short-term trends cannot be used in this context because they are contaminated with cyclical variation; longer-term trends are needed. Despite the fact that business cycle estimation is basic to the conduct of macroeconomic policy and to monitoring of the economy, many decades of attention have shown that formal modeling of economic cycles is a frustrating issue. Therefore, applied research and work at policy making institutions has had to rely heavily on ad-hoc "band-pass" filters, the most popular of which is the Hodrick-Prescott (HP) filter (see, for example, Prescott, 1986). Thus a present standard procedure to estimate economic cycles is to apply the HP filter to X11-SA series. Figures 1.1c and 1.1d display the HP long-term trend gain and estimator. Long-term trends will be called simply trends, and represented by m_t.

Figure 1.1 Short–term versus long–term trends

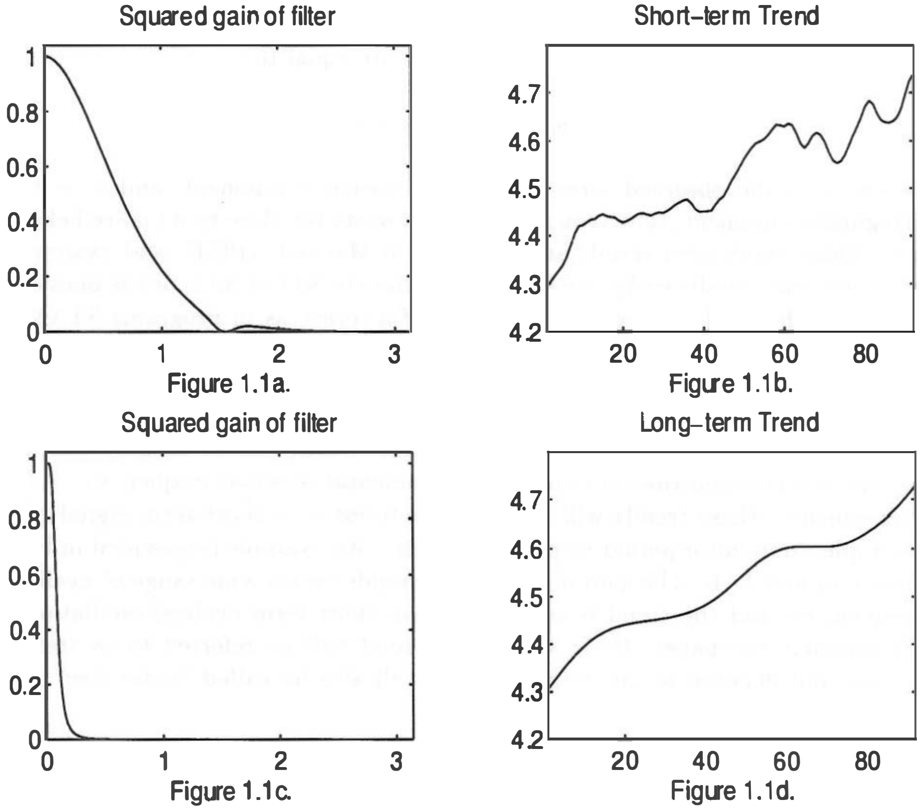


Figure 1.1a. Figure 1.1b. Figure 1.1c. Figure 1.1d.

The use of the HP filter for business-cycle estimation has been the subject of considerable academic discussion. Criticisms are found in, for example, Canova (1998), Cogley and Nason(1995), Harvey and Jaeger(1993), King and Rebelo (1993), and Maravall (1995). Norwithstanding the criticisms, its widespread use in practice may evidence (besides its simplicity) the empirical fact that, as a first (or rough) approximation, analysts find the results useful. The decision of which is the cutting point between a trend and a cycle is, ultimately, arbitrary, and to some extent depends on the purpose of the analysis. For example, from a month to month horizon, a periodic 10-year component may well be considered trend; if business cycle is the objective, it should be considered cycle.

Be that as it may, the HP filter presents some serious limitations. First, it is generally accepted that economic cycles have a non-linear structure that is not well captured with linear ones (see, for example, Hamilton,1989). In this paper we do not deal with non-linear improvements. We address, first, the well-known criticism of spurious results due to the ad-hoc character of the filter, and the (often ignored yet important) limitation implied by revisions, which produce imprecision in the cycle estimator for recent periods. Then, we show how the integration of some relatively simple ARIMA-model-based (AMB) techniques with HP filtering can produce important improvements in the performance of the cyclical signal. Finally, the complete procedure of applying the HP filter to a "clean" series is presented within a model-based methodology.(Freely available software that permits to apply the method is described in an Appendix.)

This AMB methodology displays several nice features. First, it incorporates automatically optimal treatment of end points and provides a cleaner cyclical signal. Second, it provides an internally consistent full decomposition of the series into "trend + cycle + seasonal + irregular" components, where the trend plus cycle aggregate into the standard trend-cycle component of the AMB decomposition. Third, the method is based on the AMB approach, that is, it starts with the ARIMA model for the series, which can be directly identified from the data. In this way, misspecification errors and spurious results are avoided. The procedure consists of straightforward minimum mean squared error estimation of unobserved components, modeled as ARIMA processes, which aggregate into the model identified for the observed series. The models for the trend-cycle, seasonal and irregular components are thus determined from the observed series model. The splitting of the trend-cycle into a trend plus a cycle depends on the choice of the HP-filter parameter λ. Given this parameter, the complete decomposition is then fully determined. An additional advantage is that the parametric model-based procedure provides a convenient framework for diagnostics and inference. An example of inference can be the construction of confidence intervals around the growth of the components estimators, or the computation of optimal forecasts for the components.

2 The Hodrick-Prescott filter: Wiener-Kolmogorov derivation

We start by providing an alternative representation of the HP filter which, on the one hand, provides an efficient and simple computational algorithm and, on the other, turns out to be very useful for analytical discussion. Let x_t $(t = 1, \ldots, T)$ denote an observed series. The HP filter decomposes x_t into a smooth trend (m_t) and a residual (c_t), where the trend is meant to capture the long-term growth of the series, and the residual (equal to the deviation from that growth) represents the cyclical component. Since seasonality should not contaminate the cycle, the filter is typically applied to SA series, but for the moment we shall assume that the series contains no seasonality.

The HP filter is a low-pass filter and can be seen as a Whittaker-Henderson type A filter and as a member of the Butterworth family of filters (see Gómez, 1998). The filter was derived as the solution of a problem that balances a trade-off between fit and smoothness in the following way. In the decomposition

$$x_t = m_t + c_t, \tag{2.1}$$

m_t represents the trend (the "fitted value") and c_t the cycle (the "residual"). The HP filter provides the estimator of c_t and m_t such that the expression

$$\sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=3}^{T} (\nabla^2 m_t)^2 \tag{2.2}$$

is minimized ($\nabla = 1 - B$ is the difference operator, B is the backward operator such that $B^j z_t = z_{t-j}$, and F will denote the forward operator, such that $F^j z_t = z_{t+j}$) The first summation in (2.2) penalizes bad fitting, while the second one penalizes lack of smoothness. The parameter λ regulates the trade-off between the two criteria: when $\lambda = 0, \hat{m}_t = x_t$, when $\lambda \to \infty, \hat{m}_t$ becomes a deterministic linear trend. The solution to the problem of minimizing (2.2) subject to the restriction (2.1) is given by (see Danthine and Girardin, 1989)

$$\hat{\mathbf{m}} = \mathbf{A}^{-1}\mathbf{x}, \qquad \mathbf{A} = \mathbf{I} + \lambda\mathbf{K}'\mathbf{K}, \tag{2.3}$$

where $\hat{\mathbf{m}}$ and $\mathbf{x}$ are the vectors $(\hat{m}_1, \ldots, \hat{m}_T)'$ and $(x_1, \ldots, x_T)'$ respectively, and $\mathbf{K}$ is an $(n-2) \times n$ matrix with its elements given by

$$\begin{aligned} K_{ij} &= 1 \text{ if i=j or } i = j+2, \\ K_{ij} &= -2 \text{ if } i = j+1, \\ K_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

Clearly, the estimator of the trend for a given period will depend on the length of the series. Consider the trend for period T, the last observed period. Application of (2.3) yields an estimator to be denoted $\hat{m}_{T|T}$, where the first subindex refers to the period under estimation, and the second to the last observed period. This estimator will be called the concurrent estimator. When one more quarter is observed and $\mathbf{x}$ becomes $(x_1, \ldots, x_{T+1})'$, application of (2.3) yields a new estimation of m_T, namely $\hat{m}_{T|T+1}$. As more quarters are added, the estimator is revised. It is easily seen that, for large enough k, $\hat{m}_{T|T+k}$ converges to a final or historical estimator, to be denoted $\hat{m}_T$. Therefore, for a long enough series, the final estimator may be assumed for the central periods, while estimators for the last years will be preliminary.

This two sided interpretation of the HP filter seems unavoidable. Because additional correlated new information cannot deteriorate an estimator, $\hat{m}_{T|T+1}$ should improve upon $\hat{m}_{T|T}$. Moreover, actual behavior of the US Business Cycle Dating Committee (or similar institutions) reveals in fact a two-sided filter, which starts with a preliminary estimator, and reaches the final decision with a lag of perhaps two years.

As shown in King and Rebelo (1993), the HP filter can be given a model-based interpretation. Let c_t in (2.1) be white noise with variance V_c and m_t follow the model

$$\nabla^2 m_t = a_{mt}, \tag{2.4}$$

where a_{mt} is a white noise variable (with variance V_m) uncorrelated to c_t.

Throughout the paper, the expression "white noise" will denote a zero-mean normally identically independently distributed variable. Let $\lambda = V_c/V_m$ so that, without loss of generality, we can set $V_c = \lambda, V_m = 1$. The minimum mean squared error (MMSE) estimator of m_t can be obtained in a straightforward manner via the Kalman filter (see Harvey and Jaeger, 1993).

Alternatively, the same MMSE estimator can be obtained with the so-called Wiener Kolmogorov (WK) filter. To do so, in terms of the observations, the previous model can be rewritten as

$$\nabla^2 x_t = a_{mt} + \nabla^2 c_t,$$

or the IMA (2,2) model

$$\nabla^2 x_t = (1 + \theta_1 B + \theta_2 B^2) b_t = \theta_{HP}(B) b_t, \tag{2.5}$$

where b_t are the innovations in the x_t series. The variance of b_t, V_b, and the θ_1,θ_2-parameters are found by factorizing the spectrum from the identity

$$(1 + \theta_1 B + \theta_2 B^2) b_t = a_{mt} + \nabla^2 c_t. \tag{2.6}$$

As an example, for quarterly series the standard value of λ is 1600, in which case

$$\theta_{HP}(B) = 1 - 1.77709B + .79944B^2; \quad V_b = 2001.4. \tag{2.7}$$

For an infinite realization of the series, the MMSE estimator of m_t is given by (see, for example, Maravall, 1995)

$$\hat{m}_t = k_{m(HP)} \frac{1}{\theta_{HP}(B)\theta_{HP}(F)} x_t = \nu^m_{HP}(B, F) x_t. \tag{2.8}$$

where $k_{m(HP)} = V_m/V_b$. The filter $\nu^m_{HP}(B, F)$ is symmetric and, since (2.6) implies that $\theta_{HP}(B)$ is invertible, also convergent. Following Cleveland and Tiao (1976), for a finite series, expression (2.8) can still be applied, with x_t replaced by the series extended with forecast and backcasts. A simple and efficient algorithm to apply the filter, similar to that in Burman (1980), is detailed in Appendix A.

For the estimator of the cycle,

$$\hat{c}_t = [1 - \nu^m_{HP}(B,F)]x_t = \nu^c_{HP}(B,F)x_t, \tag{2.9}$$

where $\nu^c_{HP}(B,F)$ is also a two-sided centered, symmetric, and convergent linear filter, which can be rewritten as,

$$\hat{c}_t = \nu^c_{HP}(B,F)x_t = \left[k_{c(HP)}\frac{\nabla^2\bar{\nabla}^2}{\theta_{HP}(B)\theta_{HP}(F)}\right]x_t, \tag{2.10}$$

where $k_{c(HP)} = V_c/V_b$, and a bar over an operator denotes the same operator with B replaced by F. When properly applied, the Danthine and Girardin, the Kalman filter, and the WK solutions are numerically identical (see Gómez, 1999). The last two are considerably more efficient than the first, and can be applied to series of any length. The WK filter turns out to be convenient for analytical discussion. It will prove useful to present the frequency domain version of the filter (2.10), to be denoted $\tilde{\nu}^c_{HP}(\omega)$, where ω is the frequency measured in radians. Using (2.6), the filter (2.10) can be rewritten as

$$\nu^c_{HP}(B,F) = \frac{\nabla^2\bar{\nabla}^2 V_c}{V_m + \nabla^2\bar{\nabla}^2 V_c} = \frac{\nabla^2\bar{\nabla}^2}{\lambda^{-1} + \nabla^2\bar{\nabla}^2},$$

with Fourier transform given by

$$\tilde{\nu}^c_{HP}(\omega) = \frac{4(1-cos\omega)^2}{\lambda^{-1} + 4(1-cos\omega)^2}. \tag{2.11}$$

For seasonal series, since the seasonal variation should not contaminate the cycle, the HP filter is typically applied to X11 SA quarterly series. Throughout the paper, "X11" will denote the default linear filter for an additive decomposition, as in Ghysels and Perron (1993). To adjust a series, the filter X11 will always be applied (in the X11ARIMA spirit) to the series extended at both extremes with ARIMA forecast and backcasts.

We shall center attention, first, on historical (or final) estimation. If $\nu_{X11}(B,F)$ denotes the X11-SA filter, and $\nu^c_{HP}(B,F)$ the HP filter (2.9), let

$$\nu^c_{HPX}(B,F) = \nu^c_{HP}(B,F)\nu_{X11}(B,F) \tag{2.12}$$

denote the convolution of the two. Because both, the X11 and the HP filters, are symmetric, centered, and convergent, so will their convolution. For seasonal series, the estimator of the cycle (2.9) should thus be replaced by

$$\hat{c}_t = \nu^c_{HPX}(B, F)x_t. \tag{2.13}$$

Throughout the paper we assume quarterly series and denote by S the annual aggregation operator,

$$S = 1 + B + B^2 + B^3.$$

Further, in all decompositions of a series into unobserved stochastic components, the components will be assumed orthogonal, and innovations in their stochastic models will be assumed normally distributed.

3 Revisions

3.1 Preliminary estimation of end points and revisions

In section 2 we presented the HP filter as a symmetric two-sided filter. Given that the concurrent estimator is a projection on a subset of the set of information that provides the final estimator, the later cannot be less efficient. Besides, concurrent estimators, obtained with a one-sided filter, induce phase effects that harm early detection of turning points.

If $\hat{c}_{T|T}$ denotes the estimator of the cycle for the last observed period (i.e., the concurrent estimator,) as new periods are observed the estimator will be revised to $\hat{c}_{T|T+1}, \hat{c}_{T|T+2}, \ldots$ until it converges to the final estimator $\hat{c}_T$. The difference between the final estimator and the concurrent estimator measures the total revision the concurrent estimator will undergo, and can be interpreted as a measurement error contained in the concurrent (more generally, preliminary) estimator.

Although the poor behavior of the HP filter for recent periods has been often pointed out (see Baxter and King, 1995), the revisions implied by HP filtering have not been analyzed . Two main features of the revision are of interest: a) the magnitude, and b) the duration of the revision process (i.e., the value of k

for which $\hat{c}_{T|T+k}$ has, in practice, converged). To look at these features we use the WK version of the filter (i.e., the ARIMA model based filter,) and proceed as follows.

Assume the observed series follows the ARIMA model

$$\phi(B)x_t = \theta(B)a_t, \tag{3.1}$$

with $x_t \sim I(d)$, $0 \leq d \leq 4$ (hence $\phi(B)$ contains the factor ∇^d.) Because the numerator of ν_{HP}^c in (2.10) cancels the unit roots in x_t, the estimator of the cycle can be expressed as

$$\hat{c}_t = \xi(B,F)a_t, \tag{3.2}$$

where the weights of the polynomial $\xi(B,F)$ can be obtained through the identity $\xi(B,F)\phi(B) = \nu_{HP}^c(B,F)\theta(B)$. Expression (3.2) can be rewritten as

$$\hat{c}_t = \xi^-(B)a_t + \xi^+(F)a_{t+1}, \tag{3.3}$$

where

$$\xi^-(B) = \sum_{j\geq 0} \xi_{-j}B^j,$$

and

$$\xi^+(F) = \sum_{j\geq 0} \xi_j F^j$$

are convergent polynomials. The first one contains the effect of the innovations up to and including period t, and the second one includes the effect of innovations posterior to period t. Because

$$E_t(a_{t-j}) = a_{t-j} \text{ when } j \geq 0,$$

$$E_t(a_{t-j}) = 0 \text{ when } j < 0,$$

the concurrent estimator equal to the expectation at time t of the estimator (3.3), is given by the first term in the right hand side of the equation. The revision in the concurrent estimator will thus be given by

$$r_{t|t} = \xi^{+}(F)a_{t+1} = \sum_{j=1}^{k} \xi_j a_{t+j}, \tag{3.4}$$

where the second equality uses a finite approximation based on the convergence of $\xi^{+}(F)$. From (3.4), it is straightforward to compute the variance and autocorrelations of the revision process. (We have focussed on the concurrent estimator; the analysis is trivially extended to any preliminary estimator $\hat{c}_{t|T}$.)

Although the filter $\nu_{HP}^{c}(B, F)$ is fixed, the coefficients of the forward filter $\xi^{+}(F)$ depend on the ARIMA model for the observed series. Without loss of generality, we set $Var(a_t) = 1$, so that the variance of the revision

$$Var(r_{t|t}) = \sum_{j=1}^{k} (\xi_j)^2 \tag{3.5}$$

is then expressed as a fraction of the variance of the series innovation V_a. Table 3.1 exhibits, for three models, the size of the revision and the number of periods needed for the concurrent estimator to converge to the final one (convergence is defined in practice as having removed more than 95% of the revision variance). The first example is for the case of a white noise series ($x_t = a_t$), and illustrates thus the "pure filter" effect. The second is the random walk model $\nabla x_t = a_t$, and the third example is the model for which the HP filter is optimal, namely $\nabla^2 x_t = \theta_{HP}(B)a_t$. The three examples represent, thus, an I(0), I(1) and I(2) variable, respectively.

Variable	Standard deviation of revision in concurrent estimator (as a percent of σ_a)	Periods needed for convergence
White noise	13.9	12
Random walk	91.3	9
HP-IMA(2,2)	34.0	9

Table 3.1. Revisions implied by the HP filter.

Even for the case in which the model is the one associated with optimality of the filter, the size of the revision is not negligible and the revision period lasts in practice more than 2 years.

As already mentioned, the HP filter is often applied to X11-SA series, and the convolution of the two filters was earlier denoted $\nu_{XHP}(B, F)$. It is well known that X11, another two-sided filter, also produces revisions. Therefore, the revisions associated with the filter ν_{XHP} will reflect the combined effect of the two filters. To look at this effect, we consider the case of white-noise input. Proceeding as before, it is found that 95% of the revision variance disappears after 13 quarters, and that the revision standard deviation is 1/4 of the standard deviation of a_t. Comparing these results with the first row of Table 3.1, the addition of X11 substantially increases the revision size; but the revision period barely changes.

To illustrate the revisions for series of more applied relevance we select the so-called "Airline model", discussed in Box and Jenkins (1970), given by the expression

$$\nabla\nabla_4 x_t = (1+\theta_1 B)(1+\theta_4 B^4)a_t. \tag{3.6}$$

The model fits well many series with trend and seasonality, and has became a standard example. For the most relevant range for the parameters θ_1 and θ_4, Table 3.2 presents the fraction $\sigma(\text{revision})/\sigma(a_t)$ and the number of periods (τ) needed for a 95% convergence in variance. The standard deviation of the revision represents between .4 and 1.5 of $\sigma(a_t)$, and convergence takes, roughly, between 2 and 5 years. Given that θ_1 close to -1 implies very stable trends, while θ_4 close to -1 implies very stable seasonals, what Table 3.2 shows is that series with highly moving trends and seasonals will be subject to bigger, longer lasting, revisions. It is worth pointing out that, for the range of values most often found in practice (see the study on more than 14000 real series from 17 countries in Fischer and Planas, 1998) which is the botton right corner, the revision period is equal to 9 quarters.

	$\theta_4=0$		$\theta_4=-.2$		$\theta_4=-.4$		$\theta_4=-.6$		$\theta_4=-.8$	
	σ_r/σ_a	τ	σ_r/σ_a	τ	σ_r/σ_a	τ	σ_r/σ_a	τ	σ_r/σ_a	τ
$\theta_1=.4$	1.53	19	1.44	18	1.36	17	1.28	9	1.21	9
$\theta_1=.2$	1.34	19	1.26	18	1.18	17	1.12	9	1.06	9
$\theta_1=0$	1.15	19	1.08	18	1.02	16	0.96	9	0.90	9
$\theta_1=-.2$	0.97	19	0.91	18	0.85	15	0.80	9	0.76	9
$\theta_1=-.4$	0.79	18	0.74	17	0.70	14	0.65	9	0.61	9
$\theta_1=-.6$	0.64	15	0.60	14	0.55	9	0.51	9	0.47	9
$\theta_1=-.8$	0.52	9	0.48	9	0.44	9	0.40	9	0.36	9

Table 3.2. Revisions implied by the HP-X11 filter.

3.2 An example

An application, that will also be used in later sections, will complete the discussion. We consider four quarterly Spanish short-term economic indicators that can be reasonably suspected of being related to the business cycle. The series are the industrial production index (IPI), cement consumption (CC), car registration (CR) and airline passengers (AP), for the period 1972/1 - 1997/4, and contain 104 observations. (For the IPI series, the first 12 observations were missing and the period was completed using backcasts). The series were log transformed (following proper comparison of the BIC criteria), and the application will be discussed for the additive decomposition of the logs. Moreover, so as to facilitate comparisons, we standardize the 4 logged series to have zero mean and unit variance. The 4 series are represented in Figure 3.1; their trend and seasonal behavior is clearly discernible.

ARIMA modeling of the 4 series produced similar results: the models were of the type (3.6) (i.e., of the Airline type) and a summary of results is given in Table 3.3; none of the series appeared to be in need of outlier adjustment. (Estimation was made with the program TRAMO run in an automatic mode, see Gómez and Maravall, 1996). Using the ARIMA models to extend the series, the HP ($\lambda = 1600$) filter was applied to the X11-SA series, and the 4 trends and 4 cycles obtained are displayed in Figure 3.2 and 3.3. For the series CC and CR the short-term contribution of the cyclical variation is relatively more important than for the series IPI and, in particular, AP.

	Parameter Estimates		Residual	BL test	Normality
	θ_1	θ_4	Variance V_a	$Q(< \chi^2_{14})$	$N(< \chi^2_2)$
CC	-.405	-.957	.175	18.4	.32
IPI	-.299	-.721	.054	23.3	.14
CR	-.387	-.760	.156	18.7	.79
AP	-.392	-.762	.017	21.1	2.76

Table 3.3. Summary of ARIMA estimation results.

Figure 3.1. Short–term economic indicators:original series

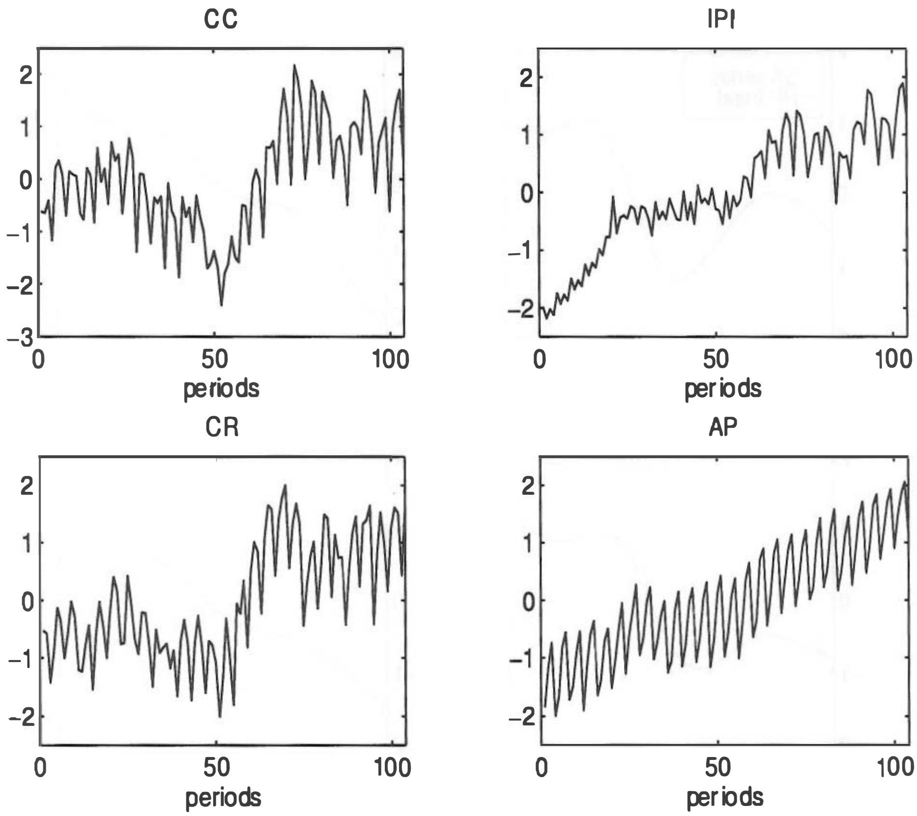


Figure 3.2. X11–SA series and HP trend

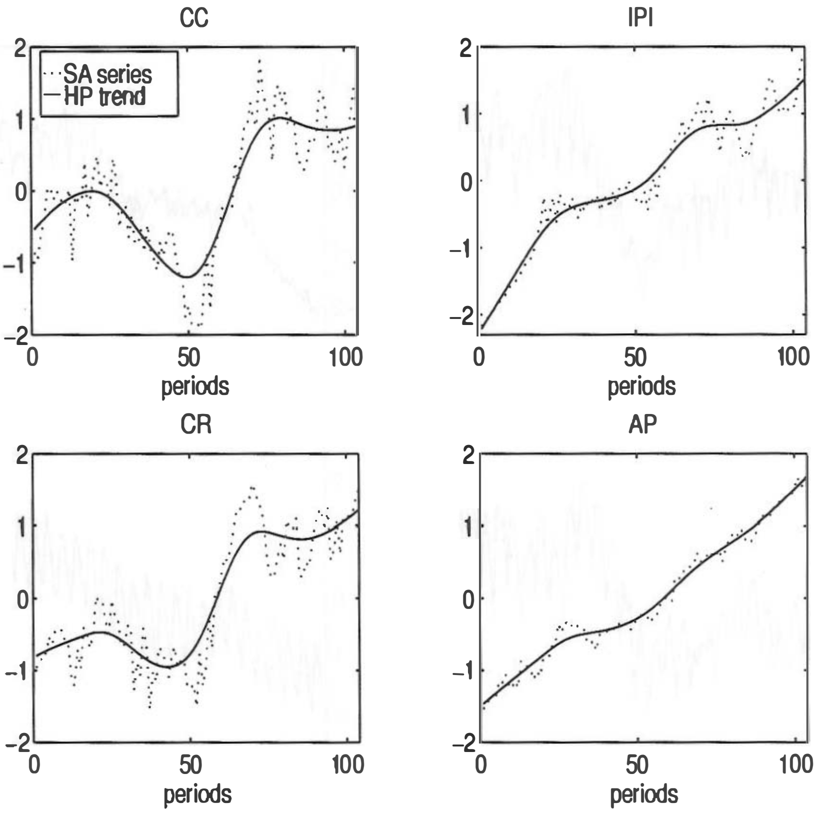
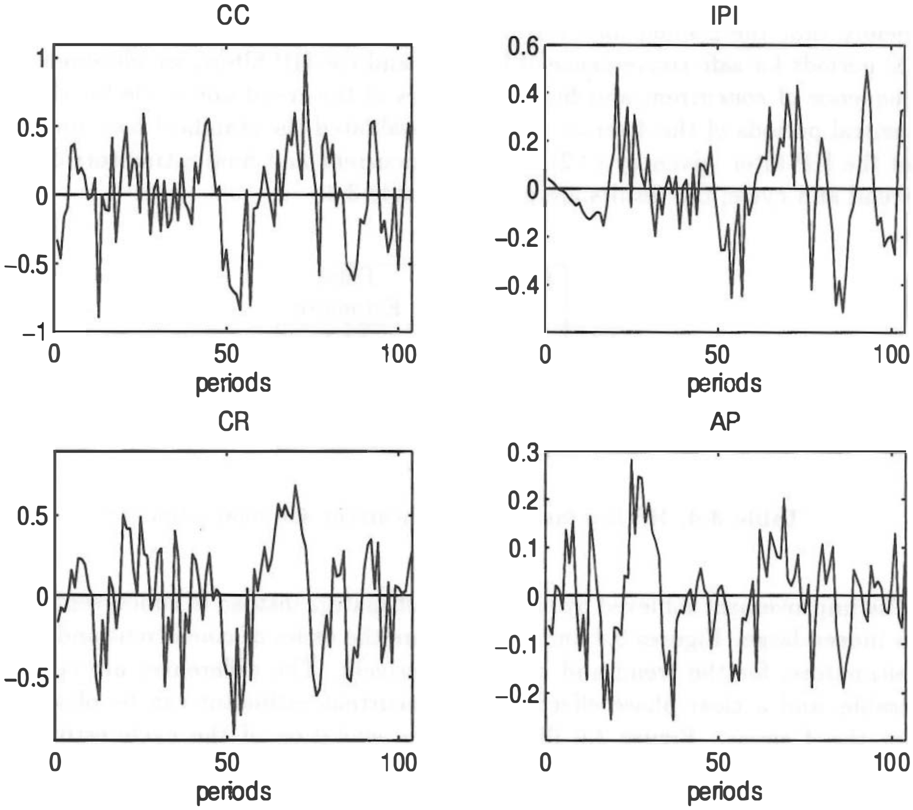


Figure 3.3. X11–HP cycles



Short-term monitoring focuses on recent periods, that is, on the concurrent estimator and its first revisions and, in fact, it is often the case that the HP filter is treated as a one-sided filter (see Prescott, 1986). We have argued before that the 2-sided final estimator is preferable. Using the first and last 22 periods for safe convergence of the X11 and the HP filters, we obtained the sequence of concurrent and final estimators of the trend and cycle for the 60 central periods of the 4 series. Then, we evaluated the standard loss function of the HP filter, given by (2.2), for the concurrent and final estimators of the trend and cycle; the results are given in Table 3.4.

	Concurrent Estimator	Final Estimator
CC	624.7	13.3
IPI	172.8	2.9
CR	513.4	11.9
AP	43.2	1.0

Table 3.4. HP loss-function for concurrent and final estimator

The improvement achieved by using final estimators instead of concurrent ones is indeed large. Figures 3.4 and 3.5 compare the series of concurrent and final estimators, for the trend and cycle respectively. The differences are considerable, and a clear phase effect in the concurrent estimator can be observed for the 4 series. Figure 3.6 illustrates the evolution of the cycle estimator from concurrent to final for three periods ($t = 61, 65$ and 70,) and Table 3.5 compares these two estimators for the 3 periods. Considering that the original series were standardized ($\mu = 0, \sigma = 1$), the revision in the estimator of the cycle is, in many cases, remarkable. For the 4 series, 95% of the revision is completed in 9 quarters, in agreement with the results of Table 3.2. The standard error of the revision is in the order of $.5\sigma_a$, certainly nonnegligible.

Figure 3.4. Concurrent versus final trend estimator

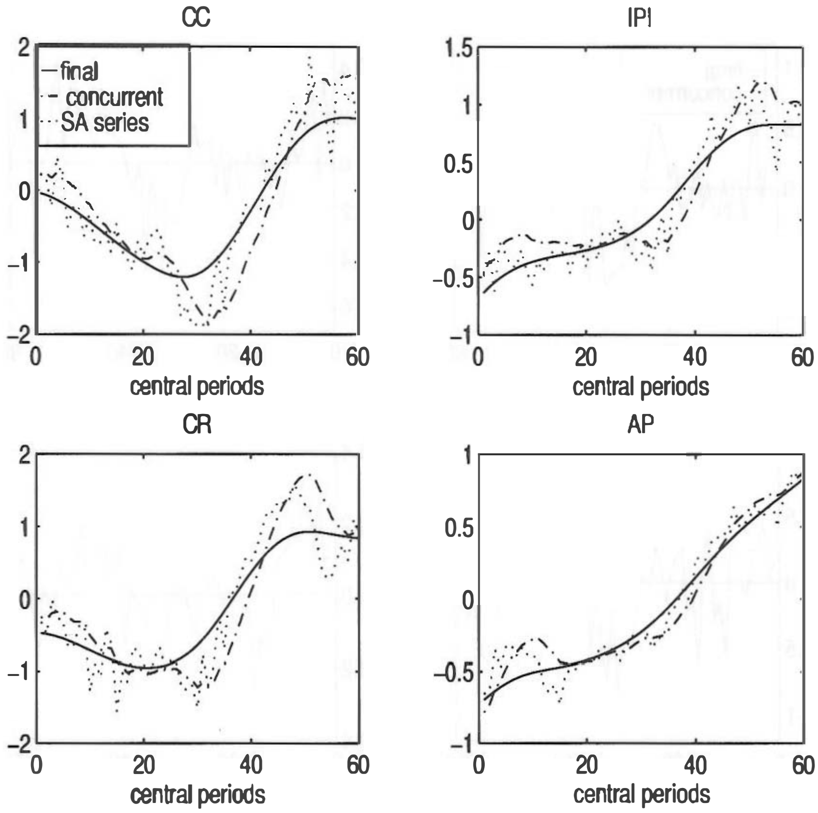


Figure 3.5. Concurrent versus final cycle estimator

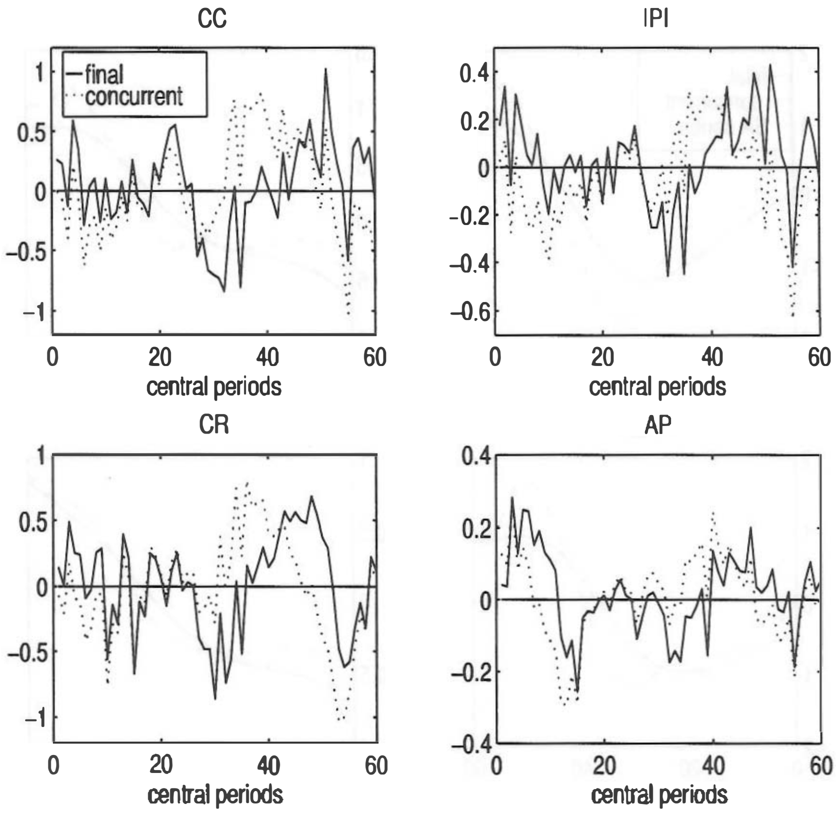
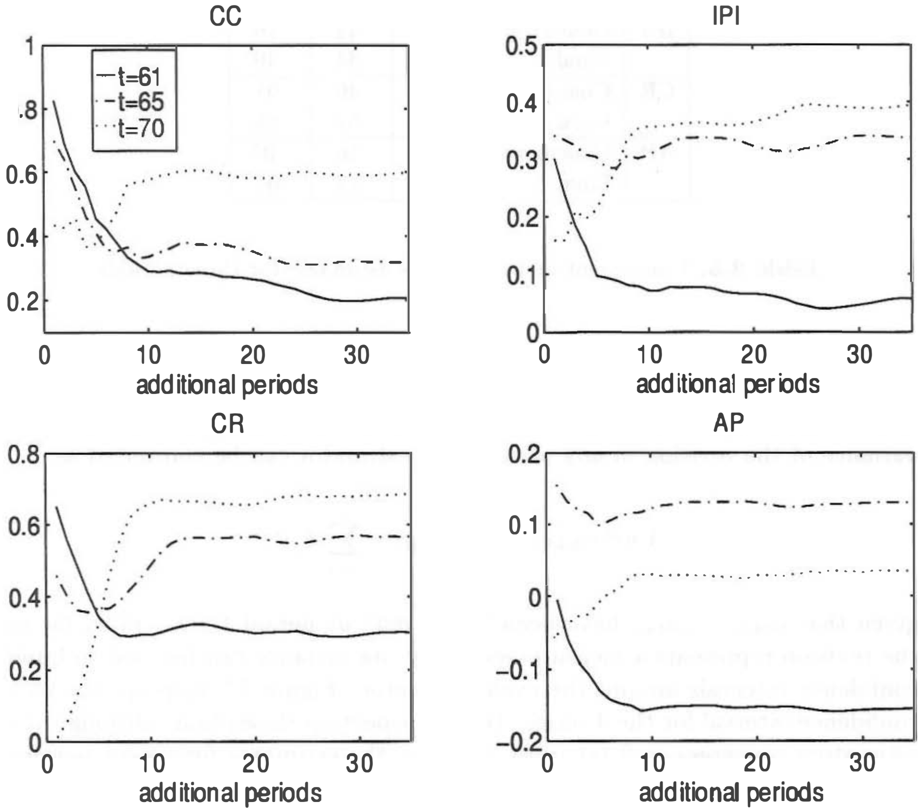


Figure 3.6. Revisions in concurrent estimator



	Period	t=61	t=65	t=70
CC	Concurrent	.83	.70	.44
	Final	.21	.32	.60
IPI	Concurrent	.30	.34	.16
	Final	.06	.34	.39
CR	Concurrent	.65	.46	.01
	Final	.30	.57	.69
AP	Concurrent	-.01	.16	-.05
	Final	-.16	.13	.04

Table 3.5. Concurrent and final cycle estimator for three periods

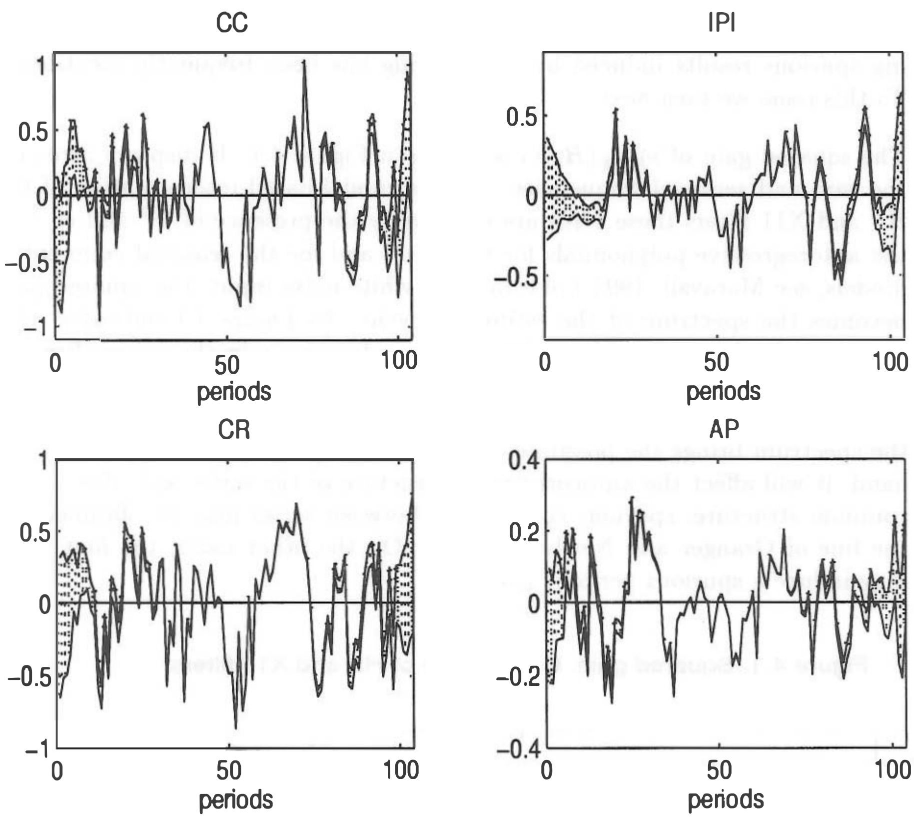
A point of applied relevance is to asses the imprecision of the estimator of the cycle for recent periods, as measured by the standard error of the revision. Computing the ξ-weights of the filter (3.4), and $Var(r_{t|t})$ as in (3.5), the variance of the revision in any preliminary estimator can be computed as

$$Var(r_{t|t+k}) = Var(r_{t|t}) - \sum_{j=1}^{k}(\xi_j)^2,$$

given that $a_{t+1}, \ldots, a_{t+k}$ have been "observed" at period $t+k$. In so far as the revision represents a measurement error, its variance can be used to build confidence intervals around the cycle estimator. Figure 3.7 displays the 95% confidence interval for the 4 series. Direct inspection shows that, although the estimators converges in 2 (at most 3) years, the estimator for recent periods is unreliable. This fast and large increase in the measurement error of the most recent signals implies that, although straightforward to obtain, forecasts would be close to useless. (In computing revisions, the X11-SA series for the full sample of 104 observations has remained constant. The revisions we have computed are thus those implied solely by the HP filter.)

We shall come back to the issue of revisions in Section 5; until then we shall center our attention on final estimators.

Figure 3.7. 95% confidence intervals for cycle (based on revisions)

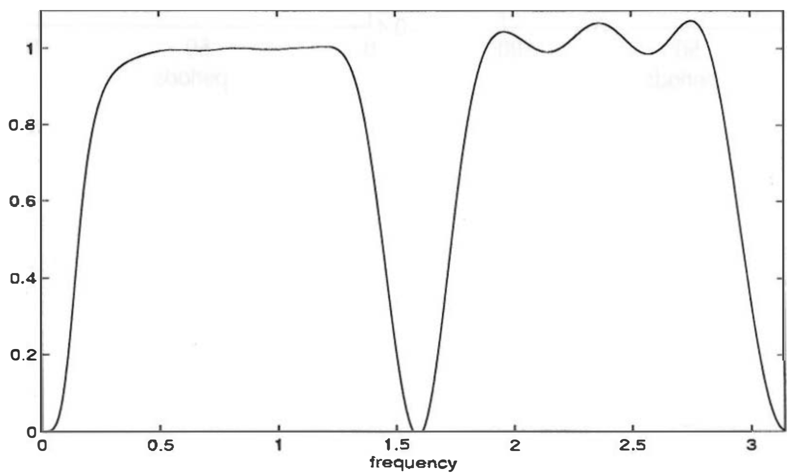


4 Spurious results

While the problem of revisions has been often overlooked, the danger of obtaining spurious results induced by HP filtering has been frequently mentioned. To this issue we turn next.

The squared gain of $\nu^c_{HPX}(B, F)$ is shown in Figure 4.1. It displays zeros for the zero and seasonal frequencies. (In the model-based interpretation of the HP and X11 filters these zeros are implied by the presence of ∇^2 and of S in the autoregressive polynomials for the trend and for the seasonal component models; see Maravall, 1995.) Assuming a white-noise input, the squared gain becomes the spectrum of the estimated cycle. As Figure 4.1 indicates, this spectrum displays two wide peaks, one for a frequency in the range $(0, \pi/2)$, i.e., the range of cyclical frequencies; the other for a frequency in the range $(\pi/2, \pi)$, the range of intraseasonal frequencies. This two-peak structure of the spectrum brings the possibility of obtaining spurious results. On the one hand, it will affect the autocorrelation structure of the series and, due to the common structure, spurious correlations between series may be obtained (in the line of Granger and Newbold, 1974). On the other hand, the first peak may induce a spurious periodic cycle.

Figure 4.1. Squared gain: Convolution of HP and X11 filters



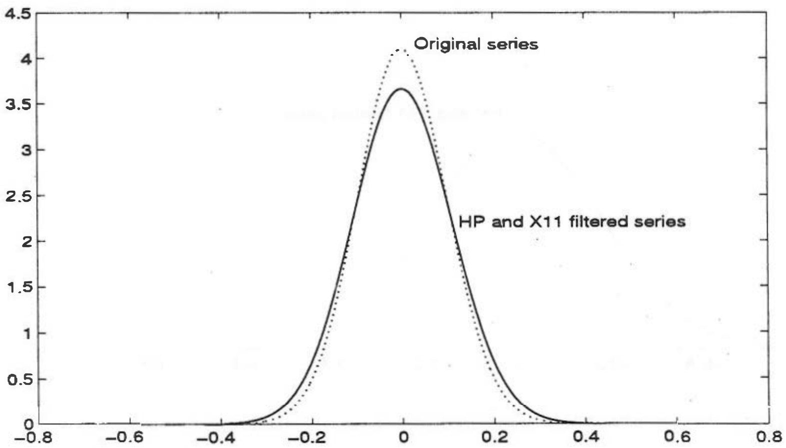
4.1 Spurious crosscorrelation

We performed a simulation in MATLAB, whereby 10.000 independent random samples of 600 observations each were drawn from a $\mathcal{N}(0,1)$ distribution. Each white-noise series was filtered through the X11 and HP filters and the last 100 values were selected. Next, 10.000 lag-zero crosscorrelation between two series were sampled (in what follows, all crosscorrelations are lag-zero ones). The average of the absolute value of the crosscorrelation between the white noise input series was .08 (SE=.06), for the seasonally adjusted series, .09 (SE=.06), and for the cycle, .09 (SE=.07). Not much crosscorrelation seems to have been induced. Table 4.1 presents the first four moments of the distribution of $\hat{\rho}_0$, the crosscorrelation estimator (including the sign) for the original series and the cycle, Figure 4.2 plots the two densities.

	Mean	Std.deviation	Skewness	Kurtosis
Original (white noise)	-.001	.11	-.03	2.9
Cycle	-.001	.11	-.05	2.9

Table 4.1. Crosscorrelation; Filtered White Noise Case

Figure 4.2. Density for correlation coefficient: white noise case



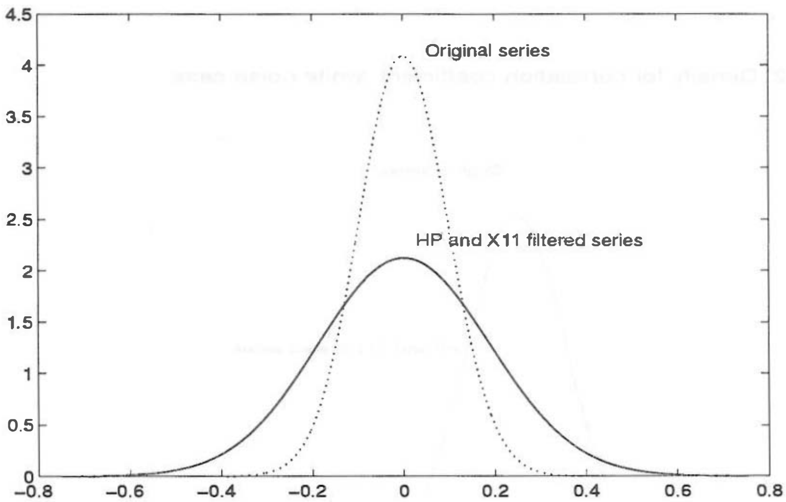
The two distributions are very close and are well approximated by a $N(0, 1/T)$. Clearly, no spurious crosscorrelations has been induced.

When the input follows the random walk model $\nabla x_t = a_t$, using the same simulation as in the previous section, the average over the 10.000 absolute value of the crosscorrelation between the differenced series is .08(.06), and between the differenced SA series, .09 (.07), the same values as before. For the cycle, however, the average increases to .16 (.11), still a small value. Table 4.2 presents the first four moments of $\hat{\rho}_0$ (with sign included) for the original series and for the cycle; the two densities are plot in Figure 4.3.

	Mean	Std. deviation	Skewness	Kurtosis
Original (random walk)	-.000	.10	-.04	2.9
Cycle	.000	.19	-.01	2.8

Table 4.2. Crosscorrelation; Filtered Random Walk Case

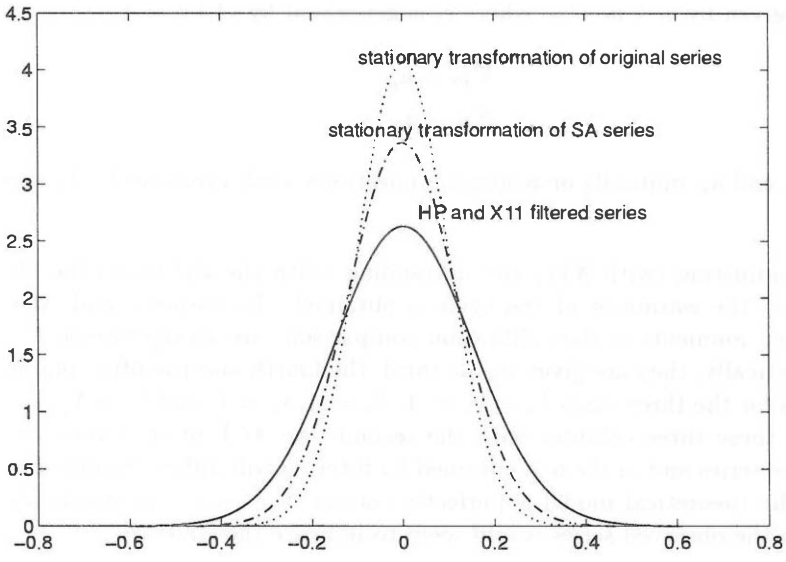
Figure 4.3. Density for correlation coefficient: random walk case



The zero-mean normality assumption can still be accepted comfortably, but the spread of the distribution of $\hat{\rho}_0$ for the cycle becomes wider. In fact, the proportion of ρ_0 estimates that lie outside the 95% significance level is 32%. For the random walk case, thus, a spurious crosscorrelation effect can be detected. Altogether, the effect is nevertheless moderate.

A similar simulation was performed for the more complex airline model (3.6), with the parameter values set at $\theta_1 = -.4$ and $\theta_4 = -.6$. Figure 4.4 plots the densities of the crosscorrelation estimator for the stationary transformation of the original and SA series and of the X11-SA and HP detrended series. The filter X11 is seen to have virtually no effect while, as before, the HP filter induces a small increase in the spread of the distribution. In summary, from the point of view of spurious crosscorrelation, the HP-X11 filter seems to induce a small amount of spuriousness and hence the detection of relatively large crosscorrelation between cycles obtained with it are unlikely to be spurious. (Although the filter will have some distorting effect on the crosscorrelations when the series are indeed correlated; see Cogley and Nason, 1995.)

Figure 4.4. Density for correlation coefficient: Airline model



4.2 Spurious autocorrelation; calibration

Assume that a theoretical economic model implies that a particular variable follows a 4-year cycle given by the AR(2) process:

$$(1 - 1.293B + .490B^2)c_t = a_{ct} \tag{4.1}$$

with a_{ct} a white-noise innovation, with variance $V_c = 1$. Assume that a large number of simulations of the model yield in fact an ACF for the variable equal to the theoretical ACF of (4.1), shown in the second column of Table 4.3. The basic idea behind calibration is to validate the economic model by comparing the previous ACF with the one implied by the observed economic variable. To compute the latter, the non-stationary trend and seasonal component need to be removed. (Besides, seasonality and often the trend are typically excluded from the theoretical economic model.)

Assume the observed series is generated precisely by the cycle given by (4.1), contaminated by a random walk trend (p_t) and a seasonal component (s_t) as in the Basic Structural Model of Harvey and Todd (1983). Thus the observed series x_t is given by $c_t + p_t + s_t$, where c_t is generated by (4.1), and

$$\nabla p_t = a_{pt}$$
$$S s_t = a_{st}$$

with a_{ct}, a_{pt} and a_{st} mutually orthogonal innovations, with variances V_c, V_p and V_s.

Seasonally adjusting (with X11) and detrending (with the HP filter) the observed series, the estimator of the cycle is obtained. Its variance and ACF (the observed moments in the calibration comparison) are straightforward to derive analytically; they are given in the third, the fourth and the fifth column of Table 4.3 for the three cases $V_p = V_s = .1$; $V_p = .1, V_s = 1$; and $V_p = V_s = 1$. Comparing these three columns with the second, the ACF of the cycle contained in the series and of the one obtained by filtering will differ considerably. Although the theoretical model is perfectly correct, the second moments obtained from the observed series would seem to indicate the contrary.

The distortion that seasonal adjustment and detrending induces in the second moments of the "observed" series is a general property which also occurs when the components are estimated as MMSE estimators in a model based approach; the ACF of the cycle obtained in this case is given in the 6^{th} row of Table 4.3. Still, the distortion induced by MMSE estimation is considerably smaller than that induced by HP-X11 filtering.

Calibration of models using filtered series seems, thus, an unreliable procedure. If the theoretical economic model is correct, then calibration should not look for similarities between the ACF of the theoretical model and of the empirical series. It should compare instead the empirical moments with the theoretical ones that include the effect of filtering the data. Performing this comparison, however, requires incorporating in some way into the model trend and seasonality (in its simplest way, as unobserved components.)

Lag-k ACF	True component	X11-HP filtered component			MMSE estimator
		$V_p = .1$, $V_s = .1$	$V_p = .1$, $V_s = 1$	$V_p = 1$ $V_s = 1$	$V_p = .1$, $V_s = 1$
k=1	.87	.71	.19	.37	.83
k=2	.63	.44	.22	.30	.43
k=3	.39	.10	-.06	.00	-.02
k=4	.20	-.05	.22	.18	-.35
k=5	.06	-.25	-.23	-.15	-.45
k=6	-.01	-.30	-.19	-.16	-.43
k=7	-.05	-.34	-.27	-.26	-.31
k=8	-.06	-.27	-.01	-.07	-.20
k=9	-.05	-.25	-.18	-.20	-.10
k=10	-.04	-.19	-.12	-.16	-.04
k=11	-.02	-.16	-.17	-.20	-.00
k=12	-.01	-.09	.06	-.03	-.02
k=13	-.00	-.08	-.07	-.12	-.03
k=14	-.00	-.05	-.03	-.08	-.03
k=15	-.03	-.04	-.11	-.13	-.02
k=16	-.00	-.01	-.13	.05	-.01

Table 4.3 Theoretical ACF of the component model and of its estimators

4.3 Spurious periodic cycle

As mentioned at the beginning of this section, the HP filter has often been accused of inducing spurious cycles. To this issue we turn next.

4.3.1 White-noise input

A priori, one can think of capturing the two spectral peaks of Figure 4.1 through an AR(4) model with two pair of complex conjugate roots. We performed the same simulation of Section 4.1 (10.000 random samples from a $N(0,1)$ distribution were filtered through the X11 and HP filters, and the last 100 values were selected from each series). Then, an AR(4) model was fit to the filtered series (i.e., to the "cycle"). Averaging the AR parameter estimates yields very approximately the model

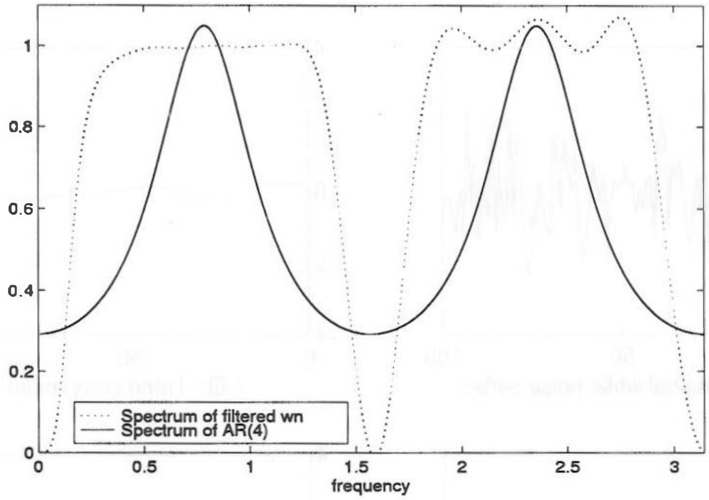
$$(1 + .31B^4)c_t = a_t, \tag{4.2}$$

and a test for the significance of the AR(4) regression yields an average F value of 3.6 (critical value 2.7). Figure 4.5 plots the spectrum of model (4.2) and it is clearly seen how the peaks of the AR approximation attempt to capture the peaks of the spectrum of the filtered white noise. It is also seen that the AR format does not permit a good approximation, reflecting the fact that the invertible AR model cannot approximate well the spectral zeros of the noninvertible cycle. The two spectral AR peaks correspond to two pairs of complex conjugate roots for the AR polynomial. The average value of the modulus and period for the two roots are given in Table 4.4, where the standard errors are given in parenthesis. (It is easily seen that factorization of the AR(4) polynomial in (4.2) yields nearly identical roots.)

	Average Modulus	Average Period
First root	.75 (.06)	7.95(.79)
Second root	.74 (.06)	2.68(.08)

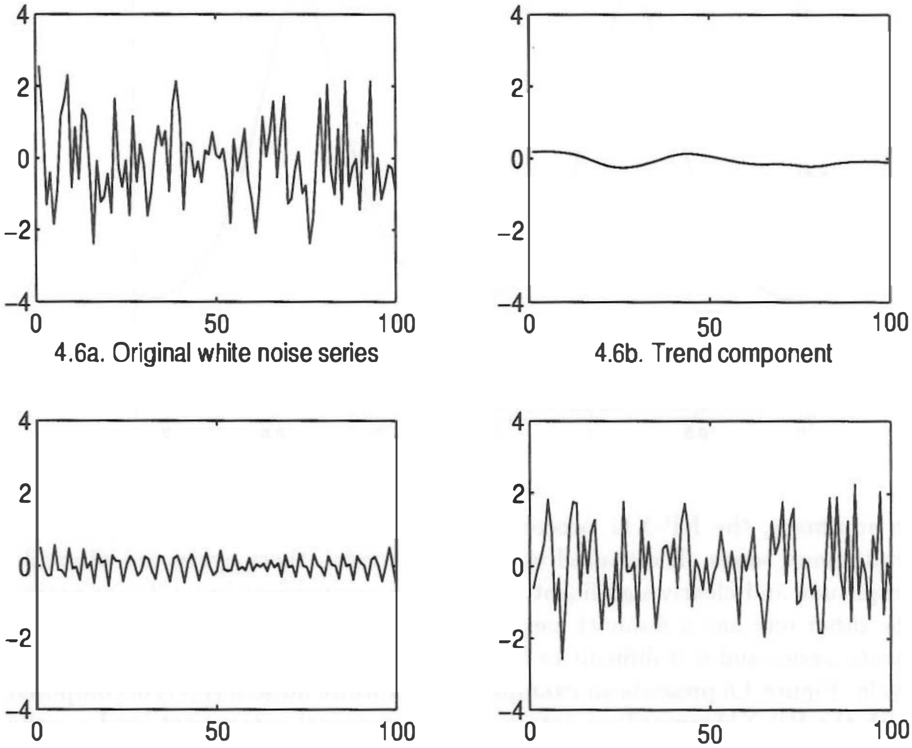
Table 4.4. AR roots: Filtered White Noise

Figure 4.5. Spectrum of AR(4) for the white noise case



In summary, the HP-X11 procedure is likely to induce spurious cycles in a white-noise series. If captured with an AR model, these cycles are of similar amplitude and clearly significant. One of the components has a 2-year period; the other one has a 8-month period. The sum of the two produces a highly erratic series and it is difficult to link this behavior to the concept of a business cycle. Figure 4.6 presents an example where a white noise series is decomposed, with the HP-X11 procedure, into a trend, a seasonal component, and a cycle. In so far as white noise is not subject to trends, nor to seasonal fluctuations, the HP-X11 decomposition is spurious, purely filter-induced. The spurious trend and seasonality removed from the series are nevertheless moderate.

Figure 4.6. Decomposition of a white noise series



4.6a. Original white noise series

4.6b. Trend component

4.6c. Seasonal component

4.6d. Cycle component

4.3.2 Random-walk input

The previous discussion, based on a white-noise input, illustrates the effect of the filter. It is of interest to see how this effect interacts with input that displays some trend structure. We consider the simplest case of the random-walk model,

$$\nabla x_t = a_t; \tag{4.3}$$

as before, the cycle is estimated through (2.13).

Letting ω denote the frequency in radians, if $\tilde{\nu}^c_{HPX}(\omega)$ is the Fourier transform of $\nu^c_{HPX}(B, F)$, the spectrum of the estimator of the cycle is given by,

$$\hat{g}^c_{HPX}(\omega) = [\tilde{\nu}^c_{HPX}(\omega)]^2 \, g_x(\omega), \tag{4.4}$$

where $g_x(\omega)$ is the pseudospectrum of x_t (see Harvey, 1989); thereafter the term spectrum will also be used to refer to a pseudospectrum. (To simplify notation, all spectra will be implicitly expressed in units of $1/2\pi$.) Figure 4.7 plots the spectra of the series (dotted line) and of the cycle (continuous line). The difference with respect to the cycle obtained with white noise (Figure 4.1) is remarkable. The peak for the high frequency is hardly noticeable, while the peak for the frequency in the cyclical range is associated with a longer period of 31-32 quarters, or, approximately, 8 years.

Performing the same simulation as before, an AR(4) model was fit to 10.000 generated random walks of 100 observations each, filtered through the X11 and HP filters, and the average F-test was equal to 38.35, overwhelmingly significant. For the random walk series, the HP-X11 filter induces a cycle dominated by an 8 year period, and hence more in line with the frequencies of interest to business cycle analysts. Figure 4.9 presents an example of a random walk decomposed by the X11 and HP filters into a trend, a seasonal component and a cycle. Although the trend and cycle are, as before, moderately small, by its own definition, does it make sense to see a random walk as generated by a trend, a seasonal component, and a 8-year cycle? Is it not rather a case of "overreading" the data? The answer to this question is not quite so obvious, as we proceed to discuss.

Figure 4.7. Spectrum of cycle component in a random walk

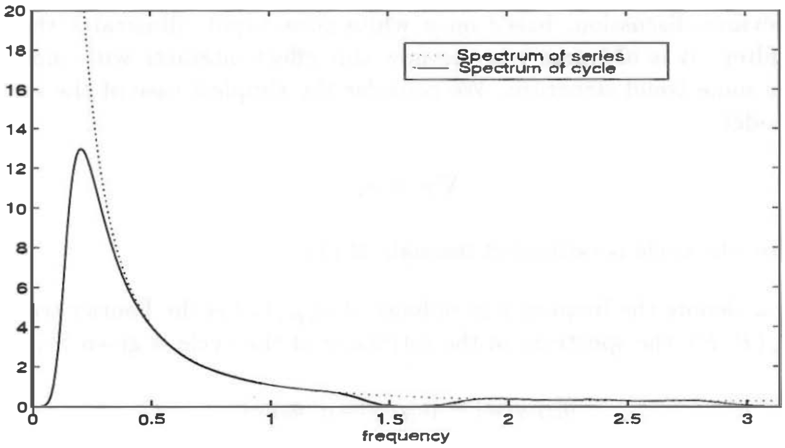


Figure 4.8. Spectrum of cycle in IMA(1,1) as a function of theta

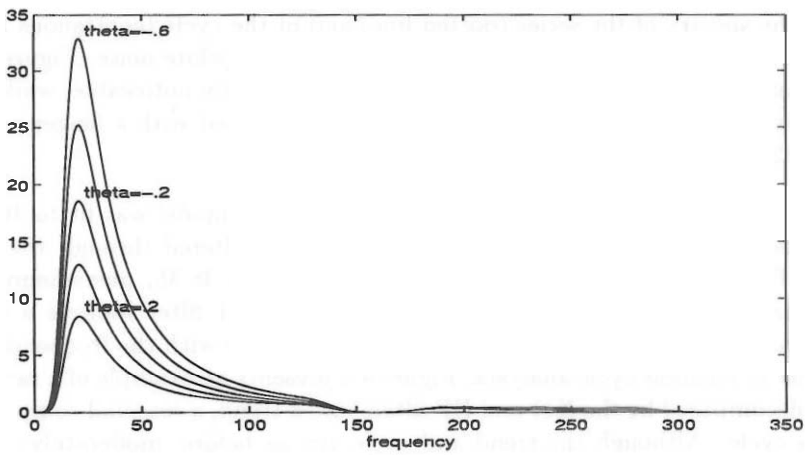
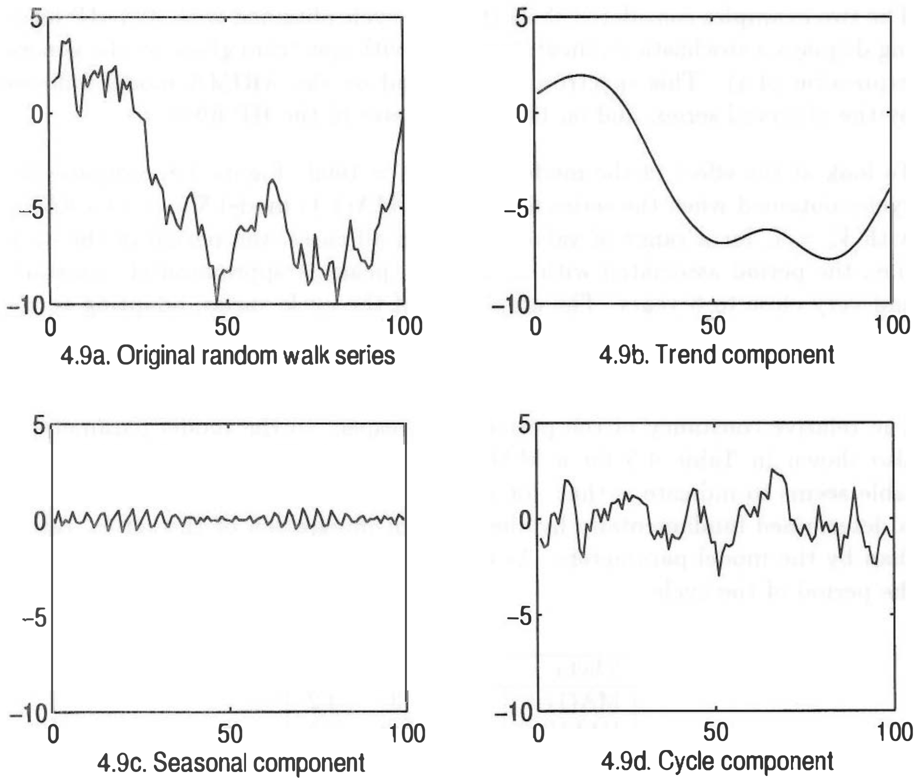


Figure 4.9. Decomposition of a random walk series



4.3.3 Spectral characteristics of the cycle; spuriousness reconsidered

The two examples considered show that the cycle obtained with X11-HP filtering displays a stochastic cyclical structure, with spectrum given by the general expression (4.4). This spectrum will depend on the ARIMA model followed by the observed series, and on the λ-parameter of the HP filter.

To look at the effect of the model, we set $\lambda = 1600$. Figure 4.8 compares the cycles obtained when the series follows the IMA(1,1) model $\nabla x_t = (1 + \theta B) a_t$, with $V_a = 1$, for a range of values for θ. In all cases, the period of the cycle (i.e., the period associated with its spectral peak) is approximately constant, and very close to 8 years. The amplitude of the cycle varies, adapting to the width of the spectral peak for $\omega = 0$ in the series model, which is determined by the parameter θ.

The relative constancy of the period with respect to the model parameter is also shown in Table 4.5 for a MA(1) and an IMA(2,1) models. What the table seems to indicate is that, for a fixed value of λ, the period of the cycle is determined fundamentally by the order of integration of the series, rather than by the model parameters. As the order of integration increases, so does the period of the cycle.

Theta	0	-.3	-.6
MA(1)	2	3	3.2
IMA(1,1)	7.9	7.9	7.9
IMA(2,1)	10.5	10.5	10.5

Table 4.5. Period of cycle (in years).

When the HP filter is applied to an X11 SA series, a similar effect is seen to occur. For the Airline model (3.6) we computed the period associated with the spectral peak of the cycle for the range $-.9 < \theta_1 < .5$ and $-.9 < \theta_4 < 0$, and in all cases the period was equal to approximately 10 years. For fixed parameter λ, three conclusions emerge:

- Given the type of ARIMA model for the series, the associated cyclical period becomes roughly fixed.
- The period seems to be mostly determined by the order of integration at the zero frequency; the stationary part of the model has little influence.

- For most actual time series containing a trend (d=1 or 2), the standard value of $\lambda = 1600$ implies a period between 8 and 10 years.

As for the parameter λ of the HP filter, its interpretation varies according to the rationalization of the filter. We saw that it regulates the trade-off between fitness and smoothness when the function (2.2) is minimized, and that it is also equal to the ratio of the cycle and trend innovations in the model-based approach. When expressed as a Butterworth type filter, its gain is expressed as

$$G(\omega) = \left[1 + \left(\frac{sin(\omega/2)}{sin(\omega_0/2)}\right)^4\right]^{-1}, \quad 0 \leq \omega \leq \pi$$

where ω_0 is the frequency for which 50% of the gain has been completed, that is $G(\omega_0) = 1/2$. As seen in Gómez and Maravall (1998),

$$\lambda = [4sin^2(\omega_0/2)]^{-2},$$

which gives a frequency interpretation for λ. Accordingly, λ plays an important role in determining the period associated with the cycle spectral peak.

Fixing the series model to that of a random walk, we proceed to analyze the dependence of the cycle period on λ. For the random-walk series (4.3), it is found that $g(x) = [2(1 - cos\omega)]^{-1}V_a$, and considering (4.4) -with $\tilde{\nu}^c_{HPX}(\omega)$ replaced by $\tilde{\nu}^c_{HP}(\omega)$- and (2.11), the spectrum of the HP-filtered cycle is equal

$$g^c_{HP}(\omega) = \frac{8(1 - cos\omega)^3}{[\lambda^{-1} + 4(1 - cos\omega)^2]^2} V_a. \tag{4.5}$$

It is straigthforward to find that, within the interval $0 \leq \omega \leq \pi$, (4.5) attains a single maximum at

$$\lambda = \frac{3}{4(1 - cos\omega)^2}.$$

For the range of frequencies associated with periods between 2 and 25 years, this function is represented in Fig. 4.10. It can be seen that the convolution with X11 has little effect on the period of the cycle peak (in fact the two figures would be indistinguishable). This was to be expected, given that, for the range of frequencies where the spectral peak is located, the gain of the X11 filter is close to 1. The values for the periods equal to an integer number of years are displayed in the second and fourth columns of Table 4.6. The relationship between λ and the period of the cycle spectral peak is seen to be highly nonlinear. When λ is small (and cycles are short), small increases in λ affect very strongly the period of the cycle; for long cycles, very large values of λ need to be used.

Period (in years)	λ	Period (in years)	λ
2	8.7	14	18970
3	41.8	15	24992
4	129.4	16	32346
5	313.1	17	41215
6	646	18	51794
7	1193	19	64291
8	2031	20	78924
9	3250	21	95923
10	4948	22	115532
11	7239	23	138004
12	10247	24	163605
13	14108	25	192614

Table 4.6. Values of λ for different cycles (period in years).

The effect of λ is illustrated in Figure 4.11, which compares the spectra of the cycles obtained with $\lambda = 1600$ and $\lambda = 25000$ for the same random walk series (the periods associated with the spectral peaks are about 8 and 15 years, respectively). The figure shows that the longer period implies a stochastic cycle that is more concentrated around its peak (i.e., a more stable cycle). The estimators of the trend and of the cycle for the two λ values are compared in Figures 4.12 and 4.13, respectively. The difference between the two trends is seen to consist of a cycle with a relatively long period. Comparison of the cycles shows that the short-term profile of the cycle is basically unaffected, and the main effect is a "pulling away" from the zero line, which allows for longer cycles.

As a consequence, the use of the X11-HP filter (or simply the HP filter) to measure the cycle implies an a-priori choice of the cycle period. Before using the HP filter to estimate a cycle, the analyst should decide the length of the period around which he wishes to measure cyclical activity. Then, given d (the number of unit roots at the zero frequency in the series at hand), he can choose the appropiate value of λ. To some extent, this may be reasonable. For example, a business cycle analyst involved in policy making may be interested in using 8 or 10-years cycles; an economic historian looking at several centuries, may be interested in spreading activity over longer periods. Viewed in this way, the HP cycle cannot be seen as spurious but as a rather particular yet possibly sensible way to look at the data. This statement will be made more precise at the end of Section 6.2.

Be that as it may, the filter presents shortcomings; the next sections address two important ones.

Fig. 4.10. Period of cycle as a function of lambda

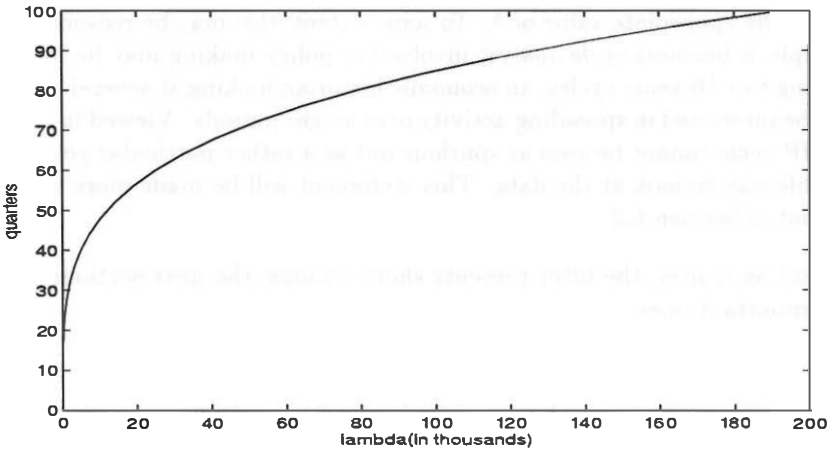


Figure 4.11. Spectrum of a cycle in a random walk

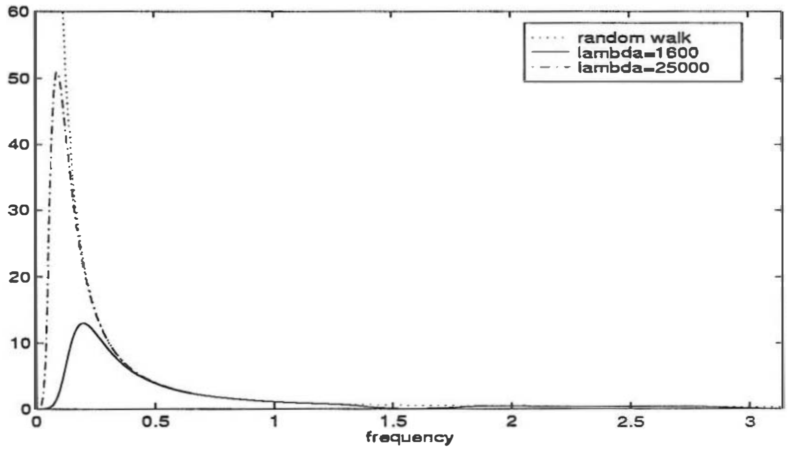


Figure 4.12. Estimated trends

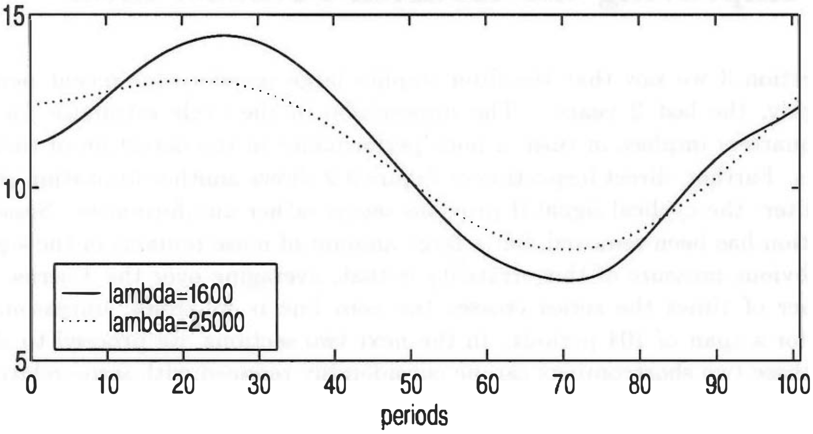
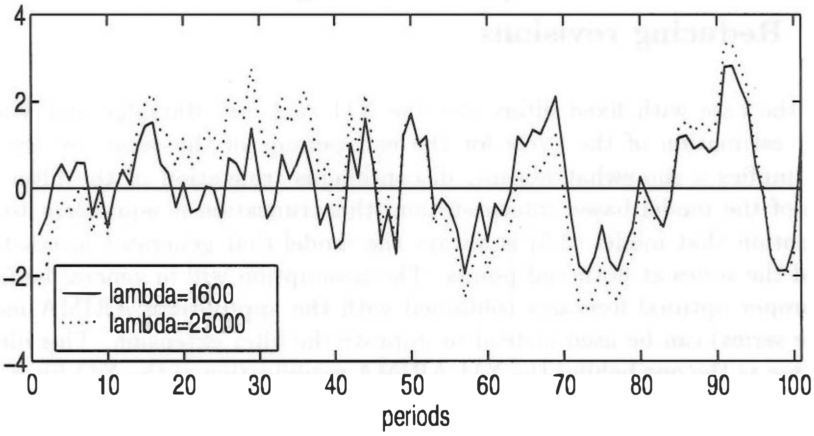


Figure 4.13. Estimated cycles



5 Improving the Hodrick-Prescott filter

In Section 3 we saw that the filter implies large revisions for recent periods (roughly, the last 2 years). The imprecision in the cycle estimator for the last quarters implies, in turn, a poor performance in the detection of turning points. Further, direct inspection of Figure 3.2 shows another limitation of the HP filter: the cyclical signal it provides seems rather uninformative. Seasonal variation has been removed, but a large amount of noise remains in the signal. An obvious measure of this erraticity is that, averaging over the 4 series, the number of times the series crosses the zero line is 31 times, unreasonably high for a span of 104 periods. In the next two sections, we proceed to show how these two shortcomings can be considerably reduced with some relatively simple modifications.

5.1 Reducing revisions

As is the case with fixed filters (for the X11 case, see Burridge and Wallis, 1984,) estimation of the cycle for the end periods of the series by the HP filter implies a somewhat abrupt, discontinuous truncation of the filter. In terms of the model based interpretation, this truncation is equivalent to the assumption that model (2.5) is always the model that generates forecasts to extend the series at both end points. The assumption will in general be false, and proper optimal forecasts (obtained with the appropriate ARIMA model for the series) can be used instead to improve the filter extension. This idea is the same as the one behind the X11 ARIMA modification of the X11 filter (see Dagum, 1980) and the HP filter applied to the series extended with ARIMA forecasts will be referred as the Hodrick-Prescott ARIMA (HPA) filter. The poor performance of the HP filter at the end of the series has been often pointed out by business cycle analysts (see, for example, Apel et al, 1996 and Baxter and King, 1995) and application of the filter to series extended with forecasts is often recommended in practice (see EU Commission, 1995).

For any positive integer k, write the final estimator of the cycle as

$$\hat{c}_t = \nu^c_{HP}(B,F)x_t = \sum_{j=0}^{\infty} \nu_{j+k} x_{t+k-j} + \sum_{j=1}^{\infty} \nu_{j+k} x_{t+k+j}, \tag{5.1}$$

and assume a series long enough so as to ignore starting values. Because the preliminary estimator $\hat{c}_{t|t+k}$ is a projection onto a subset of the set onto which $\hat{c}_t$ is projected, it follows that

$$\hat{c}_{t|t+k} = E_{t+k}(c_t) = E_{t+k}(\hat{c}_t),$$

or

$$\hat{c}_{t|t+k} = \sum_{j=0}^{\infty} \nu_{j+k} x_{t+k-j} + \sum_{j=1}^{\infty} \nu_{j+k} E_{t+k}(x_{t+k+j}), \tag{5.2}$$

which expresses the preliminary estimator as a function of the series extended with forecasts. Substracting (5.2) from (5.1), the revision in $\hat{c}_{t|t+k}$ is equal to

$$r_{t|t+k} = \sum_{j=1}^{\infty} \nu_{j+k}\, e_{t+k+j|t+k},$$

where $e_{t+k+j|t+k}$ denotes the forecasts error associated with forecasting x_{t+k+j} at time $t+k$. It follows that, reducing these forecasts errors, revisions should decrease (and early detection of turning points should improve).

To check this result, and to get an idea of the improvement that can be expected from the use of the HPA versus the HP filter we performed a simulation exercise. First, we consider the IMA(1,1) model for different values of the θ-parameter. Then, we consider the ARIMA(2,1,1) model, where the AR(2) polynomial is given by $(1 - .16B + .35B^2)$. This polynomial is the one found in Jenkins(1975) for the mink-muskrat Canadian data, and contains a cycle of period 4.4. The AR(2) structure will therefore produce an increase in the number of turning points. Again, different values of the θ-parameter were considered. A total of 14.000 series of length 100 each were simulated, and for each series the HP filter was compared to the HPA one extended with 16 ARIMA forecasts and backcasts. Table 5.1 compares the variances of the revision in the concurrent estimator and in the estimator revised after 1, 2, 3 and 4 more years of data are added. It is seen that, in all 70 cases, the HPA filter reduces considerably the revisions. This is particularly noticeable for the ARIMA(2,1,1) model, where the use of the standard HP filter more than triplicates the revision variance.

Model	Concurrent HPA	Concurrent HP	1 year rev. HPA	1 year rev. HP	2 year rev. HPA	2 year rev. HP	3 year rev. HPA	3 year rev. HP	4 year rev. HPA	4 year rev. HP
IMA(1,1)										
$\theta = -.8$	.31	.41	.08	.10	.02	.03	.01	.01	.00	.01
$\theta = -.5$	.94	1.34	.24	.33	.07	.11	.03	.06	.02	.04
$\theta = -.3$	1.58	2.54	.39	.63	.11	.19	.06	.12	.04	.08
$\theta = 0$	2.86	4.84	.71	1.18	.21	.39	.12	.24	.08	.15
$\theta = .3$	4.51	8.29	1.11	2.03	.32	.64	.19	.38	.13	.24
$\theta = .5$	5.44	11.02	1.40	2.62	.44	.86	.26	.50	.17	.32
$\theta = .8$	7.33	14.89	1.87	3.70	.60	1.17	.34	.70	.22	.46
ARIMA(2,1,1)										
$\theta = -.8$	.12	.55	.05	.14	.01	.03	.00	.01	.00	.01
$\theta = -.5$	.41	1.27	.15	.33	.04	.09	.01	.04	.00	.03
$\theta = -.3$	.74	2.23	.25	.56	.06	.15	.02	.08	.01	.05
$\theta = 0$	1.35	4.26	.44	1.09	.12	.29	.03	.15	.02	.10
$\theta = .3$	2.06	7.00	.71	1.77	.18	.46	.05	.25	.03	.18
$\theta = .5$	2.75	9.68	.95	2.44	.23	.64	.06	.33	.03	.22
$\theta = .8$	3.70	12.95	1.20	3.25	.31	.88	.09	.47	.05	.33

Table 5.1. Variance of the revision in estimator. Values are multiplied by 100.

As for the detection of turning points, we use the following simple criterion (along the lines of method B discussed in Boldin, 1994): a turning point is the first of at least two successive periods of negative/positive growth. Table 5.2 compares the performance of the HP and HPA filters in the first and last 8 observations of the simulated series, both in terms of the mean number of turning points that are dated on the original series and missed by the filtered one, and in terms of the mean number of turning points detected on the filtered series but not present in the original one ("peaks" and "throughs" are considered separately). Of the 56 comparisons, in 53 cases the gain from using the HPA filter is substantial.

Tables 5.3 and 5.4 compare the performance of the two filters when all observations in each series are considered. Table 5.3 compares the performance in detecting turning points present in the original series, and Table 5.4 looks at the false turning points indicated by the filtered series (and not present in the original one). For both tables, F_0 is the relative frequency of cases in which the two filters coincide, F_1 denotes the relative frequency of cases in which HPA performs better, while F_{-1} denotes the relative frequency of cases in which HP performs better. Both tables show that the HPA filter performs (in all 56 cases) remarkably better.

	Original		Missed				False Alarms			
	Peaks	Throughs	Peaks		Throughs		Peaks		Throughs	
IMA(1,1)			HPA	HP	HPA	HP	HPA	HP	HPA	HP
$\theta = -.8$	1.4	1.50	.10	.16	.10	.15	.18	.19	.19	.20
$\theta = -.5$	1.51	1.52	.18	.22	.19	.23	.29	.31	.26	.28
$\theta = -.3$	1.52	1.52	.22	.28	.24	.30	.36	.37	.38	.39
$\theta = 0$	1.62	1.59	.23	.32	.25	.37	.49	.51	.47	.49
$\theta = .3$	1.69	1.72	.29	.40	.29	.41	.49	.59	.55	.63
$\theta = .5$	1.77	1.79	.32	.46	.29	.41	.54	.68	.51	.63
$\theta = .8$	1.83	1.86	.30	.43	.34	.48	.52	.65	.54	.69
ARIMA(2,1,1)										
$\theta = -.8$	1.78	1.79	.05	.13	.05	.13	.21	.18	.19	.18
$\theta = -.5$	1.78	1.77	.09	.19	.11	.19	.25	.24	.26	.26
$\theta = -.3$	1.84	1.77	.12	.23	.10	.20	.25	.31	.25	.30
$\theta = 0$	1.87	1.84	.13	.25	.16	.27	.27	.38	.26	.36
$\theta = .3$	1.93	1.89	.17	.27	.20	.31	.25	.40	.25	.41
$\theta = .5$	1.94	1.96	.18	.31	.17	.32	.24	.40	.25	.42
$\theta = .8$	2.02	1.95	.18	.32	.16	.31	.22	.38	.24	.41

Table 5.2. Mean number of turning points (First and last 8 observations).

	Capt. peaks			Capt. Throughs		
Model	F_1	F_0	F_{-1}	F_1	F_0	F_{-1}
IMA(1,1)						
$\theta = -.8$	.07	.92	.01	.07	.93	.00
$\theta = -.5$	.07	.91	.02	.07	.89	.04
$\theta = -.3$	.11	.84	.05	.10	.86	.04
$\theta = 0$	.14	.80	.06	.16	.80	.04
$\theta = .3$	.17	.78	.05	.17	.78	.05
$\theta = .5$	.18	.76	.06	.19	.74	.07
$\theta = .8$	.18	.76	.06	.20	.75	.05
ARIMA(2,1,1)						
$\theta = -.8$	.11	.88	.01	.11	.88	.01
$\theta = -.5$	.14	.84	.02	.13	.84	.03
$\theta = -.3$	.17	.79	.04	.16	.80	.04
$\theta = 0$	.18	.77	.04	.18	.77	.05
$\theta = .3$	.18	.77	.05	.20	.73	.07
$\theta = .5$	.21	.73	.06	.23	.71	.06
$\theta = .8$	.21	.74	.05	.22	.73	.05

Table 5.3. Relative performance of HP vs HPA: captured turning points.

	False peaks			False Throughs		
Model	F_1	F_0	F_{-1}	F_1	F_0	F_{-1}
IMA(1,1)						
$\theta = -.8$	.06	.91	.03	.05	.91	.04
$\theta = -.5$	.09	.85	.06	.10	.84	.06
$\theta = -.3$	.12	.81	.07	.13	.78	.09
$\theta = 0$	.15	.74	.11	.15	.74	.11
$\theta = .3$	.22	.68	.10	.19	.71	.10
$\theta = .5$	.23	.68	.09	.20	.70	.10
$\theta = .8$	.25	.64	.11	.24	.65	.11
ARIMA(2,1,1)						
$\theta = -.8$	.07	.86	.07	.06	.84	.10
$\theta = -.5$	.10	.80	.10	.10	.81	.09
$\theta = -.3$	.15	.77	.08	.14	.78	.08
$\theta = 0$	.19	.74	.07	.19	.75	.06
$\theta = .3$	.28	.66	.06	.25	.67	.08
$\theta = .5$	.23	.70	.07	.23	.70	.07
$\theta = .8$	.28	.66	.06	.25	.69	.06

Table 5.4 Relative performance of HP vs HPA: spurious turning points.

In summary, the results of the simulation exercise strongly suggest that applying the HP filter to the series extended at both ends with appropriate ARIMA forecasts and backcasts is likely to provide a more precise cycle estimator for recent periods, that requires considerably smaller revisions and improves thereby detection of turning points.

5.2 Improving the cyclical signal

Concerning seasonality, its removal implies the removal of the spectral peaks associated with seasonal frequencies. Since the width of this peak varies across series, fixed filters such as X11 may over or underestimate seasonality. Having obtained an ARIMA model for the series, one could use, instead of X11, an ARIMA-model-based (AMB) type of adjustment, following the approach of Burman (1980) and Hillmer and Tiao (1982). We use the program SEATS (Gómez and Maravall, 1996) to seasonally adjust the 4 series of the example in Section 3. For the airline model (3.6), appropriate for the 4 series, the AMB method decomposes the series x_t as in

$$x_t = n_t + s_t \tag{5.3}$$

where n_t denotes the SA series and s_t the seasonal component, which follow models of the type

$$\nabla^2 n_t = \theta_n(B) a_{nt}, \tag{5.4}$$
$$S s_t = \theta_s(B) a_{st}, \tag{5.5}$$

where $\theta_m(B)$ and $\theta_s(B)$ are of order 2 and 3, respectively. The estimator of n_t is the conditional expectation $\hat{n}_{t|T} = E(n_t \mid x_1, \ldots, x_T)$. If $\theta(B) = (1+\theta_1 B)(1+\theta_4 B^4)$, the final estimator is given by the expression

$$\hat{n}_t = \left[k_n \frac{\theta_n(B) S}{\theta(B)} \frac{\theta_n(F) \bar{S}}{\theta(F)} \right] x_t. \tag{5.6}$$

The expression in brackets is the WK filter and it avoids over/underestimation by adjusting itself to the width of the spectral peaks present in the series (see, for example, Maravall, 1998).

Figure 5.1 compares the cycles obtained by applying the HP filter to the AMB and X11 SA series, and Figure 5.2 exhibits the spectra of the two cycles for the 4 series. It is seen that the estimates of the cycle produced using the two SA series are close, and no improvement results from applying the AMB method: turning points remain basically unchanged and the cyclical signal remains very noisy. (Figure 5.2 illustrates the overestimation of seasonality implied by the X11 filter for the case of the CC series: seasonality is very stable and consequently the width of the spectral peaks for the seasonal frequencies is very narrow. It is seen how the "holes" that X11 induces for these frequencies are excessively wide.)

Given that the SA series produces a cyclical signal with too much noise it would seem that this signal could be improved by removing the noise from the SA series. Thus we replace the decomposition (5.3) by

$$x_t = p_t + s_t + u_t, \tag{5.7}$$

where s_t is a before and u_t, for the case of the Airline model, is white noise.

Figure 5.1. HP cycle based on X11 and SEATS SA series

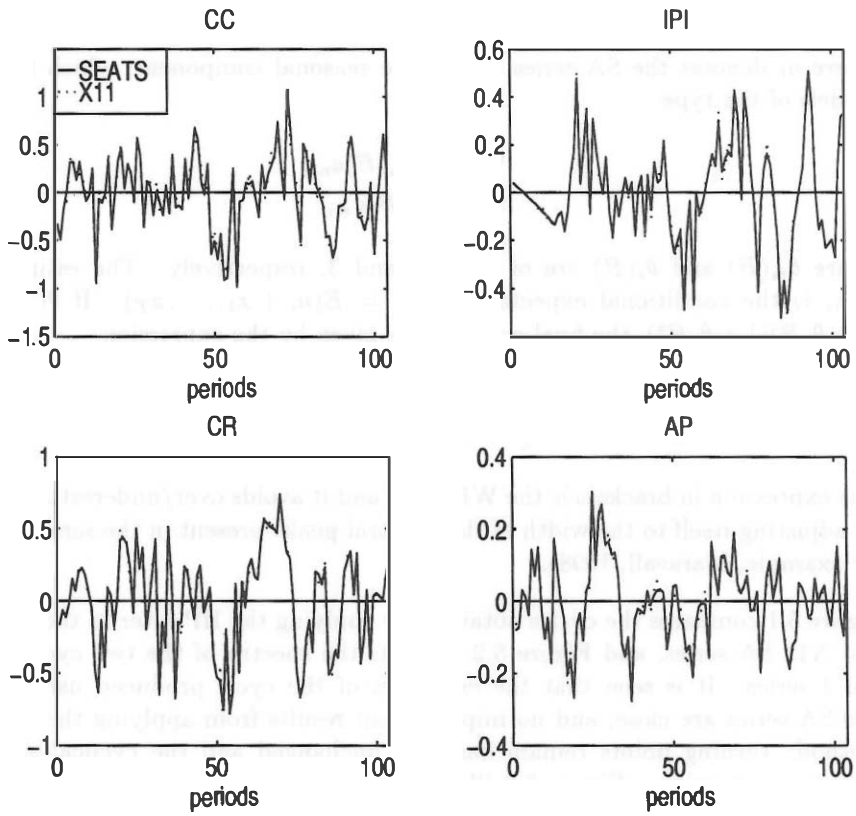
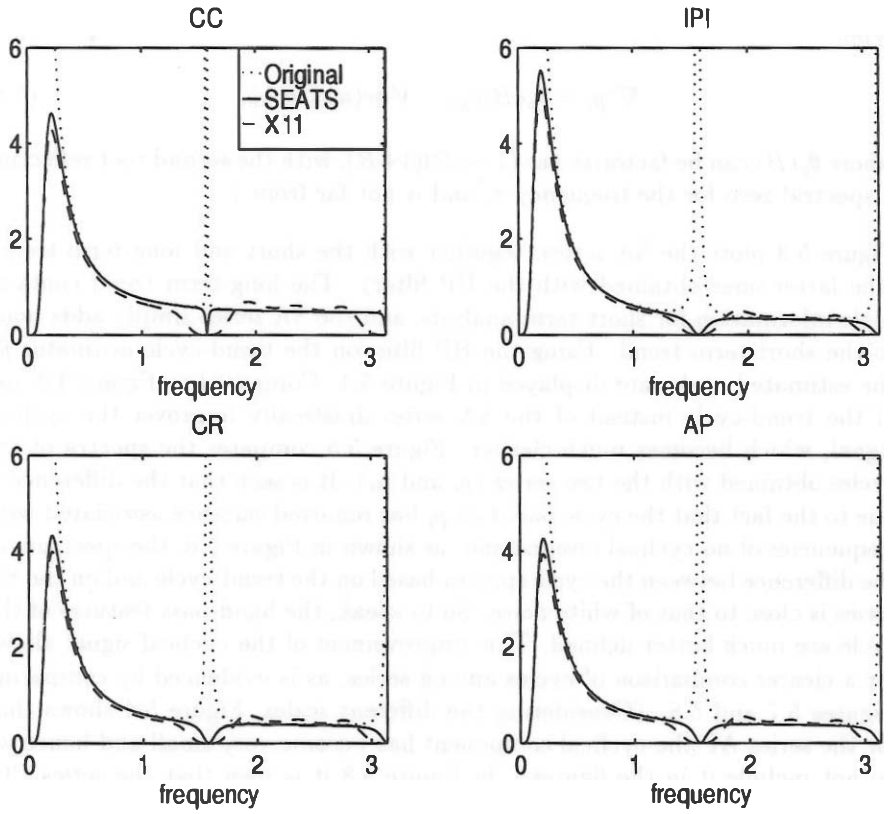


Figure 5.2. Spectrum of cycle based on X11 and SEATS SA series



From (5.3) and (5.7), $n_t = p_t + u_t$, so that the component p_t is the noise-free SA series. This component p_t is usually referred to as the trend-cycle component or the short-term trend. It follows in the four cases an IMA(2,2) model of the type,

$$\nabla^2 p_t = \theta_p(B) a_{pt}, \quad Var(a_{pt}) = V_p, \tag{5.8}$$

where $\theta_p(B)$ can be factorized as $(1-\alpha B)(1+B)$, with the second root reflecting a spectral zero for the frequency π, and α not far from 1.

Figure 5.3 plots the SA series, together with the short and long-term trends (the latter ones obtained with the HP filter). The long-term trend contains little information for short-term analysis, and the SA series simply adds noise to the short-term trend. Using the HP filter on the trend-cycle estimator $\hat{p}_t$, the estimated cycles are displayed in Figure 5.4. Compared to Figure 3.2, use of the trend-cycle instead of the SA series drastically improves the cyclical signal, which becomes much cleaner. Figure 5.5 compares the spectra of the cycles obtained with the two series (p_t and n_t). It is seen that the difference is due to the fact that the cycle based on p_t has removed variance associated with frequencies of no cyclical interest and, as shown in Figure 5.6, the spectrum of the difference between the cycle spectra based on the trend-cycle and on the SA series is close to that of white noise. So to speak, the band-pass features of the cycle are much better defined. This improvement of the cyclical signal allows for a clearer comparison of cycles among series, as is evidenced by comparing Figures 5.7 and 5.8. (Considering the different scales, Figure 5.4 shows that for the series AP the cyclical component has become very small and hence we do not include it in the figures.) In Figure 5.8 it is seen that the series CC, IPI, and CR have fairly similar cyclical patterns, moving roughly in phase.

Figure 5.3. Trend and trend-cycle components

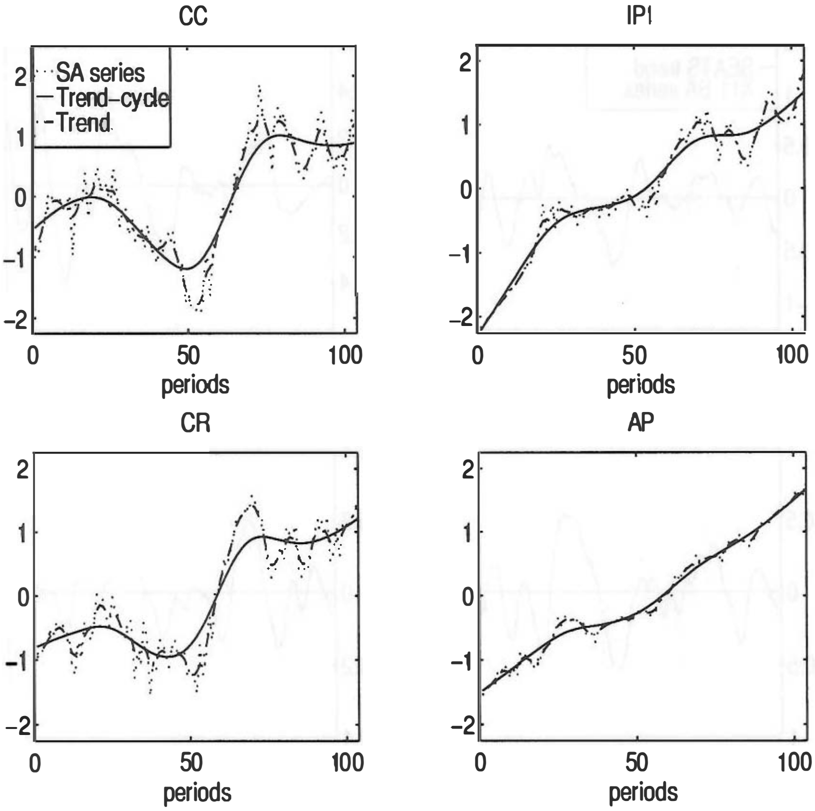


Figure 5.4. HPcycle based on SEATS trend an on X11 SA series

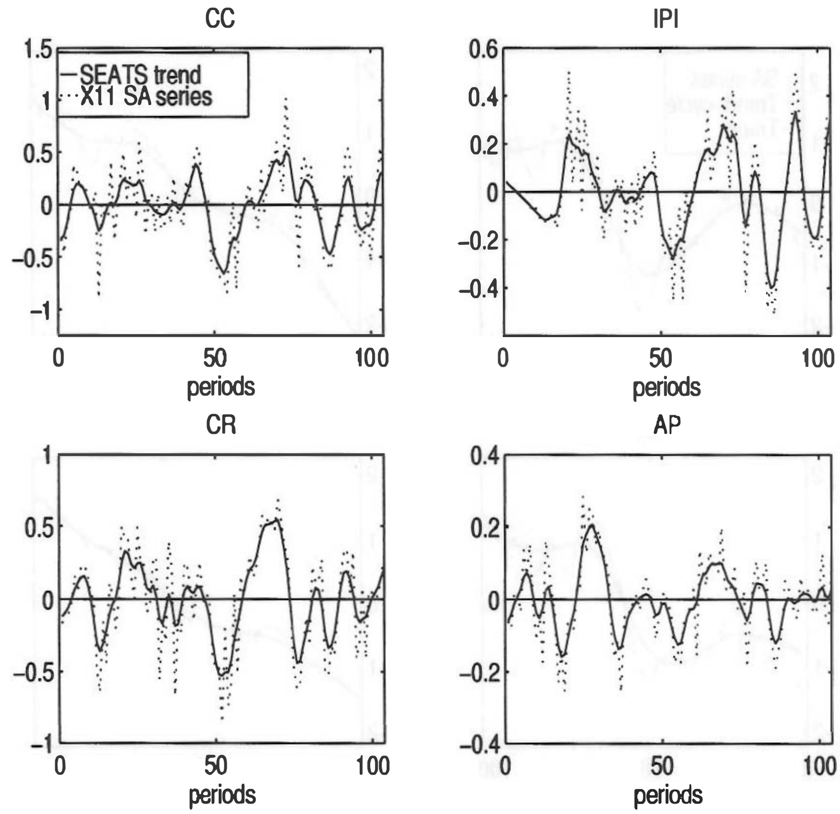


Fig. 5.5. Spectrum of cycle (SEATS trend and X11 SA series)

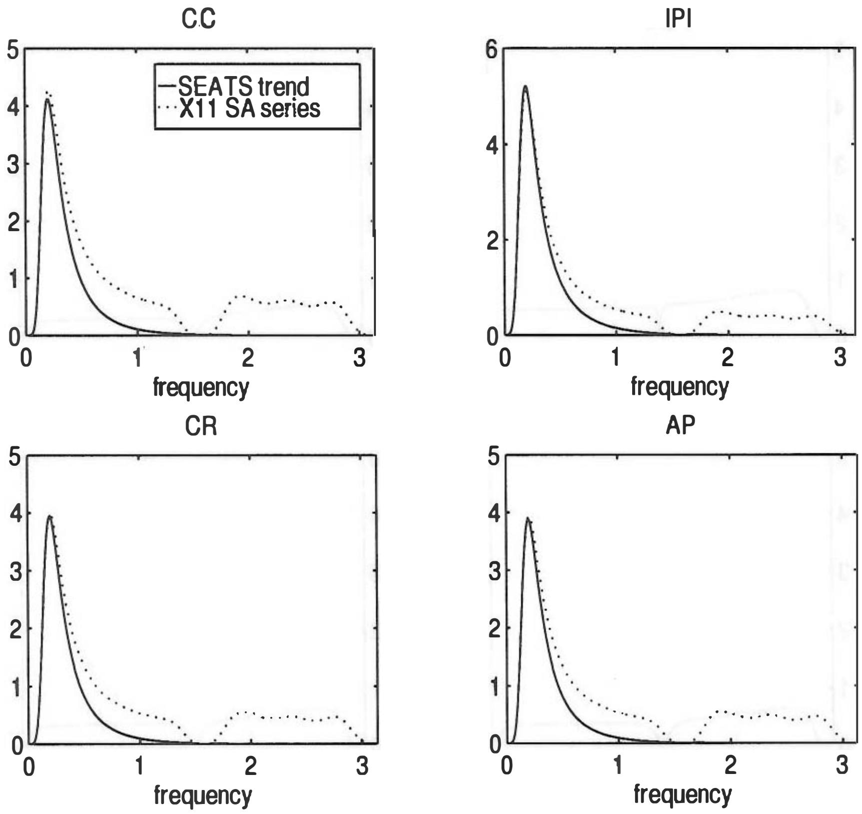


Fig. 5.6. Difference between cycles (SEATS trend and X11 SA series)

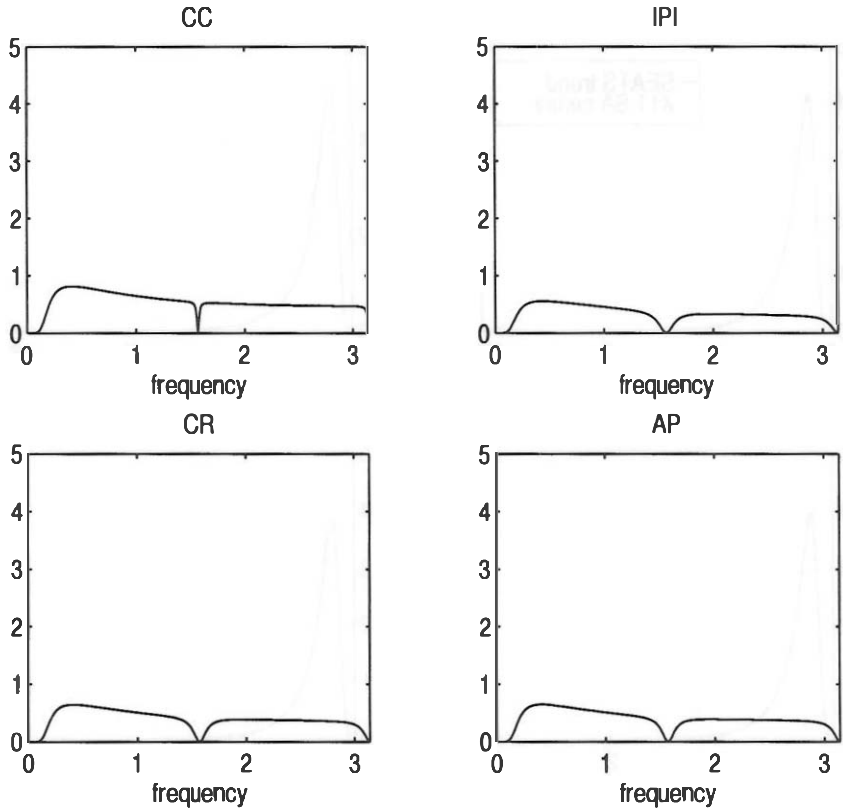


Figure 5.7. HP cycles based on X11 SA series

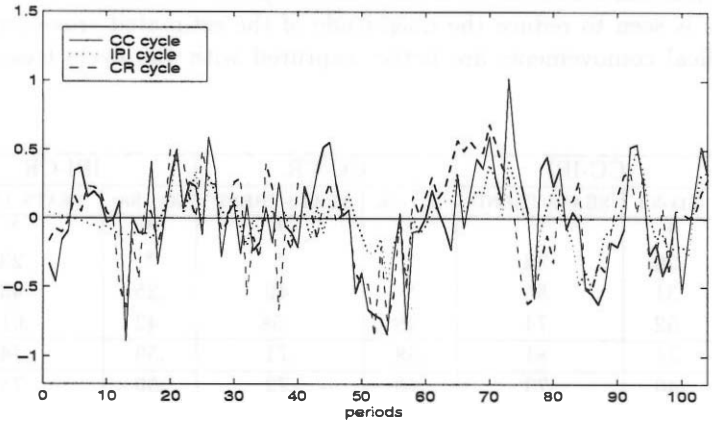


Figure 5.8. HP cycles based on SEATS trend

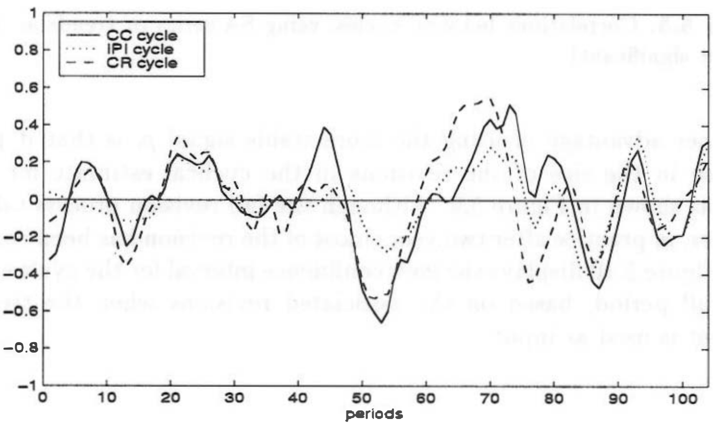


Table 5.5 compares the crosscorrelations between the cycles in the 3 series when the SA series and the trend-cycle are used as inputs. The noise contained in the SA series is seen to reduce the magnitude of the estimated crosscorrelations. The cyclical comovements are better captured with the cycle based on the trend.

Lag	CC-IPI		CC-CR		IPI-CR	
	X11-SA	SEATS-TREND	X11-SA	SEATS-TREND	X11-SA	SEATS-TREND
-4	*	*	*	*	*	*
-3	.20	.36	*	*	*	.23
-2	.31	.55	*	.40	.25	.43
-1	.52	.74	.38	. 58	.42	.61
0	.74	.81	.58	.71	.59	.74
1	.40	.73	.46	.72	.50	.75
2	.31	.57	.44	.67	.46	.69
3	.23	.40	.31	.56	.36	.57
4	*	.25	*	.44	*	.44

Table 5.5. Correlations between cycles, using SA series or trends as input. (* not significant)

One further advantage of using the more stable signal p_t is that it produces a decrease in the size of the revisions in the cyclical estimate for the last periods, as shown in Figure 5.9. Although the full revision process takes close to 10 years, in practice after two years most of the revision has been completed. Finally, Figure 5.10 displays the 95% confidence interval for the cycle estimator for the full period, based on the associated revisions when the trend-cycle component is used as input.

Fig 5.9. Standard deviation of revision from concurrent to final estimation

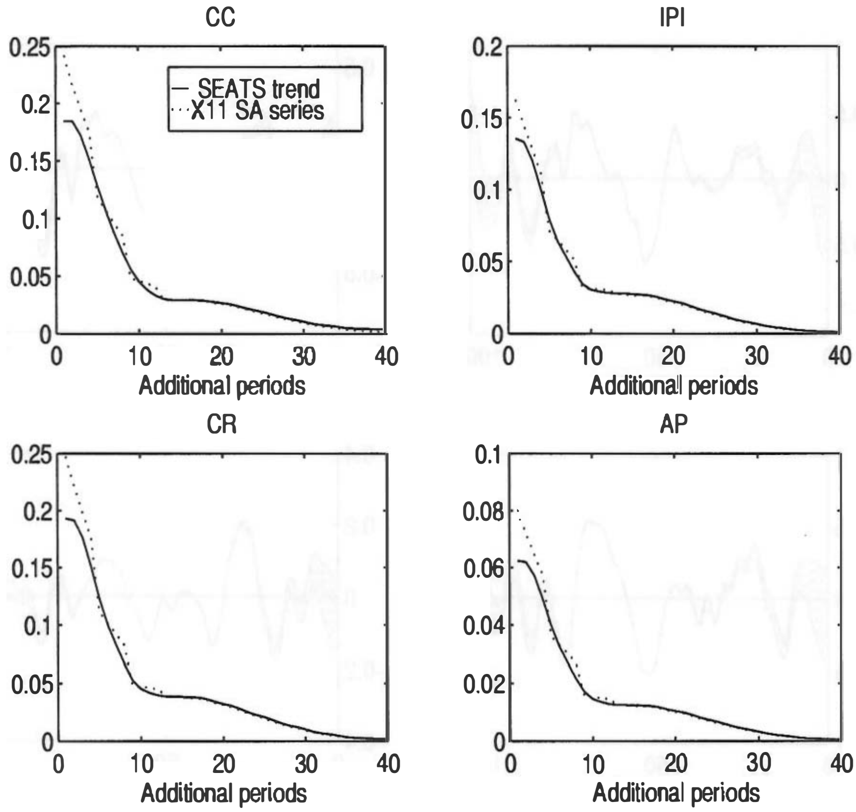
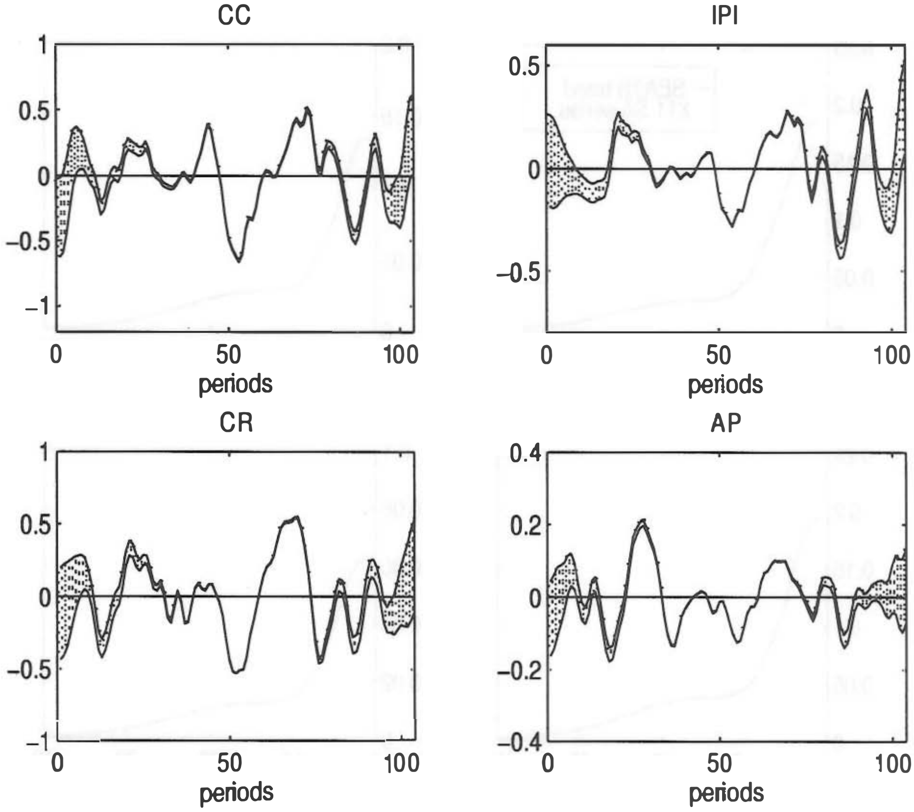


Fig 5.10. 95% CI for HP cycle (based on revisions)



6 Hodrick-Prescott filtering within a model based approach

6.1 A simple model-based algorithm

What we have suggested in the previous section is to estimate the cycle in two steps. First, the AMB method is used to obtain the trend-cycle estimator $\hat{p}_t$ (i.e., the noise-free SA series). In a second step, the HP filter is applied to $\hat{p}_t$.

Assume the observed quarterly series follows the (most often encountered in practice) ARIMA model

$$\nabla\nabla_4 x_t = \theta(B)a_t \tag{6.1}$$

with $\theta(B)$ an invertible polynomial. (The discussion extends to other AR structures, but it is greatly simplified using (6.1).) Let (5.8) denote the model for the trend-cycle component obtained from the AMB decomposition (this model is provided in the output of SEATS). The MMSE estimator of p_t is given by the WK filter (see, for example, Maravall 1995)

$$\hat{p}_t = \left[k_p \frac{\theta_p(B)S}{\theta(B)} \frac{\theta_p(F)\bar{S}}{\theta(F)}\right] x_t,$$

where $k_p = V_p/V_a$ and V_p denotes the variance of a_{pt}. The second step consists in obtaining

$$\hat{c}_t = \nu^c_{HP}(B,F)\hat{p}_t, \tag{6.2}$$

where the ν^c_{HP} filter is as in (2.10). Without loss of generality, let us standarize the units of measurement by setting $V_a = 1$, and let $k = V_p k_{c(HP)}$ where $k_{c(HP)}$ was defined in (2.10). Then it is obtained that, in terms of the observed series,

$$\hat{c}_t = \left[k \frac{\theta_p(B)\nabla\nabla_4}{\theta_{HP}(B)\theta(B)} \frac{\theta_p(F)\bar{\nabla}\bar{\nabla}_4}{\theta_{HP}(F)\theta(F)}\right] x_t. \tag{6.3}$$

Direct inspection shows that the filter in (6.3) is the ACVF of the model

$$\theta_{HP}(B)\theta(B)z_t = \theta_p(B)\nabla\nabla_4 b_t, \tag{6.4}$$

with $Var(b_t) = k$.

It is well-known (see, for example, Bell and Hillmer, 1984) that if a series following the general ARIMA model (6.1) is decomposed into signal plus white noise, the MMSE estimator of the noise is given by a filter equal to the autocovariance function (ACVF) of the inverse model (multiplied by the variance of the noise). The inverse model is the one that results from interchanging the AR and MA parts, that is,

$$\theta(B)z_t = \phi(B)a_t.$$

Since,

$$\theta_p(B)\nabla\nabla_4 x_t = \theta(B)\theta_{HP}(B)d_t \tag{6.5}$$

is the inverse model of (6.4), it follows that $\hat{c}_t$, given by (6.3), is the estimator of the noise in the decomposition of (6.5) into signal plus white-noise when the variance of the latter is k.

In this way the cycle estimator can be obtained as follows. Let x_t follow the ARIMA model (6.1), and let $\theta_p(B)$ and V_p be the MA polynomial and innovation variance of the model for the trend-cycle p_t in the standard AMB decomposition $x_t = p_t + s_t + u_t$, with s_t and u_t denoting the seasonal and irregular components. To obtain the cycle estimator:

- Multiply the AR part of the model for x_t by $\theta_p(B)$, i.e.,

$$\alpha(B) = \theta_p(B)\nabla\nabla_4$$

- Multiply the MA part of the model for x_t by $\theta_{HP}(B)$, i.e.,

$$\beta(B) = \theta_{HP}(B)\theta(B)$$

Then the WK filter that yields $\hat{c}_t$ is the estimator of the noise in the decomposition of the model

$$\alpha(B)x_t = \beta(B)a_t \tag{6.6}$$

into signal plus white noise with the variance of the noise equal to k. This filter is directly obtained as the ACVF of the model

$$\beta(B)x_t = \alpha(B)a'_t,$$

with $Var(a'_t) = k$.

This way of proceeding relies on a white-noise assumption for the cyclical component c_t, which is not very appealing. The procedure, thus, offers a simple algorithm, not a useful model-based interpretation. It is worth noticing that this algorithm will produce estimators for the end points different from the ones obtained in the previous procedure (computing first $\hat{p}_t$, and then using the HP filter). This difference is implied by the fact that, in this latter procedure, the forecasts and backcasts, used to extend the series in the Burman-Wilson algorithm described in Appendix A.1, are obtained with model (2.5), while in the signal plus noise decomposition of (6.6), they are obtained with model (6.5). The difference between the two procedures of course vanishes if the ad-hoc forecasts and backcasts are replaced by the appropriate ARIMA ones, that is, if the HPA filter is used instead of the HP one.

6.2 A complete model-based method

Looking again at expression (6.3), that provides the cycle estimator for the general model (6.1), another more appealing model-based interpretation is immediately obtained. Expression (6.3) provides the MMSE estimator of the cycle in model (6.1), when the model for the cycle is of the type

$$\theta_{HP}(B)c_t = \theta_p(B)a_{ct}, \tag{6.7}$$

with $Var(a_{ct})/Var(a_t) = k$. This model-based interpretation of the complete X11-HP filter, whereby the cycle obtained can be seen as the MMSE estimator of an unobserved component c_t that follows model (6.7), when the observed series follows the general model (6.1), is of some interest. The AR part of the model for the cycle is always the same, and equal to $\theta_{HP}(B)$; it incorporates the "fixed" character of the HP filter. On the other hand, the MA part, equal to $\theta_p(B)$, as well as the variance of the innovation (k), will depend on the particular series at hand, and will adapt the filter to the series model.

Therefore, the model for the cycle mixes the band-pass desirable features of the filter with the need to respect the series stochastic structure. Model (6.7) is based on the cycle obtained using the trend-cycle component p_t as input. If, instead, the SA series n_t is used, replacing $\theta_p(B)$ and V_p by $\theta_n(B)$ and V_n (the MA polynomial and the innovation variance in the model for the SA series), the interpretation remains the same.

For the 4 Spanish series, now standarized to have $V_a = 1$, Figure 6.1 plots the spectra of the series (dotted line), of its trend-cycle p_t (dashdot line) and of the cycle component c_t (shaded area), when the standard value $\lambda = 1600$ is used. The latter component is seen to have well defined band-pass features which adjusts to the width of the spectral peak of the trend-cycle component in the series. Figure 6.2 shows the spectra of the difference between the original series and the cycle spectra (solid line). This difference is clearly made of a long-term trend, a seasonal component, and white noise. The figure also displays the spectra of the original series (dotted line) so that the shaded area represents the series variation captured by the cycle. The decomposition of the series into the two components c_t and $x_t - c_t = m_t + s_t + u_t$, represented in Figure 6.3 seems perfectly legitimate if interest centers in optimal estimation of the series variation associated with the shaded area of Figure 6.1.

Figure 6.1. Spectra in the model–based interpretation

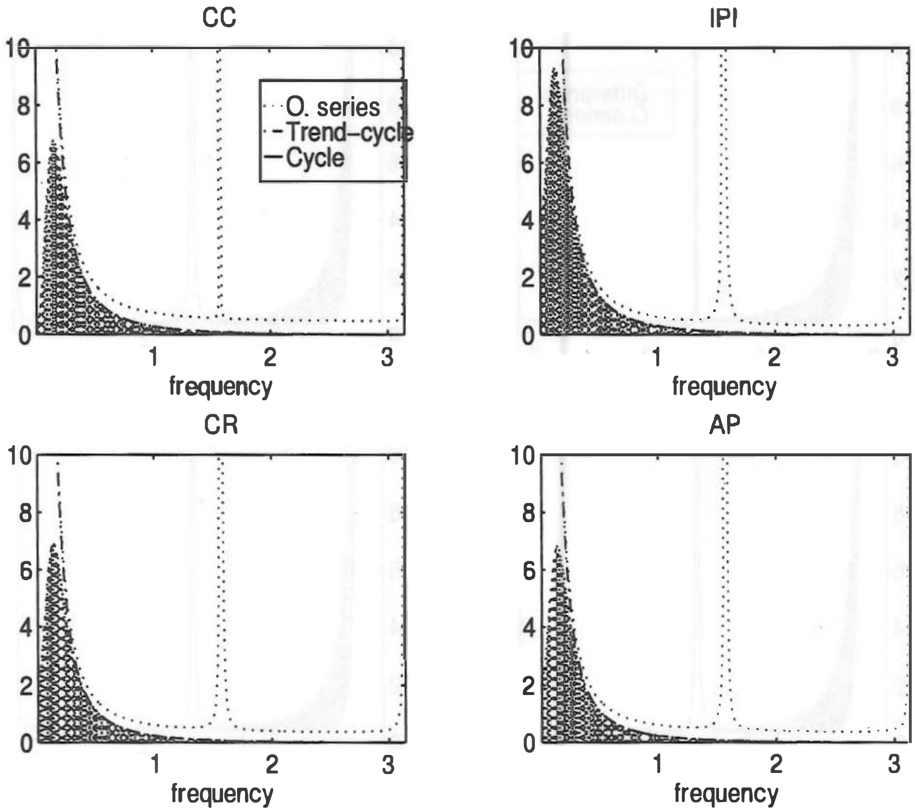


Figure 6.2. Spectra of the difference (original series minus cycle)

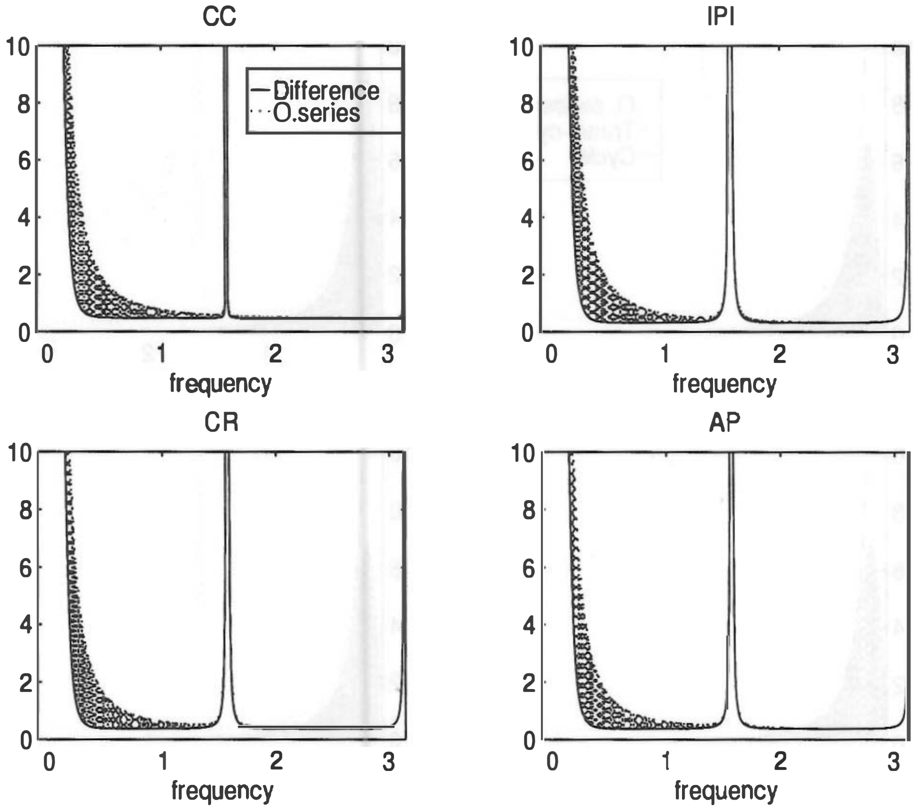
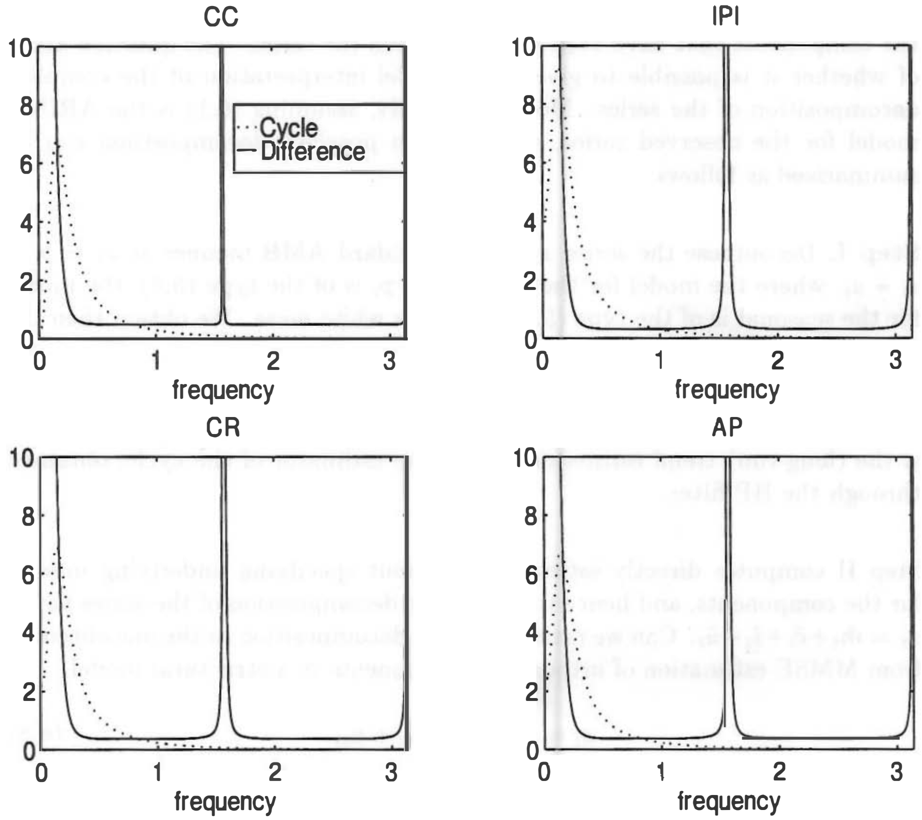


Figure 6.3. Decomposition of the series



Interpretation of the HP filter applied to the trend-cycle component as the optimal estimator of the theoretical cyclical component (6.7), when the observed series follows model (6.1), does not specify the models for the rest of the components that have been extracted from the series. The question arises of whether it is possible to give a full model interpretation of the complete decomposition of the series. More specifically, assuming (6.1) is the ARIMA model for the observed series, our two-step previous decomposition can be summarized as follows:

Step I. Decompose the series x_t in the standard AMB manner as $x_t = p_t + s_t + u_t$, where the model for the trend-cycle p_t is of the type (5.8), the model for the seasonal is of the type (5.5), and u_t is white noise. We obtain then the MMSE estimators $\hat{p}_t, \hat{s}_t$ and $\hat{u}_t$.

Step II. Next, the estimator $\hat{p}_t$ is decomposed as in $\hat{p}_t = \hat{m}_t + \hat{c}_t$, where $\hat{m}_t$ is the (long-run) trend estimator, and $\hat{c}_t$ the estimator of the cycle, obtained through the HP filter.

Step II computes directly estimators, without specifying underlying models for the components, and hence the complete decomposition of the series yields $x_t = \hat{m}_t+\hat{c}_t+\hat{s}_t+\hat{u}_t$. Can we rationalize this decomposition as the one obtained from MMSE estimation of orthogonal components in a structural model,

$$x_t = m_t + c_t + s_t + u_t, \tag{6.8}$$

where each component has a sensible model expression, and for which the reduced form (i.e., the model for the aggregate series) is of the type (6.1)? The answer to this question is in the affirmative. First, if the observed series follows the model (6.1), then the standard AMB decomposition of x_t yields a trend-cycle p_t, a seasonal component s_t, and an irregular component u_t, such that $x_t = p_t + s_t + u_t$, and the models for the components are of the type

$$\nabla^2 p_t = \theta_p(B)a_{pt}, \quad Var(a_{pt}) = V_p, \tag{6.9}$$
$$Ss_t = \theta_s(B)a_{st}, \quad Var(a_{st}) = V_s, \tag{6.10}$$
$$u_t = \theta_u(B)a_{ut}, \quad Var(a_{ut}) = V_u, \tag{6.11}$$

where $\theta_p(B)a_{pt}, \theta_s(B)a_{st}$, and $\theta_u(B)a_{ut}$ are stationary processes. (If the order of $\theta(B)$ in (6.1) is not larger than 5, then $\theta_u(B) = 1$ and u_t is a white noise irregular. This is indeed the case for the 4 series considered in the example.) In the seconsd step of the procedure, the HP filter is applied to the MMSE estimator of p_t in the above model, and this yields the estimator of the cycle, $\hat{c}_t$, and of the trend, $\hat{m}_t$.

Consider now the following unobserved component model,

$$\theta_{HP}(B)\nabla^2 m_t = \theta_p(B)a_{mt}, \quad Var(a_{mt}) = V_m, \tag{6.12}$$

$$\theta_{HP}(B)c_t = \theta_p(B)a_{ct}, \quad Var(a_{ct}) = V_c, \tag{6.13}$$

plus equations (6.10) and (6.11) for the seasonal and irregular components, where $V_m = V_p k_{m(HP)}$, $V_c = V_p k_{c(HP)}$, and all components are mutually orthogonal. As seen in Appendix B.2, $p_t = m_t + c_t$, and hence $x_t = m_t + c_t + s_t + u_t$. Thus the sum of the components (6.10), (6.11), (6.12), and (6.13) yields the model (6.1) for the observed series. The MMSE estimators of the components will thus be identical to the ones obtained with the two-step procedure. Notice, further, that adding m_t and c_t exactly yields the AMB standard decomposition of x_t into a trend-cycle, seasonal, and irregular components (the latter two remain, of course, unchanged.) Therefore, the AMB trend-cycle component accepts in turn a sensible AMB decomposition into trend plus (orthogonal) cycle. The 2-step procedure, thus, is seen to collapse into direct optimal estimation of the components in an unobserved component model. Notice that in the complete model-based representation, the two components m_t and c_t are canonical, because their models contain the MA root (1+B), present in the polynomial $\theta_p(B)$, which implies a spectral zero for $\omega = \pi$ (of course p_t also presents this feature.)

As an example, we show the unobserved component model for the car registration (CR) series, one of the 4 Spanish indicators previously used. The models for the trend-cycle, seasonal and irregular components in the AMB decomposition are given in Appendix B.3. It is seen that, for the CR case, and using the standard value $\lambda = 1600$, for which $\theta_{HP}(B) = 1 - 1.7771B + .7994B^2$, it is obtained that $k_{c(HP)} = .7994$, and $k_{m(HP)} = 1/2001.4$. The full model specification becomes

$$(1 - 1.7771B + .7994B^2)\nabla^2 m_t = (1 + \ 0662B - .9338B^2)a_{mt}$$

$$(1 - 1.7771B + .7994B^2)c_t = (1 + .0662B - .9338B^2)a_{ct},$$

$$Ss_t = (1 + .0383B - .4967B^2 - .4650B^3)a_{st},$$

and u_t white noise, with the components innovation variances given by $V_m = .0386 * 10^{-3}, V_c = .0628, V_s = .0685 * 10^{-1}, V_u = .3695$. It is straightforward to verify that the model for the components sum is the Airline model

$$\nabla\nabla_4 x_t = (1 - .387B)(1 - .760B^4)a_t,$$

with $V_a = 1$, which coincides with the model identified for CR in Section 3.

If one were to use the seasonally adjusted series $\hat{n}_t$ as the input to the HP filter in the 2-step procedure, the unobserved component model interpretation would slightly vary. Given that,

$$n_t = p_t + u_t, \tag{6.14}$$

from expressions (B.9) and (B.10) in Appendix B.2 it is found that n_t follows a model of the type

$$\nabla^2 n_t = \theta_n(B)a_{nt}, \quad Var(a_{nt}) = V_n,$$

where the polynomial $\theta_n(B)$ and the innovation variance V_u are straightforward to find from the identity

$$\theta_n(B)a_{nt} = \theta_p(B)a_{pt} + \nabla^2\theta_u(B)a_{ut}.$$

The irregular component disappears since it is absorbed by the seasonally adjusted series. Replacing V_p by V_n, so that now $k_c = V_n k_{c(HP)}$ and $k_m = V_n k_{m(HP)}$, the unobserved component model is given by

$$x_t = m_t + c_t + s_t,$$

where m_t and c_t are as in (6.12) and (6.13), with $\theta_p(B)$ replaced by $\theta_n(B)$, and s_t is as in (6.10). The model for x_t remains unchanged. The effect of these replacements is to add noise to the HP filter input, part of which is passed on to the cyclical signal (as was seen in Section 5.2), so that the cycle obtained from the seasonally adjusted series is in fact equal to the one obtained from the trend-cycle plus some added noise. For the CR series example, considering Appendix B.3, the unobserved component model becomes

$$(1 - 1.7771B + .7994B^2)\nabla^2 m_t = (1 - 1.3215B + .3621B^2)a_{mt},$$

$$(1 - 1.7771B + .7994B^2)c_t = (1 - 1.3215B + .3621B^2)a_{ct},$$

with $V_m = .00041, V_c = .6562$; the model for s_t and the variance V_s remain unchanged, as does the model for x_t.

For the 4 series of the application (CC, IPI, CR, and AP), Figures 6.4 to 6.7 display the spectral decomposition into the trend, cycle, seasonal, and irregular components. All of them have sensible shapes, and their sum yields the spectrum of the series, also shown in the figure. Figures 6.8 to 6.11 display the squared gains of the components filter; it is seen how they adapt to the spectral characteristics of each series. The complete specification of the unobserved component model for the 4 series in the example is given in appendix B.3.

An interesting remark concerns the spuriouness question discussed in Section 4, which can now be answered in a more precise manner. In so far as the overall ARIMA model for the observed series fits reasonably well the data, it is worth stressing that, because this model and the unobserved components model we have derived from it are observationally equivalent, the latter will also fit equally well the data. The two models, by construction, imply identical joint distributions functions generating the data (assuming appropiate starting conditions are set). If the ARIMA model for the observed series is acceptable on empirical grounds, so should be the unobserved component formulation. One may or may not agree, on a priori grounds, with the models specified for the components, but in no way can the unobserved components model be called spurious.

Fig. 6.4. Spectra for original series and components; series CC

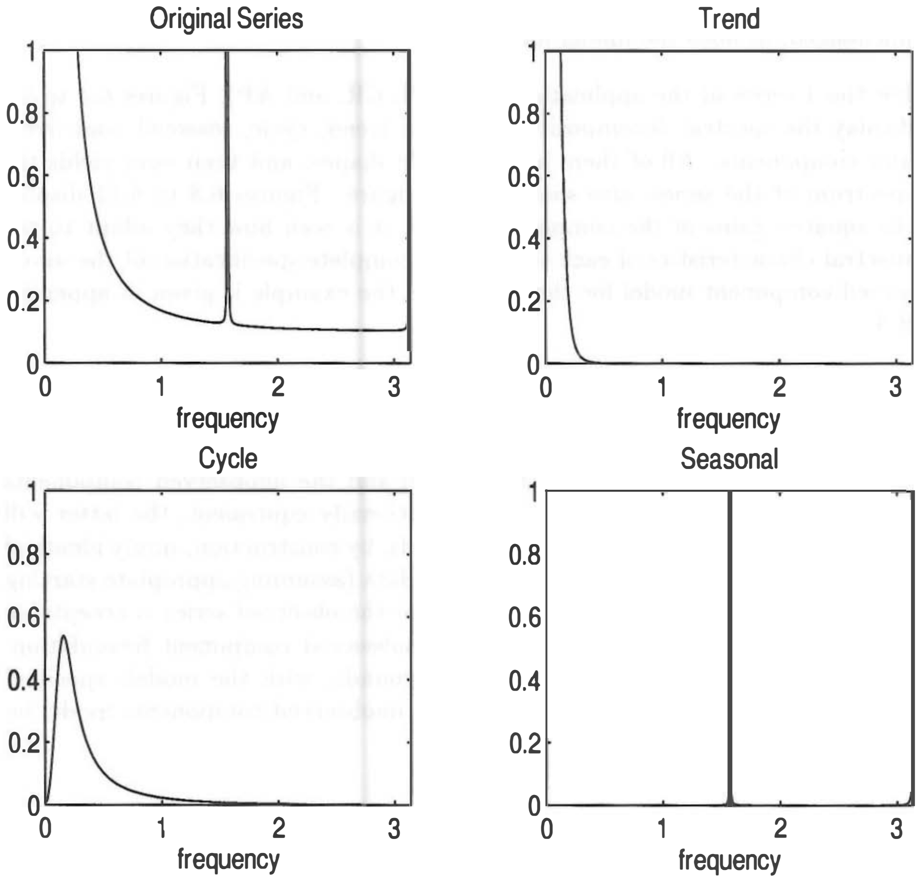


Fig. 6.5. Spectra for original series and components; series IPI

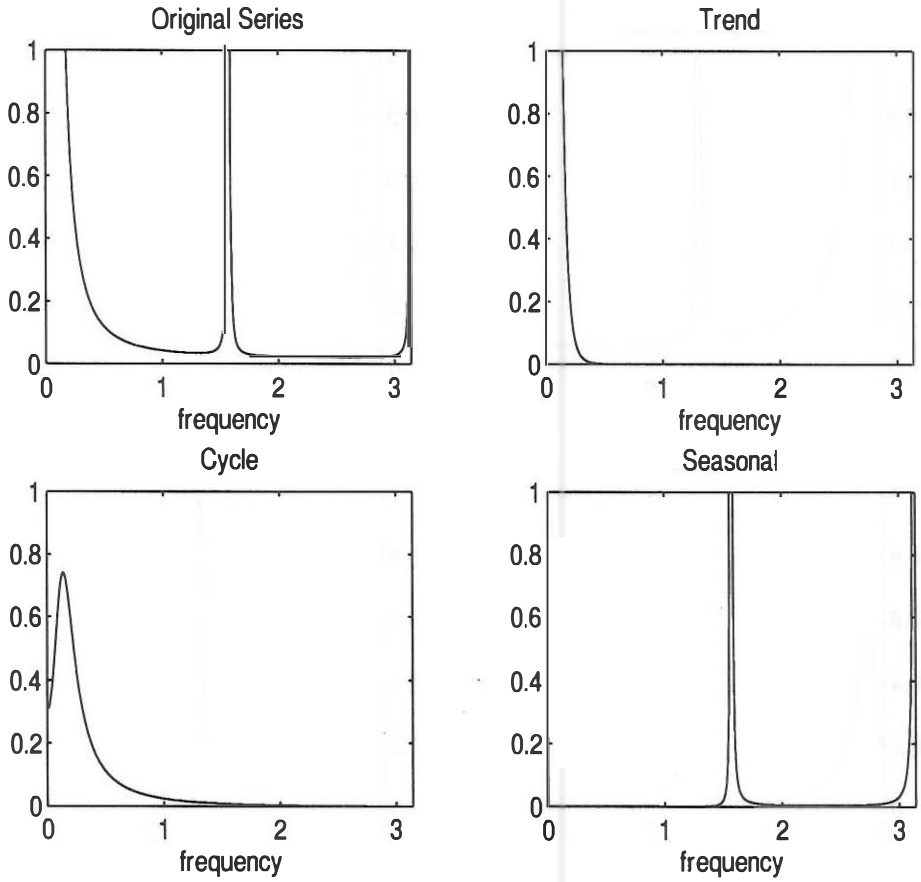


Fig. 6.6. Spectra for original series and components; series CR

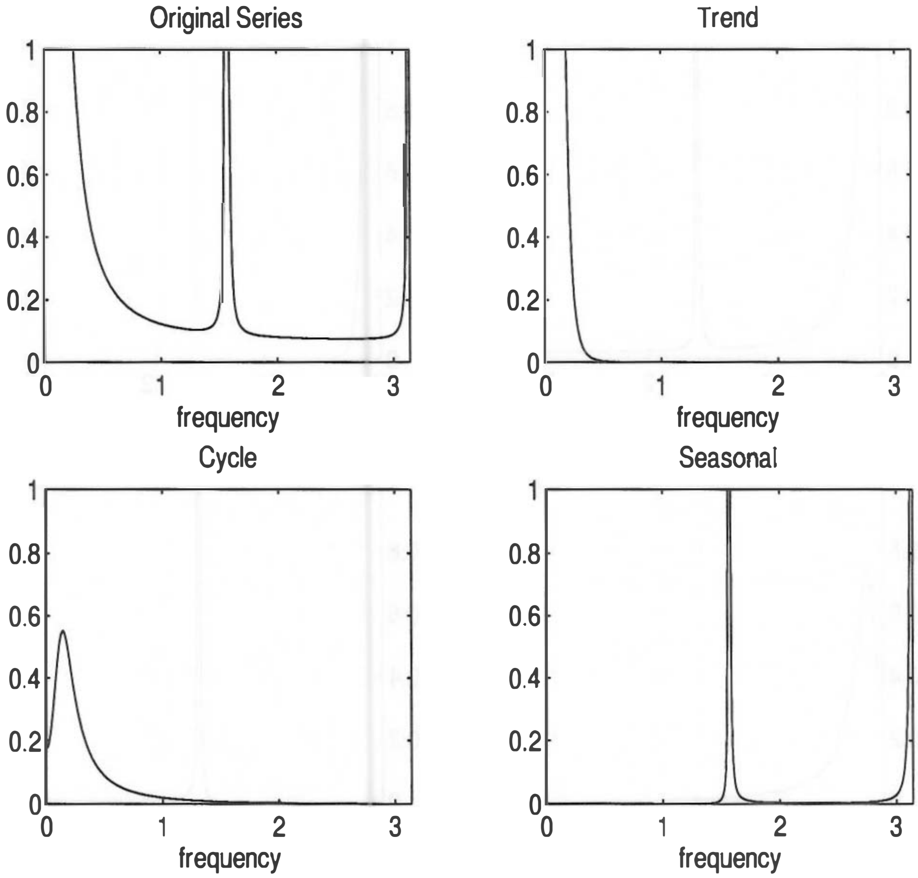


Fig. 6.7. Spectra for original series and components; series AP

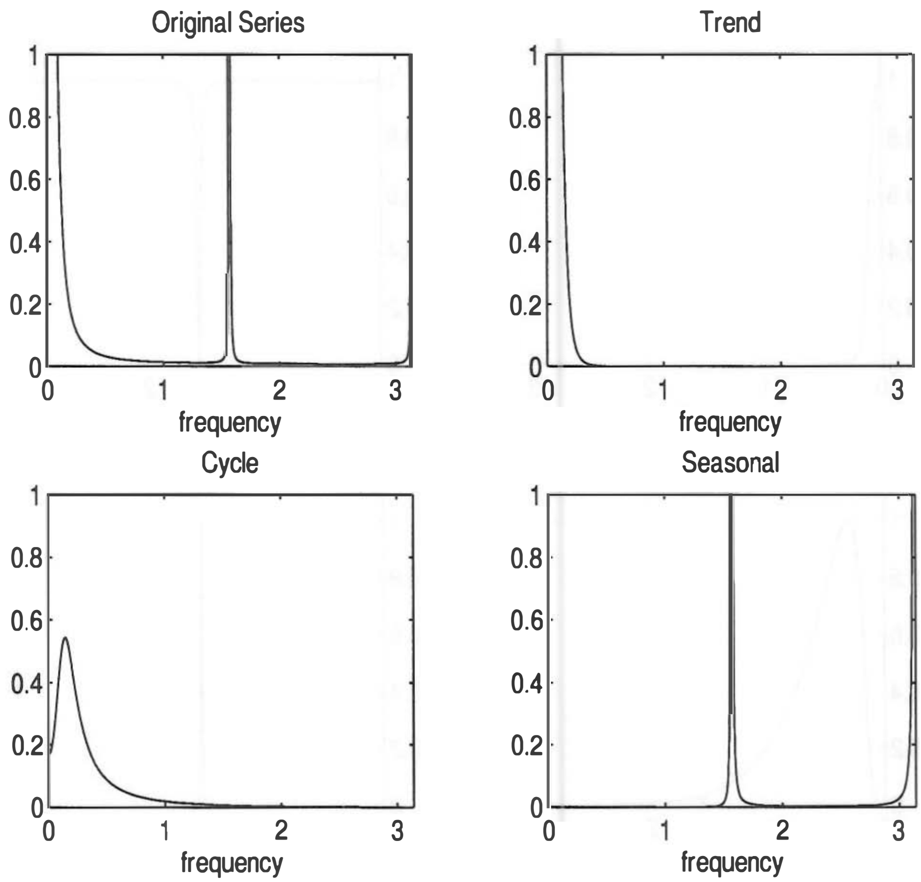


Fig. 6.8. Squared gain of filters for components; series CC

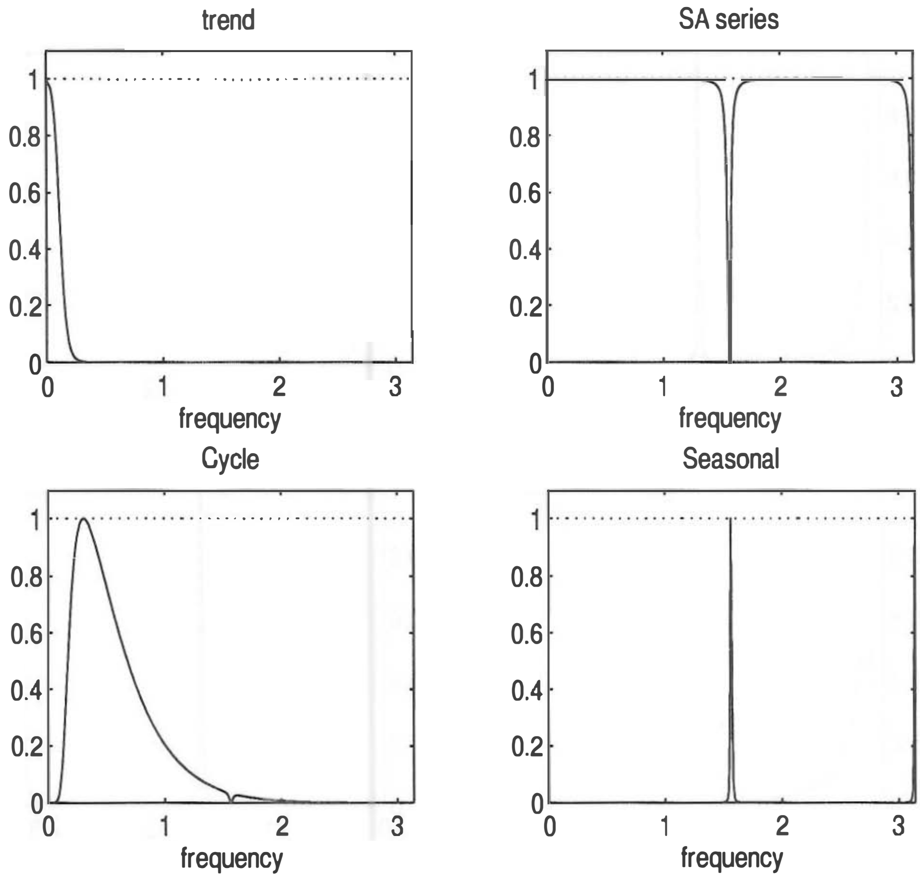


Fig. 6.9. Squared gain of filters for components; series IPI

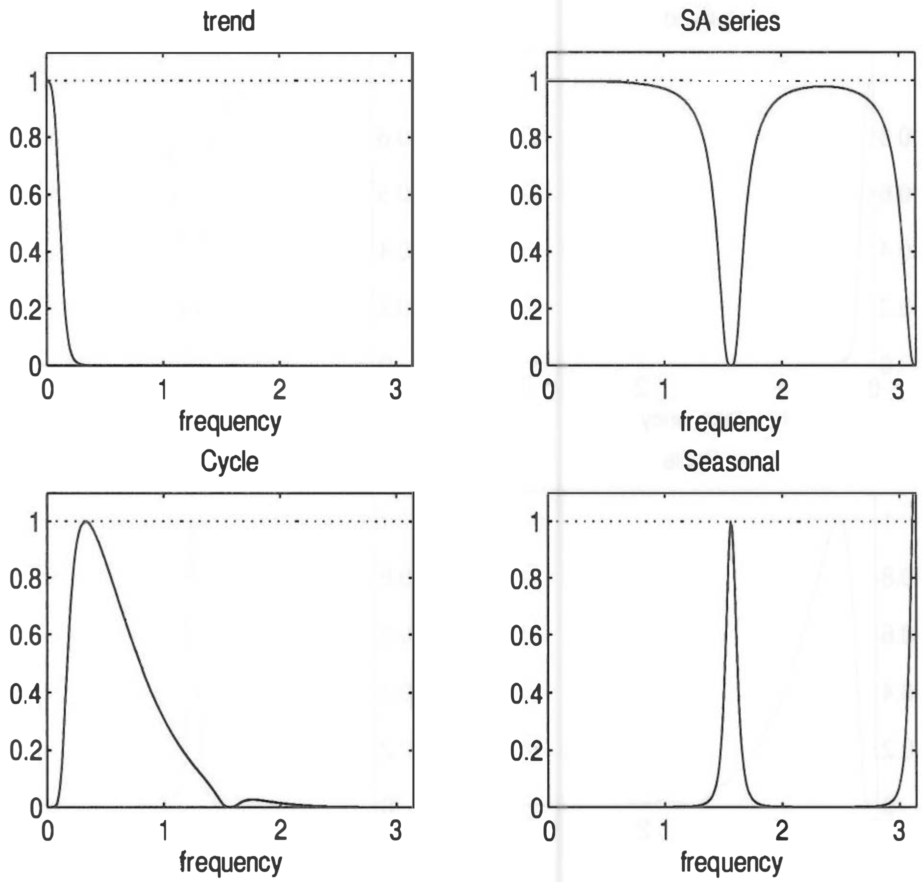


Fig. 6.10. Squared gain of filters for components; series CR

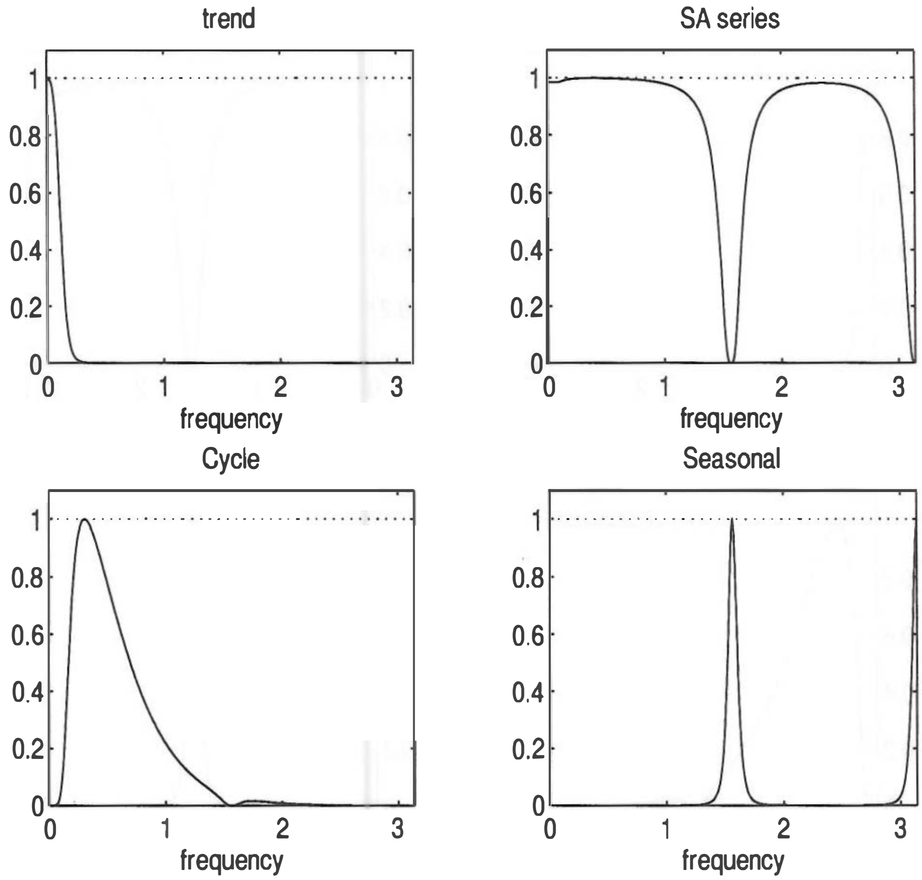
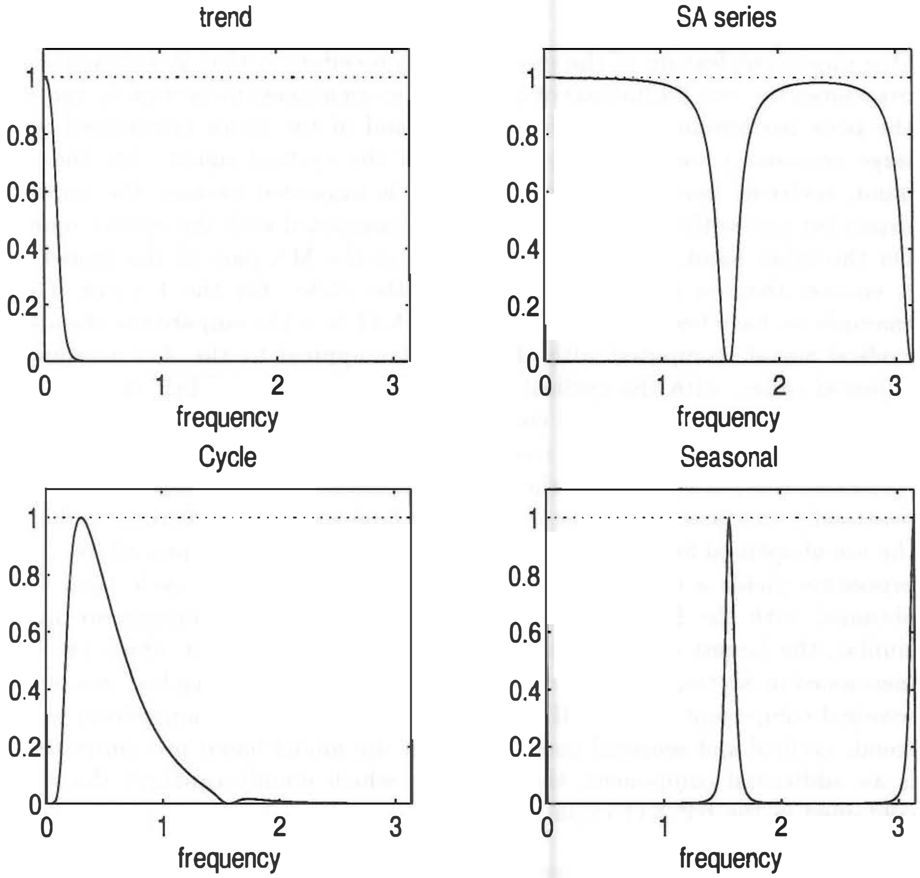


Fig. 6.11. Squared gain of filters for components; series AP



6.3 Some comments on model-based diagnostics and inference

One important feature of the model-based procedure is that it automatically overcomes the two limitations of the HP filter mentioned in Section 5, namely, the poor performance of the filter at the end of the series (associated with large revisions,) and the noisy behavior of the cyclical signal. On the one hand, revisions (and end-point treatment) is improved because the series is expanded now with forecasts and backcasts computed with the correct model. On the other hand, the presence of $\theta_p(B)$ in the MA part of the model for c_t ensures that no noise will contaminate the cycle. For the 4 series of the example we have been considering, Figures 6.12 to 6.15 compare the standard cyclical signal, computed with the HP filter applied to the X11 seasonally adjusted series, with the cyclical signal obtained as the MMSE estimator of c_t in the complete unobserved components model, and serve to illustrate the improvement. (The complete model for each one of the series is detailed in Appendix B.3.) It is seen that the trends obtained with the two procedures are practically identical, except for the first and final years; this difference is due to the use of optimal forecasts and backcasts in the model-based procedure. This procedure yields a considerably smoother (much less noisy) cycle than that obtained with the HP-X11 procedure. The seasonal components are quite similar, the largest difference occurring for the CC series, for which (as was mentioned in Section 5.1) X11 clearly overestimates the moving features of the seasonal component. For the HP-X11 procedure, the full decomposition yields trend, cyclical and seasonal components; in the model-based procedure there is an additional component, the irregular, which mainly captures the noise contained in the HP-X11 cycle.

From a more general perspective, while (blind) application of the HP-X11 filter can be seen as a black-box-type procedure, the model-based approach sets a convenient framework to analyze results, by using well defined (sensible) models and estimation criterion (MMSE). The models contain "ad-hoc" features, reflected in the polynomial $\theta_{HP}(B)$, and in $k_{m(HP)}$ and $k_{c(HP)}$ (all three determined from λ, as was seen in Section 2), and series-dependent features (such as the polynomial $\theta_p(B)$ and the variance V_p', derived from the overall model for the observed series). Ad-hoc features are thus easily incorporated into the AMB approach.

The model-based structure permits us to asses the statistical properties of the cycle such as, for example, its theoretical distribution, as well as the distribution of its optimal estimator. From equation (6.13), the distribution of c_t is easily derived; replacing x_t in expression (6.3) yields the estimator $\hat{c}_t$ as a function of the innovations in x_t,

$$\hat{c}_t = \left[k \frac{\theta_p(B)}{\theta_{HP}(B)} \frac{\theta_p(F)\bar{\nabla}\bar{\nabla}_4}{\theta_{HP}(F)\theta(F)} \right] a_t \tag{6.15}$$

from which the distribution of the MMSE estimator (a linear stationary stochastic process) is trivially obtained. Further, since expressions similar to (6.15) can be derived for all component estimators, their joint distribution is easily found. The knowledge of these distributions facilitates diagnostics and inference.

Consider first diagnostics. Our parametric model fully specifies the distributional features of the processes. Therefore, a natural diagnostic tool is to compare those features (derived from the ARIMA model for the observed series) with the corresponding sample estimates. We choose as illustration an example of some applied concern: the covariance between component estimators.

It is a well known result (see, for example, Nerlove, Grether and Carvalho, 1979) that, although the theoretical model specifies orthogonal components, the covariance between the components MMSE estimators will be nonzero. This covariance can be computed as follows.

Equation (6.15) expressed $\hat{c}_t$ as a convergent filter of the innovations a_t. For the trend estimator $\hat{m}_t$, using (2.8) instead of (2.10) with x_t replaced by $\hat{p}_t$, a similar derivation yields

$$\hat{m}_t = \left[V_m \frac{\theta_p(B)}{\theta_{HP}(B)\nabla^2} \frac{\theta_p(F)\bar{S}}{\theta_{HP}(F)\theta(F)} \right] a_t, \tag{6.16}$$

so that the crosscovariance generating function (CCVF) between the two estimators, after simplification, is found to be given by

$$CCVF(\hat{m}_t, \hat{c}_t) = V_m V_c \frac{[\theta_p(B)\theta_p(F)]^2 \, S\bar{S}}{[\theta_{HP}(B)\theta_{HP}(F)]^2 \, \theta(B)\theta(F)}. \tag{6.17}$$

Direct inspection of (6.17) shows that this CCVF is equal to the ACVF of the model

$$[\theta_{HP}(B)]^2 \, \theta(B) z_t = [\theta_p(B)]^2 \, S b_t, \quad Var(b_t) = V_m V_c. \tag{6.18}$$

The model is stationary and hence covariances are bounded. The lag-0 cross-covariance, being equal to the variance of model (6.18), will always be positive.

Be that as it may, in so far as the I(2) component m_t has no proper variance, the theoretical crosscorrelation between the estimators $\hat{m}_t$ and $\hat{c}_t$ is not well defined. For the 4 series of the example, Table 6.1 displays the sample cross-correlations between the different components: they are seen to be small with the largest one ocurring, as could be expected, between the two components that are stationary, c_t and u_t.

	CC	IPI	CR	AP
$\hat{m}_t, \hat{c}_t$	.18	-.01	.12	.02
$\hat{m}_t, \hat{s}_t$	.02	-.03	-.02	.00
$\hat{m}_t, \hat{u}_t$	.02	.01	.02	.00
$\hat{c}_t, \hat{s}_t$	-.01	.00	.00	.01
$\hat{c}_t, \hat{u}_t$	.32	.33	.32	.35
$\hat{s}_t, \hat{u}_t$	-.06	.10	.09	.10

Table 6.1. Crosscorrelations between component estimators: levels.

Having bounded variances, however, stationary transformations of the components (namely, $\nabla^2 m_t, c_t, S s_t$, and u_t) will be crosscorrelated, even asymptotically, in conflict with the theoretical assumptions made for the components. Proceeding as before, the "theoretical" crosscovariance between the MMSE estimators of the different components can be derived, and, from that, the crosscorrelation. Therefore, an element for diagnostic of model adequacy in the model-based preocedure can be to compare this theoretical crosscorrelation of the MMSE estimators with the ones provided by the sample estimates. The comparison, for the 4 series of the example and the 4 components, is given in Table 6.2. The table also contains the standard errors (SE) of the crosscorrelation estimators. These standard errors have been obtained by simulating 1000 series; it is worth mentioning that the model-based framework considerably simplifies simulation because the estimators can be directly generated from the simulated series a_t using the models (6.15), (6.16), and the equivalent ones for the seasonal and irregular components.

		CC	IPI	CR	AP
$\nabla^2\hat{m}_t, \hat{c}_t$	Estimator	-.64	-.57	-.59	-.59
	Estimate	-.66	-.65	-.70	-.65
	(SE)	(.09)	(.09)	(.09)	(.09)
$\nabla^2\hat{m}_t, S\hat{s}_t$	Estimator	-.00	-.00	-.00	-.00
	Estimate	-.00	-.00	-.00	-.00
	(SE)	(.01)	(.01)	(.01)	(.01)
$\nabla^2\hat{m}_t, \hat{u}_t$	Estimator	-.08	-.05	-.06	-.06
	Estimate	-.07	-.06	-.08	-.08
	(SE)	(.02)	(.02)	(.02)	(.02)
$\hat{c}_t, S\hat{s}_t$	Estimator	.00	.01	.01	.01
	Estimate	.01	.02	-.01	.02
	(SE)	(.02)	(.01)	(.01)	(.01)
$\hat{c}_t, \hat{u}_t$	Estimator	.30	.30	.28	.27
	Estimate	.30	.31	.30	.33
	(SE)	(.04)	(.04)	(.04)	(.04)
$S\hat{s}_t, \hat{u}_t$	Estimator	.05	.13	.13	.13
	Estimate	.09	.14	-.04	.22
	(SE)	(.09)	(.07)	(.07)	(.07)
$\nabla^2\hat{p}_t, \hat{u}_t$	Estimator	-.16	-.13	-.16	-.17
	Estimate	-.09	-.10	-.16	-.16
	(SE)	(.05)	(.05)	(.05)	(.05)

Table 6.2. Crosscorrelations between component estimators: stationary transformation.

Three results seem clear:

- The seasonal component is practically orthogonal to the other components (and hence to the SA series). In fact, the AMB decomposition into trend-cycle, seasonal and irregular components provides estimators that in practice are close to satisfying the orthogonality assumption made for the theoretical components. Therefore, the conflict between orthogonal components and correlated estimators is more apparent than real and should not be the cause of much concern.

- However, splitting the trend-cycle into trend plus cycle induces negative correlation between the estimators of these two components. Heuristically, this correlation is a reminder of the artificiality of the trend-cycle decomposition. While the data, summarized in the ARIMA model identified for the series, clearly imply spectral peaks for the zero and seasonal frequencies which are well captured by the trend-cycle and seasonal components, they have nothing to say about the partition of the zero spectral peak into trend plus cycle.

- The theoretical autocorrelations are in fair agreement with their sample counterparts. In only one case out of 24 the difference (in absolute value) between the estimator theoretical and sample crosscorrelations is larger than 1.96 SE and, even in this case, the associated t-value is moderate. Table 6.2 offers thus a favorable diagnostic concerning model adequacy.

To see an example of the use of the model in inference, assume we are interested in the following two questions:

1. What is the size of the revision in the concurrent estimator and how long does the revision process last in practice?
2. Based only on the size of the revisions, how big -in absolute value- the quarterly growth in the concurrent estimator of the cycle has to be in order to reject the hypothesis of zero growth? In other words, when can we accept that the present growth of the cycle is not zero?

Both questions are of applied relevance and can be easily answered by exploiting the model structure. Letting expression (3.2) represent (6.15), the derivation of Section 3.1 can be applied in a straightforward manner. That is, we can write

$$V_c \frac{\theta_p(B)}{\theta_{HP}(B)} \frac{\theta_p(F)\bar{\nabla}\bar{\nabla}_4}{\theta_{HP}(F)\theta(F)} = \xi^-(B) + \xi_0 + \xi^+(F),$$

and, similarly to (3.4), the revision in the concurrent estimator of c_t is equal to

$$r_{t|t} = \xi^+(F)a_t = \sum_{j=1}^{k} \xi_j^+ a_{t+j},$$

where the second equality relies on the finite truncation. For the 4 series of the example we have used k=250, more than enough for convergence to the 8^{th} decimal place. The variance of $r_{t|t}$, as well as the periods it takes to have 95%

of it removed from the estimator, are straightforward to derive. Specifically, for an integer h , $0 < h < k$, from

$$r_{t|t+h} = \sum_{j=t+h+1}^{k} \xi_j^+ a_{t+j},$$

one can compute the smallest h such that $Var(r_{t|t+h}) \geq .95 Var(r_{t|t})$. Table 6.3. displays, in the first column, the duration in quarters of the revision period measured in the previous way. In all cases the revision period lasts 11 quarters, and hence close to 3 years. (Notice that in the present case the revision includes also the one associated with reestimation of the trend-cycle p_t.) The second column of the table shows the standard deviation of the revision in the concurrent estimator expressed as a percentage of the level. For example, for the IPI series, a 95% confidence interval around the concurrent measurement would be in the order of ± 3.7 percent points of the level. The third column of the table presents the standard deviation of the revision as a fraction of the residual standard error; roughly, the size of the revision is about 1/2 that of the one-quarter-ahead forecasts. Altogether, revisions are certainly nonnegligible.

	Number of periods to complete 95% of the revision	Standard deviation of the revision	
		as a percent of the level	as a proportion of σ_a
CC	11	3.34	.44
IPI	11	1.88	.58
CR	11	6.46	.49
AP	11	2.55	.48

Table 6.3. Size and duration of the revision in concurrent estimator of cycle.

As to the question of how big the last quarter growth should be in order to confidently assert it is different from zero, assume that at time t the estimator of the cycle for time t is $\hat{c}_{t|t}$. When the next observation becomes available at period (t+1), the estimator of the cycle for the period becomes $\hat{c}_{t+1|t+1}$. Since c_t is measured in logs, differences between (not too distant) periods can be seen as rates of growth. Therefore,

$$\hat{r}_{t|t} = \hat{c}_{t+1|t+1} - \hat{c}_{t|t}$$

represents the quarterly rate-of-growth of the cycle as measured by two consecutive concurrent estimators. The final estimator of the growth between these two periods is given by

$$\hat{r}_t = \hat{c}_{t+1} - \hat{c}_t,$$

so that the error associated with revisions in c_t is equal to:

$$e_t = (\hat{c}_{t+1} - \hat{c}_{t+1|t+1}) + (\hat{c}_t - \hat{c}_{t|t}).$$

Substracting from (6.15) its expectation at time t, it is found that

$$\hat{c}_t - \hat{c}_{t|t} = \xi^+(F)a_{t+1},$$

and, likewise,

$$\hat{c}_{t+1} - \hat{c}_{t+1|t+1} = \xi^+(F)a_{t+2},$$

so that the error can be expressed as

$$\begin{aligned} e_t &= \xi_1 a_{t+1} + (\xi_1 - \xi_2)a_{t+2} + (\xi_2 - \xi_3)a_{t+3} + \ldots = \\ &= \xi_1 a_{t+1} + \sum_{j=1}^{\infty}(\xi_j - \xi_{j+1})a_{t+j+1}. \end{aligned} \qquad (6.19)$$

From this expression, the variance of e_t is easily found, and hence, we can conclude that, using a 95% significance level, only rates of growth of the cycle larger than $1.96\sigma_e$ in absolute value can be assumed to be significantly different from zero.

A more accurate measure of the quarterly growth would be given by $\tilde{r}_t = \hat{c}_{t+1|t+1} - \hat{c}_{t|t+1}$, where the estimator of the cycle for period t has been revised to take into account the new observation for (t+1). The previous derivation remains valid, except for the fact that the term $\xi_1 a_{t+1}$ disappears from the r.h.s. of (6.19). For the 4 series of the example, Table 6.3 presents the borderline growth values (namely, 1.96 σ_e) below which the (absolute value of the) measured rates of growth cannot be assumed to be significantly different from zero. As before, the values are expressed in percent points. The table indicates, for example, that the last quarterly growth of the cyclical component in the IPI series needs to be bigger than 1.35%, or smaller than -1.35%, in order for us to be 95% confident that it cannot be taken as zero.

	CC	IPI	CR	AP
Measured with concurrent estimators	2.41	1.35	4.48	1.76
Measured with concurrent and revised estimators	2.39	1.17	4.18	1.65

Table 6.3. Statistical significance of the quarterly rate of growth of the cycle (in percent points).

One can easily think of many other extensions. An example could be the computation of optimal forecasts for the components as well as their SE. By construction, these forecasts would be in full agreement with the ones provided by the ARIMA model for the series, directly identified from the data.

Notice that, being a zero-mean stationary ARMA(2,2) process with $\theta_{HP}(B)$ as the AR polynomial, the forecasts of the cycle will gradually approach zero, following damped cosine-type fluctuations. Added to the limitation implied by the size of the revision error, forecasts of the cycle can be of interest only for very short horizons.

Fig. 6.12. AMB and HP−X11 procedures; estimated components of CC

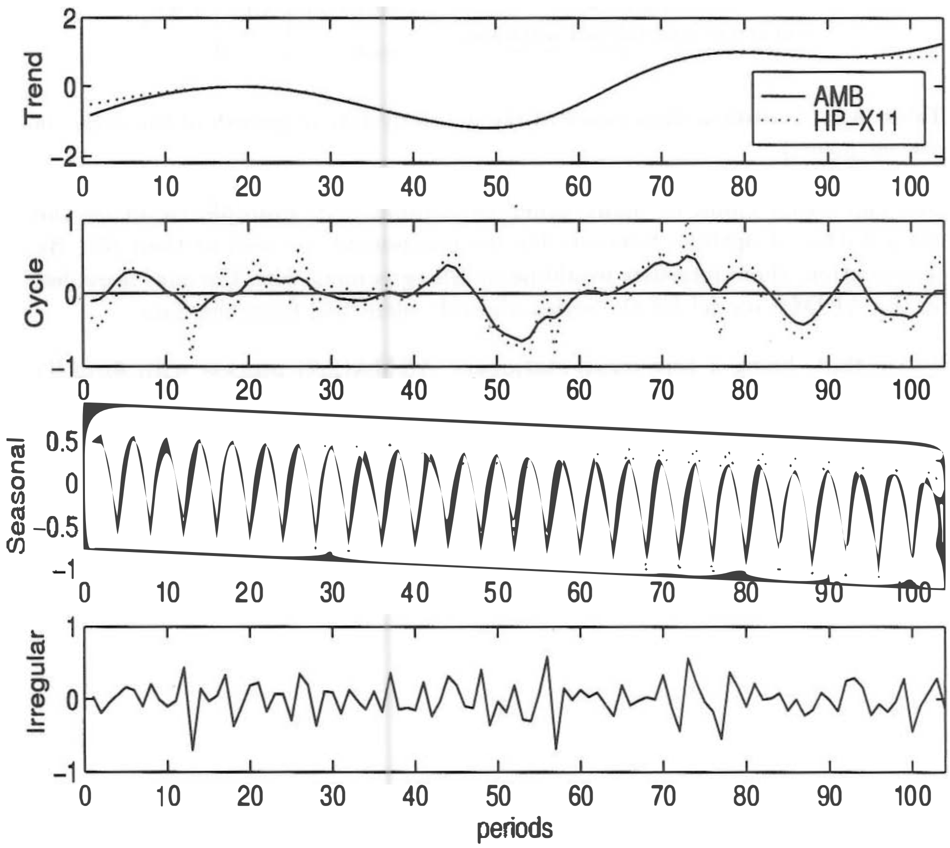


Fig. 6.13. AMB and HP−X11 procedures; estimated components of IP

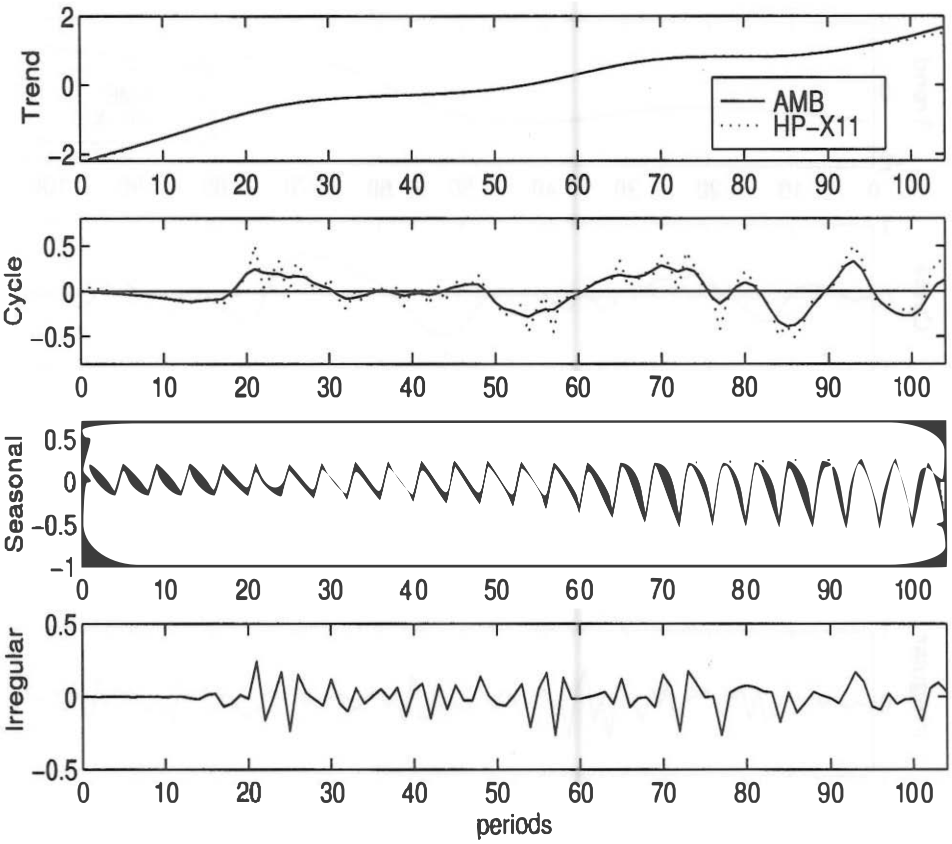


Fig. 6.14. AMB and HP−X11 procedures; estimated components of CF

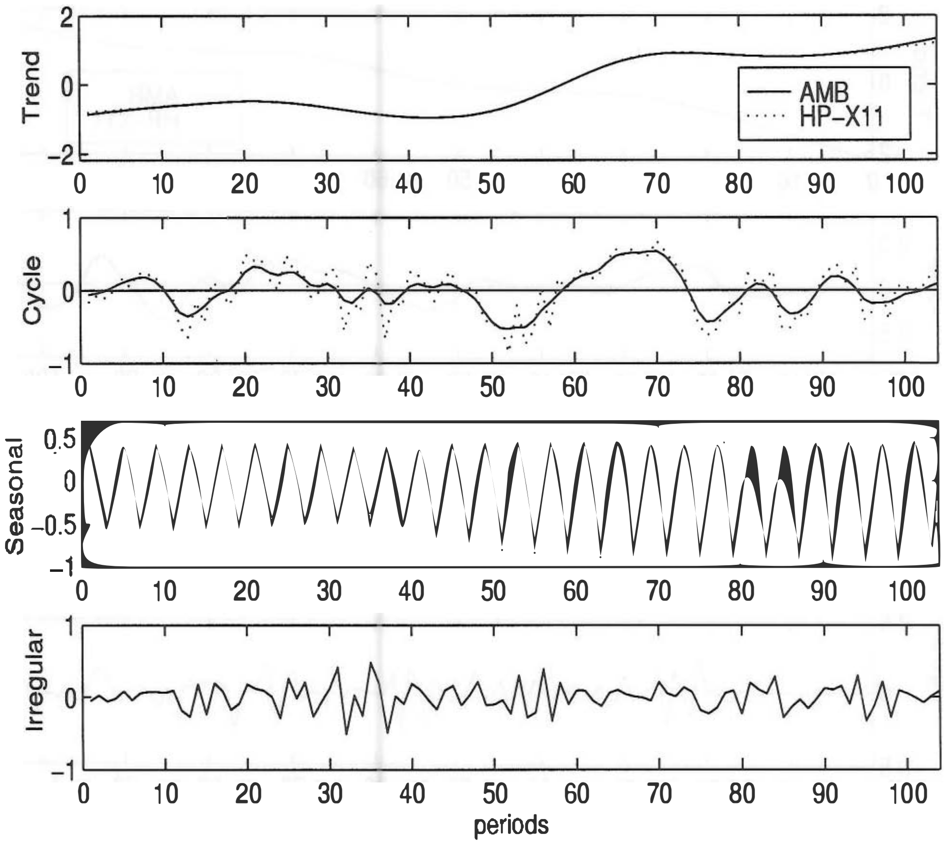
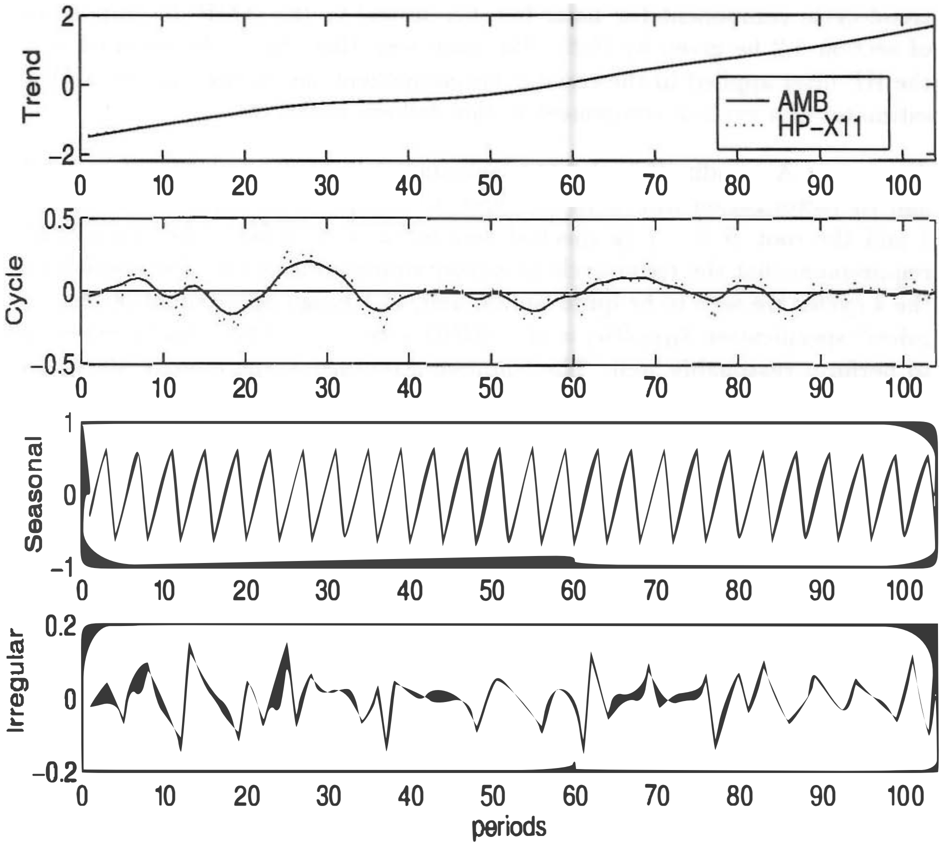


Fig. 6.15. AMB and HP–X11 procedures; estimated components of AP



6.4 MMSE estimation of the cycle: a paradox

As before, let the model for the observed series be given by (6.1) and let the trend-cycle component (or noise-free SA series) in the AMB decomposition of section 5.2 be given by (5.8). We have seen that the cycle obtained with the HP filter applied to the trend-cycle component can be seen as the MMSE estimator of a cyclical component c_t that follows model (6.7).

As seen in Appendix B.3, the MA polynomial in (5.8) is always of order 2, and its factorization will be of the type $(1 - \alpha B)(1 + B)$, where α is close to 1 and the root $B = -1$ (a spectral zero for $\omega = \pi$) reflects the "canonical" requirement that the trend-cycle be uncontaminated by noise. The models for the 4 cycles are seen to be quite similar and, to a rough approximation, the "a priori" specification $\theta_{HP}(B)c_t = (1 - .9B)(1 + B)$; $V_p = .1V_a$ could be expected to perform reasonably well. The common structure of the models illustrates how the band-pass approach to filtering can be well accommodated within the model-based approach, so that, the advantages offered by a model-based method can be exploited.

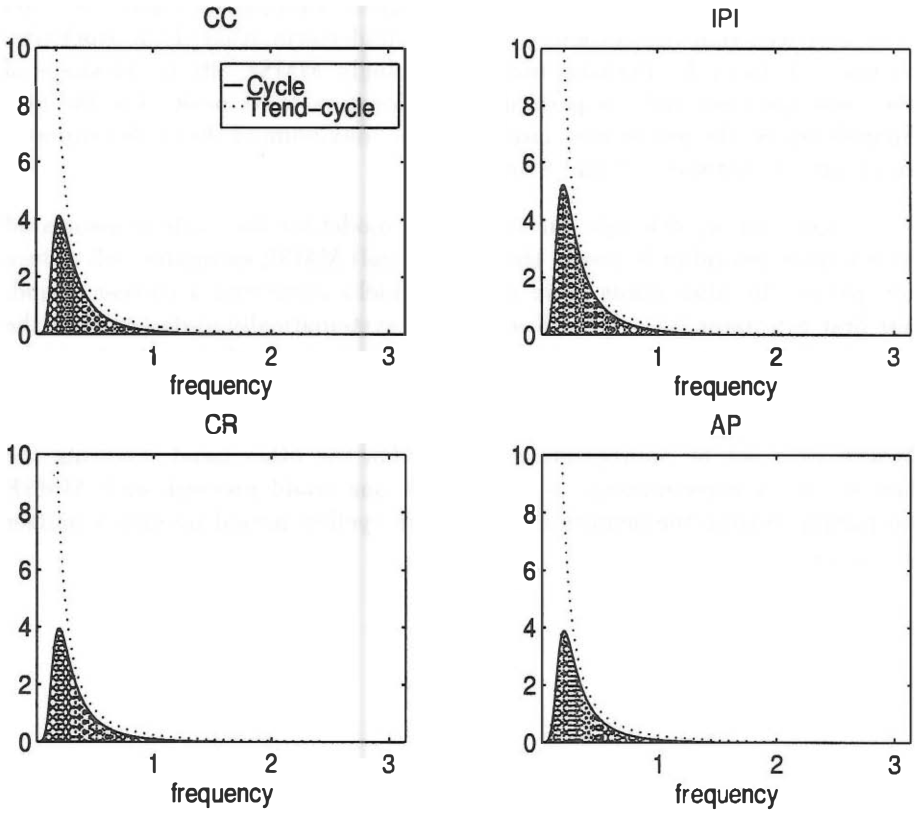
Yet the process of computing the MMSE estimator of c_t in the model-based framework presents some conceptual ambiguity, which we proceed to illustrate. If we compute the spectrum of c_t, from model (6.7), for the 4 Spanish series, and then find the frequency that corresponds to the maximum of the spectrum, the associated period is equal to 10 years. (This period is identical to the one that maximizes the cycle spectrum when X11-HP are applied to a series following the airline model.) Therefore, our model would specify a theoretical cycle dominated by the 10-year period.

As we have seen, however, MMSE estimation of a stochastic unobserved component has a distorting effect on the stochastic structure of the component. Given that (6.3) represents the MMSE estimator of c_t when (6.7) and (6.1) are the models for the cycle and for the observed series, respectively, using (6.1) in (6.3) the MMSE estimator $\hat{c}_t$ is expressed in terms of the innovations a_t in the observed series as in equation (6.15), from which its spectrum can be easily computed. Denote this spectrum by $\hat{g}_c(\omega)$. One of the major distortions induced by MMSE estimation, easily derived from (4.4), is $\hat{g}_c(\omega) \leq g_c(\omega)$ for all ω, in the same way as $\hat{g}_p(\omega) \leq g_p(\omega)$, where $\hat{g}_p(\omega)$ denotes the spectrum of the trend-cycle estimator and $g_p(\omega)$ the spectrum of (5.8). As a consequence, the MMSE of c_t will systematically underestimate the variance of c_t.

Figure 6.16 shows the spectra of the trend-cycle and cycle estimators for the 4 Spanish series. The width of the cycle estimator adjusts now to the width of the spectral peak of the trend-cycle estimator. Comparing Figures 6.1 and 6.16, it is seen that the net effect will be an underestimation of the stochastic variance of the cycle. Perhaps, more disturbingly, MMSE affects the shape of the cycle spectrum and, in particular, the location of its peak. For the four Spanish series, the period associated with the maximum of the cycle estimator spectrum lies between 7.5 and 8 years.

As a consequence, although our theoretical model for the cycle is associated with a main period of 10 years, the (theoretical) MMSE estimator will reduce this period. In other words, if we wish to model a cycle with a 10-year period, our best estimator (in a MMSE sense) will systematically underestimate the period. This creates some ambiguity in terms of which of the two should be taken as the period that characterizes the cycle. For a stubborn analyst wishing that the main period of his model for the cycle be respected by the estimator, MMSE may not be appropriate criterion. On the other hand, knowing the bias in the underestimation of the period, one could proceed with MMSE estimation, letting the definition of the main cyclical period become a matter of convention.

Figure 6.16. Spectra of the trend–cycle and cycle estimators



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Appendix A

A.1 Wiener-Kolmogorov version of the Hodrick-Prescott filter

In this appendix we apply the Burman-Wilson algorithm (Burman, 1980) to compute the HP trend with the Wiener-Kolmogorov filter. Consider the model given by (2.1) and (2.4) where c_t and b_t are uncorrelated white noises with variance V_c and V_m and let $k_m = V_m/V_a$; the reduced form model is given by (2.5) and, to simplify notation, remove subindexes from polynomials (hence $\theta_{HP}(B)$ becomes $\theta(B)$).

Due to its symmetry, the WK filter to estimate m_t, given by (2.8), can be expressed as

$$\nu(B,F) = \frac{k_m}{\theta(B)\theta(F)} = k_m \left[\frac{G(B)}{\theta(B)} + \frac{G(F)}{\theta(F)}\right], \tag{A.1}$$

where $G(B) = g_0 + g_1 B + g_2 B^2$. Removing denominators in the above identity and equating the coefficients of the terms in B^0, B^1 and B^2, yields a system of equations that can be solved for g_0,g_1 and g_2. If

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ \theta_1 & 1 & 0 \\ \theta_2 & \theta_1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \theta_2 \\ 0 & \theta_2 & \theta_1 \\ \theta_2 & \theta_1 & 1 \end{bmatrix},$$

the solution is given by

$$[g_2 \quad g_1 \quad g_0]' = A^{-1}[0 \quad 0 \quad 1]'. \tag{A.2}$$

Using (A.1), write $\hat{m}_t = \nu(B,F)x_t$ as

$$\hat{m}_t = k_m[x_t^B + x_t^F], \tag{A.3}$$

where

$$x_t^B = [G(B)/\theta(B)]x_t \tag{A.3a}$$

$$x_t^F = [G(F)/\theta(F)]x_t \tag{A.3b}$$

We shall need 4 backcast and 4 forecast of x_t; they can be computed in the usual Box-Jenkins way through model (2.5).

i) Computation of x_t^F: Differencing (A.3b) twice, and considering (2.5), yields

$$\begin{aligned}
\nabla^2 x_t^F &= \frac{G(F)}{\theta(F)}\nabla^2 x_t = \frac{G(F)}{\theta(F)}\theta(B)a_t = \\
&= (\psi_0 + \psi_1 F + \psi_2 F^2 + \ldots)(1 + \theta_1 B + \theta_2 B^2)a_t, \text{ or} \\
x_t^F - 2x_{t-1}^F + x_{t-2}^F &= (\alpha_2 B^2 + \alpha_1 B + \alpha_0 + \sum_{j=1}^{\infty} \alpha_{-j} F^j)a_t. \qquad \text{(A.4)}
\end{aligned}$$

Taking expectations at time T in both sides of (A.4), since $E_T[a_{T+k}] = 0$ for $k > 0$, for $t = T+3$ and $T+4$, it is obtained that

$$x_{T+3}^F - 2x_{T+2}^F + x_{T+1}^F = 0 \tag{A.5a}$$
$$x_{T+4}^F - 2x_{T+3}^F + x_{T+2}^F = 0. \tag{A.5b}$$

Let x_t include the four forecasts of the series, and compute the auxiliary series $y_t = G(F)x_t$, $t = 1, \ldots, T+2$. From (A.3b), $\theta(F)x_t^F = y_t$, or, for $t = T+1, T+2$,

$$x_{T+1}^F + \theta_1 x_{T+2}^F + \theta_2 x_{T+3}^F = y_{T+1} \tag{A.5c}$$
$$x_{T+2}^F + \theta_1 x_{T+3}^F + \theta_2 x_{T+4}^F = y_{T+2}. \tag{A.5d}$$

The system of four equations (A.5) can be solved for $x_{T+1}^F, \ldots, x_{T+4}^F$. The remaining x_T^F are computed recursively through

$$x_t^F = -\theta_1 x_{t+1}^F - \theta_2 x_{t+2}^F + y_t; \qquad t = T, \ldots, 1.$$

ii) Computation of x_t^B: Proceeding in a symmetric manner, compute the auxiliary series $z_t = G(B)x_t$, where x_t includes now 4 backcasts at the beginning and 4 forecasts at the end. From (A.3a),

$$(1-F)^2 x_t^B = \frac{G(B)}{\theta(B)}\theta(F)e_t, \tag{A.6}$$

where $(1-F)^2 x_t = \theta(F) e_t$, so that e_t is the forward residual. These residuals will now satisfy $E_T[e_{T-k}] = 0$ for $k > 0$. Proceeding as before, taking conditional expectations in (A.6) for $t = -3$ and -2 yields

$$x_{-3}^B - 2x_{-2}^B + x_{-1}^B = 0 \tag{A.7a}$$

$$x_{-2}^B - 2x_{-1}^B + x_0^B = 0, \tag{A.7b}$$

and, from (A.3a), $\theta(B) x_t^B = z_t$ for $t = -1, 0, 1, \ldots, T+4$. Therefore

$$x_{-1}^B + \theta_1 x_{-2}^B + \theta_2 x_{-3}^B = z_{-1} \tag{A.7c}$$

$$x_0^B + \theta_1 x_{-1}^B + \theta_2 x_{-2}^B = z_0. \tag{A.7d}$$

The system of four equations (A.7) can now be solved for $x_{-3}^B, x_{-2}^B, x_{-1}^B, x_0^B$. The rest of the x_t^B are obtained recursively from

$$x_t^B = -\theta_1 x_{t-1}^B - \theta_2 x_{t-2}^B + z_t; \qquad t = 1, \ldots, T.$$

Finally having obtained x_t^B and x_t^F, the estimator $\hat{m}_t$ is obtained through (A.3). Notice that the algorithm automatically provides four forecasts.

A.2 The algorithm

We consider the standard quarterly case with $\lambda = 1600$, for which $\theta_{HP}(B)$ and V_c/V_b are given by (2.7). From (A.2) it is found that $g_0 = -44.954$, $g_1 = 11.141$, and $g_2 = 56.235$. The matrix of coefficients in the 2 sets of equations (A.5) and (A.7) is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & -1.7771 & .7994 & 0 \\ 0 & 1 & -1.7771 & .7994 \end{bmatrix}.$$

Denote by $\mathbf{H}$ the inverse of this matrix. Let $x_t = [x_1, \ldots, x_T]$ be the series for which we wish to compute the HP trend m_t, and extend the series at both ends with 4 forecast and 4 backcast, computed with model (2.5).Then the algorithm that yields $\hat{m}_t$ is the following:

Step I. For $t = 1, \ldots, T+2$, compute (using 4 forecasts)

$$y_t = g_0 x_t + g_1 x_{t+1} + g_2 x_{t+2},$$

$$[x^F_{T+1}, \ldots, x^F_{T+4}]' = H[0, 0, y_{T+1}, y_{T+2}]',$$

and, for $t = T, \ldots, 1$, obtain recursively

$$x^F_t = -\theta_1 x^F_{t+1} - \theta_2 x^F_{t+2} + y_t.$$

Step II. For $t = -1, 0, 1, \ldots, T+4$ compute (using 4 backcasts)

$$z_t = g_0 x_t + g_1 x_{t-1} + g_2 x_{t-2},$$

$$[x^B_0, x^B_{-1}, x^B_{-2}, x^B_{-3}]' = H[0, 0, z_0, z_{-1}]',$$

and, for $t = 1, \ldots, T+4$, obtain recursively

$$x^B_t = -\theta_1 x^B_{t-1} - \theta_2 x^B_{t-2} + z_t.$$

Step III. For $t = 1, \ldots, T+4$, obtain

$$\hat{m}_{t|T} = k_m[x^F_t + x^B_t].$$

This yields the trend estimated for the sample period $t = 1, \ldots, T$, and forecasted for the periods $t = T+1, \ldots, T+4$. The algorithm consists of a few convolutions and some minor matrix multiplications. It is fast and reliable, even for a series with (say) a million observations.

A.3 A note on computation

The procedure assumes that the models are linear stochastic processes, in which case the optimal estimators are obtained with the linear filters we dereived earlier. In practice, many series may need prior treatment before the linearity assumption can be made. Examples are determination of the series transformation, detection and correction of outliers, and correction for special effects. Program TRAMO ("Time Series Regression with ARIMA noise, Missing values, and Outliers") can be used for automatic (or manual) pretreatment. The program outputs the series for which an ARIMA model can be assumed.

This ARIMA model can then be decomposed (automatic or manually) with program SEATS ("Signal Extraction in ARIMA Time Series"), and the trend-cycle estimator, together with its forecasts and backcasts can be obtained. Running SEATS again on the extended trend-cycle series, with the fixed specifications of the HP filter (for the standard quarterly series given by (2.7),) the estimator of the cycle, as well as its forecasts, is obtained.

Both (documented) programs can be freely downloaded from the site "http://www.bde.es", and are described in Gómez and Maravall (1997). They can also be supplied by the second author upon request.

Appendix B

B.1 ARIMA-model-based decomposition of a time series

For the type of quarterly series considered in this work ,we briefly summarize the AMB decomposition method, as originally developed by Burman (1980) and Hillmer and Tiao (1982). The method starts by identifying an ARIMA model for the observed series. Assume this model is given by an expression of the type:

$$\nabla\nabla_4 x_t = \theta(B)a_t, \quad a_t \sim niid(0, V_a), \tag{B.1}$$

where, for simplicity, we assume that q (the order of $\theta(B)$) ≤ 5 and that the model is invertible. Next, components are derived, such that they conform to the basic features of a trend, a seasonal, and an irregular component, and that they aggregate into the observed model (6.1). Considering that $\nabla\nabla_4$ factorizes into $\nabla^2 S$, the series is seen to contain nonstationary trend (or trend-cycle) and seasonal components. The series is decomposed, then, into

$$x_t = p_t + s_t + u_t, \tag{B.2}$$

where p_t, s_t, and u_t denote the trend-cycle, seasonal, and irregular components, respectively, the latter being a stationary process.

The following models are assumed for the components

$$\nabla^2 p_t = \theta_p(B)a_{pt}, \quad a_{pt} \sim niid(0, V_p) \tag{B.3a}$$
$$Ss_t = \theta_p(B)a_{st}, \quad a_{st} \sim niid(0, V_s) \tag{B.3b}$$
$$u_t \sim niid(0, V_u) \tag{B.3c}$$

where a_{pt}, a_{st} and u_t are mutually uncorrelated white noise variables. We refer to (B.3) as the (unobserved component) "structural model" associated with the reduced form model (B.1). Applying the operator $\nabla\nabla_4$ to both sides of (B.2), the identity

$$\theta(B)a_t = S\theta_p(B)a_{pt} + \nabla^2\theta_s(B)a_{st} + \nabla\nabla_4 u_t \tag{B.4}$$

is obtained. The l.h.s. of (B.4) is an MA(5) process. Setting the order of $\theta_p(B)$, q_p, equal to 2, and that of $\theta_s(B)$, q_s, equal to 3, all terms of the sum in the r.h.s. of (B.4) are also MA(5). Thus we assume, in general, $q_p = 2, q_s = 3$ and equating the ACVF of both sides of (B.4), a system of 6 equations are obtained (one for each nonzero covariance). The unknowns in the system are the 2 parameters in $\theta_p(B)$, the 3 parameters in $\theta_s(B)$, plus the variances V_p, V_s, and V_u; a total of 8 unknowns. There is, as a consequence, an infinite number of solutions to (B.4).

Denote a solution that implies components as in (B.3) with nonnegative spectra an admissible decomposition. The structural model (B.3) will not be identified, in general, because an infinite number of admissible decompositions are possible. The AMB method solves this underidentification problem by maximizing the variance of the noise V_u, which implies inducing a zero in the spectra of (B.3a) and (B.3b). The spectral zero translates into a unit root in $\theta_p(B)$ and in $\theta_s(B)$, so that the two components p_t and s_t are noinvertible. This particular solution to the identification problem is referred to as the "canonical" decomposition (see Box, Hillmer and Tiao, 1978, and Pierce, 1979); from all infinite solutions of the type (B.3), the canonical one maximizes the stability of the trend-cycle and seasonal components that are compatible with the model (B.1) for the observed series. Notice that the spectrum of p_t should display the zero at the frequency π, since it should be a decreasing function of ω in the interval $(0, \pi)$. Thus the trend-cycle MA polynomial can be factorized as $\theta_p(B) = (1+\alpha B)(1+B)$. The zero in the spectrum of s_t may occur at $\omega = 0$ or at a frequency roughly halfway between the two seasonal frequencies $\omega = \pi/2$ and $\omega = \pi$.

The AMB method computes the trend-cycle, seasonal, and irregular component estimators as the MMSE ("optimal") estimators based on the available series $[x_t] = [x_1, \ldots, x_T]$. Under our assumptions, these estimators are also conditional expectations of the type E(component| $[x_t]$), and they are obtained using the Wiener-Kolmogorov filter (see Bell, 1984). For a series extending from $t = -\infty$ to $t = \infty$, that follows model (B.1), assume we are interested in estimating a component, which we refer to as the "signal". the model for the signal can be expressed as

$$\phi_s(B)s_t = \theta_s(B)a_{st}, \quad a_{st} \sim niid(0, V_s).$$

An easy way to derive the WK filter for s_t is the following. Group the components that are not s_t into a "non-signal" (or noise) component n_t, an let n_t follow the model

$$\phi_n(B)n_t = \theta_n(B)a_{nt}, \quad a_{nt} \sim niid(0, V_n),$$

then the signal estimator is given by

$$\hat{s}_t = \left[\frac{V_s}{V_a}\frac{\theta_s(B)\phi_n(B)}{\theta(B)}\frac{\theta_s(F)\phi_n(F)}{\theta(F)}\right] x_t,$$

where the brackets contain the WK filter. This filter is symmetric and can be seen as the ACVF of the model

$$\theta(B)z_t = [\theta_s(B)\phi_n(B)]b_t, \quad b_t \sim niid(0, \frac{V_s}{V_a}). \tag{B.5a}$$

Applying this result to model (B.3), the WK filter to estimate the trend-cycle component is equal to the ACVF of the model

$$\theta(B)z_t = [\theta_p(B)S]b_t, \quad b_t \sim niid(0, \frac{V_p}{V_a}); \tag{B.5b}$$

for the seasonal component it is given by the ACVF of

$$\theta(B)z_t = [\theta_s(B)\nabla^2]b_t, \quad b_t \sim niid(0, \frac{V_s}{V_a}); \tag{B.5c}$$

and for the irregular component, by the ACVF of

$$\theta(B)z_t = \nabla\nabla_4 b_t, \quad b_t \sim niid(0, \frac{V_u}{V_a}).$$

Notice that this last model is the "inverse" model of (B.1), which is assumed to be known. Also invertibility of (B.1) guarantees stationarity of the models in (B.5) and hence the 3 WK filters will converge in B and in F.

For a finite realization, as seen in Cleveland and Tiao (1976), the optimal estimator of the signal is equal to the WK filter applied to the available series extended with optimal forecasts and backcasts obtained with (B.1). An extremely efficient way to compute this estimator is the Burman-Wilson algorithm described in Burman (1980).

B.2 Equivalence of the full Hodrick-Prescott and ARIMA-based approaches

We consider an observed time series that follows the model

$$\nabla\nabla_4 x_t = \theta(B)a_t, \quad Var(a_t) = 1, \tag{B.6}$$

where the polynomial $\theta(B)$ is invertible, and a_t is a zero mean niid (i.e., white noise) innovation. The units are standardarized by setting $V_a = 1$. For a quarterly economic series model (B.6) is quite general, and further generalizations would complicate notation unnecessarily; moreover, the 4 series considered in the example are particular cases of (B.6), where $\theta(B)$ is a finite, 2-parameter, MA term.

The AMB decomposition of x_t yields a trend-cycle component p_t, a seasonal component s_t, and an irregular component u_t, such that

$$x_t = p_t + s_t + u_t \tag{B.7}$$

where the models for the components are of the type

$$\nabla^2 p_t = \theta_p(B)a_{pt}, \quad Var(a_{pt}) = V_p, \tag{B.8}$$

$$Ss_t = \theta_s(B)a_{st}, \quad Var(a_{st}) = V_s, \tag{B.9}$$

$$u_t = \theta_u(B)a_{ut}, \quad Var(a_{ut}) = V_u, \tag{B 10}$$

and $\theta_p(B)a_{pt}, \theta_s(B)a_{st}$ and $\theta_u(B)a_{ut}$ are mutually orthogonal stationary processes. The optimal estimator of p_t, expressed with the WK filter, is given by

$$\hat{p}_t = \left[V_p \frac{\theta_p(B)S}{\theta(B)} \frac{\theta_p(F)\bar{S}}{\theta(F)}\right] x_t. \tag{B.11}$$

Consider now the WK expression of the HP filter explained in Section 2. Given λ (the HP-filter parameter in its standard version,) setting $\lambda = V_c/V_m$, and considering (2.6), one obtains the polynomial $\theta_{HP}(B)$ and the variance V_d from the identity

$$\theta_{HP}(B)\theta_{HP}(F)V_b = V_m + \nabla^2\bar{\nabla}^2 V_c \tag{B.12}$$

(the easiest procedure is to factorize the spectrum of the r.h.s.; see the Appendix in Maravall and Mathis, 1994). Define

$$k_{c(HP)} = V_c/V_b; \quad k_{m(HP)} = V_m/V_b$$

Expression (B.12) implies that the following constraint will be satisfied

$$\theta_{HP}(B)\theta_{HP}(F) = k_{m(HP)} + \nabla^2\bar{\nabla}^2 k_{c(HP)} \tag{B.13}$$

From (2.10), application of the HP filter to obtain the cycle from the trend-cycle estimator $\hat{p}_t$ yields

$$\hat{c}_t = \left[k_{c(HP)} \frac{\nabla^2}{\theta_{HP}(B)} \frac{\bar{\nabla}^2}{\theta_{HP}(F)}\right] \hat{p}_t.$$

(For the usual case $\lambda = 1600$, as was seen earlier, standarizing by setting $V_m = 1, V_c = 1600$, it is obtained that $k_{c(HP)} = 1/2001.4, k_{m(HP)} = .7994$, and $\theta_{HP}(B) = 1 - 1.7771B + .7994B^2$.) Using (B.11) and letting

$$k_c = V_p k_{c(HP)}; \quad k_m = V_p k_{m(HP)}, \tag{B.14}$$

the estimator $\hat{c}_t$ can be expressed in terms of the observed series as

$$\hat{c}_t = \left[k_c \frac{\theta_p(B)\nabla\nabla_4}{\theta_{HP}(B)\theta(B)} \frac{\theta_p(F)\bar{\nabla}\bar{\nabla}_4}{\theta_{HP}(F)\theta(F)}\right] x_t =$$

$$= \left[k_c \frac{\dfrac{\theta_p(B)}{\theta_{HP}(B)}}{\dfrac{\theta(B)}{\nabla\nabla_4}} \frac{\dfrac{\theta_p(F)}{\theta_{HP}(F)}}{\dfrac{\theta(F)}{\bar{\nabla}\bar{\nabla}_4}}\right] x_t.$$

This last expression can be seen as the ratio of two pseudo-ACVF (see Hatanaka and Suzuki, 1967), the one in the denominator being that of the observed series, and the one in the numerator that of the component. This shows that $\hat{c}_t$ is the WK (MMSE) estimator of a cycle c_t that follows the model

$$\theta_{HP}(B)c_t = \theta_p(B)a_{ct}, \quad Var(a_{ct}) = V_c = k_c, \tag{B.15}$$

when x_t is the output of model (B.6)

In an analoguos manner, replacing x_t by $\hat{p}_t$ in expression (2.8) and using (B.11), the long-term trend estimator $\hat{m}_t$, obtained by applying the HP trend filter to the trend-cycle estimator $\hat{c}_t$, can be expressed in terms of the observed series x_t as

$$\hat{m}_t = \left[k_m \frac{\theta_p(B)S}{\theta_{HP}(B)\theta(B)} \frac{\theta_p(F)\bar{S}}{\theta_{HP}(F)\theta(F)} \right] x_t =$$

$$= \left[k_m \frac{\dfrac{\theta_p(B)}{\theta_{HP}(B)\nabla^2}}{\dfrac{\theta(B)}{\nabla\nabla_4}} \frac{\dfrac{\theta_p(F)}{\theta_{HP}(F)\bar{\nabla}^2}}{\dfrac{\theta(F)}{\bar{\nabla}\bar{\nabla}_4}} \right] x_t.$$

The expression can be seen as a ratio of pseudo-ACVF, which directly shows that $\hat{m}_t$ is the WK (MMSE) estimator of the component m_t, given by the model

$$\theta_{HP}(B)\nabla^2 m_t = \theta_p(B) a_{mt}, \quad Var(a_{mt}) = V_m = k_m, \tag{B.16}$$

when x_t is the output of model (B.6).

Consider the unobserved component model formed by equation (B.9) for the seasonal component s_t, (B.10) for the irregular component u_t, (B.15) for the cyclical component c_t, and (B.16) for the trend component m_t. Their sum $(m_t + c_t + s_t + u_t)$ is equal to x_t. To see this, let

$$y_t = m_t + c_t + s_t + u_t$$

Since, by construction, s_t and u_t are the same as those in (B.7), to show that $y_t = x_t$ it suffices to show that $p_t = m_t + c_t$, or equivalently $\nabla^2 p_t = \nabla^2(m_t + c_t)$. From (B.15) and (B.16)

$$z_t = \nabla^2(m_t + c_t) = \frac{\theta_p(B)}{\theta_{HP}(B)} a_{mt} + \frac{\theta_p(B)\nabla^2}{\theta_{HP}(B)} a_{ct}.$$

Since $\nabla^2 p_t$ and z_t are both zero-mean, normally distributed variables, they will be the same if they have identical autocovariance generating functions. The

ACVF of z_t is equal to

$$\frac{\theta_p(B)\theta_p(F)}{\theta_{HP}(B)\theta_{HP}(F)}k_m + \frac{\theta_p(B)\nabla^2\theta_p(F)\bar{\nabla}^2}{\theta_{HP}(B)\theta_{HP}(F)}k_c =$$

$$= \theta_p(B)\theta_p(F)V_p\left[\frac{k_{m(HP)} + \nabla^2\bar{\nabla}^2 k_{c(HP)}}{\theta_{HP}(B)\theta_{HP}(F)}\right],$$

where use has been made of (B.14). In view of (B.13), the term in brackets is 1, both $\nabla^2 p_t$ and z_t have identical ACVF, and hence $y_t = x_t$. That is, underlying the 2-step procedure we followed, there is a full unobserved component model whose MMSE estimators yield identical results.

It should be mentioned that the equivalence of the 2-step and direct model approach has been shown for historical estimators. For preliminary estimators (i.e., estimators at the end points of the series), the direct model approach, implemented via the WK filter, offers directly optimal treatment of end points by extending the series x_t (long enough) with the correct model (B.6). Thus the poor behavior of the estimate for recent periods would be considerably improved.

If one uses the seasonally adjusted series instead of the trend-cycle as input for the HP filter, the previous unobserved component model is trivially modified. Let n_t denote the seasonally adjusted series

$$n_t = p_t + u_t$$

From (B.9) and (B.10), the model for n_t is also of the type

$$\nabla^2 n_t = \theta_n(B)a_{nt}, \quad Var(a_{nt}) = V_n, \tag{B.17}$$

where $\theta_n(B)$ and V_n are straightforward to obtain from the factorization of $\theta_p(B)a_{pt} + \nabla^2\theta_u(B)a_{ut}$. Deletting the component u_t, the unobserved component model is given by (B.9) for the seasonal component, and (B.15) and (B.16), with $\theta_p(B)$ and V_p replaced by $\theta_n(B)$ and V_n, for the cycle and the trend. These replacements are equivalent to adding the noise u_t to the input in the HP filter, which deteriorates the signal, as was seen in Section 5.

B.3 Complete unobserved component model for the 4 series of the example

For all cases, the decomposition is given by

$$x_t = n_t + s_t,$$
$$n_t = p_t + u_t,$$
$$p_t = m_t + c_t,$$

where

- x_t = observed series,
- n_t = seasonally adjusted (SA) series,
- s_t = seasonal component,
- p_t = trend-cycle component,
- u_t = irregular component,
- m_t = trend component,
- c_t = cyclical component.

The series are standarized by setting $V_a = 1$; "w.n." denotes a white-noise variable.

The components are assumed mutually orthogonal. Given λ (the parameter of the HP filter), all component models are fully derived simply from the ARIMA model for the observed series.

We list next the models for each one of the components. (The factorization of the MA polynomials for the trend-cycle and SA series is also given.)

1 Cement Consumption (series CC)

Model for series:

$$\nabla\nabla_4 x_t = (1 - .405B)(1 - .957B^4)a_t.$$

AMB decomposition into trend-cycle, seasonal, and irregular components

Trend-cycle:

$$\nabla^2 p_t = (1 + .011B - .989B^2)a_{pt} = (1 - .989B)(1 + B)a_{pt}, V_p = .0856$$

Seasonal:

$$Ss_t = (1 - .049B - .495B^2 - .455B^3)a_{st}, V_s = .00023$$

Irregular:

$$u_t = w.n(0, V_u), \quad V_u = .4723;$$

SA series:

$$\nabla^2 n_t = (1 - 1.394B + .401B^2)a_{nt} = (1 - .405B)(1 - .989B)a_{nt}, V_n = .9675.$$

HP Decomposition of the trend-cycle into trend plus cycle.

Trend:

$$(1 - 1.777B + .799B^2)\nabla^2 m_t = (1 + .011B - .989B^2)a_{mt}, V_m = .43 * 10^{-4}$$

Cycle:

$$(1 - 1.777B + .799B^2)c_t = (1 + .011B - .989B^2)a_{ct}, V_c = .0685;$$

2 Industrial Production Index (series IPI)

Model for series:

$$\nabla\nabla_4 x_t = (1 - .299B)(1 - .721B^4)a_t.$$

AMB decomposition into trend-cycle, seasonal, and irregular components

Trend-cycle:

$$\nabla^2 p_t = (1 + .078B - .922B^2)a_{pt} = (1 - .922B)(1 + B)a_{pt}, V_p = .0975$$

Seasonal:

$$Ss_t = (1 - .029B - .502B^2 - .527B^3)a_{st}, V_s = .0083$$

Irregular:

$$u_t = w.n(0, V_u), \quad V_u = .3098;$$

SA series:

$$\nabla^2 n_t = (1 - 1.222B + .277B^2)a_{nt} = (1 - .301B)(1 - .921B)a_{nt}, V_n = .7932.$$

HP Decomposition of the trend-cycle into trend plus cycle.

Trend:

$$(1 - 1.777B + .799B^2)\nabla^2 m_t = (1 + .078B - .922B^2)a_{mt}, V_m = .49 * 10^{-4}$$

Cycle:

$$(1 - 1.777B + .799B^2)c_t = (1 + .078B - .922B^2)a_{ct}, V_c = .0779;$$

3 Car Registration (series CR)

Model for series:

$$\nabla\nabla_4 x_t = (1 - .387B)(1 - .760B^4)a_t.$$

AMB decomposition into trend-cycle, seasonal, and irregular components

Trend-cycle:

$$\nabla^2 p_t = (1 + .066B - .934B^2)a_{pt} = (1 - .934B)(1 + B)a_{pt}, V_p = .0773$$

Seasonal:

$$Ss_t = (1 - .038B - .497B^2 - .465B^3)a_{st}, V_s = .0069$$

Irregular:

$$u_t = w.n(0, V_u), \quad V_u = .369;$$

SA series:

$$\nabla^2 n_t = (1 - 1.322B + .362B^2)a_{nt} = (1 - .388B)(1 - .934B)a_{nt}, V_n = .821.$$

HP Decomposition of the trend-cycle into trend plus cycle.

Trend:

$$(1 - 1.777B + .799B^2)\nabla^2 m_t = (1 + .066B - .934B^2)a_{mt}, V_m = .39 * 10^{-4}$$

Cycle:

$$(1 - 1.777B + .799B^2)c_t = (1 + .066B - .934B^2)a_{ct}, V_c = .0618;$$

4 Airline Passengers (series AP)

Model for series:

$$\nabla\nabla_4 x_t = (1 - .392B)(1 - .762B^4)a_t.$$

AMB decomposition into trend-cycle, seasonal, and irregular components

Trend-cycle:

$$\nabla^2 p_t = (1 + .065B - .935B^2)a_{pt} = (1 - .935B)(1 + B)a_{pt}, V_p = .0763$$

Seasonal:

$$Ss_t = (1 - .041B - .496B^2 - .463B^3)a_{st}, V_s = .0067$$

Irregular:

$$u_t = w.n(0, V_u), \quad V_u = .3730;$$

SA series:

$$\nabla^2 n_t = (1 - 1.327B + .367B^2)a_{nt} = (1 - .393B)(1 - .934B)a_{nt}, V_n = .823.$$

HP Decomposition of the trend-cycle into trend plus cycle.

Trend:

$$(1 - 1.777B + .799B^2)\nabla^2 m_t = (1 + .065B - .935B^2)a_{mt}, V_m = .38 * 10^{-4}$$

Cycle:

$$(1 - 1.777B + .799B^2)c_t = (1 + .065B - .935B^2)a_{ct}, V_c = .0610;$$

The results can be summarized as follows. All four models for the series are relatively close, in particular for the series CR and AP.

The trend-cycle component is always an IMA(2,2) model, where the MA polynomial can be factorized as

$$\theta_p(B) = (1 - \theta B)(1 + B),$$

with θ close to .95. The model is thus very close to a noninvertible IMA(1,1) model with mean, and with MA root B=-1 which implies a monotonically decreasing spectrum with a zero for the frequency $\omega = \pi$ rads. (This model is a particular case of the tangent family of Butterworth filters.)

The seasonal component always follows an ARMA(3,3) model, with the nonstationary AR polynomial $S = 1 + B + B^2 + B^3$. The seasonal innovation variance of the series CC indicates a highly stable seasonal component, while the series IPI contains the most unstable one. All MA polynomials for the seasonal component contain the root B=1 and hence the component spectrum will present a zero for the zero frequency.

The irregular component is always white noise. Between 30% and 50% of the series uncertainty (as measured by the variance of the one-step-ahead forecast error) is caused by the presence of noise. The remaining uncertainty is associated with the stochastic features of the trend-cycle and seasonal components.

The seasonally adjusted series is always an IMA(2,2) model, and the factorization of the MA shows that one of the roots is very close to B=.95. As was the case with the trend-cycle, the model is, thus, very close to an IMA(1,1) model with mean. Since the other MA root is moderately small, the model for the SA series is, in the 4 cases, not far from the popular "random walk plus drift" model.

The models for the previous components are fully derived from the ARIMA model for the observed series. To split the trend-cycle into trend plus cycle we need the value of λ (or, equivalently, as was seen in Section 4.3.3, the main period of the cycle.)

The models for the trend and for the cycle both preserve the MA polynomial $\theta_p(B)$ of the trend-cycle model, and hence will display a spectral zero for the π frequency. The model for the cycle is always a stationary ARMA(2,2) model with the AR polynomial determined by λ (λ therefore determines the period of the spectral peak of the cycle). It is also seen that most of the variance of the trend-cycle innovation is absorbed by the cycle itself. Finally, the model for the trend is an ARIMA(2,2,2) process, with the stationary AR polynomial equal to that of the cycle, and its innovation variance strongly reduced (this reduction is also determined by λ.)

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