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Asymptotics of partial sums of the Dirichlet series of the arithmetic derivative

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Abstract. For $\emptyset \neq P \subseteq \mathbb{P}$, let D_P be the arithmetic subderivative function with respect to P on $\mathbb{Z}_+$, let ζ_{D_P} be the function defined by the Dirichlet series of D_P, and let σ_{D_P} denote its abscissa of convergence. Under certain assumptions concerning s and P, we present asymptotic formulas for the partial sums of $\zeta_{D_P}(s)$ and show that $\sigma_{D_P} = 2$. We also express $\zeta_{D_P}(s)$, $s > 2$, using the Riemann zeta function.

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1. Introduction

Let $n \in \mathbb{Z}_+$. There exists a unique sequence of nonnegative integers (with only finitely many positive terms)

$$(\nu_p(n))_{p\in\mathbb{P}},$$

where $\mathbb{P}$ stands for the set of primes, such that

$$n = \prod_{p\in\mathbb{P}} p^{\nu_p(n)}.$$

We use the approach mostly from [1, 3, 5, 7]. Let $\emptyset \neq P \subseteq \mathbb{P}$. The *arithmetic subderivative* of n with respect to P is

$$D_P(n) = n'_P := \sum_{p\in P} n'_p,$$

where n'_p is the *arithmetic partial derivative* of n with respect to $p \in \mathbb{P}$, defined by

$$D_p(n) = n'_p = n'_{\{p\}} := \frac{\nu_p(n)}{p} n.$$

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The *arithmetic derivative* of n is

$$D(n) = n' := n'_{\mathbb{P}} = \sum_{p\in\mathbb{P}} n'_p.$$

We define the (*arithmetic*) *logarithmic subderivative*, *logarithmic partial derivative*, and *logarithmic derivative* of n, respectively, as follows:

$$\mathrm{ld}_P(n) = \frac{n'_P}{n}, \quad \mathrm{ld}_p(n) = \frac{n'_p}{n}, \quad \mathrm{ld}(n) = \frac{n'}{n}.$$

Let f be an arithmetic function. There exists $\sigma_f \in \mathbb{R} \cup \{\pm\infty\}$ such that its *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C},$$

converges if $\Re(s) > \sigma_f$ ($\Re$ denotes the real part) and diverges if $\Re(s) < \sigma_f$ (see [6, p. 108, Theorem 3]). We call σ_f the *abscissa of convergence* of this series and define the function ζ_f by

$$\zeta_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \Re(s) < \sigma_f.$$

For example, let the function u be identically one. The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta_u(s), \quad \text{and} \quad \sigma_u = 1.$$

Our paper originates from three results due to Barbeau [1]. The first one gives an upper bound for n' using n:

Lemma 1 (see [1, p. 118] or [7, Theorem 9])**.** *Let $n \in \mathbb{Z}_+$. Then*

$$n' \leq \frac{n \log n}{2 \log 2}.$$

In the next theorem, the first and second formula describe the asymptotic behavior of

$$\sum_{1\leq n\leq x} \mathrm{ld}(n) \quad \text{and} \quad \sum_{1\leq n\leq x} n' :$$

Theorem 1 (see [1, pp. 119–121] or [7, Theorem 24])**.** *Asymptotically,*

$$\sum_{1\leq n\leq x} \mathrm{ld}(n) = Cx + O(\log x \log\log x)$$

and

$$\sum_{1\leq n\leq x} n' = C\frac{x^2}{2} + O(x^{1+\delta}).$$

Here

$$C = \sum_{p\in\mathbb{P}} \frac{1}{p(p-1)} = 0.749\ldots, \tag{1}$$

and $\delta > 0$ is arbitrary.

In the proofs cited above, actually $x \in \mathbb{Z}_+$, but they can easily be extended to hold for $x \in \mathbb{R}$, $x \geq 1$.

Our goal is to find asymptotic formulas for the partial sums of $\zeta_{D_P}(s)$, in other words, for the sums

$$\sum_{1\leq n\leq x} \frac{n'}{n^s} \text{ and } \sum_{1\leq n\leq x} \frac{n'_p}{n^s} \text{ and, more generally, for } \sum_{1\leq n\leq x} \frac{n'_P}{n^s}, \tag{2}$$

where $s \in \mathbb{R}$. As a corollary, we will see that $\sigma_D = \sigma_{D_p} = \sigma_{D_P} = 2$. For $s = 1$ and $s = 0$, the formulas concerning the first sum are already given in Theorem 1. Lastly, we express $\zeta_{D_P}(s)$, $s > 2$, using the values of ζ.

Our main tool is the following Abel's summation formula:

Lemma 2 (see [6, p. 3, Theorem 1])**.** *Let (a_n) be a sequence of complex numbers, let $x > 1$, and let $g : [1, x] \to \mathbb{C}$ be a continuously differentiable function. Then*

$$\sum_{1\leq n\leq x} a_n g(n) = \Big(\sum_{1\leq n\leq x} a_n\Big) g(x) - \int_1^x \Big(\sum_{1\leq n\leq t} a_n\Big) g'(t)\mathrm{d}t.$$

2. Partial sums of $\zeta_D(2)$

In this section, we consider the first sum of (2) with $s = 2$. We obtain the following result:

Theorem 2. *Asymptotically,*

$$\sum_{1\leq n\leq x} \frac{n'}{n^2} = C\log x + O(1).$$

Proof. Applying Lemma 2 to

$$a_n = \frac{n'}{n}, \quad g(x) = \frac{1}{x},$$

we obtain

$$\sum_{1\leq n\leq x} \frac{n'}{n^2} = \sum_{1\leq n\leq x} \frac{n'}{n}\frac{1}{n} = H(x) + K(x),$$

where

$$H(x) = \Big(\sum_{1\leq n\leq x} \frac{n'}{n}\Big)\frac{1}{x}, \quad K(x) = \int_1^x \Big(\sum_{1\leq n\leq t} \frac{n'}{n}\Big)\frac{1}{t^2}\mathrm{d}t.$$

By Theorem 1,

$$H(x) = C + O(x^{-1}\log x \log\log x) = O(1) \tag{3}$$

and

$$\begin{aligned}K(x) &= \int_1^x (Ct + O(\log t \log\log t))\frac{1}{t^2}\mathrm{d}t \\ &= \int_1^x C\frac{1}{t}\mathrm{d}t + \int_1^x O(t^{-2}\log t \log\log t)\mathrm{d}t \\ &= C\log x + O\Big(\int_1^x t^{-2}\log t\log\log t\,\mathrm{d}t\Big).\end{aligned}$$

Further, since

$$\log t \log\log t = O(t^\delta)$$

for any $\delta \in (0,1)$, we have

$$\begin{aligned}K(x) = C\log x + O\Big(\int_1^x t^{\delta-2}\mathrm{d}t\Big) &= C\log x + O(x^{\delta-1}) + O(1) \\ &= C\log x + O(1). \end{aligned} \tag{4}$$

Now, the claim follows from (3) and (4). □

Corollary 1. *It holds that* $\sigma_D = 2$.

Proof. By Lemma 1,

$$0 \le \frac{n'}{n^s} \le \frac{n\log n}{2n^s \log 2} = \frac{\log n}{2n^{s-1}\log 2}. \tag{5}$$

If $s > 2$, then the series

$$\sum_{n=1}^\infty \frac{\log n}{n^{s-1}}$$

converges. By using (5), we conclude that the series

$$\sum_{n=1}^\infty \frac{n'}{n^s}$$

converges, too. Hence $\sigma_D \ge 2$. On the other hand, since by Theorem 2 the series

$$\sum_{n=1}^\infty \frac{n'}{n^2}$$

diverges, we have $\sigma_D \le 2$. □

3. Partial sums of $\zeta_D(s)$, $1 \ne s < 2$

Next, we study the first sum of (2) in the case of $1 \ne s < 2$.

Theorem 3. *Let $1 \neq s < 2$. Asymptotically,*

$$\sum_{1 \le n \le x} \frac{n'}{n^s} = \frac{C}{2-s} x^{2-s} + R(x),$$

where $R(x)$ is defined as follows: If $1 < s < 2$, then $R(x) = O(1)$. If $s < 1$, then $R(x) = O(x^{\delta-(s-1)})$ for any $\delta > 0$.

Proof. Assume first that $1 < s < 2$. We proceed as in the proof of Theorem 2 but take

$$g(x) = \frac{1}{x^{s-1}}.$$

Then

$$\sum_{1 \le n \le x} \frac{n'}{n^s} = H(x) + K(x),$$

where

$$H(x) = Cx^{2-s} + O(x^{1-s} \log x \log\log x) = Cx^{2-s} + O(1)$$

and

$$\begin{aligned} K(x) &= \int_1^x (Ct + O(\log t \log\log t)) \frac{s-1}{t^s} \mathrm{d}t \\ &= C(s-1) \int_1^x \frac{\mathrm{d}t}{t^{s-1}} + (s-1) \int_1^x O(t^{-s} \log t \log\log t) \mathrm{d}t \\ &= C\frac{s-1}{2-s} x^{2-s} + O(1) + O\Big(\int_1^x t^{-s} \log t \log\log t \, \mathrm{d}t \Big) \\ &= C\frac{s-1}{2-s} x^{2-s} + O(1) + O\Big(\int_1^x t^{\delta-s} \mathrm{d}t \Big) \\ &= C\frac{s-1}{2-s} x^{2-s} + O(1) + O(x^{\delta-(s-1)}) + O(1). \end{aligned}$$

We can restrict ourselves to $0 < \delta \le s-1$. Then $\delta - (s-1) \le 0$, which implies that

$$K(x) = C\frac{s-1}{2-s} x^{2-s} + O(1)$$

and further,

$$H(x) + K(x) = C\Big(1 + \frac{s-1}{2-s}\Big) x^{2-s} + O(1) = \frac{C}{2-s} x^{2-s} + O(1),$$

completing the proof in this case.

If $s < 1$, then

$$K(x) = C\frac{s-1}{2-s} x^{2-s} + O(x^{\delta-(s-1)}),$$

and we can proceed as above. □

Note that this theorem is a generalization of the latter part of Theorem 1; just set $s = 0$.

4. Partial sums of $\zeta_{D_p}(1)$

We show that the asymptotic formulas for the partial sums of $\zeta_D(s)$ given in Theorems 1–3 have variants for those of $\zeta_{D_p}(s)$. In these variants, the coefficient C given in (1) is replaced by C_p defined as

$$C_p = \frac{1}{p(p-1)}, \quad p \in \mathbb{P}.$$

Note that $C = \sum_{p\in\mathbb{P}} C_p$.

We begin the study of the partial sums of $\zeta_{D_p}(s)$ with $s = 1$.

Theorem 4. *Let $p \in \mathbb{P}$. Asymptotically,*

$$\sum_{1\le n\le x} \mathrm{ld}_p(n) = C_p x + O(\log x).$$

Proof. It is easy to see that it is enough to consider the sum

$$\sum_{k=1}^{n} \mathrm{ld}_p(k) = \mathrm{ld}_p \prod_{k=1}^{n} k = \mathrm{ld}_p(n!).$$

We modify the proof of the first part of Theorem 1. By [2, Theorem 416],

$$n! = \prod_{q\in\mathbb{P}} q^{\mu_q(n)}, \tag{6}$$

where

$$\mu_q(n) = \sum_{m=1}^{\infty} \left\lfloor \frac{n}{q^m} \right\rfloor = \sum_{m=1}^{\alpha(n)} \left\lfloor \frac{n}{q^m} \right\rfloor, \quad \alpha(n) = \left\lfloor \frac{\log n}{\log 2} \right\rfloor. \tag{7}$$

Now, denoting by $\overset{(i)}{=}$ that the equation follows from the formula (i), we obtain

$$\begin{aligned}
\mathrm{ld}_p(n!) &\overset{(6)}{=} \mathrm{ld}_p \prod_{q\in\mathbb{P}} q^{\mu_q(n)} = \frac{\mu_p(n)}{p} \overset{(7)}{=} \frac{1}{p} \sum_{m=1}^{\alpha(n)} \left\lfloor \frac{n}{p^m} \right\rfloor \\
&= \frac{1}{p} \sum_{m=1}^{\alpha(n)} \frac{n}{p^m} + \frac{1}{p} \sum_{m=1}^{\alpha(n)} O(1) \overset{(7)}{=} n \sum_{m=2}^{\alpha(n)+1} \frac{1}{p^m} + O(\log n) \\
&= n \sum_{m=2}^{\infty} \frac{1}{p^m} - n \sum_{m=\alpha(n)+2}^{\infty} \frac{1}{p^m} + O(\log n) \\
&= C_p n - \frac{n}{p^{\alpha(n)+1}(p-1)} + O(\log n).
\end{aligned}$$

It remains to study the complexity of

$$A(n) = \frac{n}{p^{\alpha(n)+1}(p-1)}.$$

Since

$$p^{\alpha(n)+1} \ge 2^{\alpha(n)+1} > n$$

by (7), it follows that $A(n) = O(1)$, and the proof is complete. □

5. Partial sums of $\zeta_{D_p}(s)$ and $\zeta_{D_P}(s)$

In this section, we continue by studying the second sum of (2), where $1 \neq s \leq 2$. We first assume that $s = 2$.

Theorem 5. *Let $p \in \mathbb{P}$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'_p}{n^2} = C_p \log x + O(1).$$

Proof. The proof is analogous to that of Theorem 2. We apply Lemma 2 to

$$a_n = \frac{n'_p}{n}, \quad g(x) = \frac{1}{x},$$

and use Theorem 4. □

Corollary 2. *Let $p \in \mathbb{P}$. Then $\sigma_{D_p} = 2$.*

Proof. Clearly, $0 \leq n'_p \leq n'$ for all $n \in \mathbb{Z}_+$. Since $\sigma_D = 2$ by Corollary 1, we have $\sigma_{D_p} \geq 2$. On the other hand, since by Theorem 5 the series

$$\sum_{n=1}^{\infty} \frac{n'_p}{n^2}$$

diverges, it follows that $\sigma_{D_p} \leq 2$. □

Next, we consider the case of $1 \neq s < 2$.

Theorem 6. *Let $p \in \mathbb{P}$ and $1 \neq s < 2$. Asymptotically,*

$$\sum_{1 \leq n \leq x} \frac{n'_p}{n^s} = \frac{C_p}{2-s} x^{2-s} + R(x),$$

where $R(x)$ is as in Theorem 3.

Proof. The proof is a simple modification of that of Theorem 3. □

Corollary 3 (see Theorem 1)**.** *Let $p \in \mathbb{P}$. Then*

$$\sum_{1 \leq n \leq x} n'_p = C_p \frac{x^2}{2} + O(x^{\delta+1})$$

for any $\delta > 0$.

Our results about $\zeta_{D_p}(s)$ can be extended to concern $\zeta_{D_P}(s)$ if $P \subset \mathbb{P}$ is nonempty and finite (or if $P = \mathbb{P}$, see Theorem 3). Then C_p is replaced by

$$C_P = \sum_{p \in P} \frac{1}{p(p-1)}.$$

For example, Theorem 4 and Theorem 6 ($s = 0$) extend to

$$\sum_{1 \leq n \leq x} \mathrm{ld}_P(n) = C_P x + O(\log x), \qquad \sum_{1 \leq n \leq x} n'_P = C_P \frac{x^2}{2} + O(x^{\delta+1}),$$

and Corollary 2 extends to $\sigma_{D_P} = 2$.

6. Reducing ζ_{D_P} to ζ

It is natural to expect that ζ_{D_P} has a close relation to the Riemann zeta function ζ. For ζ_{D_p}, this relation is already known in the following lemma (originally with different terminology and notation):

Lemma 3 (see [4, Lemma 6]). *Let $p \in \mathbb{P}$ and $s > 2$. Then*

$$\zeta_{D_p}(s) = \frac{\zeta(s-1)}{p^s - p}.$$

We extend this to ζ_{D_P}.

Theorem 7. *Let $\emptyset \neq P \subseteq \mathbb{P}$ and $s > 2$. Then*

$$\zeta_{D_P}(s) = \zeta(s-1) \sum_{p \in P} \frac{1}{p^s - p}.$$

Proof. We have

$$\zeta_{D_P}(s) = \sum_{n=1}^{\infty} \frac{n'_P}{n^s} = \sum_{n=1}^{\infty} \frac{n \sum_{p \in P} \frac{\nu_p(n)}{p}}{n^s} = \sum_{n=1}^{\infty} \sum_{p \in P} \frac{\nu_p(n)}{pn^{s-1}}. \tag{8}$$

Since the series (8) converges and all its terms are nonnegative, we can change the order of summation. Therefore, by the simple calculation and applying Lemma 3 we obtain

$$\begin{aligned} \zeta_{D_P}(s) &= \sum_{p \in P} \sum_{n=1}^{\infty} \frac{\nu_p(n)}{pn^{s-1}} = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{n\nu_p(n)}{pn^s} = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{n'_p}{n^s} \\ &= \sum_{p \in P} \zeta_{D_p}(s) = \sum_{p \in P} \frac{\zeta(s-1)}{p^s - p}, \end{aligned}$$

completing the proof. □

In particular,

$$\zeta_D(s) = \zeta(s-1) \sum_{p \in \mathbb{P}} \frac{1}{p^s - p}.$$

7. Three further questions

In the case of $s \leq 2$, Theorems 1–3 give asymptotic formulas for the first sum of (2), and Theorems 4–6 give those for the second. What about the case of $s > 2$? Theorems 3 and 6 with $R(x) = O(1)$ hold also then, but since the main term has a smaller complexity than the error term, we get nothing reasonable out of them. The question about a nontrivial asymptotic formula for the second (and third) sum of (2) in the case of $s > 2$ therefore remains open.

As noted at the end of Section 5, our results about $\zeta_{D_p}(s)$ can be extended to $\zeta_{D_P}(s)$ if $P \subset \mathbb{P}$ is nonempty and finite or if $P = \mathbb{P}$. Can they be extended also if $P \subset \mathbb{P}$ is infinite? This question remains open, too.

Using advanced number-theoretic methods, the error terms of our asymptotic formulas can probably be improved, i.e., their complexity can be decreased. How could this be done? This is our third question.

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