Benemérita Universidad Autónoma de Puebla

Facultad de Ciencias Físico Matemáticas

Relación de Perturbación Perturbación en Mecánica Cuántica
(Disturbance-Disturbance Relation in Quantum Mechanics)

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asesorado por

Dr. Luis Manuel Arévalo Aguilar

Puebla Pue.
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Les agradezco a todos aquellos que me han ayudado a realizar este trabajo, en particular a Elena y a toda mi familia y profesores.

Summary

In this thesis we propose a new Uncertainty Relation, that we call *Entropic Uncertainty Relation for Disturbance-Disturbance*, which relates the uncertainty associated with a property of a system with the uncertainty of other property of the same system. This relation talks about the impossibility to obtain a completely accurate probability distribution of an observable when it is measured and the fact that measuring a system yields a disturbance that produces a collapse of the state of the system, therefore changing the probability distribution and the statistics of the other observables (properties) of the system.

To make understandable this topic, in the Introduction of this work we exhibit many important ideas of Quantum Mechanics that give the background needed to realize our work, like the postulates of Quantum Mechanics and the idea of uncertainty. In Chapter 1 we show the difference between the Uncertainty Raltions and the Uncertainty Principles and we present a summary of the Entropic and non-Entropic Relations associated to each of the Uncertainty Principles. Also, in order to show the importance of the postulates, we use them to analyze the famous Stern-Gerlach experiment. In Chapter 2 we introduce the development of our work, in which we establish the adequate theoretical framework by means of the postulates of quantum mechanics and what is know as a *probability metric* in order to measure the uncertainty, obtaining as our final product the new formulation of an Uncertainty Relation, the *Entropic Uncertainty Relation for Disturbance-Disturbance.* The obtained relation is discussed and it is proposed an statement associated to it. Finally, the general conclusions of this work are presented in Chapter 3.

List of Publications

The work done in order to obtain this degree produced two articles. The first one is a paper where we study the Stern-Gerlach Experiment considering it completely as a quantum system and using all four postulates of Quantum Mechanics [1], as discussed in the Introduction. This paper has been already accepted by the European Journal of Physics.

- E Benítez Rodríguez and L M Arévalo Aguilar. A full quantum analysis of the Stern-Gerlach Experiment using the evolution operator method: Analising current issues on the teaching quantum mechanics. To be published.

There is also another paper in the process of submission to a journal based on the work done in this thesis about our new Entropic Uncertainty Relations and Entropic Uncertainty Principle. This paper is tentatively called Uncertainty Principle of Disturbance-Disturbance.

General Index

Introduction

Physics is a science that has had an extraordinary development through the centuries and decades. The knowledge about nature has increased thanks to the investigation and experimentation in Particle, Statistical, Optical, and Quantum Physics. All this kwnowledge needs to convey to new generations of physicists in an acceptable form, thus, the importance of teaching Physics should be a main topic into our academic routine[2]. In particular, Quantum Mechanics is somewhat rough for theaching. Many bad habits in teaching have affected the general understanding of this area and has effects such as a general misunderstanding of many main topics like the Uncertainty Principle and the measurement in Quantum Mechanics.

A very important example of these misunderstandings is the Stern-Gerlach experiment (SGE). In many textbooks the SGE is studied in a semi- classical manner[3–5]. This gives a wrong idea about what is the meaning of the SGE and the way that we understand it. In a recent work [1] we studied the SGE by means of a complete quantum analysis. In particular, we found that the SGE is a creator of entangled states with an easy treatment (simply employing the general solution of the Schrödinger equation), a result that is not unknown to the Quantum Physics community, but which is not mentioned on usually textbooks.

In particular, we want to focus on one of the most controversial topics on Quantum Mechanics. Even nowadays, the Uncertainty Principle is the center of many works around the globe, due to its theoretical and experimental importance, from the stability of matter[6] to electromagnetic field modes enclosed in a cavity[7]. The uncertainty principle is a fundamental part of Quantum Mechanics, which has had a great development in the last years. The understanding of the nature of quantum systems shown in the recent works on the area has implied an exhaustive revision of the fundamentals of the theory in general, and, in particular, on the uncertainty principle[8–11]. To understand this changes we review the basic axioms or postules of Quantum Mechanics.

1 Postulates of Quantum Mechanics

Quantum Mechanics has a set of postulates, which are a base for the theory and establish a logical frame in order to enlighten the theory. In the literarture there are a lot of presentations of these postulates [5, 12–17], so we show here a basic set of four postulates that we consider suitable for the following chapters.

Postulate 1. Any quantum system is associated with a Hilbert space, and the state of the system is represented by a vector on that Hilbert space.

Postulate 2. The dynamic evolution of a quantum system is described by an unitary operator.

Postulate 3. The measurable physical variables are represented by self-adjoint operators. The probability of finding in a measurement that the state of the system is an eigenvector of the operator $\hat{A}$, $|a_k\rangle$, is $|\langle\psi|a_k\rangle|^2$, where $|\psi\rangle$ is the initial state of the system.

Postulate 4. The measurement of observable $\hat{A}$ collapses the state of the system to one of the eigenvectors of $\hat{A}$.

With this four main postulates one can begin to study any quantum system. The adequate treatment of a particular quantum system depends on the intrinsic characteristics of that system.

An example of the use of the postulates in the treatment of a quantum system: the SGE

The SGE was made by Stern and Gerlach in 1922 [18]. This experiment was proposed to show the space quantization of the Debye-Somerfeld old quantum theory, and curiously, three years later it was used to prove the existence of spin [19, 20].

The SGE consists on an oven from which a beam of silver atoms leave. Next, with the use of a slit or a pinhole, the direction of the atoms is selected, and then, the resultant beam passes through an inhomogeneus magnetic field. Finally, a screen is placed after the magnetic field to see the result of the interaction between the atoms and the magnetic field. A scheme is shown in Figure 1.

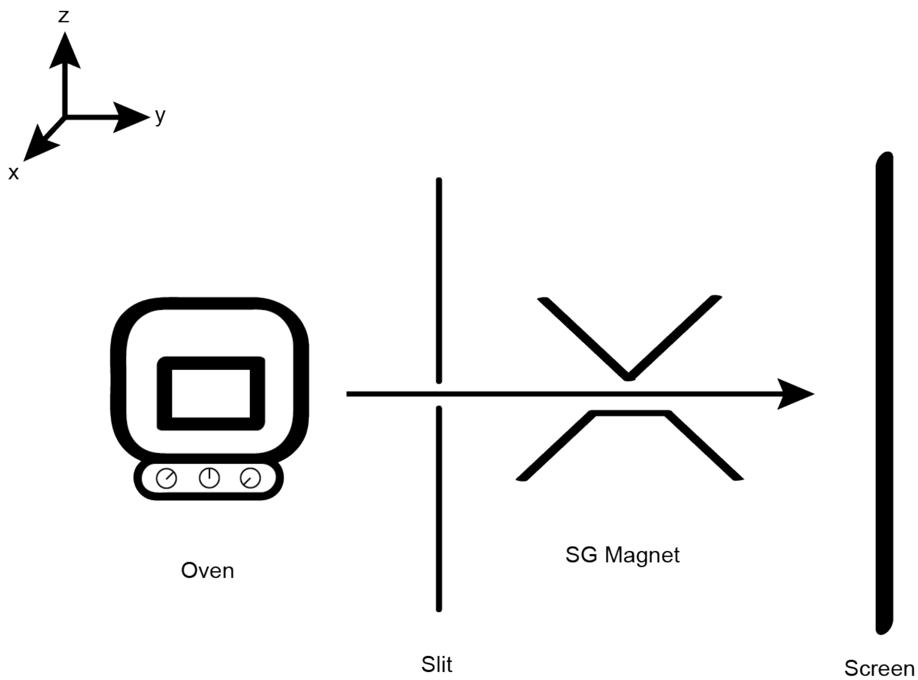

Figure 1: Scheme of the Stern Gerlach Experiment.

In order to describe the initial state of the atoms that leave the oven, we associate the state of spin of these (Postulate 1) whith a linear combination of the kets of spin up, $|z_1\rangle$, and down, $|z_2\rangle$, in z-direction:

$$\alpha|z_1\rangle + \beta|z_2\rangle. \tag{1}$$

Moreover, we can consider the position of the beam of atoms and represent it by a Gaussian wave packet, and then, put a general state of the system as

$$|\psi(0)\rangle = \psi_0 \left(\alpha|z_1\rangle + \beta|z_2\rangle\right), \tag{2}$$

where ψ_0 is given by

$$\psi_0 = \frac{1}{(2\pi\sigma_0^2)^{3/4}} \exp\left(-\frac{\mathbf{r}^2}{4\sigma_0^2} + i\mathbf{k}\cdot\mathbf{r}\right) \tag{3}$$

with σ_0 the width of the wave packet, $\mathbf{r}$ the position of the particle and $\mathbf{k}$ the wave vector.

Now, to know the behavior of the system over time, we recall the Hamiltonian of the system,

$$\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + \mu_c(\boldsymbol{\sigma}\cdot\mathbf{B}), \tag{4}$$

where $\mu_c = g\frac{e\hbar}{4m_e}$, g is the gyromagnetic ratio, $\boldsymbol{\sigma}$ is the Pauli spin matrices vector and $\mathbf{B}$ is the inhomogeneus magnetic field with the form $\mathbf{B} = -bx\hat{\imath} + (B_0 + bz)\hat{k}$.

Next, to obtain the dynamics of the system we apply simply the unitary evolution operator (Postulate 2) to the initial state [21]

$$\begin{aligned} |\psi(t)\rangle &= \hat{U}\left[\psi_0\left(\alpha|z_1\rangle + \beta|z_2\rangle\right)\right] \\ &= \exp\left(\frac{-it}{\hbar}\hat{H}\right)\left[\psi_0\left(\alpha|z_1\rangle + \beta|z_2\rangle\right)\right]. \end{aligned} \tag{5}$$

On this way we obtain the state of the system at time t [1]

$$\begin{aligned} |\psi(t)\rangle &= \exp\left[\left(\frac{-it}{\hbar}\right)^3 \frac{\hbar^2\mu_c^2 b^2}{6m}\right][\sigma_0/(2\pi)^{1/2}]^{3/2}\left[\sigma_0^2 + \frac{i\hbar t}{2m}\right]^{-3/2}\exp\left(-\sigma_0^2 k_y^2\right) \\ &\times \exp\left(\frac{-1}{4[\sigma_0^2 + i\hbar/2m]}(-i4y\sigma_0^2 k_y)\right)\exp\left(\frac{-1}{4[\sigma_0^2 + i\hbar/2m]}(x^2 + y^2 - 4\sigma_0^4 k_y^2)\right) \\ &\times \left\{\alpha\exp\left(\frac{-it}{\hbar}(B_0 + bz)\right)\exp\left[\frac{-1}{4[\sigma_0^2 + i\hbar/2m]}\left(z - \frac{t^2\mu_c b}{2m}\right)^2\right]|z_1\rangle\right. \\ &\left. + \beta\exp\left(\frac{it}{\hbar}(B_0 + bz)\right)\exp\left[\frac{-1}{4[\sigma_0^2 + i\hbar/2m]}\left(z + \frac{t^2\mu_c b}{2m}\right)^2\right]|z_2\rangle\right\}. \end{aligned} \tag{6}$$

Now, suppose we want to obtain the probability to find an electron of the beam in spin up state, $|z_1\rangle$, at time t. By means of Postulate 3, we have

$$\begin{aligned} |\langle z_1|\psi(t)\rangle|^2 &= |\alpha|^2[\sigma_0/(2\pi)^{1/2}]^3\left[\sigma_0^4 + \left(\frac{\hbar t}{2m}\right)^2\right]^{-3/2}\exp(-2\sigma_0^2 k_y^2)\exp\left[\frac{\hbar t y\sigma_0 k_y}{m\left(\sigma_0^4 + \left(\frac{\hbar t}{2m}\right)^2\right)}\right] \\ &\times \exp\left\{\frac{-2\sigma_0^2}{4[\sigma_0^4 + \left(\frac{\hbar t}{2m}\right)^2]}\left[x^2 + y^2 - 4\sigma_0^4 k_y^2 + \left(z - \frac{t^2\mu_c b}{2m}\right)^2\right]\right\}. \end{aligned} \tag{7}$$

In this case the probability depends on the time, therefore, when one puts a screen after the magnet of the SGE (see Figure 1), one collapses the state of the system, represented by Equation 6, to an eigenstate of the position of the system (Postulate 4), and we can see two areas of concentration of silver atoms. See Figure 2 below.

Thus, we see clearly that it is possible to study any particular quantum system with the use of the Postulates of Quantum Mechanics. Once we have introduced ourselves in the Quantum Mechanic context we can talk about the Uncertainty Principle in Quantum Mechanics.

2 Uncertainty

The Uncertainty Principle in Quantum Mechanics was proposed by Werner Heisenberg in 1927 [22]. The principal implication of this proposal was that it exists an upper bound for the accuracy with which we can identify the state of a quantum system [8]. This caused theorethical, experimental and philosophical implications [23] that changed the way in which physicists think about Quantum Mechanics and the fact that we can not predict with certainty the result of experiments in the quantum world.

Thus, the notion of disturbance in quantum mechanics is introduced naturally due to the interaction between the observer and the system. The first work that fully characterizes disturbance in Quantum Mechanics was made by Ozawa in 2003, [11]. In this work Ozawa proposed a new Uncertainty Principle which talks about the noise and disturbance in a measurement. In particular the disturbance in a system is related to a measure made by an apparatus.

Due to the closeness between Quantum Mechanics and Information Theory in the second half of the 20th century, new measures of uncertainty were needed. The Shannon Entropy was the first natural measure of uncertainty [24–26],

$$H(P) = -\sum_{i=1}^{N} p_i \ln p_i. \tag{8}$$

In equation (8), p_i is the probability associated with a particular probability distribution, where $\sum_{i=1}^{N} p_i = 1$. It was natural that different uncertainty relations related with the Shannon Entropy appeared through the years[8, 27–30]. But Shannon Entropy is not the only measure of uncertainty related with entropy, so a new research branch called *Entropic Uncertainty Relations* was formed[25, 30–34]. In 2014, for the first time, an article that combines the concepts of entropy and disturbance came to light [35]. This paper proposes a Quantum Entropic Uncertanty Relation for noise and disturbance and it uses the conditional entropy[25, 26, 36] as a measure of that uncertainty. This thesis is also a proposal of a new Entropic Uncertainty Relation, but with other type of measure of uncertainty and a different physical meaning.

To this point we have discussed the importance of the postulates of Quantum Mechanics and the notion of uncertainty. By doing so, we tried to establish a solid base in order for us to be able to study a quantum system. We have proposed a set of postulates, Postulates 1, 2, 3 and 4, which is composed of necessary statements to describe completely and logically a quantum system. In addition, we show here an example of the application

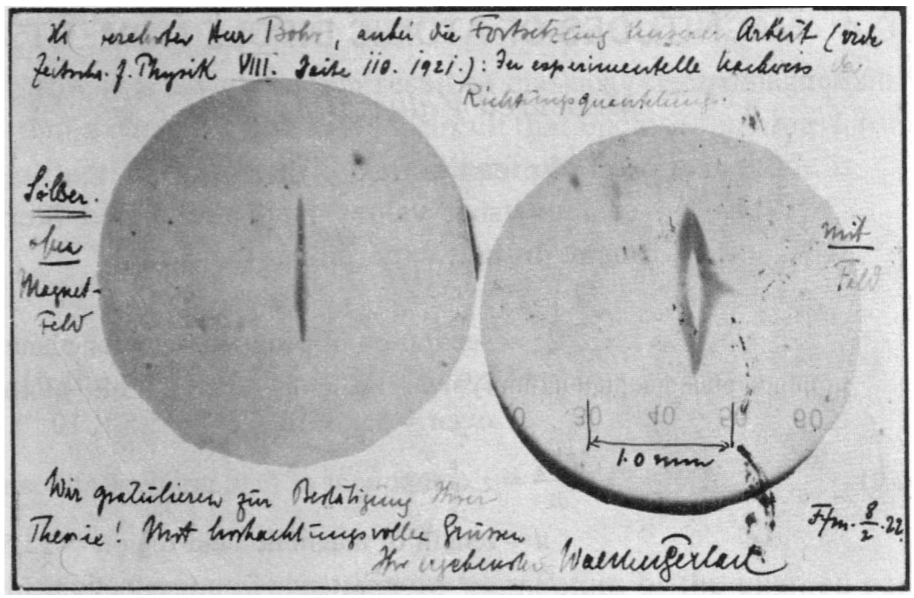

Figure 2: Gerlach's postcard to Niels Bohr, where we can see the state of the beam at a given time [18].

of this set of postulates to an important topic in Quantum Mechanics, the Stern-Gerlach experiment. On the other hand, the uncertainty plays an important role in the formalism of Quantum Mechanics due to the probabilistic nature of the theory, and never should be diminished. In this introduction we talked briefly about the concept of uncertainty and uncertainty measures with the intention to give context to the work to be done in this thesis.

Chapter 1

Uncertainty Principles and Uncertainty Relations

First, we can talk about three different *Uncertainty Principles* [37], each one with its own *Uncertainty Relation*. It is clear that we can not associate one Uncertainty Relation with two or more sentences (Uncertainty Principles) due to the fact that sentences are not equivalent. We should note that it is impossible to prove that the sentences are equivalent (see below). To this day we can find in the literature three well-established Uncertainty Principles with their own Uncertainty Relation.

The first one reads: It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized [37]. The Uncertainty Relation associated with this sentence is the Robertson-Kennard Uncertainty Relation [27, 38]

$$\sigma_{\hat{A}}(\psi)\sigma_{\hat{B}}(\psi) \geq \frac{|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|}{2}, \tag{1.1}$$

where $|\psi\rangle$ is the initial state of the quantum system, $\hat{A}$ and $\hat{B}$ two observables of the system, $\sigma_{\hat{A}}(\psi)$ and $\sigma_{\hat{B}}(\psi)$ the standar deviations of $\hat{A}$ and $\hat{B}$, and $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, the commutator between the two observables. It is important to note that this Uncertainty Relation is the one appearing in most of the textbooks [3, 6, 13, 17, 39], giving an incomplete picture of the Quantum Mechanics Uncertainty Principles and Relations [27]. The Heisenberg's Uncertainty Relation [22, 27] is a particular case of this Uncertainty Relation where $\hat{A} = x$ and $\hat{B} = p$ are the canonical position and momentum respectively,

$$\sigma_x(\psi)\sigma_p(\psi) \geq \hbar/2. \tag{1.2}$$

Equation (1.2) has an irreducible lower bound, in this sense, is more correct than (1.1), because the lower bound does not depend on the state of the system, something that in Quatum Mechanics is more correct due to the constant evolution of states into others [8].

The second well-known Uncertatiny Principle says that it is impossible to know with complete certainty the value of position and momentum of a system that are simultaneously measured [37]. The Uncertainty Relation for this case only exists for the case of the canonical conjugate observables $\hat{x}$ and $\hat{p}$ [27]

$$\sigma_{\hat{x}}(\psi)\sigma_{\hat{p}}(\psi) \geq 1. \tag{1.3}$$

The third Uncertainty Principle establishes that it is impossible to measure one property of a system without disturbing other property of the same system [37]. The most accepted Uncertainty Relation associated to this principle is the Uncertainty Relation of noise and disturbance. This was demonstrated by Ozawa in 2003 [11] and says that it is impossible to know with total certainty the value of an observable $\hat{A}$ due to the noise of the measurement device and it is also impossible to measure this property without disturbing the other property of the system, $\hat{B}$. The Uncertainty Relation of Noise-Disturbance is expressed as

$$\epsilon_{\hat{A}}(\psi)\eta_{\hat{B}}(\psi) + \epsilon_{\hat{A}}(\psi)\sigma_{\hat{B}}(\psi) + \sigma_{\hat{A}}(\psi)\eta_{\hat{B}}(\psi) \geq \frac{|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|}{2}. \tag{1.4}$$

The sigmas are the standar deviations of $\hat{A}$ and $\hat{B}$, $\epsilon_{\hat{A}}(\psi)$ is the noise and $\eta_{\hat{B}}(\psi)$ the disturbance. For more details see [11].

The logic of mathematical propositions rules out the possibility to relate many propositions with only a mathematical relation, which in general provide us with a proof that these Uncertainty Principles are not equivalent and neither are its uncertainty relations. Furthermore, from this point of view we can say that it is relatively easy to see the difference between the three Uncertainty Principles.

Similarly, we can talk about three different Entropic Uncertainty Principles and their Entropic Uncertainty Relations. Here we present a summary.

The first Uncertanty Principle talks about preparation of states, and says that it is impossible to prepare with total accuracy two quantum states. The most general Uncertainty Relation associated with this principle is [40]

$$S_{\hat{A}}(\psi) + S_{\hat{B}}(\psi) \geq -2\ln c, \tag{1.5}$$

where $S(\psi) = -\sum_i^N |\langle\psi|a_j\rangle|^2 \ln|\langle\psi|a_j\rangle|^2$, and we have a similar expression for $\hat{B}$, and

$c = \max_{j,k} |\langle a_j | b_k \rangle|$, is the maximum overlap between the observables $\hat{A}$ and $\hat{B}$.

The second Entropic Uncertainty Principle says that it is impossible to know with total certainty the value of two properties of a system if we make a simultaneous measure on it. In this case, to the best of our knowledge, there is not an Entropic Uncertainty Relation associated with this principle.

The third Entropic Uncertainty Relation is the Entropic Noise-Disturbance Uncertainty Relation. This relation was proposed in 2014 [35] and relates the noise and the disturbance with relative entropies. The Entropic Uncertainty Principle of Noise-Disturbance says that it is impossible to measure a property of a quantum system without obtaining a result with noise nor without disturbing the system. The general Entropic Uncertainty Relation of Noise-Disturbance is

$$N(M.X) + D(M.Z) \geq -\log_2 c. \tag{1.6}$$

Here X and Z are two observables of the system and M is the measuring apparatus, and we have

$$c = \max_{x,z} \| \langle \varphi^x | \psi^z \rangle |^2 \tag{1.7}$$

the maximum overlap between the sets of eigenstates of observables X, $\{|\varphi^x\rangle\}$, and Z, $\{|\psi^z\rangle\}$.

In equation (1.6) $N(M.X)$ is associated with the noise and $D(M.Z)$ with the disturbance. Both are related with what is known as conditional entropy [25, 26, 35].

In this section we introduce the three well known Uncertainty Principles and their corresponding non-Entropic and Entropic Uncertainty Relations. The historic development of the Uncertainty Principles and Relations is presented intrinsically. A summary of the principles and relations can be found on Table 1.1.

Uncertainty Priciple	Uncertainty Relation	Entropic Uncertainty Relation
It is impossible to prepare states in which position and momentum are simultaneously arbitrary well localized	Eq. (1.1) Robertson-Kennard Uncertainty Relation	Eq. (1.5)
It is impossible to measure position and momentum simultaneously	Eq. (1.3)	-
It is impossible to measure position without disturbing momentum and viceversa	Eq. (1.4) Uncertainty Relation of Noise-Disturbance	Eq. (1.6) Entropic Uncertainty Relation of Noise-Disturbace

Table 1.1: Summary of Uncertainty Principles

Chapter 2

Uncertainty Principle of Disturbance-Disturbance

2.1 An Uncertainty Relation using all the axioms of Quantum Mechanics

The Robertson-Kennard Uncertainty Principle talks about preparation of states and its Uncertainty Relation is easily proved by using only three of the basic postulates of Quantum Mechanics and the Cauchy-Schwarz Inequality. We believe that a complete Uncertainty Relation should be proved using the four basic postulates, included the fourth postulate of quantum measurement.

To prove the Robertson-Kennard Generalized Uncertainty Relation we need two main results [39]:

Standard Deviation: It is defined as the second statistical moment. In Dirac notation we have the standard deviation associated to an observable $\hat{Q}$ in some determined state as

$$\sigma_{\hat{Q}}^2 = \langle(\hat{Q} - \langle\hat{Q}\rangle)^2\rangle = \langle(\hat{Q} - \langle\hat{Q}\rangle)\Psi|(\hat{Q} - \langle\hat{Q}\rangle)\Psi\rangle. \quad (2.1)$$

Cauchy-Schwarz Inequality: $\forall v, w \in V$, V a vector space, we have,

$$|\langle v, w\rangle| \leq \|v\|\|w\|. \quad (2.2)$$

Proof of the Cauchy-Schwarz Inequality [41]: When $w = 0$ the relation is trivial. Now, suppose that $w = e$ is an unitary vector, this is $e \in V$ and $\|e\| = 1$. If c is the component of v along e, then $v - ce$ is perpendicular to e, and so, perpendicular to ce.

Using the Pythagorean theorem we have

$$\begin{aligned}\|v\|^2 &= \|v-ce\|^2+\|ce\|^2 \\ &= \|v-ce\|^2+c^2. \qquad (2.3)\end{aligned}$$

Then, $c^2 \le \|v\|^2$, so $|c| \le \|v\|$. Finally, if w is an arbitrary, $\neq 0$, vector, we have the unitary vector $e = \frac{w}{\|w\|}$ and now,

$$\left|\left\langle v, \frac{w}{\|w\|}\right\rangle\right| \le \|v\|. \qquad (2.4)$$

Then

$$|\langle v,w\rangle| \le \|w\|\|v\|, \qquad (2.5)$$

as we wanted to prove.

We may now prove the Robertson-Kennard Uncertainty Relation [12]:

Take two self-adjoint operators $\hat{A}$ and $\hat{B}$, and a quantum state $|\psi\rangle$. Let us suppose $\langle\psi|\hat{A}\hat{B}|\psi\rangle = x+iy$, where x and y are real numbers. We note that $\langle\psi|[\hat{A},\hat{B}]|\psi\rangle = 2iy$ and $\langle\psi|\{\hat{A},\hat{B}\}|\psi\rangle = 2x$. This implies

$$|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|^2+|\langle\psi|\{\hat{A},\hat{B}\}|\psi\rangle|^2 = 4|\langle\psi|\hat{A}\hat{B}|\psi\rangle|^2. \qquad (2.6)$$

By applying the Schwarz inequality for vectors $\hat{A}\psi$ and $\hat{B}\psi$ we get

$$|\langle\hat{A}\psi,\hat{B}\psi\rangle|^2 = |\langle\psi|\hat{A}\hat{B}|\psi\rangle|^2 \le \langle\psi|\hat{A}^2|\psi\rangle\langle\psi|B^2|\psi\rangle, \qquad (2.7)$$

where we should remember that the norm of a vector v is $\|v\| = \sqrt{\langle v,v\rangle}$. Combining these last two expressions we have

$$|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|^2 \le 4\langle\psi|\hat{A}^2|\psi\rangle\langle\psi|\hat{B}^2|\psi\rangle. \qquad (2.8)$$

Now assume that $\hat{C}$ and $\hat{D}$ are two observables. Replacing $\hat{A} = \hat{C}-\langle\hat{C}\rangle$ and $\hat{B} = \hat{D}-\langle\hat{D}\rangle$ in the last equation, and remembering the definition of standard deviation, we obtain the Robertson-Kennard Generalized Uncertainty Relation

$$\sigma_{\hat{C}}\sigma_{\hat{D}} \ge \frac{|\psi|[\hat{C},\hat{D}]|\psi\rangle|}{2}. \qquad (2.9)$$

2.1.1 A thought experiment and the Jensen-Shannon uncertainty relation for disturbance

The main goal of this thesis is to find an Uncertainty Principle with its own Uncertainy Relation, making use of all of the four postulates as a principal background, and a relation between the disturbance of one of the properties of the system and the disturbance of another property of the same system. We conclude that a *real* Uncertainty Principle should have a one to one relation with its Uncertainty Relation. We want to develop an Uncertainty Relation in the style of the *distribution error* estimation, in the sense of [42]. In [42] the *value-comparison error* is discussed as well. The difference between these two ways of comparing results is that the first one compares probability distributions while the second one compares the results individually.

In this case we want to know how a system with two properties, represented by two operators $\hat{A}$ and $\hat{B}$, behaves when a measurement is applied on it, and to observe how the distance between the probability distributions of its properties changes. It is known to all that when one makes a measurement of any kind on a quantum system one produces a disturbance on it. Moreover, as it is well known, the observables do not have a pre-existing value prior to measurement, therefore, we take as a departure that the disturbance is produced in the state of the system. Then, this disturbance affects only the probability distributions. In order to describe the resulting disturbance we wish to measure the distance between the probability distributions of two properties of the system at different times, i.e. before and after the measurement.

The thought experiment which we consider is as follows. We have a quantum system, that has two properties which we want to know. We realize a projective measurement on one property, say $\hat{A}$, then we disturb the system, and, due to the measurement, the system collapses to an eigenstate of the observable $\hat{A}$ and the value of the observable $\hat{B}$ is now unknow to us. However, we know that this has been disturbed by our measurement. Now we want to compare the probability distribution of the observable $\hat{A}$ before the measurement and the probability distribution of the same observable after the measurement (in this case we compare a probability distribution versus only a possible state). Also, we compare the distance between the distributon of observable $\hat{B}$ before and after the measurement of observable $\hat{A}$.

To measure the distance between two probability distributions there exist the so-called *probability metrics* [29, 32, 36, 43–46]. In physics the most common probability metric is the relative entropy or Kullback-Leibler divergence [25, 26]. Between two probability distributions $P(x)$ and $Q(x)$, we have

$$S(P,Q) = \sum_x p(x) \log \frac{p(x)}{q(x)} = E_P \log \frac{p(x)}{q(x)}. \tag{2.10}$$

The last equality tells us that the relative entropy can be interpreted as the expectation value of the logarithm function of $\frac{p(x)}{q(x)}$, commonly identified as the most simple uncertainty measure.

In the present work we will use a different metric of those used in some other works [25, 35, 42]. A good measure of the distance between two discrete probability distributions is the symmetric Jensen Shannon Divergence [44],

$$D_{PQ} = \sqrt{\sum_{i=1}^{N} \left(p_i \ln \frac{2p_i}{p_i + q_i} + q_i \ln \frac{2q_i}{q_i + p_i} \right)}, \tag{2.11}$$

where p_i and q_i are two probability distributions. This metric has never before been related to physics.

Additionally, there exists a relation between the Jensen Shannon divergence (2.11) and the Kullback-Leibler divergence (2.10) and Shannon Entropy (8) [32],

$$\begin{aligned} D^2_{P,Q} &= S\left(P, \frac{P+Q}{2}\right) + S\left(Q, \frac{P+Q}{2}\right) \\ &= H\left(\frac{P+Q}{2}\right) - \frac{1}{2}H(P) - \frac{1}{2}H(Q). \end{aligned} \tag{2.12}$$

Then, we have the Symmetric Jensen Shannon Entropy in terms of the eigenstates of the observables. In this case we want to compare the distribution of observable $\hat{B}$ before the measurement versus the distribution of the same observable after the measurement, we take $D_{\hat{B}}$ as the disturbance in the statistics of $\hat{B}$ due to the measurement of $\hat{A}$, and we find it given as,

$$D_{\hat{B}} = \sqrt{\sum_{j=1}^{N} \left(|\langle b_j|a_s\rangle|^2 \ln \frac{2|\langle b_j|a_s\rangle|^2}{|\langle b_j|a_s\rangle|^2 + |\langle b_j|\psi\rangle|^2} + |\langle b_j|\psi\rangle|^2 \ln \frac{2|\langle b_j|\psi\rangle|^2}{|\langle b_j|\psi\rangle|^2 + |\langle b_j|a_s\rangle|^2} \right)}, \tag{2.13}$$

where $|\langle b_j|\psi\rangle|^2$ is the probability of finding the state in a eigenstate of the observable $\hat{B}$, and $|\langle b_j|a_s\rangle|^2$ is the probability of finding the system in an eigenstate of the observable $\hat{B}$, given that the state after the measurement is an eigenvector of $\hat{A}$, $|a_s\rangle$.

Now, we want to investigate how the distance between the probability distributions behaves when one of the probability distribitions tends to the other, i. e. we want to study the next limit,

$$\lim_{P\to Q} D^2_{PQ}, \tag{2.14}$$

which, expanded term by term, tends in first order of p_i to [44]

$$\frac{1}{4}\chi^2 = \frac{1}{4}\sum_i \frac{(p_i - q_i)^2}{q_i}. \tag{2.15}$$

This is the distance χ^2 between p_i and q_i. This distance is not symmetric, thus we can express it in terms of the probability distributions of $\hat{B}$. The two χ^2 distances that we can have are the following,

$$\frac{1}{4}\chi^2(|\langle b_j|a_s\rangle|^2, |\langle b_j|\psi\rangle|^2) = \frac{1}{4}\sum_j |\langle b_j|a_s\rangle|^2 \left(1 - \frac{|\langle b_j|\psi\rangle|^2}{|\langle b_j|a_s\rangle|^2}\right)^2, \tag{2.16}$$

$$\frac{1}{4}\chi^2(|\langle b_j|\psi\rangle|^2, |\langle b_j|a_s\rangle|^2)\frac{1}{4} = \sum_j |\langle b_j|\psi\rangle|^2 \left(1 - \frac{|\langle b_j|a_s\rangle|^2}{|\langle b_j|\psi\rangle|^2}\right)^2. \tag{2.17}$$

Now, we want to find the minimum value of this distance, different of zero. To obtain a result we make the next considerations. First, we want that the minimun does not depend on the state of the system, so we want to consider a little change over the initial state of the system. Thus, we can write the initial state as an eigenvector $|a_s\rangle$ of the operator $\hat{A}$ plus a variation due to the fact that the distance χ^2 is zero when the initial state is an eigenvalue of $\hat{A}$. So, we take $|\psi\rangle = |a_s\rangle + \delta(|a_s\rangle)$, normalized. Then we can write respectively the equations (2.16) and (2.17) as

$$\frac{1}{4}\chi^2(|\langle b_j|a_s\rangle|^2, |\langle b_j|\psi\rangle|^2) \geq \frac{1}{4}\sum_j |c_j|^2 \left(1 - \frac{|c_j|^2 + \delta(|c_j|^2)}{|c_j|^2}\right)^2 = \chi^{(1)}_{\hat{B},min}, \tag{2.18}$$

$$\frac{1}{4}\chi^2(|\langle b_j|\psi\rangle|^2, |\langle b_j|a_s\rangle|^2) \geq \frac{1}{4}\sum_j \left(|c_j|^2 + |\delta c_j|^2\right) \left(1 - \frac{|c_j|^2}{|c_j|^2 + \delta(|c_j|^2)}\right)^2 = \chi^{(2)}_{\hat{B},min}, \tag{2.19}$$

making $c_j = \langle b_j|a_s\rangle$.

Taking into consideration the aproximation we get relations for the minimum distance between two different probability distributions of a quantum system,

$$D_{\hat{B}} \geq \frac{1}{2}\sqrt{\chi^2(|\langle b_j|a_s\rangle|^2, |\langle b_j|\psi\rangle|^2)} \geq \sqrt{\chi^{(1)}_{\hat{B},min}}, \tag{2.20}$$

$$D_{\hat{B}} \geq \frac{1}{2}\sqrt{\chi^2(|\langle a_s|b_j\rangle|^2, |\langle \psi|b_j\rangle|^2)} \geq \sqrt{\chi^{(2)}_{\hat{B},min}}, \tag{2.21}$$

where $D_{\hat{B}}$ is expresed in equation (2.13). This last two expressions are the same due to the simmetry of Jensen Shannon Entropy.

The existence of the two equations, 2.20 and 2.21, is because the distance χ^2 is not symmetric. For practical purposes we can take the minimum between $\sqrt{\chi^{(1)}_{\hat{B},min}}$ and $\sqrt{\chi^{(2)}_{\hat{B},min}}$ as $\min\left\{\sqrt{\chi^{(1)}_{\hat{B},min}}, \sqrt{\chi^{(2)}_{\hat{B},min}}\right\}$, and take in consideration only one of these relations.

Therefore, we can show our first result as the following equation, that we can name the *Jensen Shannon Uncertainty Relation for Disturbance*

$$D_{\hat{B}} \geq \min\left\{\sqrt{\chi^{(1)}_{\hat{B},min}}, \sqrt{\chi^{(2)}_{\hat{B},min}}\right\}. \tag{2.22}$$

Now, we want to obtain a relation between the two observables of the system $\hat{A}$ and $\hat{B}$. For this purpose we can make a treatment of the observable $\hat{A}$ before and after realizing a proyective measurement of $\hat{A}$.

2.1.2 Example of the Jensen-Shannon Uncertainty Relation for disturbance

Suppose that we have a particle of $\frac{1}{2}$ spin, consider that the initial state is

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle + |x_2\rangle), \tag{2.23}$$

then, when we realize a projective measure on it, we collapse its state to one of the components of the superposition state. Whithout loss of generality, suppose that we collapse its state to $|x_1\rangle$. In this way we can apply directly the equation and obtain

$$D_{(S_z)} \geq \min\left\{\sqrt{\chi^{(1)}_{\hat{B},min}(S_z)}, \sqrt{\chi^{(2)}_{\hat{B},min}(S_z)}\right\} \tag{2.24}$$

where

$$D_{(S_z)} = \sqrt{\sum_{j=1}^{N}\left(|\langle z_j|x_1\rangle|^2 \ln \frac{2|\langle z_j|x_1\rangle|^2}{|\langle z_j|x_1\rangle|^2 + |\langle z_j|\phi\rangle|^2} + |\langle z_j|\phi\rangle|^2 \ln \frac{2|\langle z_j|\phi\rangle|^2}{|\langle z_j|\phi\rangle|^2 + |\langle z_j|x_1\rangle|^2}\right)}; \tag{2.25}$$

$$\sqrt{\chi^{(1)}_{\hat{B},min}(S_z)} = \frac{1}{2}\sqrt{\sum_j |\langle z_j|x_1\rangle|^2 \left(1 - \frac{|c_j|^2 + \delta(|c_j|^2)}{|\langle z_j|x_1\rangle|^2}\right)^2}, \tag{2.26}$$

$$\sqrt{\chi^{(2)}_{\hat{B},min}(S_z)} = \frac{1}{2}\sqrt{\sum_j \left(|c_j|^2 + \delta(|c_j|^2)\right)\left(1 - \frac{|\langle z_j|x_1\rangle|^2}{|c_j|^2 + \delta(|c_j|^2)}\right)^2}. \tag{2.27}$$

We can write $|x_1\rangle = \frac{1}{\sqrt{2}}(|z_1\rangle + |z_2\rangle)$ and then, the equations (2.26),(2.27) can be written as

$$\sqrt{\chi^{(1)}_{\hat{B},min}(S_z)} = \frac{1}{2}\sqrt{2(\delta(|c_1|^2))^2 + 2(\delta(|c_2|^2))^2}, \tag{2.28}$$

$$\sqrt{\chi^{(2)}_{\hat{B},min}(S_z)} = \frac{1}{2}\sqrt{\frac{(\delta(|c_1|^2))^2}{\frac{1}{2} + \delta(|c_1|^2)} + \frac{(\delta(|c_2|^2))^2}{\frac{1}{2} + \delta(|c_2|^2)}}. \tag{2.29}$$

With the condition $\delta(|c_1|^2) = -\delta(|c_2|^2)$ due to the normalization this expressions are

$$\sqrt{\chi^{(1)}_{\hat{B},min}(S_z)} = |\delta(|c_1|^2)|, \tag{2.30}$$

$$\sqrt{\chi^{(2)}_{\hat{B},min}(S_z)} = \frac{1}{2}\sqrt{\frac{(\delta(|c_1|^2))^2}{\frac{1}{4} - (\delta(|c_1|^2))^2}}. \tag{2.31}$$

To obtain a numeric result we can give small values to $\delta(|c_1|^2)$, such as $|\delta(|c_1|^2)| << \frac{1}{2}$, see Figure 2.1.

2.2 The Uncertainty Relation of Disturbance-Disturbance

Now, to complete our study, we calculate the distance between the probability distributions of observable $\hat{A}$. Remember that in our though experiment we are measuring the observable $\hat{A}$ and we obtain as a result that the system collapses to an eigenstate of the same observable. So the probability of finding the system in an eigenstate of $\hat{A}$ before the

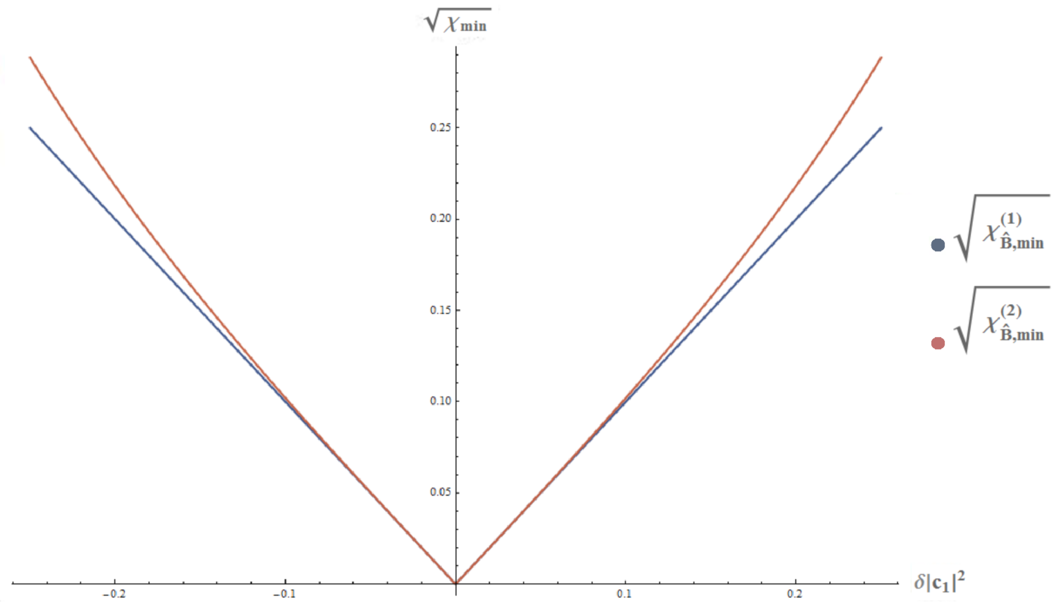

Figure 2.1: Values of $\sqrt{\chi^{(1)}_{\hat{B},min}}$, $\sqrt{\chi^{(2)}_{\hat{B},min}}$ when $\delta(|c_1|^2)$ takes values in the interval $[-.25, .25]$.

measurement is $|\langle a_j|\psi\rangle|^2$ and after the measurement we can say with total accuracy that the system is in an eigenstate of $\hat{A}$, say $|a_s\rangle$. This is, the probability of finding the system in $|a_s\rangle$ is 1.

We know from equations (2.14), (2.15) that the Jensen-Shannon entropy tends to χ^2 when one distribution tends to the other, so we have

$$D_{\hat{A}} \geq \frac{1}{2}\sqrt{\chi^2(|\langle a_j|\psi\rangle|^2, \delta_{j,s})}, \tag{2.32}$$

$$D_{\hat{A}} \geq \frac{1}{2}\sqrt{\chi^2(\delta_{j,s}, |\langle a_j|\psi\rangle|^2)}, \tag{2.33}$$

where

$$\chi^2(|\langle a_j|\psi\rangle|^2, \delta_{j,s}) = \sum_j |\langle a_j|\psi\rangle|^2 \left(1 - \frac{\delta_{j,s}}{|\langle a_j|\psi\rangle|^2}\right)^2, \tag{2.34}$$

$$\chi^2(\delta_{j,s}, |\langle a_j|\psi\rangle|^2) = \sum_j \delta_{j,s} \left(1 - \frac{|\langle a_j|\psi\rangle|^2}{\delta_{j,s}}\right)^2. \tag{2.35}$$

To obtain a trade-off relation we take into consideration the same aproximation for the initial state: $|\psi\rangle = |a_s\rangle + |da_s\rangle$, where $|\psi\rangle$ is normalized, as it should be. We can make this because χ^2 is minimum when ψ is an eigenstate of $\hat{A}$, that is $|a_s\rangle$.

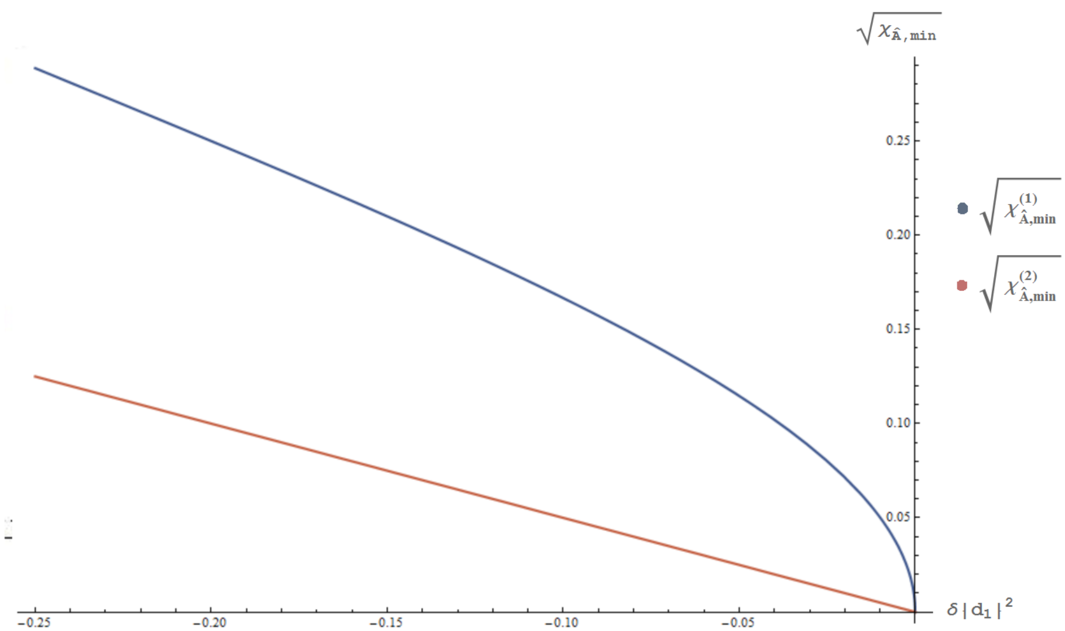

Figure 2.2: Values of $\sqrt{\chi^{(1)}_{\hat{A},min}(S_x)}$, $\sqrt{\chi^{(2)}_{\hat{A},min}(S_x)}$ when $\delta(|d_1|^2)$ takes values in the interval $[-.25, 0]$.

So, with this approximation we have now the minimum values for equations (2.34), (2.35),

$$\frac{1}{4}\chi^2(|\langle a_j|\psi\rangle|^2, \delta_{j,s}) \geq \frac{1}{4}\sum_j \left(\delta_{j,s} + \delta(|d_j|^2)\right)\left(1 - \frac{\delta_{j,s}}{\delta_{j,s} + \delta(|d_j|^2)}\right)^2 = \chi^{(1)}_{\hat{A},min}, \tag{2.36}$$

$$\frac{1}{4}\chi^2(\delta_{j,s}, |\langle a_j|\psi\rangle|^2) \geq \frac{1}{4}\sum_j \delta_{j,s}\left(1 - \frac{\delta_{j,s} + \delta(|d_j|^2)}{\delta_{j,s}}\right)^2 = \chi^{(2)}_{\hat{A},min}, \tag{2.37}$$

where $d_j = \langle a_j|a_s\rangle$ is the probability distribution. We should note that $\delta(|d_s|^2)$ is always negative, because we are measuring the distance between $\delta_{s,j}$ and a probability distribution close to it, and any probability must take values in the interval $[0, 1]$.

These last two equations can be reduced, so that

$$\chi^{(1)}_{\hat{A},min} = \frac{1}{4}\left(1 + \delta(|d_s|^2)\right)\left(1 - \frac{1}{1 + \delta(|d_s|^2)}\right)^2 + \frac{1}{4}\sum_{k\neq s}\delta(|d_k|^2), \tag{2.38}$$

$$\chi^{(2)}_{\hat{A},min} = \frac{1}{4}(\delta(|d_s|^2))^2. \tag{2.39}$$

For our example we have

$$\sqrt{\chi^{(1)}_{\hat{A},min}(S_x)} = \frac{1}{2}\sqrt{(1+\delta(|d_1|^2))\left(1-\frac{1}{1+\delta(|d_1|^2)}\right)^2 - \delta(|d_1|^2)} \tag{2.40}$$

$$\sqrt{\chi^{(2)}_{\hat{A},min}(S_x)} = \frac{1}{2}|\delta(|d_1|^2)|, \tag{2.41}$$

under the condition $\delta(|d_1|^2) = -\delta(|d_2|^2)$. These results are plotted in Figure 2.2.

Now, we can obtain the *Entropic Uncertainty Relation of Disturbance-Disturbance* as the sum of the uncertainties of observables $\hat{A}$ and $\hat{B}$,

$$D_{\hat{A}} + D_{\hat{B}} \geq \min\left\{\sqrt{\chi^{(1)}_{\hat{A},min}}, \sqrt{\chi^{(2)}_{\hat{A},min}}\right\} + \min\left\{\sqrt{\chi^{(1)}_{\hat{B},min}}, \sqrt{\chi^{(2)}_{\hat{B},min}}\right\}. \tag{2.42}$$

Thus, we have found a new Entropic Uncertainty Relation. This relation relates the disturbance of two observables of a system, one of which is measured. Its important to say that there is not a similar relation in the literature, and because of this we need to associate it to a new statement, namely, a new Uncertainty Principle. We can say that:

It is impossible to measure an observable without disturbing simultaneously its statistic distribution and the statistic distribution of other observable.

Therefore we have found an Uncertainty Relation using the four postulates of quantum mechanics, Postulates 1, 2, 3 and 4. One of its most important properties is that this is an Uncertainty Relation comparing probability distributions.

Once we have shown the most important result of our work, we want to emphasize the role of $\delta(|c_j|^2)$ and $\delta(|d_j|^2)$ in the above relations. This quantities are small and they never are zero; its size depends entirely on the quantum sytem and the observer. It is physically impossible that this numbers are zero because it is impossible (until now) to know with total accuracy the state of the system, or, equivalently, to obtain the same probability distribution as the expected.

In this Chapter there were proposed two new Entropic Uncertainty Relations, to which the four postulates of Quantum Mechanics are utterly important. On the one hand, we have that the important components of said relations are, first, that the measure associated to the uncertainty is the Jensen-Shannon Entropy, which in turn is a probabilistic metric. In consequence what we are comparing, or measuring, are probability distrubutions of quantum systems. The second component is the fact that we tried to find a lower bound for the uncertainty of the system which does not depend on the state of the system. On the other hand, we proposed a new Uncertainty Principle related to our Uncertainty Relation. To the best of our knowledge this new statement is not found on the literature. We can say that our new Uncertainty Principle and Uncertainty Relation talk about the disturbance that one caused when one interacts with a quantum system.

Chapter 3

Conclusions

In this work we have seen a new proposal of an Entropic Uncertainty Relation completely related with the act of measurement, which is a fact that we can not exclude of any quantum treatment of a system. Also, we propose a new Uncertainty Principle associated with our Uncertainty Relation, somethig that is very important to establish a complete treatment to the topic.

This Entropic Uncertainty Relation of Disturbance-Disturbance is established with the use of the four Postulates of Quantum Mechanics, specially, the fourth postulate is very important to our relation because it depends completely on the state obtained when one makes a projective measurement. The intermediate result, that we called the *Jensen Shannon Uncertainty Relation for Disturbance* was shown and it is by itself a well established uncertainty relation, which can be utilized in a variety of escenaries. Also, we introduce an example of the use of our Entropic Uncertainty Relation.

In the future we plan to give to our relation a more general use, in particular we are interested in the case of the canonical observables, position and momentum. Also we want to show more examples of our relation on many physical systems. It is important to mention that we are working on a new paper where we will show a quantum treatment of a series of rotated SGE magnets.

The lack of adequate measures of uncertainty has made the area of Entropic Uncertainty Relations grow and actually there is a growing number of attemps to relate it with fundamental physics, in particular with Quantum Mechanics and the Quantum Mechanical Uncertainty Relations in their diverse presentations. A main topic in this thesis is the correct understanding of the difference between the Uncertainty Relations and Uncertainty

Principles, because it plays a central role in the basic courses of Quantum Mechanics, therefore it is important to present a comprehensive overview to undergraduate students.

The uncertainty in Quantum Mechanics must be understood as something inherent to the theory and practice due to our inability to interact with matter in its simplest expression.

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