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Permutation notations for the exceptional Weyl group F_4

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Permutation notations for the exceptional Weyl group F_4

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This paper describes a permutation notation for the Weyl groups of type F_4 and G_2. The image in the permutation group is presented as well as an analysis of the structure of the group. This description enables faster computations in these Weyl groups which will prove useful for a variety of applications.

1. Introduction

Weyl groups, or finite Coxeter groups, are widely used in mathematics and in applications (some examples are given in Section 2). They are most commonly represented by generators and relations. The disadvantage of that representation is that elements are not uniquely represented by strings or even minimal strings of generators. For the classical Weyl groups combinatorialists use one-line permutation notation, which corresponds to the orbit of the standard basis vectors under the Weyl group. This combinatorial representation provides unique representation, which makes it efficient for computation (see [Haas and Helminck 2012]). Many properties of the elements, such as length and order, can be quickly read from the combinatorial representation (see, for example, [Haas et al. 2007]). Further, the unique representation provides insight into more complex structures such as involution and twisted involution posets; see [Haas and Helminck 2011].

For the exceptional Weyl groups of type G_2, F_4, E_7 and E_8, the orbit of the standard basis vectors includes not just the positive and negative axes but additional vectors, making description by permutation somewhat less obvious. Nonetheless,

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similar representations can be made in these cases as well. In this paper we give a permutation representation for the Weyl group of type F_4 and discuss a number of properties of this representation. We also give a similar presentation for G_2.

2. Motivation

Given a field k, symmetric k-varieties are the homogenous spaces G/H, where G is the set of k-rational points of reductive group $\overline{G}$ defined over k and H the set of k-rational points of the set of fixed points of an automorphism σ (defined over k) of the group $\overline{G}$. For k the real or p-adic numbers these are also known as reductive symmetric spaces. These symmetric k-varieties have a detailed fine structure of root systems and Weyl groups, similar to that of the group G itself. This fine structure involves 4 (restricted) root systems and Weyl groups. To study the structure of symmetric k-varieties one needs detailed descriptions of this fine structure and how they act on the various types of elements of these root systems and Weyl groups. For example, to study the representations associated with these symmetric k-varieties one needs a detailed description of the orbits of (minimal) parabolic k-subgroups acting on these symmetric k-varieties. A characterization of these orbits was given in [Helminck and Wang 1993]. They showed that these orbits can be characterized by $\bigcup_{i \in I} W_G(A_i)/W_H(A_i)$, where $\{A_i \mid i \in I\}$ is a set of representatives of the H-conjugacy classes of the σ-stable maximal k-split tori, $W_G(A_i)$ is the set of Weyl group elements that have a representative in $N_G(A_i)$, the normalizer of A_i in G, and $W_H(A_i)$ is the set of Weyl group elements that have a representative in $N_H(A_i)$. To fully classify these orbits one needs to compute the subgroups $W_H(A_i)$ of $W_G(A_i)$. This requires a detailed analysis of the structure of the Weyl groups and their subgroups.

Another example is that the classification of Cartan subspaces can be reduced to a classification of $W_H(A)$-conjugacy classes of σ-singular involutions. The $W_G(A)$-conjugacy classes of involutions were classified in [Helminck 1991]. A detailed analysis of the Weyl groups and their subgroups will enable one to determine how a $W_G(A)$-conjugacy class breaks up in $W_H(A)$-conjugacy classes. There are many other problems related to symmetric k-varieties for which one needs a detailed description of the various Weyl groups and their subgroups. The detailed combinatorial analysis of the structure of the Weyl groups of types F_4 and G_2 in this paper enables us to compute the necessary data to solve those problems for those symmetric k-varieties that have a restricted Weyl group of type F_4 and G_2.

The classical text on Weyl groups is [Bourbaki 2002], while a good modern treatment to Weyl groups and their uses in Lie theory can be found in [Humphreys 1972]. The Weyl groups of type F and G are two of the exceptional Coxeter groups; see [Humphreys 1990] for a basic treatment of these groups.

3. The Weyl group of type F_4

The root system of type F_4 has the following characteristics. There are $n = 48$ roots. The usual basis is the set

$$\left\{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\}.$$

The complete set of roots is $\{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$. The positive roots are $\{e_i, e_i \pm e_j, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. Recall the associated Weyl group is generated by the reflections over the hyperplanes orthogonal to the basis roots. These are usually denoted s_{α_i}, which we abbreviate to s_i. Here we label the short positive roots with the numbers 1- 12 and describe how the Weyl group of type F_4 is associated with a subgroup of the permutation group on $[-12, \ldots, 12]$. I.e., each element in $W(F_4)$ will be associated with a signed permutation on $\{1, \ldots, 12\}$.

To begin compute the images of the short roots under the basis. These are given in Table 1. Each column in the table describes where each root goes under each basis reflection, so when read from top to bottom, the column gives one-line notation for the generators of the Weyl group. These generators are

$$\begin{aligned}
s_1 &= (1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12),\\
s_2 &= (1, 2, 4, 3, 5, 6, 8, 7, 9, 10, 12, 11),\\
s_3 &= (1, 2, 3, -4, 12, 11, 10, 9, 8, 7, 6, 5),\\
s_4 &= (9, 10, 11, 12, -5, 6, 7, 8, 1, 2, 3, 4).
\end{aligned}$$

	root r	$s_{\alpha_1}(r)$	$s_{\alpha_2}(r)$	$s_{\alpha_3}(r)$	$s_{\alpha_4}(r)$
1	e_1	1	1	1	9
2	e_2	3	2	2	10
3	e_3	2	4	3	11
4	e_4	4	3	-4	12
5	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$	5	5	12	-5
6	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$	7	6	11	6
7	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4)$	6	8	10	7
8	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$	8	7	9	8
9	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	9	9	8	1
10	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$	11	10	7	2
11	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$	10	12	6	3
12	$\frac{1}{2}(e_1 - e_2 - e_3 + e_4)$	12	11	5	4

Table 1. Generators of $W(F_4)$.

In cycle notation, they can be expressed as products of transpositions as follows:

$$s_1 = (2,3)(6,7)(10,11), \qquad s_3 = (8,9)(7,10)(6,11)(5,12)(4,-4),$$
$$s_2 = (3,4)(7,8)(11,12), \qquad s_4 = (1,9)(2,10)(3,11)(4,12)(5,-5).$$

Note that the elements of $W(F_4)$ are in one-to-one correspondence with only a subset of signed permutations on $[1,\ldots,12]$. In particular, since the first 4 elements give the image of the standard basis of $\mathbb{R}^4$, they determine the other 8 positions uniquely. In Section 3.10 we will see that there are further restrictions on what can occur in the first four places.

3.1. ***A minimal word algorithm.*** We develop a method for the important task of converting from this one-line notation to the standard representation of an element as a minimal word. For $x \in W(F_4)$, recall that the length of x, $l(x)$, is the number of letters in the minimal word of x. It is well-known that the length of x equals the number of positive roots mapped to negative roots by x.

Lemma 3.2. *Any nontrivial element of $W(F_4)$ maps at least one of e_4, $e_2 - e_3$, $e_3 - e_4$, and $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ to a negative root.*

Proof. This set of roots is exactly the set of roots which get mapped to negative roots under the basis reflections. □

Lemma 3.3. *Let $x \in W(F_4)$. Then x maps α_i to a negative root if and only if $l(xs_i) < l(x)$.*

Proof. This follows directly from the definitions. □

Algorithm 3.4. Given an element $x = (a_1, a_2, \ldots, a_{12}) \in W(F_4)$, the following algorithm will output a minimal word for x.

1. If all $a_i > 0$, go to step 6. Otherwise, go to step 2.
2. If $a_4 < 0$, right multiply by s_3 and go to step 1. Otherwise, go to step 3.
3. If $a_5 < 0$, right multiply by s_4 and go to step 1. Otherwise, go to step 4.
4. If $a_3 < 0$, right multiply by s_2 and go to step 1. Otherwise, go to step 5.
5. Right multiply by s_1 and go to step 1.
6. If the resulting element is not the identity, compare it to the following list in order to determine the final step(s).
   (a) $\{1,3,2,4,5,7,6,8,9,11,10,12\} = s_1$.
   (b) $\{1,2,4,3,5,6,8,7,9,10,12,11\} = s_2$.
   (c) $\{1,3,4,2,5,7,8,6,9,11,12,10\} = s_2 s_1$.
   (d) $\{1,4,2,3,5,8,6,7,9,12,10,11\} = s_1 s_2$.
   (e) $\{1,4,3,2,5,8,7,6,9,12,11,10\} = s_1 s_2 s_1$.

Theorem 3.5. *Algorithm 3.4 produces a minimal word for x.*

Proof. Note that even if $a_i > 0$ for all i, the length of x may not be zero because not all positive roots are represented in the list of twelve roots. In particular, there are six elements of $W(F_4)$ such that $a_i > 0$ for all i. They are precisely those listed in Step 6. of the algorithm together with the identity. Clearly steps 2 and 3 reduce the length of x. If we arrive at step 4, i.e., $a_4 > 0$, and $a_3 < 0$, then one can check that $e_3 - e_4$ maps to a negative root under x, so multiplying by s_2 will reduce the length of x.

If we arrive at step 5, i.e., $a_3, a_4, a_5 > 0$, but some other a_i is negative, then we show that a_2 must be negative. Suppose instead that $a_2 > 0$. Let $\langle i\rangle + \langle j\rangle$ denote the root which is the vector sum of roots i and j. E.g., $\langle 5\rangle = \frac{1}{2}(\langle 1\rangle - \langle 2\rangle - \langle 3\rangle - \langle 4\rangle)$ and since x is a linear map this implies $\langle a_5\rangle = \frac{1}{2}(\langle a_1\rangle - \langle a_2\rangle - \langle a_3\rangle - \langle a_4\rangle)$. Rearranging gives $\langle a_1\rangle = \langle a_2\rangle + \langle a_3\rangle + \langle a_4\rangle + 2\langle a_5\rangle$, with all terms on the right positive by assumption. Therefore, $a_1 > 0$. Similar calculations done in the correct order show that all other a_i must be positive. Explicitly: $\langle a_{10}\rangle = \langle a_5\rangle + \langle a_2\rangle$; $\langle a_{12}\rangle = \langle a_5\rangle + \langle a_4\rangle$; $\langle a_{11}\rangle = \langle a_5\rangle + \langle a_3\rangle$; $\langle a_6\rangle = \langle a_{12}\rangle + \langle a_3\rangle$; $\langle a_7\rangle = \langle a_{12}\rangle + \langle a_2\rangle$; $\langle a_8\rangle = \langle a_{10}\rangle + \langle a_3\rangle$; $\langle a_9\rangle = \frac{1}{2}(\langle a_1\rangle + \langle a_2\rangle + \langle a_3\rangle + \langle a_4\rangle)$.

Thus $a_2 < 0$. In this case $e_2 - e_3$ will be mapped to a negative root, so right multiplication by s_1 will reduce the length.

If we arrive at step 6 then all $a_i > 0$. Clearly these must be products of s_1 and s_2 only. The 5 elements listed above plus the identity are all the possibilities. $\square$

One can determine the length of any $x \in W(F_4)$ by finding a reduced word as above. In what follows we give a combinatorial description of length. Partition the short roots of F_4 into the three sets

$$\alpha = \{\pm 1, \pm 2, \pm 3, \pm 4\}, \quad \beta = \{\pm 5, \pm 6, \pm 7, \pm 8\}, \quad \gamma = \{\pm 9, \pm 10, \pm 11, \pm 12\}.$$

Lemma 3.6. *For all $x \in W(F_4)$, $\{x(\alpha), x(\beta), x(\gamma)\} = \{\alpha, \beta, \gamma\}$. In other words, x permutes the sets α, β and γ.*

Theorem 3.7. *For an element $x = (a_1, a_2, \cdots, a_{12})$, define*

$$N(x) = |\{i : a_i < 0\}|$$

and

$$p(a_i, a_j) = \begin{cases} 0 & \text{if } |a_i| < |a_j| \text{ and } a_i > 0, \\ 2 & \text{if } |a_i| < |a_j| \text{ and } a_i < 0, \\ 1 & \text{if } |a_i| > |a_j|. \end{cases}$$

Find k such that $\{\pm a_{4k+1}, \pm a_{4k+2}, \pm a_{4k+3}, \pm a_{4k+4}\} = \alpha$. If $k = 1$,

$$l(x) = \sum_{i>j} p(a_{4k+i}, a_{4k+j}) + N(x).$$

Otherwise,

$$l(x) = \sum_{i<j} p(a_{4k+i}, a_{4k+j}) + N(x).$$

Proof. The length counts the number of positive roots mapped to negative roots under x. The function $N(x)$ counts the number of short roots mapped to negative roots, while the $p(a_{4k+i}, a_{4k+j})$ terms account for the number of long roots mapped to negative roots. There are three cases depending on which set is mapped to α.

Suppose $x(\alpha) = \alpha$. Each of the positive long roots $e_i \pm e_j$, $i < j$ is the sum or difference of the roots $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$; where the difference is taken as $\langle i\rangle - \langle j\rangle$ where $i < j$. Thus to determine which of these is mapped to a negative long root, we need only consider the sum and difference of $\langle a_i\rangle$ for $i = 1, \dots, 4$. It is easy to check that $\langle a_i\rangle + \langle a_j\rangle$ is a negative root exactly when either $|a_i| > |a_j|$ and $a_j < 0$ or when $|a_i| < |a_j|$ and $a_i < 0$. As well, $\langle a_i\rangle - \langle a_j\rangle$ is negative exactly when $|a_i| < |a_j|$ and $a_i < 0$ or when $|a_i| > |a_j|$ and $a_j > 0$.

Suppose $x(\beta) = \alpha$. Each of the positive long roots $e_i \pm e_j$, $j < i$ is the sum or difference of the roots $\langle 5\rangle$, $\langle 6\rangle$, $\langle 7\rangle$, $\langle 8\rangle$ where the difference is taken as $\langle i\rangle - \langle j\rangle$ where $j < i$. With this reversed order the same conditions for when $\langle a_i\rangle + \langle a_j\rangle$ and $\langle a_i\rangle - \langle a_j\rangle$ are negative will still hold.

Suppose $x(\gamma) = \alpha$. Each of the positive long roots $e_i \pm e_j$, $i < j$ is the sum or difference of the roots $\langle 9\rangle$, $\langle 10\rangle$, $\langle 11\rangle$, $\langle 12\rangle$; where the difference is taken as $\langle i\rangle - \langle j\rangle$ where $i < j$. Again the same conditions hold. $\square$

3.8. ***Group structure and notation properties.*** It is also useful to consider the images of the three sets of short roots as permutations in signed S_4. Refer to the elements in positions 1-4 as set A, the elements in positions 5-8 as set B, and the elements in positions 9-12 as set C. Formally, let $f \in F_4$ such that $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12})$. For $1 \le i \le 4$ let $a_i = f_i \pmod 4$, using the representatives $\{1, 2, 3, 4\}$ for $\mathbb{Z}_4$. Similarly, $b_i = f_{4+i} (\text{mod } 4)$, and $c_i = f_{8+i} \pmod 4$, for $1 \le i \le 4$ again using the representatives $\{1, 2, 3, 4\}$ for $\mathbb{Z}_4$. Let $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$, and $C = (c_1, c_2, c_3, c_4)$. Denote $(|a_1|, |a_2|, |a_3|, |a_4|)$ by $|A|$, and define $|B|$ and $|C|$ analogously. For example, if $f = (6, -8, 5, -7, 9, 11, -10, 12, -2, 4, 1, 3)$, then $|A| = (2, 4, 1, 3)$, $|B| = (1, 3, 2, 4)$, and $|C| = (2, 4, 1, 3)$.

Theorem 3.9. *The parity of the negations in each block, given the order of the sets α, β, and γ, is the following*:

set order	block A (1–4)	block B (5–8)	block C (9–12)
$\alpha\beta\gamma$	even	even	even
$\alpha\gamma\beta$	odd	even	even
$\beta\alpha\gamma$	odd	odd	odd
$\beta\gamma\alpha$	even	odd	odd
$\gamma\alpha\beta$	odd	even	odd
$\gamma\beta\alpha$	even	odd	even

Proof. Note that generators s_1 and s_2 do not change the parity of negations in any set, nor do they change the order of the sets. Therefore it suffices to inductively show that this table holds after operating by generators s_3 and s_4 on the right. It is simple to compute using the following rules. s_3 swaps the second and third blocks, and adds or subtracts one negative from the first block. s_4 swaps the first and third blocks, and adds or subtracts one negative from the second block. □

3.10. ***Restrictions on the values of*** $|a_i|$**.** Let $\mathbb{V}$ be the subset of S_4 generated by $(12)(34)$ and $(13)(24)$, and K be the subset of S_4 generated by (23) and (34).

For $X \in S_4$ define $v(X)$ to be the unique element of $\mathbb{V}$ in the coset KX.

Theorem 3.11. *Let $f \in W(F_4)$ with sets A_f, B_f and C_f as defined above. Then $|C_f| = |B_f|v(|A_f|)$, $|B_f| = |A_f|v(|C_f|)$, and $|A_f| = |C_f|v(|B_f|)$.*
Alternative statement: *Let $f \in W(F_4)$ with sets f_α, f_β, and f_γ as defined above. Then $f_\gamma = f_\beta v(f_\alpha)$, $f_\beta = f_\alpha v(f_\gamma)$, and $f_\alpha = f_\gamma v(f_\beta)$.*

Proof. We proceed by induction. The statement is true for $f = identity$. Assume its true for f, we show its true for $s_i f$ for each s_i. Note that $v((s_i f)_\mu) = v(f_\mu)$ when $i = 1, 2, 4$ and $\mu = \alpha, \beta, \gamma$; $v((s_3 f)_\alpha) = v(f_\alpha)$, $v((s_3 f)_\beta) = (14)(23)v(f_\gamma)$ and $v((s_3 f)_\gamma) = (14)(23)v(f_\beta)$. The cases for s_i where $i \neq 3$ are straightforward.

Furthermore, $(s_3 f)_\beta = (14)(23) f_\gamma$ and $(s_3 f)_\gamma = (14)(23) f_\beta$. These equations provide all of the required components for the proof. For example assume $f_\beta = f_\gamma v(f_\alpha)$. Since $v((s_3 f)_\alpha) = v(f_\alpha)$ and $(s_3 f)_\gamma = (14)(23) f_\beta$ and $(s_3 f)_\beta = (14)(23) f_\gamma$ it follows that $(s_3 f)_\gamma = (s_3 f)_\beta v((s_3 f)_\alpha)$. □

3.12. ***$W(F_4)$ as a semidirect product.*** Let F_D denote the subgroup of $W(F_4)$ containing all $d \in F_D$ where $d(\alpha) = \alpha$, $d(\beta) = \beta$, and $d(\gamma) = \gamma$, and let F_S be the subgroup of $W(F_4)$ generated by the generators s_3 and s_4. Let T be the group representing the order of the sets α, β and γ. Define $\tau : W(F_4) \mapsto T$ in the obvious way. Note that $\tau(f) = id$ if and only if $f \in F_D$. Now by Theorem 3.9, the sets α, β and γ occur in order $\alpha\beta\gamma$ in the bottom row notation of f if and only if the permutation A_f contains an even number of negative signs.

Lemma 3.13. *F_D is isomorphic to D_4.*

Proof. The map $\psi : F_D \to D_4$ such that $\psi(f) = A_f$ for $f \in F_D$ provides the isomorphism. □

Theorem 3.14. $W(F_4) = F_D \rtimes F_S$.

Proof. We can represent $f \in W(F_4)$ by a pair (d, s) where $f = ds$ and s is the unique element of F_S such that $\tau(s) = \tau(f)$. Define $\phi_s : F_D \mapsto F_D$ where $\phi_s(d) = sds^{-1}$ for $d \in F_D$ and $s \in F_S$. One can check that if $f_1 = d_1 s_1$, represented by the pair (d_1, s_1), and $f_2 = d_2 s_2$, represented by the pair (d_2, s_2), then $f_1 f_2 = d_1 \phi_{s_1}(d_2) s_1 s_2$, represented by the pair $(d_1 \cdot \phi_{s_1}(d_2), s_1 \cdot s_2)$. □

One might hope that this semidirect product would provide an efficient notation for computation in $W(F_4)$. A road block to this seems to be finding a combinatorial description of the multiplication.

4. The Weyl group of type G_2

The root system of type G_2 has the following characteristics. There are $n = 12$ roots. The usual basis is the set $\{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$. The complete set of roots is $\{\pm(e_i - e_j)\}$, where $i < j$ and $i, j \in \{1, 2, 3\}$, and $\{\pm(2e_i - e_j - e_k)\}$, where $\{i, j, k\} = \{1, 2, 3\}$. The positive roots are $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. Again we let s_i denote the reflection over the hyperplane orthogonal to α_i. We label the short positive roots $2\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2$, and α_1, with the numbers 1-3 respectively, and describe how the Weyl group of type G_2 is associated with a subgroup of the permutation group on $[-3, \dots, 3]$. Here are the images of roots 1, 2, and 3 under the generators of $W(G_2)$:

	root r	$s_{\alpha_1}(r)$	$s_{\alpha_2}(r)$
1	$-e_2 + e_3$	2	1
2	$-e_1 + e_3$	1	3
3	$e_1 - e_2$	-3	2

Reading from top to bottom in each column gives one-line notation for the generators, namely $s_1 = (2, 1, -3)$ and $s_2 = (1, 3, 2)$.

As with $W(F_4)$ we can give a simple combinatorial length formula for $W(G_2)$.

Theorem 4.1. *The length of an element $x = (a_1, a_2, a_3)$ in $W(G_2)$ is given by $l(x) = \sum_{i<j} p(a_i, a_j)$ where $p(a_i, a_j)$ is defined as follows*:

$$p(a_i, a_j) = \begin{cases} 0 & \text{if } |a_i| < |a_j| \text{ and } a_i > 0, \\ 2 & \text{if } |a_i| < |a_j| \text{ and } a_i < 0, \\ 1 & \text{if } |a_i| > |a_j|. \end{cases}$$

Proof. The length counts the number of positive roots mapped to negative roots under x. One can check that the positive roots in $W(G_2)$ are of the form $\langle i\rangle \pm \langle j\rangle$ where $i < j$ and $i, j \in \{1, 2, 3\}$. To determine which of $\langle i\rangle \pm \langle j\rangle$ are mapped to negative roots, we need to determine when $\langle a_i\rangle \pm \langle a_j\rangle$ is a negative root. One can check that $\langle a_i\rangle + \langle a_j\rangle$ is negative when $|a_i| < |a_j|$ and $a_i < 0$, or when $|a_i| > |a_j|$ and $a_j < 0$. Similarly $\langle a_i\rangle - \langle a_j\rangle$ is negative when $|a_i| < |a_j|$ and $a_i < 0$, or when $|a_i| > |a_j|$ and $a_j > 0$. $\square$

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