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On torus homeomorphisms semiconjugate to irrational rotations

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Abstract. In the context of the Franks–Misiurewicz conjecture, we study homeomorphisms of the two-torus semiconjugate to an irrational rotation of the circle. As a special case, this conjecture asserts uniqueness of the rotation vector in this class of systems. We first characterize these maps by the existence of an invariant 'foliation' by essential annular continua (essential subcontinua of the torus whose complement is an open annulus) which are permuted with irrational combinatorics. This result places the considered class close to skew products over irrational rotations. Generalizing a well-known result of Herman on forced circle homeomorphisms, we provide a criterion, in terms of topological properties of the annular continua, for the uniqueness of the rotation vector. As a byproduct, we obtain a simple proof for the uniqueness of the rotation vector on decomposable invariant annular continua with empty interior. In addition, we collect a number of observations on the topology and rotation intervals of invariant annular continua with empty interior.

1. *Introduction*

Rotation theory, as a branch of dynamical systems, goes back to Poincaré's celebrated classification theorem for circle homeomorphisms. It states that given an orientation-preserving circle homeomorphism f with lift $F : \mathbb{R} \to \mathbb{R}$, the limit

$$\rho(F) = \lim_{n\to\infty} (F^n(x) - x)/n,$$

called the *rotation number* of F, exists and is independent of x. Furthermore, $\rho(F)$ is rational if and only if f has a periodic orbit and $\rho(F)$ is irrational if and only if f is semiconjugate to an irrational rotation.

Since both cases of the above dichotomy are easy to analyse, this result provides a complete description of the possible long-term behaviour for a whole class of systems without any additional *a priori* assumptions, a situation which is still rare even nowadays in the theory of dynamical systems. In addition, the rotation number can be viewed as an element of the first homological group of the circle and thus provides a link between the dynamical behaviour of homeomorphisms and the topological structure of the manifold. It is not surprising that the consequences of this result have found numerous applications in the sciences, ranging from quantum physics to neural biology [**1**, **2**]. Hence, the attempt to apply this approach to higher-dimensional manifolds, in order to obtain a classification of possible dynamics in terms of rotation vectors and rotation sets, is most natural. However, despite impressive contributions over the last few decades, fundamental problems still remain open even in dimension two.

Already in the case of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, a unique rotation vector does not have to exist. Instead, given a torus homeomorphism f homotopic to the identity and a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$, the *rotation set* is defined as

$$\rho(F) = \big\{\rho \in \mathbb{R}^2 \mid \exists z_i \in \mathbb{R}^2,\ n_i \nearrow \infty : \lim_{i\to\infty} (F^{n_i}(z_i) - z_i)/n_i = \rho\big\}.$$

This is always a compact and convex subset of the plane [**3**]. Consequently, three principal cases can be distinguished according to whether the rotation set (1) has non-empty interior, (2) is a line segment of positive length or (3) is a singleton, that is, f has a unique rotation vector. Existing results on each of the three cases suggest that a classification approach is indeed feasible: for example, in case (1) the dynamics are ‘rich and chaotic’, in the sense that the topological entropy is positive [**4**] and all of the rational rotation vectors in the interior of $\rho(F)$ are realized by periodic orbits [**5**]; in case (3) a Poincaré-like classification exists under the additional assumption of area preservation and a certain *bounded mean motion* property [**6**], and the consequences of *unbounded mean motion* are being explored recently as well [**7**–**9**]. In the case where the rotation set is a segment of positive length, examples can be constructed whose rotation set is either (a) a segment with rational slope and infinitely many rational points or (b) a segment with irrational slope and one rational endpoint [**10**]. Recent results on torus homeomorphisms with this type of rotation segment indicate that these examples can be seen as good models for the general case [**11**–**13**]. In addition, there exist many further results that provide more information on each of the three cases. Just to mention some of the contributions in this direction, we refer to [**14**–**19**].

In the light of these advances, it seems reasonable to say that the outline of a complete classification emerges. Yet, there is still a major blank spot in the current state of knowledge. It is not known whether any rotation segment other than the two cases (a) and (b) mentioned above can occur, and if so, hardly anything is known about the dynamical consequences of rotation segments of such exceptional type. Actually, it was conjectured by Franks and Misiurewicz in [**10**] that these cannot occur. However, while this conjecture has been the focus of attention for more than two decades, it has defied all experts and to date there are still only very partial results on the problem. A deeper reason for this may lie in the fact that it concerns dynamics without any periodic points, in particular in the case where the rotation segments do not contain any rational points†, and therefore many standard techniques in topological dynamics based on the existence of periodic orbits fail to apply. Independent of whether the conjecture is true or false, this highlights the need for a better understanding of periodic-point-free dynamics, which seems a worthy task in a broader context as well.

† Note that a periodic orbit always has a rational rotation vector.

We believe that in this situation the systematic investigation of suitable subclasses of periodic-point-free torus homeomorphisms is a good way to obtain further insight. In fact, there are some classes that have been studied intensively already. First, Franks and Misiurewicz proved that for time-one maps of flows the rotation set is either a singleton or an interval of type (a) or (b) [**10**]. Second, Kwapisz considered torus homeomorphisms that preserve the leaves of an irrational foliation and showed that the rotation set is either a segment with a rational endpoint or a singleton [**20**]. Finally, for skew products over irrational rotations on the torus, Herman proved the uniqueness of the rotation vector [**21**]. Hence, in these cases the conjecture was confirmed for the particular subclasses, which are certainly very restrictive compared with general torus homeomorphisms. However, since these are the only existing partial results on the problem, they are the only obvious starting point for further investigations. The aim of this article is to make a first step in this direction by studying torus homeomorphisms which are semiconjugate to a one-dimensional irrational rotation. For obvious reasons these do not have any periodic orbits, but apart from this little is known about the dynamical implications of this property. We first provide an analogous characterization of these systems.

Denote by $\mathrm{Homeo}_0(\mathbb{T}^d)$ the set of homeomorphisms of the d-dimensional torus that are homotopic to the identity. Recall that an *essential annular continuum* $A \subseteq \mathbb{T}^2$ is a continuum whose complement $\mathbb{T}^2 \setminus A$ is homeomorphic to the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$. An *essential circloid* is an essential annular continuum which is minimal with respect to inclusion amongst all essential annular continua. We refer to §2 for the corresponding definitions in higher dimensions. Note that for any family of pairwise disjoint essential continua in $\mathbb{T}^d$ there exists a natural circular order. We say a wandering† essential continuum has irrational combinatorics (with respect to $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$) if its orbit is ordered in $\mathbb{T}^d$ in the same way as the orbit of an irrational rotation on $\mathbb{T}^1$. See §3 for more details.

THEOREM 1. *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$. Then the following statements are equivalent:*
(i) *f is semiconjugate to an irrational rotation R of the circle;*
(ii) *there exists a wandering essential circloid with irrational combinatorics;*
(iii) *there exists a wandering essential continuum with irrational combinatorics;*
(iv) *there exists a semiconjugacy h from f to R such that for all $\xi \in \mathbb{T}^1$ the fibre $h^{-1}\{\xi\}$ is an essential annular continuum.*

The proof is given in §3. Issues concerning the uniqueness of the semiconjugacy in the above situation are discussed in §4. In general, the semiconjugacy is not unique, but there exist important situations where it is unique up to post-composition with a rotation. In this case every semiconjugacy has only essential annular continua as fibres.

For the two-dimensional case, the implication '(iii) $\Rightarrow$(i)' in Theorem 1 is contained in [**22**], and the proof easily extends to higher dimensions. In our context, the most important fact will be the equivalence '(i) $\Leftrightarrow$ (iv)', which says that the semiconjugacy can always be chosen such that its fibres are annular continua. This places the considered systems very close to skew products over irrational rotations, with the only difference that the

† We call $A \subseteq \mathbb{T}^d$ *wandering*, if $f^n(A) \cap A = \emptyset$ for all $n \geq 1$.

topological structure of the fibres can be more complicated. For this reason, one may hope to generalize Herman's result to this larger class of systems, thus proving the existence of a unique rotation vector. To that end, however, we here have to make an additional assumption on the topological regularity of the fibres of the semiconjugacy.

An essential annular continuum $A \subseteq \mathbb{T}^2$ admits essential simple closed curves in its complement. The homotopy type of such curves is unique, and we define it to be the *homotopy type of* A. We say A is *horizontal* if its homotopy type is $(1, 0)$. Given a horizontal annular continuum A, we denote by $\widehat{A}$ a connected component of $\pi^{-1}(A)$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection. Let $T : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x + 1, y)$. Then we say A is *compactly generated* if there exists a compact connected set $G_0 \subseteq \widehat{A}$ such that $\widehat{A} = \bigcup_{n\in\mathbb{Z}} T^n(G_0)$. In this case G_0 is called a *compact generator* of A. An essential annular continuum with arbitrary homotopy type is said to be compactly generated if there exists a homeomorphism of $\mathbb{T}^2$ which maps it to a compactly generated horizontal one.

THEOREM 2. *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is semiconjugate to an irrational rotation of the circle and the semiconjugacy h is chosen such that its fibres $h^{-1}(\xi)$ are all essential annular continua. Further, assume that there exists a measurable set $\Omega \subseteq \mathbb{T}^1$ of positive Lebesgue measure such that $h^{-1}\{\xi\}$ is compactly generated for all $\xi \in \Omega$. Then f has a unique rotation vector.*

The proof is given in §5.

Remark 1.1.

(i) We say an annular continuum is *thin* if it has empty interior. Note that in the situation of Theorem 2, all but at most countably many of the fibres are thin in this sense.

(ii) Since the set Ω is measurable and compactly generated fibres are mapped to compactly generated ones, ergodicity of the irrational rotation implies that almost all fibres have this property.

(iii) A thin annular continuum A contains a unique circloid C_A (see [**6**, Lemma 3.4]). If the fibre $h^{-1}\{\xi\}$ of the semiconjugacy h over ξ is thin, we denote this circloid by C_ξ. It turns out that the assertion of Theorem 2 remains true if the fibres $h^{-1}(\xi)$ are replaced by the circloids C_ξ in the statement. This is not completely obvious, since in general a thin annular continuum A may not be compactly generated even if this is true for the circloid C_A it contains. Only the converse implication is true, as we show in Proposition 6.5.

However, in the situation of the theorem, it turns out that having a set of positive measure on which fibres are compactly generated is equivalent to having a set of positive measure on which the corresponding circloids are compactly generated. A precise statement is given in Proposition 6.7.

(iv) The notion of a compact generator is closely related to the more classical one of decomposability of an annular continuum. Here, a continuum C is called *decomposable* if it can be written as the union of two proper subcontinua. It is rather easy to show that the existence of a compact generator of a circloid C is equivalent to the fact that a lift of C to a finite covering of $\mathbb{T}^2$ is decomposable. Likewise, a compactly generated annular continuum always has a decomposable finite covering, although the converse is not true anymore in this case, even if the annular continuum is thin. However, we will not make use of these facts and just work with compact generators, which are most convenient for our purposes.

As a byproduct of our methods, we also obtain the uniqueness of the rotation vector for invariant compactly generated thin annular continua, thus obtaining a variation of a result by Barge and Gillette [**23**].

THEOREM 3. *Suppose* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *and* A *is a thin annular continuum that is compactly generated and* f*-invariant. Then* $f_{|A}$ *has a unique rotation vector, that is, there exists a vector* $\rho \in \mathbb{R}^2$ *such that* $\lim_{n\to\infty}(F^n(z) - z/n) = \rho$ *for all* $z \in \mathbb{R}^2$ *with* $\pi(z) \in A$. *Moreover, the convergence is uniform in* z.

Barge and Gillette stated the result for decomposable cofrontiers, which includes the case of thin circloids, but their argument can be adapted to thin annular continua without too much effort. Our proof is essentially a variation of theirs. An alternative proof by Le Calvez [**24**] uses Caratheodory's prime ends, which is a classical approach to study the rotation theory of continua [**25**–**27**].

It should be noted that there exist important examples of invariant thin annular continua which are not compactly generated. One example is the Birkhoff attractor [**28**], which does not have a unique rotation vector and therefore cannot have compact generator due to the above statement. Another well-known example is the pseudo-circle, which was constructed by Bing in [**29**] and latter shown to occur as a minimal set of smooth surface diffeomorphisms [**30**, **31**]. Whether pseudocircles admit dynamics with non-unique rotation vectors is still open.

We close by collecting some observations on the topology and dynamics of invariant thin annular continua in §6. It is known that any thin annular continuum A contains a unique circloid C_A (see Lemma 2.3). We show that if A is compactly generated, then so is the circloid C_A. Conversely, if C_A is compactly generated then either A is compactly generated as well or A contains at least one *infinite spike*, that is, an unbounded connected component of $A \setminus C_A$. Finally, reproducing some examples due to Walker [**32**] we show that thin annular continua can have any compact interval as rotation segment, even in the absence of periodic orbits.

2. *Notation and preliminaries*

The following notions are usually used in the study of dynamics on the two-dimensional torus or annulus. For convenience, we stick to the same terminology also in higher dimensions. We let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\mathbb{A}^d = \mathbb{T}^{d-1} \times \mathbb{R}$ the d-dimensional annulus. If $d = 2$, we simply write $\mathbb{A}$ instead of $\mathbb{A}^2$. We will often compactify $\mathbb{A}$ by adding two points $-\infty$ and $+\infty$, thus making it a sphere. As long as no ambiguities can arise, we will always denote canonical quotient maps such as $\mathbb{R} \to \mathbb{T}^1$, $\mathbb{R}^d \to \mathbb{T}^d$, $\mathbb{R}^d \to \mathbb{A}^d$ by π. Likewise, on any product space π_i denotes the projection to the ith coordinate. We call a subset $A \subseteq \mathbb{A}^d$ or $A \subseteq \mathbb{R}^d$ bounded from above (from below) if $\pi_d(A)$ is bounded from above (from below). By $\overline{A}$ or $\mathrm{cl}(A)$ we denote the closure of a set A. A closed set is called *thin* if it has empty interior.

We say a continuum (that is, a compact and connected set) $E \subseteq \mathbb{A}^d$ is *essential* if $\mathbb{A}^d \setminus E$ contains two unbounded connected components. In this case, one of these components will be unbounded above and bounded below, and we denote it by $\mathcal{U}^+(A)$. The second unbounded component will be bounded above and unbounded below, and we denote it by $\mathcal{U}^-(A)$. The set A is called an *essential annular continuum* if $\mathbb{A}^d \setminus A = \mathcal{U}^+(A) \cup \mathcal{U}^-(A)$. Note that in dimension two, one can show by using the Riemann mapping theorem that both unbounded components are homeomorphic to $\mathbb{A}$ and A is the intersection of a decreasing sequence of topological annuli. This is not true anymore in higher dimensions, but at least we have the following.

LEMMA 2.1. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of essential annular continua, then $A = \bigcap_{n\in\mathbb{N}} A_n$ is an essential annular continuum as well.*

Proof. As a decreasing intersection of essential continua, A is an essential continuum. Further, we have that $\mathbb{T}^d \setminus A = \bigcup_{n\in\mathbb{N}} \mathbb{T}^d \setminus A_n$ is the union of the two sets

$$U^+ = \bigcup_{n\in\mathbb{N}} \mathcal{U}^+(A_n) \quad \text{and} \quad U^- = \bigcup_{n\in\mathbb{N}} \mathcal{U}^-(A_n).$$

As the union of an increasing sequence of open connected sets is connected, both these sets are connected. Hence, $\mathbb{T}^d \setminus A$ consists of exactly two connected components $\mathcal{U}^+(A) = U^+$ and $\mathcal{U}^-(A) = U^-$, both of which are unbounded. □

Given a set $S \subseteq \mathbb{R}^d$, we say S is *horizontal* if $\pi_d(S)$ is bounded and $\mathbb{R}^d \setminus \overline{S}$ contains two different connected components $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$ whose image under π_d is unbounded. Note that in this case one of the two components, which we always denote by $\mathcal{U}^+(S)$, is bounded below whereas the other component, denoted by $\mathcal{U}^-(S)$, is bounded above. By definition, $\mathcal{U}^\pm(S)$ are always open sets. Similarly, given a set $B \subseteq \mathbb{A}^d$ bounded above (below) we denote by $\mathcal{U}^+(B)$ ($\mathcal{U}^-(B)$) the unique connected component of $\mathbb{A}^d \setminus \overline{B}$ which is unbounded above (below). The same notation is used on $\mathbb{R}^d$. A horizontal connected closed set S is called a *horizontal strip*, if $\mathbb{R}^d \setminus S = \mathcal{U}^+(S) \cup \mathcal{U}^-(S)$. Note that, thus, the lift of an essential annular continuum $A \subseteq \mathbb{A}^d$ to $\mathbb{R}^d$ is a horizontal strip. More generally, we say a *strip* is a set which can be obtained from a horizontal strip by a linear coordinate change.

In any d-dimensional manifold M, we say A is an *annular continuum* if it is contained in a topological annulus $\mathcal{A} \simeq \mathbb{A}^d$ and it is an essential annular continuum in the above sense when viewed as a subset of $\mathcal{A}$. In this situation, we say A is *essential* if essential loops in $\mathcal{A}$ are also essential in M. We call $C \subseteq \mathbb{A}^d$ an *essential circloid* if it is an essential annular continuum and does not contain any other essential annular continuum as a strict subset. Circloids in general manifolds are then defined in the same way as annular continua. Finally, we call a strip S *minimal* if it is a minimal element of the set of strips with the partial ordering by inclusion.

LEMMA 2.2. *An annular essential continuum in $\mathbb{A}^d$ or $\mathbb{T}^d$ is a circloid if and only if its lift to $\mathbb{R}^d$ is a minimal strip.*

Proof. We give the proof for $\mathbb{A}^d$, the case of $\mathbb{T}^d$ is more or less the same. Let $C \subseteq \mathbb{A}^d$ be a circloid and denote its lift to $\mathbb{R}^d$ by S. Suppose S' is a closed connected strict subset of S. Then there exists $x \in S$ and $\delta \in (0, 1/4)$ such that $B_\delta(x) \cap S' = \emptyset$. Let $x_0 = \pi(x)$. Then $C' = C \setminus B_\delta(x_0)$ is non-essential, and we can find a proper curve $\gamma : \mathbb{R} \to \mathbb{A}^d$ in C' that goes from $-\infty$ to $+\infty$, that is, $\lim_{t\to\pm\infty} \pi_d \circ \gamma(t) = \pm\infty$. Further, we may assume that γ takes values in $B_\delta(x_0)$ only on a single open interval. This allows us to choose a suitable lift $\widehat{\gamma} : \mathbb{R} \to \mathbb{R}^d$ of γ that takes values in $B_\delta(x)$, but not in $B_\delta(x) + (0, n)$ for any $n \in \mathbb{Z} \setminus \{0\}$. Then $\widehat{\gamma}$ is a proper curve in the complement of S' connecting $\mathcal{U}^-(S)$ and $\mathcal{U}^+(S)$, and therefore S' cannot be a strip. This shows that the lift of a circloid is a minimal strip. The converse implication is proved in a similar way. □

LEMMA 2.3. [**6**, Lemma 3.4] *Every thin annular continuum $A \subseteq \mathbb{A}^d$ contains a unique circloid C_A, which is given by*

$$C_A = \overline{\mathcal{U}^+(A)} \cap \overline{\mathcal{U}^-(A)}. \tag{2.1}$$

The same statement applies to thin strips and to thin annular continua in $\mathbb{T}^d$.

The proof in [**6**] is given for essential annular continua and for $d = 2$, but it literally goes through in higher dimensions and for strips. The same is true for the following result, which describes an explicit construction to obtain essential circloids from arbitrary essential continua. Given an essential set $A \subseteq \mathbb{R}^d$ which is bounded above, we write $\mathcal{U}^{+-}(A)$ instead of $\mathcal{U}^-(\mathcal{U}^+(A))$ and use analogous notation for other concatenations of these procedures.

LEMMA 2.4. [**6**, Lemma 3.2] *If $A \subseteq \mathbb{A}^d$ is an essential continuum, then*

$$\mathcal{C}^+(A) = \mathbb{T}^d \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A)) \quad \text{and} \quad \mathcal{C}^-(A) = \mathbb{T}^d \setminus (\mathcal{U}^{-+}(A) \cup \mathcal{U}^{-+-}(A))$$

are circloids. Further, we have $\partial\mathcal{C}^\pm(A) \subseteq A$.

The circloids $\mathcal{C}^+(A)$ and $\mathcal{C}^-(A)$ are the 'highest' and the lowest circloids, respectively, whose boundary is contained in A. The same construction works for strips in $\mathbb{R}^d$, and for essential continua in $\mathbb{T}^d$ as long as they are not doubly essential, that is, they admit an essential curve in their complement. However, in these cases an orientation has to be fixed in order to distinguish between the upper and the lower minimal strip, respectively circloid.

Given two horizontal essential continua $A_1, A_2 \subseteq \mathbb{T}^d$, we say $\hat{A}_i \subseteq \mathbb{A}^d$ is a lift of A_i if it is a connected component of $\pi^{-1}(A_i)$. We write $\widehat{A}_1 \prec \widehat{A}_2$ if $\widehat{A}_2 \subseteq \mathcal{U}^+(\widehat{A}_1)$. When A_1 and A_2 are disjoint, we choose lifts $\widehat{A}_1 \prec \widehat{A}_2$ such that no integer translate of $\widehat{A}_1$ or $\widehat{A}_2$ is contained in $\mathcal{U}^+(\widehat{A}_1) \cap \mathcal{U}^-(\widehat{A}_2)$. Then we let $(A_1, A_2) = \pi(\mathcal{U}^+(\widehat{A}_1) \cap \mathcal{U}^-(\widehat{A}_2))$ and $[A_1, A_2] = \pi(\mathbb{A}^d \setminus (\mathcal{U}^-(\widehat{A}_1) \cup \mathcal{U}^+(\widehat{A}_2)))$.

With these notions, we define a circular order on pairwise disjoint essential continua $A_1, A_2, A_3 \subseteq \mathbb{T}^d$ by

$$A_1 \prec A_2 \prec A_3 \quad \Leftrightarrow \quad A_2 \in (A_1, A_3).$$

Using these notions, we now say a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint essential continua in $\mathbb{T}^d$ has *irrational combinatorics* if there exists $\rho \in \mathbb{R} \setminus \mathbb{Q}$ such that for arbitrary $y_0 \in \mathbb{T}^1$ the sequence $y_n = y_0 + n\rho \bmod 1$ satisfies

$$A_k \prec A_m \prec A_n \quad \Leftrightarrow \quad y_k < y_m < y_n$$

for all $k, m, n \in \mathbb{Z}$. We complete the topological preliminaries with two applications of Mayer–Vietoris sequences.

LEMMA 2.5. *Let A, B be compact subsets of $\mathbb{A}^d$ such that:*
(i) $A \cap B = \emptyset$;
(ii) $\mathbb{A}^d \setminus A$ *has exactly one unbounded component;*
(iii) $\mathbb{A}^d \setminus B$ *has exactly one unbounded component.*
Then $\mathbb{A}^d \setminus (A \cup B)$ has exactly one unbounded component.

Proof. We include a short proof based on the use of Mayer–Vietoris sequences. Assume for a contradiction that $\mathbb{A}^d \setminus (A \cup B)$ has two unbounded components. Let $U^\pm$ be the component of $(A \cup B)^c$ containing $\pm\infty$, where A^c denotes the complement of a set A in $\overline{\mathbb{A}^d}$. Let $\gamma^\pm$ be the 0-cycle corresponding to the point $\pm\infty$ and $\kappa^\pm = [\gamma^\pm] \in H_0((A \cup B)^c)$ its equivalence class. Note that since both A and B have only one unbounded component in their complement, the 0-cycle $\gamma^+ - \gamma^-$ represents the zero element in $H_0(A^c)$ and $H_0(B^c)$. Therefore, in the Mayer–Vietoris sequence

$$H_1(\overline{\mathbb{A}^d}) \xrightarrow{\partial_*} H_0(A^c \cap B^c) \xrightarrow{\theta_*} H_0(A^c) \oplus H_0(B^c) \xrightarrow{\xi_*} H_0(\overline{\mathbb{A}^d}) \to 0,$$

the map θ^* sends $\kappa^+ - \kappa^-$ to zero. However, as the sequence is exact and $H_1(\overline{\mathbb{A}^d}) = 0$, the map θ_* is injective. Hence, we must have $\kappa^+ = \kappa^-$, contradicting our assumption. □

A proof of the following statement can be given in a similar way.

LEMMA 2.6. [**34**, Theorem 11.5] *Suppose A, $B \subseteq \mathbb{R}^2$ are both continua, but $A \cap B$ is not connected. Then $A \cup B$ separates the plane, meaning that $\mathbb{R}^2 \setminus (A \cup B)$ has at least two connected components.*

Finally, we will frequently use the following uniform ergodic theorem (e.g. [**35**, **36**]).

THEOREM 2.7. *Suppose X is a compact metric space and $f : X \to X$ and $\varphi : X \to \mathbb{R}$ are continuous. Further, assume that there exists $\rho \in \mathbb{R}$ such that*

$$\int_X \varphi \, d\mu = \rho$$

for all f-invariant ergodic probability measures μ on X. Then

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=0}^{n-1} \varphi \circ f^i(x) \right) = \rho \quad \textit{for all } x \in X.$$

Furthermore, the convergence is uniform on X.

3. *Semiconjugacy to an irrational rotation*

We now turn to the proof of Theorem 1. The implications (ii) ⇒ (iii) and (iv) ⇒ (i) in Theorem 1 are obvious. Hence, in order to prove all the equivalences, it suffices to prove (iii) ⇒ (ii), (i) ⇒ (iii) and (ii) ⇒ (iv). We do so in three separate lemmas and start by treating the easiest of the three implications, which is (iii) ⇒ (ii).

LEMMA 3.1. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and suppose E is a wandering essential continuum. Then $\mathcal{C}^+(E)$ is a wandering essential circloid and the circular ordering of the orbits of E and $\mathcal{C}^+(E)$ are the same.*

Proof. Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and E is a wandering essential continuum with irrational combinatorics. Let $E_n = f^n(E)$ and $C_n = \mathcal{C}^+(E_n) = f^n(\mathcal{C}^+(E))$. Note that, as remarked above, Lemma 2.4 can be applied to essential continua of arbitrary 'homotopy type'. Assume for a contradiction that the C_n are not pairwise disjoint, that is, $C_i \cap C_j \neq \emptyset$ for some integers $i \neq j$. Since $\partial C_n \subseteq E_n$ for all $n \in \mathbb{Z}$ and the E_n are pairwise disjoint, C_i must intersect the interior of C_j or vice versa. Assuming the first case, C_i has to intersect some connected component U of $\mathrm{int}(C_j)$. We distinguish three cases. First, if $C_i \subseteq U$, then this contradicts the minimality of C_j. Second, if $U \subseteq \mathrm{int}(C_i)$, then $\partial U \subseteq \mathrm{int}(C_i)$ since $\partial U \subseteq \partial C_j \subseteq E_j$ and, hence, ∂U is disjoint from $\partial C_i \subseteq E_i$. This means that ∂C_j intersects $\mathrm{int}(C_i)$. However, as C_j cannot be contained in C_i we must have $\partial C_i \cap \partial C_j \neq \emptyset$, contradicting the disjointness of E_i and E_j. As a third possibility, this only leaves the case where ∂C_i intersects ∂U, and hence ∂C_j, leading to the same contradiction as before. Thus, C_i and C_j are disjoint, which shows that C_0 is wandering. The fact that circular ordering is preserved when going from $(E_n)_{n\in\mathbb{N}}$ to $(C_n)_{n\in\mathbb{N}}$ is obvious. □

The next lemma shows (ii) ⇒ (iv). Given $\rho \in \mathbb{T}^d$, we denote by $R_\rho : \mathbb{T}^d \to \mathbb{T}^d,\ x \mapsto x + \rho$ the rotation by ρ.

LEMMA 3.2. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and suppose C is a wandering essential circloid with irrational combinatorics of type ρ. Then there exists a semiconjugacy $h : \mathbb{T}^d \to \mathbb{T}^1$ from f to R_ρ such that the fibres $h^{-1}\{\xi\}$ are all essential annular continua.*

Proof. By performing a change of coordinates, we may assume that C and all its iterates are horizontal. Let $T' : A^d \to A^d,\ (x, y) \mapsto (x, y + 1)$. We let $C_n := f^n(C)$ and denote the connected components of the lifts of these circloid by $\hat{C}_{n,m}$, where the indices are chosen such that for all integers n, m we have:

(i) $\pi(\hat{C}_{n,m}) = C_n$;
(ii) $F(\hat{C}_{n,m}) = \hat{C}_{n+1,m}$;
(iii) $T'(\hat{C}_{n,m}) = \hat{C}_{n,m+1}$.

We claim that

$$H(z) = \sup\{n\rho + m \mid z \in \mathcal{U}^+(\hat{C}_{n,m})\}$$

is a lift of a semiconjugacy h with the required properties. Note that due to the irrational combinatorics we have $n\rho + m < \tilde{n}\rho + \tilde{m}$ if and only if $\hat{C}_{n,m} \prec \hat{C}_{\tilde{n},\tilde{m}}$, such that in particular $H(z)$ is well defined and finite for all $z \in \mathbb{A}^d$. Further, for any $z \in \mathbb{A}^d$ we have

$$\begin{aligned} H \circ F(z) &= \sup\{n\rho + m \mid F(z) \in \mathcal{U}^+(\hat{C}_{n,m})\} \\ &= \sup\{n\rho + m \mid z \in \mathcal{U}^+(\hat{C}_{n-1,m})\} = H(z) + \rho. \end{aligned}$$

In a similar way one can see that $H \circ T(z) = H(z) + 1$, such that H projects to a map $h : \mathbb{T}^d \to \mathbb{T}^1$ which satisfies $h \circ f = R_\rho \circ h$.

In order to check the continuity of H, suppose $U \subseteq \mathbb{R}$ is an open interval and let $z \in H^{-1}(U)$. Choose $r = n\rho + m < H(z) < \tilde{n}\rho + \tilde{m} = s$ with $r, s \in U$. Then $z \in \mathcal{U}^+(C_{n,m}) \cap \mathcal{U}^-(C_{\tilde{m},\tilde{n}}) =: V$. From the definition of H we see that $H(V) \subseteq [r, s] \subseteq U$, and thus $H^{-1}(U)$ contains an open neighbourhood of z. Since U and $z \in H^{-1}(U)$ were arbitrary, H is continuous. The fact that h is onto follows easily from the minimality of R_ρ, so that h is indeed a semiconjugacy from f to R_ρ.

It remains to prove the fact that the fibres $h^{-1}\{\xi\}$ are annular continua. In order to do so, note that for $\xi \in \mathbb{T}^1$

$$\begin{aligned} H^{-1}\{\xi\} &= \bigcap_{n\rho+m<\xi} \mathcal{U}^+(\hat{C}_{n,m}) \cap \bigcap_{\tilde{n}+\rho\tilde{m}>\xi} \mathcal{U}^-(\hat{C}_{\tilde{n},\tilde{m}}) \\ &= \bigcap_{n\rho+m<\xi} \mathbb{A}^d \setminus \mathcal{U}^-(\hat{C}_{n,m}) \cap \bigcap_{\tilde{n}\rho+\tilde{m}>\xi} \mathbb{A}^d \setminus \mathcal{U}^+(\hat{C}_{\tilde{n},\tilde{m}}). \end{aligned} \tag{3.1}$$

Note here that for all n, m, n', m' with $n\rho + m < n'\rho + m'$ we have

$$\mathcal{U}^+(\hat{C}_{n',m'}) \subseteq \mathbb{A}^d \setminus \mathcal{U}^-(\hat{C}_{n',m'}) \subseteq \mathcal{U}^+(\hat{C}_{n,m})$$

and similar inclusions hold in the opposite direction. This explains the second equality in (3.1). Choosing sequences $n_i, m_i, \tilde{n}_i, \tilde{m}_i$ with $n_i\rho + m_i \nearrow \xi$ and $\tilde{n}_i\rho + \tilde{m}_i \searrow \xi$, we can rewrite (3.1) as

$$H^{-1}\{\xi\} = \bigcap_{i\in\mathbb{N}} \mathbb{A}^d \setminus (\mathcal{U}^-(\hat{C}_{n_i,m_i}) \cup \mathcal{U}^+(\hat{C}_{\tilde{n}_i,\tilde{m}_i})).$$

Since the sets of the intersection are all essential annular continua, so is $H^{-1}\{\xi\}$ by Lemma 2.1. □

It remains to prove the implication (i) ⇒ (iii).

LEMMA 3.3. *Suppose that $h : \mathbb{T}^d \to \mathbb{T}^1$ is a semiconjugacy from $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ to an irrational rotation R_ρ. Then every fibre $h^{-1}\{\xi\}$ contains a wandering essential continuum with irrational combinatorics.*

Proof. We first show that the action $h^* : \Pi_1(\mathbb{T}^d) \to \Pi_1(\mathbb{T}^1)$ of h on the fundamental groups is non-trivial. Suppose for a contradiction that $h^* = 0$. Then any lift $H : \mathbb{R}^d \to \mathbb{R}$ of h is bounded since in this case $\sup_{z\in\mathbb{R}^d} \|H(z)\| = \sup_{z\in[0,1]^d} \|H(z)\|$. However, if $\widehat{R}_\rho$ is the lift of R_ρ which satisfies $H \circ F = \widehat{R}_\rho \circ H$, then this contradicts the unboundedness of

$$H \circ F^n(z) = \widehat{R}_\rho^n \circ H(z).$$

Consequently, h^* is non-trivial, and by composing h with a linear torus automorphism we may assume that h^* is just the projection to the last coordinate. This composition may change the rotation number, but does not effect its irrationality. We obtain a lift $\hat{h} : \mathbb{A}^d \to \mathbb{R}$ which satisfies $\hat{h}(z) \to \pm\infty$ if $z \to \pm\infty$.

As a consequence, the intermediate value theorem implies that every properly embedded line $\Gamma = \{\gamma(t) \mid t \in \mathbb{R}\}$ intersects all level sets $\widehat{E}_x = \hat{h}^{-1}\{x\}$. Hence, all $\widehat{E}_x$ are essential.

If $\widehat{E}_x$ is not connected, we consider the family of all compact and essential subsets of $\widehat{E}_x$ and choose and element $\widehat{\mathcal{E}}$ which is minimal with respect to the inclusion. Note that such minimal elements exist by the lemma of Zorn. By Lemma 2.5 $\widehat{\mathcal{E}}$ is connected.

Further, $\mathcal{E} = \pi(\widehat{\mathcal{E}})$ is wandering since $\widehat{\mathcal{E}} \subseteq h^{-1}\{x\}$. Hence, $\mathcal{E}$ is the wandering essential continuum we are looking for. The fact that $\mathcal{E}$ has irrational combinatorics can be seen from the semiconjugacy equation. □

4. *On the uniqueness of the semiconjugacy*

In light of the preceding section, it is an obvious question to ask to what extent a semiconjugacy between $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and an irrational rotation R_ρ of the circle is unique. It is easy to check that for every rigid rotation $R : \mathbb{T}^1 \to \mathbb{T}^1$ the map $R \circ h$ is a semiconjugacy between f and R_ρ as well. Hence, there is non-uniqueness of the semiconjugacy in general. Nevertheless, one could ask whether there is uniqueness up to post-composition with rotations. In brief, we will speak of *uniqueness modulo rotations*.

Consider $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ given by $f(x, y) = (x + \rho_1, y)$ with $\rho_1 \in \mathbb{Q}^c$. For any continuous function $\alpha : \mathbb{T}^1 \to \mathbb{T}^1$, we have that $h_\alpha(x, y) = x + \alpha(y)$ is a semiconjugacy from f to R_{ρ_1}. Thus, we do not have uniqueness of the semiconjugacy even modulo rotations. However, it is not difficult to see that all of the possible semiconjugacies between f and R_{ρ_1} are given by h_α for some continuous function α. This implies in particular that on every minimal set $Y_r = \{(x, y) \in \mathbb{T}^2 \mid y = r\}$, $r \in \mathbb{T}^1$, given any two semiconjugacies h_1 and h_2 we have that $h_{1|Y_r} = (R \circ h_2)_{|Y_r}$ for some rigid rotation R. This, as we will see, is a general fact.

We say that an f-invariant set Ω is *externally transitive* if for every $x, y \in \Omega$ and neighbourhoods U_x, U_y of x and y, respectively, there exists $n \in \mathbb{N}$ such that $f^n(U_x) \cap U_y \neq \emptyset$. Note that $f^n(U_x)$ and U_y do not need to intersect in Ω as in the usual definition of topological transitivity. In the above example the sets Y_r are transitive, hence externally transitive.

Given $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ semiconjugate to a rigid rotation R_ρ and a f-invariant set $\Omega \subseteq \mathbb{T}^2$, we say that the semiconjugacy is *unique modulo rotations on* Ω if for all semiconjugacies h_1, h_2 from f to R_ρ we have $h_{1|\Omega} = (R \circ h_2)_{|\Omega}$ for some rigid rotation R.

PROPOSITION 4.1. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be semiconjugate to a rigid rotation of $\mathbb{T}^1$. Further, assume that $\Omega \subset \mathbb{T}^2$ is an externally transitive invariant set of f. Then the semiconjugacy is unique modulo rotations on Ω.*

Proof. Let h_1, h_2 be two semiconjugacies between f and R_ρ. By post-composing with a rigid rotation, we may assume that $h_1(x) = h_2(x)$ for some $x \in \Omega$. Suppose for a contradiction that $h_1(y) \neq h_2(y)$ for some $y \in \Omega$.

Let $\varepsilon = \frac{1}{2} \cdot d(h_1(y), h_2(y))$ and $\delta > 0$ such that $d(h_1(x'), h_2(x')) < \varepsilon$ if $x' \in B_\delta(x)$ and $d(h_1(y'), h_2(y')) > \varepsilon$ if $y' \in B_\delta(y)$. Due to Ω being externally transitive, there exists $z \in B_\delta(x)$ and $n \in \mathbb{N}$ such that $f^n(z) \in B_\delta(y)$. However, at the same time we have that $\varepsilon < d(h_1(f^n(z)), h_2(f^n(z))) = d(R^n_\rho(h_1(z)), R^n_\rho(h_2(z))) = d(h_1(z), h_2(z)) < \varepsilon$, which is absurd. □

As a consequence, we obtain the uniqueness of the semiconjugacy modulo rotations whenever the non-wandering set of f is externally transitive. The reason is the following simple observation.

LEMMA 4.2. *If $h_1(x) = h_2(x)$ for two semiconjugacies between $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and a rigid rotation of $\mathbb{T}^1$, then $h_1(y) = h_2(y)$ for all y with $x \in \overline{\mathcal{O}(y, f)}$.*

Proof. Suppose for a contradiction that $x \in \overline{\mathcal{O}(y, f)}$ but $h_1(y) \neq h_2(y)$. Let $\varepsilon = d(h_1(y), h_2(y))/2$ and $\delta > 0$ such that if $x' \in B_\delta(x)$, then $h_1(x'), h_2(x') \in B_\varepsilon(h_1(x))$. Further, let $n \in \mathbb{N}$ be such that $z := f^n(y) \in B_\delta(x)$. Then on the one hand $h_1(z), h_2(z) \in B_\varepsilon(h_1(x))$, and on the other hand $d(h_1(z), h_2(z)) = d(h_1(y), h_2(y)) = 2\varepsilon$, which is absurd. □

Given $f \in \mathrm{Homeo}(\mathbb{T}^2)$ we denote its non-wandering set by $\Omega(f)$. Since any orbit accumulates in the non-wandering set, the combination of Proposition 4.1 and Lemma 4.2 immediately yields the following result.

COROLLARY 4.3. *Suppose that $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is semiconjugate to a rigid rotation of $\mathbb{T}^1$. Further assume that $\Omega(f)$ is externally transitive. Then the semiconjugacy is unique modulo rotations.*

For irrational pseudorotations of the torus†, external transitivity of the non-wandering set was proved by Potrie in [**37**]. Hence, applying Corollary 4.3 in both coordinates yields the following result.

COROLLARY 4.4. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be an irrational pseudo-rotation which is semiconjugate to the respective rigid translation of $\mathbb{T}^2$. Then the semiconjugacy is unique up to composing with rigid translations of $\mathbb{T}^2$.*

Finally, one may ask the following question.

Question 4.5. Does every semiconjugacy between $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and a rigid rotation on $\mathbb{T}^1$ have essential annular continua as fibres?

We note that in the example $f(x, y) = (x + \rho_1, y)$ discussed above this is true, since the fibres of the semiconjugacy h_α are the essential circles $\{(x - \alpha(y), y) \mid y \in \mathbb{T}^1\}$, $x \in \mathbb{T}^1$. By Theorem 1 it is also true whenever the semiconjugacy is unique modulo rotations, since there always exists one semiconjugacy with this property and the topological structure of the fibres is certainly preserved by post-composition with rotations.

5. *Fibred rotation number for foliations of circloids*

The aim of this section is to prove Theorem 2. In order to do so, we need some further preliminary results. Given two open connected subsets U, V of a manifold M, we say that $K \subseteq M \setminus (U \cup V)$ *separates* U and V if U and V are contained in different connected components of $M \setminus K$.

LEMMA 5.1. *Suppose $S \subseteq \mathbb{R}^d$ is a thin horizontal strip and $K \subseteq S$ is a connected closed set that separates $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$. Then $C_S \subseteq K$.*

† That is, torus homeomorphisms homotopic to the identity with unique and totally irrational rotation vector.

Proof. Suppose $C_S \nsubseteq K$ and let $z \in C_S \setminus K$. Then $B_\varepsilon(z) \subseteq \mathbb{R}^d \setminus K$. However, as $B_\varepsilon(z)$ intersects both $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$ by Lemma 2.3, this means that $\mathcal{U}^+(S) \cup B_\varepsilon(z) \cup \mathcal{U}^-(S)$ is contained in a single connected component of $\mathbb{R}^d \setminus K$, contradicting the fact that K separates $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$. □

Given an essential annular continuum $A \subseteq \mathbb{A}$, we denote its lift to $\mathbb{R}^2$ by $\widehat{A} = \pi^{-1}(A)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x + 1, y)$. Then we say A has a *compact generator*, if there exists a compact connected set $G_0 \subseteq \widehat{A}$ such that $\bigcup_{n\in\mathbb{Z}} G_n = \widehat{A}$, where $G_n = T^n(G_0)$.

LEMMA 5.2. *If $A \subseteq \mathbb{A}$ is an annular continuum with generator G_0, then $G_n \cap G_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$.*

Proof. It suffices to prove that $G_0 \cap G_1 \neq \emptyset$. Suppose for a contradiction that the intersection is empty. Then G_0 has a connected neighbourhood U such that $T(U) \cap U = \emptyset$. Since U cannot be contained in bounded connected component of $T(U)$ and vice versa, we have $T(D) \cap D = \emptyset$ where $D = \mathrm{Fill}(U)$. Owing to Frank's lemma [**38**], this implies that $T^n(D) \cap D = \emptyset$ for all $n \in \mathbb{Z} \setminus \{0\}$, contradicting the connectedness of $A \subseteq \bigcup_{n\in\mathbb{Z}} T^n(D)$. □

Given any bounded set $B \subseteq \mathbb{R}^2$, we let

$$\nu_B = \max\{n \in \mathbb{N} \mid \exists z \in B : T^n(z) \in B\}.$$

LEMMA 5.3. *Suppose that $A, A' \subseteq \mathbb{A}$ are thin essential annular continua with compact generators G_0, G_0'. Further, assume $f \in \mathrm{Homeo}_0(\mathbb{A})$ maps A to A'. Then for any lift F of f the set $F(G_0)$ intersects at most $\nu_{G_0} + \nu_{G_0'} + 1$ integer translates of G_0'.*

Proof. Suppose $F(G_0)$ intersects G_n' and G_m' for some $m > n$. Then due to Lemma 5.2, the set

$$\bigcup_{k\leq n} G_k' \cup F(G_0) \cup \bigcup_{k\geq m} G_k' \subseteq \widehat{A'}$$

is connected and therefore separates $\mathcal{U}^+(\widehat{A'})$ and $\mathcal{U}^-(\widehat{A'})$. Hence, by Lemma 5.1 it contains $C_{\widehat{A'}} = \widehat{C_{A'}}$. Let $z_0 \in G_0' \cap C_{\widehat{A}}$ and assume without loss of generality that $z_j = T^j(z_0) \notin G_0'$ for all $j \geq 1$. Then $z_n \in G_n'$ and $z_j \notin \bigcup_{k\leq n} G_k'$ for all $j > n$. Furthermore, since $z_m \in G_m'$ we have that $z_j \notin \bigcup_{k\geq m} G_k'$ for all $j < m - \nu_{G_0'}$. Thus, we must have

$$\{z_{n+1}, \ldots, z_{m-\nu_{G_0'}-1}\} \subseteq F(G_0).$$

However, since $F(G_0)$ contains at most ν_{G_0} integer translates of z_0, this implies $m - n \leq \nu_{G_0} + \nu_{G_0'} + 1$. □

As a first consequence, this yields the following variation of [**23**, Theorem 2.7].

COROLLARY 5.4. *Let $f \in \mathrm{Homeo}_0(\mathbb{A})$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ and suppose A is an f-invariant thin essential annular continuum which is compactly generated. Then $f_{|A}$ has a unique rotation number, that is,*

$$\rho_A(F) = \lim_{n\to\infty} \pi_2 \circ (F^n(z) - z)/n$$

exists for all $z \in \pi^{-1}(A)$ and is independent of z. Moreover, the convergence is uniform in z.

Proof. As $\rho(F, z) = \lim_{n\to\infty} \pi_2 \circ (F^n(z) - z)/n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(z)$ is an ergodic sum with observable $\varphi(z) = \pi_2(F(z) - z)$, we have that $\rho(f, z) = \int_A \varphi\, d\mu =: \rho(\mu)$ μ-almost surely for every f-invariant probability measure supported on A. Note here that φ is well defined as a function $\mathbb{A} \to \mathbb{R}$. Assume for a contradiction that the rotation number is not unique on A. Then Theorem 2.7 implies the existence of two f-invariant ergodic measures μ_1, μ_2 supported on A with $\rho(\mu_1) \neq \rho(\mu_2)$. Consequently, we can choose $z_1, z_2 \in A$ with $\rho(F, z_1) = \rho(\mu_1) \neq \rho(F, z_2) = \rho(\mu_2)$. However, at the same time we may choose lifts $\hat{z}_1, \hat{z}_2 \in G_A$ of z_1, z_2, where G_A is a compact generator of A. Then Lemma 5.3 implies that $F^n(\hat{z}_1)$ and $F^n(\hat{z}_2)$ are contained in the union of $2\nu_{G_A} + 1$ adjacent copies of G_A. Consequently, we have that $d(F^n(\hat{z}_1), F^n(\hat{z}_2)) \leq \mathrm{diam}(G_A) + 2\nu_{G_A} + 1$ for all $n \in \mathbb{N}$, a contradiction. The uniform convergence follows from the same argument. □

Remark 5.5.
(i) As remarked before we note that as a special case, Corollary 5.4 applies to decomposable essential thin circloids. In order to see this, recall that a continuum C is called *decomposable* if it can be written as the union of two non-empty continua C_1 and C_2. If C is a thin circloid, then due to the minimality of circloids C_1 and C_2 have to be non-essential. Hence, connected components $\widehat{C_i}$ of $\pi^{-1}(C_i) \subseteq \mathbb{R}^2$, $i = 1, 2$, are bounded. If these lifts are chosen such that their intersection is non-empty, then $G = \widehat{C}_1 \cup \widehat{C}_2$ is a compact generator of C.
(ii) Conversely, if C has a compact generator G_0 and n is chosen such that $T^n(G_0) \cap G_0 = \emptyset$, then a lift of C to the $2n$-fold covering decomposes into two continua, which are given by the projections of the sets $\bigcup_{i=0}^{n-1} T^i(G_0)$ and $\bigcup_{i=n}^{2n-1} T^i(G_0)$.
(iii) Theorem 2.7 in [**23**] is stated for the case where A is a *cofrontier*, which is defined as an irreducibly plane-separating continuum. A cofrontier is always the boundary of a circloid, and conversely an annular continuum is a circloid if and only if its boundary is a cofrontier. Since circloids may have interior, the two concepts are not the same. However, while the subtle difference may play a role in other situations (see, for example, [**6**]), it is of minor importance in our context here and the arguments in [**23**] work for both cases.
(iv) Examples of (hereditarily) non-decomposable circloids were constructed by Bing [**39**] and may occur as minimal sets of smooth surface diffeomorphisms [**30**, **31**].

In the particular case of rational rotation number, we further obtain the existence of periodic orbits.

COROLLARY 5.6. *Let $f \in \mathrm{Homeo}_0(\mathbb{A})$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$. Further, suppose that A is an f-invariant thin essential annular continuum which is compactly generated and $\rho_A(F) = \{p/q\}$. Then F has a q-periodic orbit with rotation number p/q in $\pi^{-1}(A)$.*

Proof. By going over to the qth iterate, we may assume that $\rho_A(F) = \{0\}$. Further, by replacing G_0 with $\bigcup_{i=0}^{n} G_i$ for sufficiently large $n \in \mathbb{N}$, we may assume without loss of generality that $F(G_0) \cap G_0 \neq \emptyset$. Then Lemma 5.3 implies that $C := \overline{(\bigcup_{k\in\mathbb{Z}} F^k(G_0))}$ is a compact and invariant set. Moreover, as A is thin, C is a non-separating continuum.

Therefore the Cartwright and Littlewood theorem [**40**] implies the existence of a fixed point of F in C. □

As Lemma 5.3 works for any combination of two compactly generated thin annular continua, we can prove Theorem 2 in a similar way as the above Corollary 5.4. However, what we need as a technical prerequisite is the measurable dependence of the size of the generators of fibres $h^{-1}(\xi)$ under the assumptions of the theorem. We obtain this in several steps. We place ourselves in the situation of Theorem 2 and assume again without loss of generality that the action $h^* : \Pi_1(\mathbb{T}^2) \to \Pi_1(\mathbb{T}^1)$ on the fundamental group is the projection to the second coordinate. This implies that the annular continua $\mathcal{A}_\xi = h^{-1}\{\xi\}$ are all of homotopy type (1, 0). We denote by $\hat{f}$ the lift of f to $\mathbb{A}$ and by F the lift to $\mathbb{R}^2$. Further, we denote by $\hat{h} : \mathbb{A} \to \mathbb{R}$ a lift of h to $\mathbb{A}$ and by $H : \mathbb{R}^2 \to \mathbb{R}$ a lift to $\mathbb{R}^2$.

Let $\Omega_0 = \{\xi \in \mathbb{T}^1 \mid \mathcal{A}_\xi \text{ is thin}\}$, $\Omega = \pi^{-1}(\Omega_0)$ and $A_\xi = \hat{h}^{-1}\{\xi\}$ ($\xi \in \mathbb{R}$). Then all A_ξ are essential annular continua in $\mathbb{A}$, and A_ξ is thin if and only if $\xi \in \Omega$. Further, define $A^+_\xi = \partial\mathcal{U}^+(A_\xi)$ and $A^-_\xi = \partial\mathcal{U}^-(A_\xi)$. Then for all $\xi \in \Omega$ we have $A_\xi = A^+_\xi \cup A^-_\xi$ and, by Lemma 2.3, $A^+_\xi \cap A^-_\xi = C_{A_\xi} =: C_\xi$.

We recall that for a metric space (X, d) and $C, D \subset X$, the Hausdorff distance is defined as

$$d_{\mathcal{H}}(C, D) = \max\left\{\sup_{x\in C} d(x, D), \sup_{y\in D} d(y, C)\right\}.$$

The convergence of a sequence $\{C_n\}_{n\in\mathbb{N}}$ of subsets in X to $A \subset X$ in this distance is denoted either by $C_n \to_{\mathcal{H}} A$ or by $\lim^{\mathcal{H}}_{n\to\infty} C_n = A$. Note that $d_{\mathcal{H}}(C, D) < \varepsilon$ if and only if $C \subseteq B_\varepsilon(D)$ and $D \subseteq B_\varepsilon(C)$, and that the Hausdorff distance defines a complete metric if one restricts to compact subsets.

LEMMA 5.7. *If A_ξ is thin, then $\lim^{\mathcal{H}}_{\xi'\nearrow\xi} A^-_{\xi'} = \lim^{\mathcal{H}}_{\xi'\nearrow\xi} A_{\xi'} = A^-_\xi$ and $\lim^{\mathcal{H}}_{\xi'\searrow\xi} A^+_{\xi'} = \lim^{\mathcal{H}}_{\xi'\searrow\xi} A_{\xi'} = A^+_\xi$.*

Proof. We prove $\lim^{\mathcal{H}}_{\xi'\nearrow\xi} A^-_{\xi'} = \lim^{\mathcal{H}}_{\xi'\nearrow\xi} A_{\xi'} = A^-_\xi$, the opposite case follows by symmetry. Since $A^-_{\xi_n} \subseteq A_{\xi_n}$, it suffices to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such for all $\xi' \in (\xi - \delta, \xi)$ we have

$$A_{\xi'} \subseteq B_\varepsilon(A^-_\xi) \quad \text{and} \quad A^-_\xi \subseteq B_\varepsilon(A^-_{\xi'}). \tag{5.1}$$

We start by showing the first inclusion. Fix $\varepsilon > 0$. Assume for a contradiction that there exists a sequence $\xi_n \nearrow \xi$ such that $A_{\xi_n} \subsetneq B_\varepsilon(A^-_\xi)$ for all $n \in \mathbb{N}$. Let $z_n \in A_{\xi_n} \setminus B_\varepsilon(A^-_\xi)$ and $z = \lim_{n\to\infty} z_n$. Then $z \notin B_\varepsilon(A^-_\xi)$ and, thus, since all of the z_n are below A_ξ, we have $z \notin A_\xi$. However, at the same time $h(z) = \lim_{n\to\infty} h(z_n) = \lim_{n\to\infty} \xi_n = \xi$, a contradiction.

Conversely, in order to show the second inclusion in (5.1), assume for a contradiction that there exists a sequence $\xi_n \nearrow \xi$ such that $A^-_\xi \subsetneq B_\varepsilon(A^-_{\xi_n})$ for all $n \in \mathbb{N}$. Let $K_n = A^-_\xi \setminus B_\varepsilon(A^-_{\xi_n})$ and note that $A^-_\xi \cap B_\varepsilon(A^-_{\xi_n}) = A^-_\xi \cap B_\varepsilon(\mathcal{U}^-(A^-_{\xi_n}))$, which is increasing in n. Then $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact sets, such that $K = \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$. Let $z \in K$. Then $B_\varepsilon(z) \cap A^-_{\xi_n} = \emptyset$ and thus $B_\varepsilon(z) \subseteq \mathcal{U}^+(A^-_{\xi_n})$ for all $n \in \mathbb{N}$. This implies $h(z') \geq \xi$ for all $z' \in B_\varepsilon(z)$, contradicting the fact that $B_\varepsilon(z)$ intersects $\mathcal{U}^-(A_\xi)$ and $h < \xi$ on $\mathcal{U}^-(A_\xi)$. □

Given a compactly generated thin annular continuum A, we let

$$\tau(A) = \inf\{\mathrm{diam}(G) \mid G \text{ is a compact generator of } A\}. \tag{5.2}$$

LEMMA 5.8. *The function $\xi \mapsto \tau(A^-_\xi)$ is lower semicontinuous from the left on Ω, that is,*

$$\liminf_{\xi' \nearrow \xi} \tau(A^-_{\xi'}) \geq \tau(A^-_\xi) \quad \textit{for all } \xi \in \Omega.$$

Similarly, $\xi \mapsto \tau(A^+_\xi)$ is lower semicontinuous from the right on Ω.

Proof. Let $\xi_n \nearrow \xi$ and assume without lose of generality that $\tau := \lim_{n\to\infty} \tau(A^-_{\xi_n})$ exists and is finite. Choose generators G_{ξ_n} of A_{ξ_n} of diameter smaller than $\tau(A^-_{\xi_n}) + 1/n$. Then, using Lemma 5.7, it is straightforward to verify that any limit point G of $(G_{\xi_n})_{n\in\mathbb{N}}$ in the Hausdorff metric is a compact generator of A^-_ξ of diameter smaller than τ. □

It is easy to check that real-valued functions which are lower semicontinuous from one side are also measurable. Consequently, since $\tau(A_\xi) \leq \eta(\xi) := \tau(A^-_\xi) + \tau(A^+_\xi)$, the function η provides a measurable majorant for the minimal diameter of the generators of A_ξ. Further, $A^\pm_\xi$ are compactly generated if and only if A_ξ is compactly generated, a fact which follows from the topological considerations on thin annular continua exposed in the next section, see Lemma 6.1. Altogether, this yields the following result.

COROLLARY 5.9. *The map $\xi \mapsto \tau(A_\xi)$ has a measurable majorant $\eta : \mathbb{T}^1 \to \mathbb{R}^+$ such that $\eta(\xi) < \infty$ if A_ξ is compactly generated.*

We are now in the position to complete the proof of Theorem 2 by adapting the measure-theoretic argument of Herman in [**21**, Theorem 5.4].

Proof of Theorem 2. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and suppose $h : \mathbb{T}^2 \to \mathbb{T}^1$ is a semiconjugacy to the irrational rotation R_ρ. We assume that $h^* = \pi_1^*$, such that there exist continuous lifts $\hat{h} : \mathbb{A} \to \mathbb{R}$ of h and $\hat{f} : \mathbb{A} \to \mathbb{A}$ of f which satisfy

$$\hat{h} \circ \hat{f} = R_\rho \circ \hat{h}.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Assume for a contradiction that f has no unique rotation vector. Since the semiconjugacy is homotopic to π_1, the first coordinate of any rotation vector of f must be ρ. Therefore, similar to the proof of Corollary 5.4, this implies the existence of two f-invariant ergodic probability measures μ_1 and μ_2 with

$$\rho_1 = \int_{\mathbb{T}^2} \pi_2(F(z) - z)\, d\mu_1(z) \neq \int_{\mathbb{T}^2} \pi_2(F(z) - z)\, d\mu_2(z) = \rho_2.$$

As $h^{-1}\{\xi\}$ is compactly generated for Lebesgue-almost everywhere $\xi \in \mathbb{T}^1$, Corollary 5.9 yields the existence of a finite-valued measurable majorant of $\xi \mapsto \tau(h^{-1}\{\xi\})$. Hence, we can find a constant $C > 0$ and a set $\Omega_C \subseteq \mathbb{T}^1$ of positive measure such that for all $\xi \in \Omega_C$ the annular continuum $h^{-1}\{\xi\}$ has a compact generator G_ξ with $\mathrm{diam}(G_\xi) \leq C$.

Both μ_1 and μ_2 must be mapped to the Lebesgue measure on $\mathbb{T}^1$ by h, since this is the only invariant probability measure of R_ρ. Hence, for almost every $\xi \in \mathbb{T}^1$ there exist points

$z_1, z_2 \in h^{-1}\{\xi\}$ which are generic with respect to μ_1 and μ_2, respectively. In particular, for any lift $\hat{z}_i \in \mathbb{R}^2$ of z_i we have that

$$\lim_{n\to\infty} \pi_2(F^n(\hat{z}_i) - \hat{z}_i)/n = \rho_i \quad (i = 1, 2). \tag{5.3}$$

Without loss of generality, we may assume that $h^{-1}\{\xi\}$ has compact generator G_ξ and $R^n_\rho(\xi)$ visits Ω_C infinitely many times, that is, $R^{n_i}_\rho(\xi) \in \Omega_C$ for a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ of integers. Given lifts $\hat{z}_1, \hat{z}_2 \in G_\xi$ of z_1, z_2, Lemma 5.3 implies that

$$\pi_2(F^{n_i}(\hat{z}_1)) - \pi_2(F^{n_i}(\hat{z}_2) \le \operatorname{diam}(G_{r_\rho^{n_i}(\xi)}) + \nu_{G_\xi} + \nu_{G_{r_\rho^{n_i}(\xi)}} + 1 \le \nu_{G_\xi} + 2C + 1$$

for all $i \in \mathbb{N}$. As $\rho_1 \neq \rho_2$, this contradicts (5.3). □

6. *Comments on the topology of thin annular continua*

The aim of this section is to give a basic classification for the topology of thin annular continua in the context of compact generators, and to prove the result on foliations of $\mathbb{T}^2$ into essential annular continua mentioned in Remark 1.1(c). First, we have the following.

LEMMA 6.1. *Suppose A is a thin essential annular continuum which is compactly generated. Then any thin essential annular continuum $A' \subseteq A$ is compactly generated and $\tau(A') \le \tau(A)$, with τ defined as in (5.2). In particular, this holds for the circloid C_A and for the annular continua A^- and A^+ defined in the previous section.*

Proof. Suppose G_0 is a compact generator of A and $\widehat{A}$, $\widehat{A'}$ are lifts of A and A', respectively, to $\mathbb{R}^d$. Then it suffices to show that $G'_0 = G_0 \cap \widehat{A'}$ is connected, since in this case G'_0 is a compact generator of A'.

In order to do so, consider the closed-disc-compactification D of $\mathbb{R}^2$, obtained by adding a circle at infinity. Let C be the closure of $\widehat{A'}$ in D. Note that C is just the union of $\widehat{A'}$ with two points in the unit circle. Suppose for a contradiction that $G_0 \cap \widehat{A'} = G_0 \cap C$ is not connected. Viewing D again as a subset of the plane allows to apply Lemma 2.6, which implies that the union $G_0 \cup C$ separates the plane. However, this is impossible since by assumption $\widehat{A}$ has empty interior. □

We now investigate essential annular continua which are not compactly generated in more detail. To that end, given $X \subset \mathbb{R}^2$, we denote by $[X]_y$ the connected component of y in X and define $\operatorname{width}(X) = \sup\{x_1 - x_2 \mid (x_1, y_1), (x_2, y_2) \in X\}$. For an essential annular continuum $A \subset \mathbb{A}$, we denote the lifts of A and C_A by $\widehat{A}$ and $\widehat{C}_A$. Then we define the set of *spikes* of A as

$$\mathcal{S}_A := \{[\widehat{A} \setminus \widehat{C}_A]_x \mid x \in \widehat{A} \setminus \widehat{C}_A\}$$

and say that A has an infinite spike if there exists $S \in \mathcal{S}_A$ with $\operatorname{width}(S) = \infty$. Further we let $W_{\mathcal{S}_A} := \sup\{\operatorname{width}(S) \mid S \in \mathcal{S}_A\}$. We start with a general observation.

LEMMA 6.2. *If A is a thin annular continuum and S is a spike of A with $\operatorname{width}(S) < \infty$, then $\overline{S} \cap \widehat{C}_A \neq \emptyset$.*

Proof. As in the previous proof, we consider the closed-disc-compactification D of $\mathbb{R}^2$ and the closure C of $\widehat{C}_A$ in D. Then [**41**, Theorem 2.16] implies that $\overline{S}$ intersects C, and since S is bounded the intersection must be contained in $\widehat{C}_A$. □

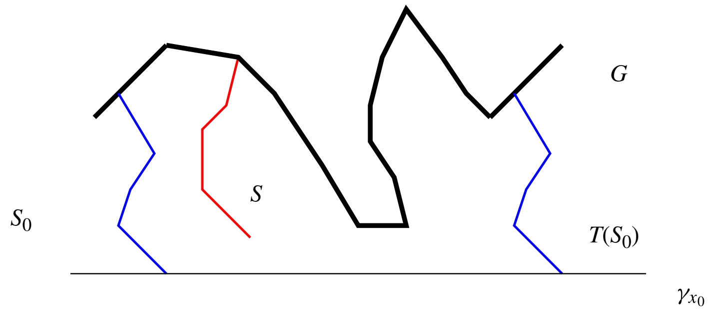

FIGURE 1. Proof of Lemma 6.4: the generator G of C_A and the two copies of S_0 bound every other spike.

COROLLARY 6.3. *Let A be an essential thin annular continuum. If C_A is compactly generated and $W_{\mathcal{S}_A} < \infty$, then A is compactly generated.*

Proof. Let G' be a generator of C_A. For every spike S choose $n \in \mathbb{Z}$ such that $S' := T^n(S)$ intersects G'. Note that this is possible due to Lemma 6.2. Since $W_{\mathcal{S}_A} < \infty$, we have that $G := \overline{(G' \cup \bigcup_{S \in \mathcal{S}_A} S')}$ is a compact generator of A. □

Our next aim is to show that if C_A is compactly generated, then $W_{\mathcal{S}_A} = \infty$ implies the existence of an infinite spike.

LEMMA 6.4. *Let A be a thin annular continuum such that C_A is compactly generated and $W_{\mathcal{S}_A} = \infty$. Then there exists an infinite spike $S \in \mathcal{S}_A$.*

We note that when C_A has no compact generator, then $W_{\mathcal{S}_A}$ may be infinite even if all spikes are bounded. An example can be produced by attaching longer and longer spikes to the pseudocircle constructed by Bing [**29**].

Proof. We assume that the supremum $W_{\mathcal{S}_A}$ is obtained by spikes in $\mathcal{U}^-(\widehat{C}_A)$, the other case is symmetric. Suppose for a contradiction that width$(S) < \infty$ for every $S \in \mathcal{S}_A$. Let $x_0 \in \widehat{A} \setminus \mathcal{U}^+(C_{\widehat{A}}) = (\widehat{A} \cap \mathcal{U}^-(\widehat{C}_A)) \cup \widehat{C}_A$ such that

$$\pi_2(x_0) = \min\left\{\pi_2(x) \;\middle|\; x \in \bigcup_{S \in \mathcal{S}_A} S \cap \mathcal{U}^-(\widehat{C}_A)\right\}.$$

By changing coordinates if necessary, we can ensure that the map π_2 on A reaches its minimum outside of $C_{\widehat{A}}$. Hence, we may assume $x_0 \notin C_{\widehat{A}}$.

Let $\gamma_{x_0}(t) = x_0 + t \cdot (1, 0)$ and $S_0 \in \mathcal{S}_A$ such that $x_0 \in S_0$. Then due to Lemma 6.2 and the fact that C_A has a compact generator, we can consider a compact generator G of C_A that verifies $G \cap \overline{S_0} \neq \emptyset$ and $G \cap \overline{T(S_0)} \neq \emptyset$ (see Figure 1).

Due to the definition of x_0, we have that given any spike $S \subset \mathcal{U}^-(\widehat{C}_A)$ different from S_0 the inclusion

$$S \subset \left(\overline{\mathcal{U}^+(\gamma_{x_0}(\mathbb{R}))} \cap \mathcal{U}^-\left(\bigcup_{n \in \mathbb{Z}} T^n(G)\right)\right) \setminus \bigcup_{n \in \mathbb{Z}} T^n(S_0)$$

holds. Therefore, width$(S) < 2 \cdot$ width$(S_0) +$ width(G). This contradicts $W_{\mathcal{S}_A} = \infty$. □

Altogether, we have now obtained the following basic classification concerning the existence of generators for essential thin annular and their circloids.

PROPOSITION 6.5. *Let $A \subset \mathbb{A}$ be an essential thin annular continuum. Then:*
(i) *if A is not compactly generated, then either*
  (a) *C_A is not compactly generated or*
  (b) *C_A is compactly generated and A contains an infinite spike;*
(ii) *if A compactly generated, then so is C_A.*

Note that Proposition 6.5 does not rule out the coexistence of an infinite spike and a compact generator. In fact, this may happen, and a way to construct such examples is the following. Let $I = [0, 1] \times \{0\}$ and $J = \{0\} \times [0, 1]$. We consider $K = J \cup I \cup T(J)$. Fix $x_0 \in J \setminus I$ and $x_1 = T(x_0)$ and let $\gamma : \mathbb{R}^+ = [0, +\infty) \to \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ y > 0\}$ be an injective curve that verifies
(i) $\gamma([n, +\infty)) \subset B_{1/n}(K)$ for every $n \in \mathbb{N}$;
(ii) $\lim_i \gamma(t_i) = x_0$, $\lim_j \gamma(t_j) = x_1$ for two strictly increasing sequences of positive integers $(t_i)_{i\in\mathbb{N}}$, $(t_j)_{j\in\mathbb{N}}$.
Now let $A = \pi(\tilde{A})$ where $\tilde{A} := \bigcup_{n\in\mathbb{Z}} T^n(K \cup \gamma(\mathbb{R}^+))$. It is easy to see that A is a thin essential annular continuum. Furthermore, the set $G = K \cup \gamma(\mathbb{R}^+)$ is compact and connected, and hence a generator of A. Finally the set $S := \tilde{A} \setminus (\mathbb{R} \times \{0\})$ is connected since $S = \bigcup_{n\in\mathbb{Z}} T^n((J \cup T(J) \setminus I) \cup \gamma(\mathbb{R}^+))$. Hence, A has compact generator G and at the same time contains the infinite spike S. What is not clear to us is whether similar examples can be produced with an infinite spike that is not T-invariant.

Question 6.6. Suppose that A is a thin annular continuum which contains an infinite spike S with $T^n(S) \cap S = \emptyset$ for all $n \in \mathbb{N}$. Does this imply that A has no compact generator?

As an example in the class of continua given in (i)(a), we have the *Birkhoff attractor*. This is an essential thin circloid which has a segment as a rotation set for some map that leaves it invariant (see e.g. [**28**]). Hence, due to Corollary 5.4 the Birkhoff attractor cannot have a compact generator. For the class given in (i)(b) we can consider the continuum given by $A = \pi(\mathbb{R} \times \{0\} \cup \{(x, 1/x) \in \mathbb{R}^2 \mid x \geq 1\})$, which contains the infinite spike $S = \{(x, 1/x) \in \mathbb{R}^2 \mid x \geq 1\}$. Again, annular continua of this type can occur as invariant sets with non-unique rotation number for annular homeomorphisms. Examples, which are basically due to Walker [**32**], will be discussed in the next section.

Finally, as mentioned in Remark 1.1(c), we close with a result on the topology of essential annular continua in foliations given by a semiconjugacy.

PROPOSITION 6.7. *In the situation of Theorem 1.1, the following are equivalent.*
(i) *There exists a set $\Omega \subseteq \mathbb{T}^1$ of positive Lebesgue measure such that for all $\xi \in \Omega$ the fibre $A_\xi = h^{-1}(\xi)$ is compactly generated.*
(ii) *There exists a set $\Omega \subseteq \mathbb{T}^1$ of positive Lebesgue measure such that for all $\xi \in \Omega$ the fibre A_ξ is thin and the circloid C_ξ it contains is compactly generated.*

The significance of this statement lies in the fact that it demonstrates that there is at least one mechanism, compactly generated circloids with infinite spikes attached, which can lead to the non-uniqueness of the rotation vector in the invariant case, but not in the case of a semiconjugacy to an irrational rotation.

Proof of Proposition 6.7. By Proposition 6.5, if A_ξ is compactly generated on a set of positive measure, then so is C_ξ. Conversely, suppose the circloids C_ξ are compactly generated for all $\xi \in \Omega$, where $\Omega \subseteq \mathbb{T}^1$ has positive measure. For all $\xi \in \Omega$, let G_ξ be a compact generator of C_ξ with $\mathrm{diam}(G_\xi) = \tau(C_\xi)$. Further, let τ be as in (5.2). Using Lemma 6.1, it can be shown in exactly the same way as in Corollary 5.9 that the mapping $\xi \mapsto \tau(C_\xi)$ has a measurable majorant η. Using this, we define

$$\Omega_n = \{\xi \in \mathbb{T}^1 \mid \eta(\xi) \leq n\}.$$

Then $\Omega' = \bigcup_{n\in\mathbb{N}} \Omega_n$ has the same measure as Ω.

Fix any $n \in \mathbb{N}$ and any Lebesgue density point in Ω_n. Then there exist sequences ξ_n^- and ξ_n^+ such that $\xi \in (\xi_n^-, \xi_n^+)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(\xi_n^-, \xi_n^+) = 0$. Exactly as in the proof of Lemma 5.8, we have that $\tilde{G}^\pm = \lim^{\mathcal{H}}_{n\to\infty} G_{\xi_n^\pm}$ is a compact generator of $A_\xi^\pm$, where we go over to subsequences if necessary in order to force convergence. Consequently $\tilde{G}_\xi = \tilde{G}_\xi^- \cup \tilde{G}_\xi^+$ is a compact generator of A_ξ. Since Lebesgue density points have full measure in Ω_n and $\lim_{n\to\infty} \mathrm{Leb}_{\mathbb{T}^1}(\Omega_n) = \mathrm{Leb}_{\mathbb{T}^1}(\Omega)$, this proves the statement. □

7. *Rotation intervals for thin annular continua: construction of examples*

Our final objective is to construct examples of invariant thin essential annular continua which have compactly generated circloid, at least one infinite spike and a non-trivial rotation interval. As mentioned before, our construction is similar to that of Walker [**32**]. It leads to the following statement.

PROPOSITION 7.1. *Given any segment $I \subset \mathbb{R}$, there exists a map $f \in \mathrm{Homeo}(A)$ which leaves invariant an essential thin annular continuum $A \subset \mathbb{A}$ such that C_A has compact generator, A has an infinite spike, and $\rho_A(f) = I$.*

Proof. Let $\mathcal{D} \subset \mathrm{Diffeo}_+(\mathbb{T}^1)$ be the set of lifts $G : \mathbb{R} \to \mathbb{R}$ of orientation-preserving circle diffeomorphisms g with a totally disconnected non-wandering set $\Omega(g)$. Note that this means g either has rational rotation number and a totally disconnected set of periodic points, or g is a Denjoy example (with irrational rotation number).

Our aim is to construct a family of examples of homeomorphisms $f_{G,\alpha}$ of $\mathbb{A}$, parametrized by $G \in \mathcal{D}$ and $\alpha \in \mathbb{R}$, such that:

(i) $f_{G,\alpha}$ leaves invariant some annular continuum $A_{G,\alpha}$ with compactly generated circloid and at least one infinite spike; and

(ii) $\rho_{A_{G,\alpha}}(F) = \mathrm{conv}(\{\alpha, \rho(G)\})$, where $\mathrm{conv}(X)$ denotes the convex hull of X, and $\rho(G)$ is the rotation number of G.

This will prove Proposition 7.1.

For any $t \in [0, \infty)$, let $\mathcal{R}_t = \mathbb{R} \times \{t\}$ and define $i : (0, +\infty) \to \mathbb{R}$ by $i(x) = 1/x$. Further, let $\mathcal{L} = \{L_p\}_{p\in\mathbb{R}}$ be the C^∞-foliation of $\mathbb{R} \times (0, +\infty)$ whose leaves are given by

$$L_p = \mathrm{graph}(i) + (p - 1, 0)$$

for every $p \in \mathbb{R}$, where $\mathrm{graph}(i) = \{(x, i(x)) \mid x > 0\}$. Hence, all leaves are horizontal translates of each other, and L_p is the leaf passing through the point $(p, 1)$, see Figure 2. Let

$$p(x, y) = x - \frac{1}{y} + 1$$

and note that thus $(x, y) \in L_{p(x,y)}$ for all $(x, y) \in \mathbb{R} \times (0, \infty)$.

Now, consider

$$F_1 : \mathbb{R} \times (0, \infty) \to \mathbb{R} \times (0, \infty), \quad (x, y) \mapsto (x + v(x, y), y),$$

where

$$v(x, y) = G(p(x, y)) - p(x, y).$$

Note that $p(F_1(x, y)) = x + v(x, y) + 1 - 1/y = p(x, y) + v(x, y) = G(p(x, y))$, such that $F_1(L_p) = L_{G(p)}$. Hence, the map F_1 permutes the leaves of the foliation $\mathcal{L}$ according to the dynamics given by G, while leaving the second coordinate invariant. In particular, this means that F_1 preserves the set

$$\mathcal{T} := \bigcup_{p \in \pi^{-1}(\Omega(g))} L_p.$$

Further, F_1 is a C^1 diffeomorphism since p is C^∞ and G is C^1.

Given $(x, y) \in \mathbb{R} \times (0, \infty)$, let $X(x, y)$ be the vector which is tangent to $L_{p(x,y)}$ in the point (x, y) and which is scaled such that its first coordinate is $\alpha - v(F_1^{-1}(x, y)) = \alpha - p(x, y) + G^{-1}(p(x, y))$. In explicit form, we have

$$X(x, y) = (\alpha - v(F_1^{-1}(x, y)), t(x, y)),$$

where

$$t(x, y) = -\frac{\alpha - v(F_1^{-1}(x, y))}{(x - p(x, y) + 1)^2}.$$

Then X defines a $\mathcal{C}^1$-vector field on $\mathbb{R} \times (0, \infty)$.

Choose an increasing $\mathcal{C}^1$-function $\eta : (0, \infty) \to [0, \infty)$ such that $\eta(y) = 0$ for $y \geq 2/3$ and $\eta(y) = 1$ if $0 \leq y \leq 1/3$ and let $\tilde{X}(x, y) = \eta(y)X(x, y)$. Then again $\tilde{X}$ is a $\mathcal{C}^1$-vector field, which induces a flow $\Phi^{\tilde{X}}$ on $\mathbb{R} \times (0, \infty)$. We denote the time-one-map of this flow by F_2 and define $F_{G,\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{G,\alpha} = \begin{cases} F_2 \circ F_1(x, y) & \text{if } y > 0, \\ (x + \alpha, y) & \text{if } y \leq 0. \end{cases}$$

See Figure 2 for a geometric intuition. By periodicity of the construction in the x-direction, $F_{G,\alpha}$ induces a map $f_{G,\alpha}$ on $\mathbb{A}$, and we claim that this has the properties stated above. In order to see this, note that F_1 preserves the horizontal lines $\mathcal{R}_t$, $t > 0$, and since $\eta(y) = 0$ if $y \geq 2/3$ this implies that $F_{G,\alpha}$ preserves all of the horizontal lines above $\mathcal{R}_{2/3}$. Further, the flow $\Phi^{\tilde{X}}$ preserves the foliation $\mathcal{L}$, even leaf by leaf, since by definition it is a flow along the leaves of this foliation. Therefore, $F_{G,\alpha}$ preserves the horizontal strip continuum $(\mathcal{R}_0 \cup \mathcal{T}) \cap (\mathbb{R} \times [0, 1])$, which projects to an annular continuum $A_{G,\alpha}$ with the properties stated above.

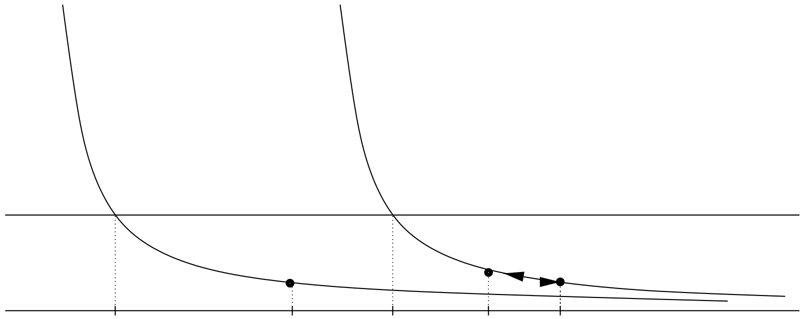

FIGURE 2. Two-step construction of the map $F_{G,\alpha}$. The flow $\Phi^{\tilde{X}}$ used in order to define F_2 moves points along the leaves of the foliation $\mathcal{L}$. Owing to the geometry of $\mathcal{L}$, orbits close to $\mathcal{R}_0$ remain near $\mathcal{R}_0$ for a long time and move with constant speed $\alpha - (\pi_1 \circ F_1(x, y) - x)$ in the x-direction, such that $\pi_1 \circ F_{G,\alpha}(x, y) - x = \alpha$.

Moreover, $F_{G,\alpha}$ is bijective and a homeomorphism when restricted to either the open upper half-plane or the closed lower half-plane. In order to show that it is a homeomorphism of $\mathbb{R}^2$, it only remains to check the continuity of $F_{G,\alpha}$ on the line $\mathcal{R}_0$. However, due to the geometry of the foliation $\mathcal{L}$ and the definition of the vector field $\tilde{X}$, which coincides with X in $\mathbb{R} \times [0, 1/3]$, points which are close to $\mathcal{R}_0$ get mapped to points close to $\mathcal{R}_0$ again. The reason is that orbits of $\Phi^{\tilde{X}}$ starting close to $\mathcal{R}_0$ travel with bounded speed along the leaves of the foliation $\mathcal{L}$, which are almost horizontal near $\mathcal{R}_0$. Furthermore, if these orbits start sufficiently close to $\mathcal{R}_0$, then they will remain in the region $\mathbb{R} \times [0, 1/3]$ until time 1. Since the first coordinate of the vector field is equal to $\alpha - v(F_1^{-1}(x, y))$, which is constant along the leaves of the foliation as it only depends on $p(x, y)$, we obtain that $\pi_1(F_2(x, y)) - x = \alpha - v(F_1^{-1}(x, y)$ for sufficiently small $y > 0$ and all $x \in \mathbb{R}$. However, this means that $\pi_1(F_{G,\alpha}(x, y)) - x = \alpha$. Altogether, this shows the continuity of $F_{G,\alpha}$ on $\mathcal{R}_0$.

Finally, we need to check that $\rho(F_{G,\alpha}) = \mathrm{conv}(\alpha, G(\alpha))$. By going over to the inverses if necessary, we may assume without loss of generality that $\rho(G) > \alpha$. In this case, the line $\mathcal{R}_0$ is a repeller, since in order to make up for the difference $\alpha - \rho(G) < 0$ of the rotation numbers, orbits close to $\mathcal{R}_0$ have to move to the left, and hence upwards, until they leave the region $\mathbb{R} \times [0, 1/3]$.

Consequently, all forward orbits starting strictly above $\mathcal{R}_0$ will remain bounded away from $\mathcal{R}_0$. This means, however, that the part of the horizontal displacement $\pi_1 \circ F^n_{G,\alpha}(x, y) - x$ which comes from the movement along the leaves is bounded independent of n. Hence, the asymptotic speed of these orbits is determined by the permutation of the leaves by F_1, which implies that they have rotation number $\rho(G)$. Hence, all rotation vectors are either $\rho(G)$ or α, and since the endpoints of the rotation interval are always realized by pointwise rotation vectors, we obtain $\rho_{A_{G,\alpha}}(F_{G,\alpha}) = [\alpha, \rho(G)]$. □

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