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Manil T. Mohan

Sivaguru S. Sritharan
Air Force Institute of Technology

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Stochastic quasilinear evolution equations in UMD Banach spaces

Manil T. Mohan[*1] and **Sivaguru S. Sritharan**[** 2]

[1] Department of Mathematics and Statistics, Air Force Institute of Technology, 2950 Hobson Way, Wright Patterson Air Force Base, OH 45433, USA

[2] Office of the Vice Chancellor & Provost, Air Force Institute of Technology, 2950 Hobson Way, Wright Patterson Air Force Base, OH 45433, USA

In this work we prove the existence and uniqueness up to a stopping time for the stochastic counterpart of Tosio Kato's quasilinear evolutions in UMD Banach spaces. These class of evolutions are known to cover a large class of physically important nonlinear partial differential equations. Existence of a unique maximal solution as well as an estimate on the probability of positivity of stopping time is obtained. An example of stochastic Euler and Navier-Stokes equation is also given as an application of abstract theory to concrete models.

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1 Introduction

In a seminal paper [10], Tosio Kato established the existence and uniqueness of local in time mild solutions of the Cauchy problem for various quasilinear equations of evolution. He showed that a wide range of important physical problems can be modeled in a unified manner by a class of quasilinear evolution equations in a Banach space. These examples include the first order symmetric hyperbolic systems, second order nonlinear wave equations, Korteweg-de Vries (KdV) equation, Navier-Stokes equations, Euler equations of fluid dynamics, equations of compressible fluid flow, compressible viscoelastic fluid flow equations, magnetohydrodynamic (MHD) equations, coupled Maxwell and Dirac equations of quantum electrodynamics, and Einstein field equations of general relativity. As a stochastic counterpart of Kato's theory, the paper [7] considered the stochastic quasilinear evolution equation in a separable Hilbert space with Gaussian cylindrical Wiener noise and outlined the ideas for existence and uniqueness of local mild solutions using fixed point arguments. In this work we consider the stochastic quasilinear evolution equation in a reflexive UMD Banach space with more general noise coefficient and prove the existence and uniqueness of local pathwise mild solution up to a stopping time as well as the existence of a maximal solution. This paper also improves several other aspects of the Hilbert space case presented in [7] such as the introduction of stopping time in the arguments along with an estimate of its probability of positivity and a clarification on the conditions on the covariance structure of the noise. We demonstrate the application of abstract theory to concrete models by giving an example of the stochastic Euler and Navier-Stokes equations perturbed by Gaussian cylindrical Wiener noise. The current literature on stochastic evolutions, for example [3, 4, 21, 27, 22, 26], do not cover linear and quasilinear evolution equation of hyperbolic type. To the best of the authors knowledge, a systematic treatment for the local solvability of stochastic quasilinear evolution equation of hyperbolic type by extending Kato's theory is only treated in [7] and this paper.

*e-mail: **?** , Phone: ? , Fax: ?
**Corresponding author: e-mail: sivaguru.sritharan@afit.edu, Phone: +1 937 255 6565 X 3315, Fax: ?

2 Stochastic quasilinear evolution equation

Let us consider the Cauchy problem for the stochastic quasilinear evolution equation

$$\left.\begin{array}{r}
d\mathbf{u}(t) + A(t, \mathbf{u}(t))\mathbf{u}(t)dt = \Phi dW_{\mathbb{H}}(t), \ 0 \le t \le T, \\
\mathbf{u}(0) = \mathbf{u}_0,
\end{array}\right\} \tag{2.1}$$

in a Banach space $\mathbb{X}$. Let $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$ and $D(A)$ denotes the domain of any operator A. Let us begin with a set of assumptions as in [10] and [7].

2.1 The main assumptions are;

(A1) Let $\mathbb{X}$ be a reflexive UMD Banach space of type 2 (defined in section 2.1). There is an another reflexive UMD Banach space $\mathbb{Y} \subset \mathbb{X}$ of type 2 which is continuously and densely embedded in $\mathbb{X}$. There exists an isomorphism $S : \mathbb{Y} \to \mathbb{X}$ and the norm of $\mathbb{Y}$ is chosen so that S becomes an isometry, i.e., $\|\mathbf{u}\|_{\mathbb{Y}} = \|S\mathbf{u}\|_{\mathbb{X}}$.

(A2) Let $\mathbb{W}$ be an open ball in $\mathbb{Y}$. Let $A(\cdot, \cdot) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ be a function on $[0, T] \times \mathbb{W}$ into $G(\mathbb{X}, 1, \beta(\mathbf{y}))$ (defined in section 2.1), where $\beta(\cdot)$ is a real number and is a constant for $\mathbf{y} \in \mathbb{W}$, i.e.,

$$\left\|e^{-sA(t, \mathbf{y})}\right\|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \le e^{\beta(\mathbf{y})s}, \ s \in [0, \infty), \ t \in [0, T], \ \mathbf{y} \in \mathbb{W}. \tag{2.2}$$

(A3) For all $t \in [0, T]$ and $\mathbf{y} \in \mathbb{W}$, we have

$$SA(t, \mathbf{y})S^{-1} = A(t, \mathbf{y}) + B(t, \mathbf{y}), \tag{2.3}$$

where

$$B(t, \mathbf{y}) \in \mathcal{L}(\mathbb{X}, \mathbb{X}) \text{ and } \|B(t, \mathbf{y})\|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \le \lambda_1(\mathbf{y}), \tag{2.4}$$

where $\lambda_1(\mathbf{y}) > 0$ is a constant for $\mathbf{y} \in \mathbb{W}$. Including the domain relation, (2.3) is satisfied in the strict sense. That is, a function $\mathbf{x} \in \mathbb{X}$ which is in $D(A(t, \mathbf{y}))$ if and only if $S^{-1}\mathbf{x} \in D(A(t, \mathbf{y}))$ with $A(t, \mathbf{y})S^{-1}\mathbf{x} \in \mathbb{Y}$.

(A4) For all $t \in [0, T]$ and $\mathbf{y} \in \mathbb{W}$, we have $A(t, \mathbf{y}) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ (in the sense that $\mathbb{Y} \subset D(A(t, \mathbf{y}))$ and the restriction of $A(t, \mathbf{y})$ to $\mathbb{Y}$ is in $\mathcal{L}(\mathbb{Y}, \mathbb{X})$) and

$$\|A(t, \mathbf{y})\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \le \lambda_2(\mathbf{y}), \tag{2.5}$$

where $\lambda_2(\mathbf{y}) > 0$ is a constant for $\mathbf{y} \in \mathbb{W}$. Also,

(i) for all $\mathbf{y} \in \mathbb{W}$, $A(t, \mathbf{y})$ is continuous in the $\mathcal{L}(\mathbb{Y}, \mathbb{X})$−norm,

(ii) for all $t \in [0, T]$, $A(t, \cdot)$ is Lipschitz continuous, that is,

$$\|A(t, \mathbf{y}_1) - A(t, \mathbf{y}_2)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \le \mu(\mathbf{y}_1, \mathbf{y}_2)\|\mathbf{y}_1 - \mathbf{y}_2\|_{\mathbb{X}}, \tag{2.6}$$

where $\mu(\mathbf{y}_1, \mathbf{y}_2) > 0$ is a constant for $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{W}$.

(A5) Let $\mathbf{y}_0$ be the center of $\mathbb{W}$. Then $A(t, \mathbf{y})\mathbf{y}_0 \in \mathbb{Y}$ for all $t \in [0, T]$, $\mathbf{y} \in \mathbb{W}$, and

$$\|A(t, \mathbf{y})\mathbf{y}_0\|_{\mathbb{Y}} \le \lambda_3(\mathbf{y}), \ t \in [0, T], \ \mathbf{y} \in \mathbb{W}, \tag{2.7}$$

where $\lambda_3(\mathbf{y}) > 0$ is a constant for $\mathbf{y} \in \mathbb{W}$.

In (2.1), $W_{\mathbb{H}}(\cdot)$ is a cylindrical Wiener process in $\mathbb{H}$, where $\mathbb{H}$ is a separable Hilbert space, and Φ is a $\gamma-$radonifying operator (defined in section 2.1) in $\mathbb{X}$ with the following properties:

(i) $\Phi \in \gamma(\mathbb{H}, \mathbb{X})$ with $\|\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})} < \infty$,

(ii) $S\Phi \in \gamma(\mathbb{H}, \mathbb{X})$ with $\|S\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})} < \infty$,

(iii) $S^2\Phi \in \gamma(\mathbb{H}, \mathbb{X})$ with $\|S^2\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})} < \infty$,

where $\gamma(\mathbb{H}, \mathbb{X})$ is the space of all $\gamma-$radonifying operators from $\mathbb{H}$ to $\mathbb{X}$. If $\mathbb{X}$ is a Hilbert space, then $\gamma(\mathbb{H}, \mathbb{X}) = \mathscr{L}_2(\mathbb{H}, \mathbb{X})$ isometrically, where $\mathscr{L}_2(\mathbb{H}, \mathbb{X})$ denotes the space of all Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{X}$.

Remark 2.2 The theory discussed in this paper works for any reflexive UMD Banach space, with some suitable modifications in the above properties (i)-(iii).

Definition 2.3 Let $\tau(\omega)$ be a given stopping time and $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. An $(\mathscr{F}_t)_{t \in [0, \tau)}-$adapted stochastic process $\{\mathbf{u}(t), t \geq 0\} \in \mathrm{C}(0, \tau(\omega); \mathbb{W})$, a.s., is a *local pathwise mild solution* in $\mathbb{Y}$ of the stochastic quasilinear evolution equation (2.1) if

  (i) $\{\mathbf{u}(t), t \geq 0\}$ is jointly measurable with respect to (t, ω) and $\mathbb{E}\left[\sup\limits_{0 \leq t \leq \tau(\omega)} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right] < \infty$,

  (ii) $(\mathscr{F}_t)_{t \in [0, \tau)}-$adapted paths of $\mathbf{u}(\cdot)$ are continuous,

 (iii) for all $t \in [0, \tau(\omega))$,

$$\mathbf{u}(t) = \mathrm{U}^{\mathbf{u}}(t, 0)\mathbf{u}_0 + \mathrm{U}^{\mathbf{u}}(t, 0) \int_0^t \Phi \mathrm{dW}_{\mathbb{H}}(t) + \int_0^t \mathrm{U}^{\mathbf{u}}(t, s) \mathrm{A}(s, \mathbf{u}(s)) \left(\int_s^t \Phi \mathrm{dW}_{\mathbb{H}}(r)\right) \mathrm{d}s, \quad (2.8)$$

holds with probability one (in (2.8) $\mathrm{U}^{\mathbf{u}}(\cdot, \cdot)$ is the random evolution operator, see Section 2.3),

 (iv) for a given $0 < \delta < 1$, $\mathbb{P}\{\tau(\omega) > \delta\} \geq 1 - \delta^2 \mathcal{M}$, where $\mathcal{M}$ is a constant dependent of $\mathbf{u}_0$ and Φ, and independent of δ.

A local pathwise mild solution $(\mathbf{u}(t))_{t \in [0, \tau)}$ is called *maximal pathwise mild solution* in $\mathbb{Y}$ consisting of $\mathrm{C}(0, \tau(\omega); \mathbb{W})-$valued admissible processes, if for any other local pathwise mild solution $(\widetilde{\mathbf{u}}(t))_{t \in [0, \widetilde{\tau})}$ in $\mathbb{Y}$, almost surely we have $\widetilde{\tau} \leq \tau$ and $\widetilde{\mathbf{u}} \equiv \mathbf{u}\big|_{[0, \widetilde{\tau})}$. Clearly, a maximal local mild solution in $\mathbb{Y}$ is always unique in $\mathbb{Y}$. We say that τ is an *explosion time* if for almost all $\omega \in \Omega$ with $\tau(\omega) < T$,

$$\lim_{t \uparrow \tau(\omega)} \left[\sup_{0 \leq s \leq t} \|\mathbf{u}(s, \omega)\|_{\mathbb{Y}}\right] = \infty. \quad (2.9)$$

The main theorem of this paper is

Theorem 2.4 *Let* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$ *be a given filtered probability space. Let Assumption 2.1 be satisfied and let* $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, *a.s. Then, for the stopping time* τ_N *defined by*

$$\tau_N := \inf_{t \geq 0} \left\{ t : \beta(\mathbf{u}(t)) \vee \lambda_1(\mathbf{u}(t)) \vee \lambda_2(\mathbf{u}(t)) \vee \lambda_3(\mathbf{u}(t)) \vee \mu(\mathbf{u}(t)) \vee \left\|\int_0^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}} \right.$$
$$\left. \vee \left\|\int_0^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}} \geq N \right\}, \quad (2.10)$$

for $N \in \mathbb{N}$, *we have*

  (i) *for* $t \in [0, \widetilde{T} \wedge \tau_N)$, $\widetilde{T} \leq T$, *there exists an* $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N)}-$*adapted stochastic process* $\mathbf{u}(\cdot)$ *having continuous trajectories satisfying (2.8) with probability one in* $\mathbb{L}^2(\Omega; \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{W}))$,

  (ii) $\mathbb{E}\left[\sup\limits_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right] \leq 3e^{4N\widetilde{T}} \left\{\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right] + N^2 \left(1 + 4N^2\widetilde{T}\right)\right\}$,

 (iii) *for a given* $0 < \delta < 1$,

$$\mathbb{P}\{\tau_N(\omega) > \delta\} \geq 1 - C\delta^2 \left\{\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right] + C\left(\delta\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})}^2 + 8\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})}^2 + 8\|\mathrm{S}^2\Phi\|_{\gamma(\mathbb{H}, \mathbb{X})}^2\right)\right\},$$

where C *is a positive constant independent of* δ,

 (iv) *there exists a unique pathwise maximal solution* $(\mathbf{u}(\cdot), \tau_\infty)$, *where* $\tau_\infty = \lim\limits_{N \to \infty} \tau_N$.

To establish the existence and uniqueness of local pathwise mild solution of (2.1), we proceed as follows:

1. For certain function $t \mapsto \mathbf{v}(t) \in \mathbb{Y}$, we consider the stochastic linear evolution equation with random drift

$$\left.\begin{array}{r} d\mathbf{u}(t) + \mathrm{A}(t, \mathbf{v}(t))\mathbf{u}(t)dt = \Phi d\mathrm{W}_{\mathbb{H}}(t), \ 0 \leq t \leq T, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{array}\right\} \tag{2.11}$$

2. We prove that (2.11) has a unique solution $\mathbf{u} = \mathbf{u}(t)$ for $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., where $\mathbb{W}$ is an open ball in $\mathbb{Y}$ of radius R (throughout this manuscript), by constructing a random evolution operator to the problem (2.11), and then we define a mapping $\mathbf{v} \mapsto \mathbf{u} = \Psi(\mathbf{v})$.

3. We show that the map $\Psi(\cdot)$ has a fixed point, which is the unique solution of (2.11), by using the contraction mapping theorem.

2.1 Preliminaries

In this subsection, we give some basic concepts of UMD Banach spaces and stable family of generators of a C_0-semigroup.

Definition 2.5 Let $(\gamma_n)_{n=1}^{\infty}$ be a sequence of independent standard Gaussian random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ (we use the notation $(\Omega, \mathscr{F}, \mathbb{P})$ for the probability space on which our process is defined). A bounded linear operator $\mathrm{R} \in \mathcal{L}(\mathbb{H}, \mathbb{X})$ is said to be $\gamma-radonifying$ if there exists an orthonormal basis $(e_n)_{n=1}^{\infty}$ of $\mathbb{H}$ such that the Gaussian series $\sum_{n=1}^{\infty} \gamma_n \mathrm{R}e_n$ converges in $\mathbb{L}^2(\widetilde{\Omega}; \mathbb{X})$. Then, we define

$$\|\mathrm{R}\|_{\gamma(\mathbb{H}, \mathbb{X})} := \left(\widetilde{\mathbb{E}} \left\| \sum_{n=1}^{\infty} \gamma_n \mathrm{R}e_n \right\|_{\mathbb{X}}^2 \right)^{\frac{1}{2}}, \tag{2.12}$$

and the number $\|\mathrm{R}\|_{\gamma(\mathbb{H}, \mathbb{X})}$ does not depend on the sequence $(\gamma_n)_{n=1}^{\infty}$ and the basis $(e_n)_{n=1}^{\infty}$. It defines a norm on the space $\gamma(\mathbb{H}, \mathbb{X})$ of all $\gamma-$radonifying operators from $\mathbb{H}$ into $\mathbb{X}$.

Definition 2.6 Let $1 \leq p \leq 2$. A Banach space $\mathbb{X}$ is called of *type* p if there exists a constant $\alpha \geq 0$ such that for all finite sequences $(x_n)_{n=1}^{N}$ in $\mathbb{X}$, we have

$$\mathbb{E}\left[\left\| \sum_{n=1}^{N} r_n x_n \right\|_{\mathbb{X}}^p \right] \leq \alpha^p \sum_{n=1}^{N} \|x_n\|_{\mathbb{X}}^p, \tag{2.13}$$

where $(r_n)_{n \geq 1}$ be a Rademacher sequence, i.e., a sequence of independent random variables taking the values ± 1 with probability $1/2$.

The least admissible constant is denoted by $\alpha_{p, \mathbb{X}}$.

Definition 2.7 Let $(\mathrm{M}_n)_{n=1}^{N}$ be an $\mathbb{X}-$valued martingale. The sequence $(d_n)_{n=1}^{N}$ defined by $d_n := \mathrm{M}_n - \mathrm{M}_{n-1}$ with $\mathrm{M}_0 = 0$ is called the *martingale difference sequence* associated with $(\mathrm{M}_n)_{n=1}^{N}$.

We call $(d_n)_{n=1}^{N}$ an $\mathbb{L}^p-$*martingale difference sequence* if it is the difference sequence of an $\mathbb{L}^p-$martingale.

Definition 2.8 Let $1 \leq p \leq 2$. A Banach space $\mathbb{X}$ is of *martingale type* p if there exists a constant $\kappa \geq 0$ such that for all all finite $\mathbb{X}-$valued martingale difference sequences $(d_n)_{n=1}^{N}$, we have

$$\mathbb{E}\left[\left\| \sum_{n=1}^{N} d_n \right\|_{\mathbb{X}}^p \right] \leq \kappa^p \sum_{n=1}^{N} \mathbb{E}\left[\|d_n\|_{\mathbb{X}}^p \right]. \tag{2.14}$$

The least admissible κ is denoted by $\kappa_{p, \mathbb{X}}$. Since every Gaussian sequence is a martingale difference sequence, every Banach space of martingale type p is type p, with constant $\alpha_{p, \mathbb{X}} \leq \kappa_{p, \mathbb{X}}$.

Definition 2.9 A Banach space $\mathbb{X}$ is called a UMD$-$space *(unconditional martingale differences)* if for some $p \in (1, \infty)$ (equivalently for all $p \in (1, \infty)$), there exists a constant $\eta \geq 0$ such that for all $\mathbb{X}-$valued $\mathbb{L}^p-$martingale difference sequences $(d_n)_{n=1}^N$ and all signs $(\varepsilon_n = \pm 1)_{n=1}^N$, we have

$$\mathbb{E}\left[\left\|\sum_{n=1}^N \varepsilon_n d_n\right\|_{\mathbb{X}}^p\right] \leq \eta^p \mathbb{E}\left[\left\|\sum_{n=1}^N d_n\right\|_{\mathbb{X}}^p\right], \tag{2.15}$$

for all $N \geq 1$.

The least admissible constant in Definition 2.9 is called the UMD$_p-$constant of $\mathbb{X}$ and is denoted by $\eta_{p,\mathbb{X}}$.

Remark 2.10 Let $1 \leq p \leq 2$ and $\mathbb{X}$ be a UMD Banach space with type p, then $\mathbb{X}$ is of martingale type p and $\kappa_{p,\mathbb{X}} \leq \eta_{p,\mathbb{X}}\alpha_{p,\mathbb{X}}$ (see Proposition 5.3, [23]).

We now consider UMD Banach spaces of type 2. A Banach space $\mathbb{X}$ has type 2 if and only if we have the inclusion

$$\mathbb{L}^2(0, T; \gamma(\mathbb{H}, \mathbb{X})) \hookrightarrow \gamma(\mathbb{L}^2(0, T; \mathbb{H}), \mathbb{X}),$$

for any $T > 0$. Let $W_\mathbb{H}$ be an $\mathbb{H}-$valued cylindrical Wiener process on $(\Omega, \mathscr{F}, \mathbb{P})$. An $\mathbb{H}-$strongly measurable process $\Phi : [0, T] \times \Omega \to \mathcal{L}(\mathbb{H}, \mathbb{X})$ satisfying the equivalent conditions of Theorem 3.6, [20] will be called $\mathbb{L}^p-$*stochastically integrable* with respect to $W_\mathbb{H}$. The *stochastic integral* of Φ with respect to $W_\mathbb{H}(\cdot)$ is denoted by $\int_0^T \Phi(t) dW_\mathbb{H}(t)$.

Theorem 2.11 *Let $\mathbb{X}$ be a* UMD$-$*space and let $p \in (1, \infty)$. If $\mathbb{X}$ has type 2, then every $\mathbb{H}-$strongly measurable and adapted process Φ which belongs to $\mathbb{L}^p(\Omega; \mathbb{L}^2(0, T; \gamma(\mathbb{H}, \mathbb{X})))$ is $\mathbb{L}^p-$stochastically integrable with respect to $W_\mathbb{H}$ and we have*

$$\mathbb{E}\left[\left\|\int_0^T \Phi(t) dW_\mathbb{H}(t)\right\|_{\mathbb{X}}^p\right] \leq C_{p,\mathbb{X}} \mathbb{E}\left[\|\Phi\|_{\mathbb{L}^2(0,T;\gamma(\mathbb{H},\mathbb{X}))}^p\right]. \tag{2.16}$$

P r o o f. See Corollary 3.10, [20], section 5, [24]. $\square$

Hence if $\mathbb{X}$ is a UMD Banach space and has type 2, then for every adapted and strongly measurable $\Phi \in \mathbb{L}^p(\Omega; \mathbb{L}^2(0, T; \gamma(\mathbb{H}, \mathbb{X})))$, the non-anticipating stochastic integral process $\left(\int_0^t \Phi(s) dW_\mathbb{H}(s)\right)_{t \in [0,T]}$ exists and is pathwise continuous. Hence, for all $p \in (1, \infty)$ there exists a constant $C_{p,\mathbb{X}}$ independent of $\Phi(\cdot)$ such that the following one-sided estimate holds:

$$\left\{\mathbb{E}\left[\sup_{t \in [0,T]}\left\|\int_0^t \Phi(s) dW_\mathbb{H}(s)\right\|_{\mathbb{X}}^p\right]\right\}^{1/p} \leq C_{p,\mathbb{X}}\left\{\mathbb{E}\left[\int_0^T \|\Phi(t)\|_{\gamma(\mathbb{H},\mathbb{X})}^2 dt\right]^{p/2}\right\}^{1/p}. \tag{2.17}$$

For the Burkholder-Davis-Gundy inequality in UMD$_p$ Banach spaces, for $1 < p < \infty$, see Theorem 4.4, [20]. For more details on $\gamma-$radonifying operators and stochastic integration in UMD Banach spaces, interested readers readers may look into [19, 20, 23].

Let $G(\mathbb{X})$ denotes the set of all negative generators of C_0-semigroup on $\mathbb{X}$, i.e., a linear operator A in a Banach space $\mathbb{X}$ is in $G(\mathbb{X})$ if it generates a semigroup $U = \{U(t) = e^{-tA}; 0 \leq t < \infty\}$ of class C_0 ([11]). Following Kato [10], let us give some basic definitions:

Definition 2.12 Let $G(\mathbb{X}, M, \beta)$ denotes the set of all linear operators A in $\mathbb{X}$ such that $-A$ generates a C_0-semigroup $\{e^{-tA}\}$ with

$$\|e^{-tA}\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \leq Me^{\beta t}, 0 \leq t < \infty, \beta \in \mathbb{R}.$$

The operator A is $m-$*accretive* if $A \in G(\mathbb{X}, 1, 0)$, in which case $\{e^{-tA}\}$ is a contraction semigroup. A is said to be *quasi$-m-$accretive* if $A \in G(\mathbb{X}, 1, \beta)$.

Definition 2.13 A family $A = \{A(t)\}$ of elements of $G(\mathbb{X})$, is said to be *stable* if there are constants M and β such that

$$\left\| \prod_{j=1}^{k} (A(t_j) + \lambda)^{-1} \right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \leq M(\lambda - \beta)^{-k}, \ \lambda > \beta, \tag{2.18}$$

for every finite family $0 \leq t_1 \leq \cdots \leq t_k \leq T, k = 1, 2, \cdots$. The pair (M, β) is called the *stability index* for $\{A(t)\}$. In (2.18), the operator product is *time-ordered*, i.e., if $t_i > t_j$, then the operator $A(t_j)$ will be on the left of the operator $A(t_i)$.

It can be shown that (2.18) is equivalent to ([8])

$$\left\| \prod_{j=1}^{k} e^{-s_j A(t_j)} \right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \leq M e^{\beta(s_1 + \cdots + s_k)}, \tag{2.19}$$

for all t_j such that $0 \leq t_1 \leq \cdots \leq t_k \leq T, k = 1, 2, \cdots$ and all $s_j \geq 0$, and the product on the left is time-ordered. The family $\{A(t)\}$ is trivially stable with stability index $(1, \beta)$ if $A(t) \in G(\mathbb{X}, 1, \beta)$.

Definition 2.14 ([25, 6]) Let A be a linear operator in $\mathbb{X}$ such that $-A$ generates a C_0−semigroup and $\mathbb{Y}$ be a subspace of $\mathbb{X}$ which is closed with respect to the norm $\|\cdot\|_{\mathbb{Y}}$. Then $\mathbb{Y}$ is called A−admissible if $e^{-tA} \mathbb{Y} \subseteq \mathbb{Y}$, for $t \geq 0$, i.e., $\mathbb{Y}$ is an invariant subspace of e^{-tA} and the restriction of e^{-tA} to $\mathbb{Y}$ is a C_0−semigroup in $\mathbb{Y}$, i.e., it is strongly continuous in the norm $\|\cdot\|_{\mathbb{Y}}$.

2.2 The Class $\mathscr{S}$

Let us assume that $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., where $\mathbb{W}$ is an open ball in $\mathbb{Y}$ of radius R (here R is deterministic). Since $\mathbb{W}$ is an open ball in $\mathbb{Y}$ containing $\mathbf{u}_0$, we can choose the $R > 0$ such that $\|\mathbf{u}_0 - \mathbf{y}_0\|_{\mathbb{Y}} < R$, a.s., where $\mathbf{y}_0$ is the center of the ball $\mathbb{W}$. Let $\mathscr{S}$ be the set of all functions $\mathbf{v}(\cdot, \cdot)$ from $[0, \widetilde{T} \wedge \tau_N] \times \Omega$ to $\mathbb{Y}$ such that

$$(i) \ \ \|\mathbf{v}(t, \omega) - \mathbf{y}_0\|_{\mathbb{Y}} \leq R, \ \text{a.s. so that } \mathbf{v} \in \mathbb{W}, \ \text{a.s.,} \tag{2.20}$$

$$(ii) \ \ \mathbf{v}(\cdot, \cdot) \text{ is continuous from } [0, \widetilde{T} \wedge \tau_N] \times \Omega \text{ to } \mathbb{X}, \tag{2.21}$$

$$(iii) \ \ \mathbf{v}(\cdot, \cdot) \text{ is } (\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]} - \text{adapted}, \tag{2.22}$$

where $\widetilde{T}$ is a positive number and $\widetilde{T} \leq T$, which will be determined later, and τ_N is the stopping time defined below (see (2.25)).

For $\mathbf{v} \in \mathscr{S}$, let us denote $A^{\mathbf{v}}(t) = A(t, \mathbf{v}(t)) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ for all $t \in [0, \widetilde{T}]$. With this notation, one can reduce (2.11) to

$$\left. \begin{aligned} d\mathbf{u}(t) + A^{\mathbf{v}}(t)\mathbf{u}(t)dt &= \Phi dW_{\mathbb{H}}(t), \ 0 \leq t \leq \widetilde{T}, \\ \mathbf{u}(0) &= \mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}, \end{aligned} \right\} \tag{2.23}$$

for $0 < \widetilde{T} \leq T$. Since $\Phi \in \gamma(\mathbb{H}, \mathbb{X})$, from (2.17), it can be easily seen that, for all $p \in (1, \infty)$, there exists a constant $C_{p,\mathbb{X}}$ independent of Φ such that the following one-sided estimate holds:

$$\mathbb{E}\left[\sup_{t \in [0,\widetilde{T}]} \left\| \int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}}^p \right] \leq C_{p,\mathbb{X}} \int_0^{\widetilde{T}} \|\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^p \, dt = C_{p,\mathbb{X}} \widetilde{T} \|\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^p. \tag{2.24}$$

Also, it is clear that (2.23) is a stochastic linear evolution equation in $\mathbf{u}(t)$. But unlike the linear evolution system described in [10] for the deterministic setting, here the drift is random due to the dependence of $\mathbf{v}(t, \omega)$ and we need to construct a *random evolution operator* in this context. Let us first define the sequence of stopping times τ_N to be

$$\tau_N := \inf_{t \geq 0} \left\{ t : \beta(\mathbf{v}(t)) \vee \lambda_1(\mathbf{v}(t)) \vee \lambda_2(\mathbf{v}(t)) \vee \lambda_3(\mathbf{v}(t)) \vee \mu(\mathbf{v}(t)) \vee \left\| \int_0^t S\Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \right.$$

$$\vee \left\| \int_0^t S^2 \Phi d W_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \geq N \right\},\qquad(2.25)$$

for $N \in \mathbb{N}$.

Remark 2.15 Since $\mathbf{v}(\cdot, \cdot)$ is random, the constants β, λ_1, λ_2, λ_3 and μ in Assumption 2.1 depends not only on the radius of the open ball $\mathbb{W}$, but also on N, where N is given in the definition of the stopping time (2.25). Hence, by making use of the stopping time, we have

(i) $\left\| e^{-s A(t, \mathbf{v}(t))} \right\|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \leq e^{N s}$, for all $s \in [0, \infty)$, $\mathbf{v}(\cdot) \in \mathbb{W}$ and $t \in [0, \widetilde{T} \wedge \tau_N]$,

(ii) $\| A(t, \mathbf{v}(t)) \|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \leq N$, for all $\mathbf{v}(\cdot) \in \mathbb{W}$ and $t \in [0, \widetilde{T} \wedge \tau_N]$,

(iii) $\| B(t, \mathbf{v}(t)) \|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \leq N$, for all $\mathbf{v}(\cdot) \in \mathbb{W}$ and $t \in [0, \widetilde{T} \wedge \tau_N]$,

(iv) $\| A(t, \mathbf{v}_1(t)) - A(t, \mathbf{v}_2(t)) \|_{\mathcal{L}(\mathbb{Y}, \mathbb{X})} \leq N \| \mathbf{y}_1 - \mathbf{y}_2 \|_{\mathbb{X}}$ for all $\mathbf{v}_1(\cdot), \mathbf{v}_2(\cdot) \in \mathbb{W}$ and $t \in [0, \widetilde{T} \wedge \tau_N]$,

(v) $\| A(t, \mathbf{v}(t)) \mathbf{y}_0 \|_{\mathbb{Y}} \leq N$, for all $\mathbf{v}(\cdot) \in \mathbb{W}$ and $t \in [0, \widetilde{T} \wedge \tau_N]$, where $\mathbf{y}_0$ is the center of the open ball $\mathbb{W}$.

By Assumption 2.1−(A2), $A^{\mathbf{v}}(t) \in G(\mathbb{X}, 1, \beta(\mathbf{v}))$ and hence the family $\{ A^{\mathbf{v}}(t) \}$ is stable with stability index $(1, \beta(\mathbf{v}))$.

Lemma 2.16 *Let τ_N be the stopping time defined in (2.25), then the map $A^{\mathbf{v}}(\cdot) : [0, \widetilde{T} \wedge \tau_N] \to \mathcal{L}(\mathbb{Y}, \mathbb{X})$ is continuous in its norm.*

Proof. See Lemma 9.1, [10]. □

By Assumption 2.1−(A3), we have

$$S A^{\mathbf{v}}(t) S^{-1} = A^{\mathbf{v}}(t) + B^{\mathbf{v}}(t), \ B^{\mathbf{v}}(t) = B(t, \mathbf{v}(t)) \in \mathcal{L}(\mathbb{X}, \mathbb{X}), \ \| B^{\mathbf{v}}(t) \|_{\mathcal{L}(\mathbb{X}, \mathbb{X})} \leq N.\qquad(2.26)$$

Lemma 2.17 *Let τ_N be the stopping time defined in (2.25), then the map $B^{\mathbf{v}}(\cdot) : [0, \widetilde{T} \wedge \tau_N] \to \mathcal{L}(\mathbb{X}, \mathbb{X})$ is weakly continuous and hence strongly measurable.*

Proof. See Lemma 9.2, [10]. □

2.3 Random Evolution Operator

Let us now construct the *random evolution operator* to the problem (2.11). We consider the homogeneous random evolution equation

$$\left. \begin{aligned} \frac{d\mathbf{u}(t)}{dt} + A(t, \mathbf{v}(t)) \mathbf{u}(t) = 0, \ 0 \leq t \leq \widetilde{T}, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{aligned} \right\}\qquad(2.27)$$

$\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., where $\mathbf{u}(t, \omega)$ and $\mathbf{v}(t, \omega) \in \mathscr{S}$ are random. We construct the random evolution operator with the help of Assumption 2.1 $(A1) - (A4)(i)$, Lemma 2.16 and Lemma 2.17.

Theorem 2.18 *Let τ_N be the stopping time defined in (2.25). Under Assumption 2.1 $(A1) - (A4)(i)$, there exists a unique evolution operator $\{ U^{\mathbf{v}}(t, s) \} := \{ U(t, s, (\mathbf{v}(r, \omega))_{s \leq r \leq t}) \}$ defined on the triangle $\widetilde{\triangle} := 0 \leq s \leq t \leq \widetilde{T} \wedge \tau_N$, with the following properties:*

1. $U^{\mathbf{v}}(t, s)$ *is strongly continuous on $\widetilde{\triangle}$ to $\mathcal{L}(\mathbb{X}, \mathbb{X})$, with $U^{\mathbf{v}}(s, s) = I$.*

2. $U^{\mathbf{v}}(t, s) U^{\mathbf{v}}(s, r) = U^{\mathbf{v}}(t, r)$.

3. $U^{\mathbf{v}}(t, s) \mathbb{Y} \subset \mathbb{Y}$, *and $U^{\mathbf{v}}(t, s)$ is strongly continuous on $\widetilde{\triangle}$ to $\mathcal{L}(\mathbb{Y}, \mathbb{Y})$.*

4. *For each $\mathbf{y} \in \mathbb{Y}$, $U^{\mathbf{v}}(\cdot, \cdot) \mathbf{y}$ satisfies the following:*

$$U^{\mathbf{v}}(t, s) \mathbf{y} - \mathbf{y} = - \int_s^t A(r, \mathbf{v}(r)) U^{\mathbf{v}}(r, s) \mathbf{y} dr,\qquad(2.28)$$

$$\mathrm{U}^{\mathbf{v}}(t,s)\mathbf{y} - \mathbf{y} = -\int_s^t \mathrm{U}^{\mathbf{v}}(t,r)\mathrm{A}(r,\mathbf{v}(r))\mathbf{y}\mathrm{d}r, \tag{2.29}$$

so that $\dfrac{\partial \mathrm{U}^{\mathbf{v}}(t,s)}{\partial t} = -\mathrm{A}(t,\mathbf{v}(t))\mathrm{U}^{\mathbf{v}}(t,s)$, $\dfrac{\partial \mathrm{U}^{\mathbf{v}}(t,s)}{\partial s} = \mathrm{U}^{\mathbf{v}}(t,s)\mathrm{A}(s,\mathbf{v}(s))$ *exist pointwise in* Ω *in the strong sense in* $\mathcal{L}(\mathbb{Y},\mathbb{X})$ *and are strongly continuous on* $\widetilde{\triangle}$ *to* $\mathcal{L}(\mathbb{Y},\mathbb{X})$.

P r o o f. Let us construct the piecewise constant families $\{\mathrm{A}_n(t,\mathbf{v}(t))\}_{t\in[0,\widetilde{T}]}, n = 1, 2, \cdots$ for approximating the family $\{\mathrm{A}(t,\mathbf{v}(t))\}_{t\in[0,\widetilde{T}\wedge\tau_N]}$ as follows: Let $t_k^n = \dfrac{k}{n}(\widetilde{T}\wedge\tau_N)$, $k = 0, 1, \cdots, n$ and let

$$\mathrm{A}_n(t,\mathbf{v}(t)) = \mathrm{A}(t_k^n,\mathbf{v}(t_k^n)) \text{ for } t_k^n \le t \le t_{k+1}^n, \quad k = 0, 1, \cdots, n-1, \tag{2.30}$$

$$\mathrm{A}_n(\widetilde{T}\wedge\tau_N,\mathbf{v}(\widetilde{T}\wedge\tau_N)) = \mathrm{A}(\widetilde{T}\wedge\tau_N,\mathbf{v}(\widetilde{T}\wedge\tau_N)). \tag{2.31}$$

From these piecewise constant families, we can construct the random evolution approximations $\mathrm{U}_n^{\mathbf{v}}(\cdot,\cdot)$ to $\mathrm{U}^{\mathbf{v}}(\cdot,\cdot)$ in $[0,\widetilde{T}\wedge\tau_N]$ using the similar methods described in Theorem I, [9], Theorem 5.3.1, [25] such that

$$\frac{\partial}{\partial t}\mathrm{U}_n^{\mathbf{v}}(t,s)\mathbf{y} = -\mathrm{A}_n^{\mathbf{v}}(t)\mathrm{U}_n^{\mathbf{v}}(t,s)\mathbf{y} \text{ for } t \ne t_k^n, k = 0, 1, \cdots, n, \tag{2.32}$$

$$\frac{\partial}{\partial s}\mathrm{U}_n^{\mathbf{v}}(t,s)\mathbf{y} = \mathrm{U}_n^{\mathbf{v}}(t,s)\mathrm{A}_n^{\mathbf{v}}(s)\mathbf{y} \text{ for } s \ne t_k^n, k = 0, 1, \cdots, n, \tag{2.33}$$

for $\mathbf{y} \in \mathbb{Y}$ and

$$\mathrm{U}^{\mathbf{v}}(t,s)\mathbf{x} = \lim_{n\to\infty}\mathrm{U}_n^{\mathbf{v}}(t,s)\mathbf{x}, \text{ for } \mathbf{x}\in\mathbb{X}, 0 \le s \le t \le \widetilde{T}\wedge\tau_N, \tag{2.34}$$

where $\mathrm{A}_n^{\mathbf{v}}(\cdot) = \mathrm{A}_n(\cdot,\mathbf{v}(\cdot))$ and $\mathrm{U}_n^{\mathbf{v}}(t,s) = \mathrm{U}_n^{\mathbf{v}}(t,s,(\mathbf{v}(r,\omega))_{s\le r\le t})$. $\square$

Theorem 2.19 *Let τ_N be the stopping time defined in (2.25). Under Assumption 2.1 (A1)−(A4)(i), let*

$$\{\mathrm{U}^{\mathbf{v}}(t,s)\} := \{\mathrm{U}(t,s,(\mathbf{v}(r,\omega))_{s\le r\le t})\}$$

be the unique evolution operator defined on the triangle $\widetilde{\triangle} := 0 \le s \le t \le \widetilde{T}\wedge\tau_N$ with the properties described in Theorem 2.18. Then, we have

$$\sup_{t,s\in\widetilde{\triangle}}\|\mathrm{U}^{\mathbf{v}}(t,s)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \le e^{\beta(\mathbf{v})\widetilde{T}} \le e^{N\widetilde{T}}, \tag{2.35}$$

$$\sup_{t,s\in\widetilde{\triangle}}\|\mathrm{U}^{\mathbf{v}}(t,s)\|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})} \le e^{(\beta(\mathbf{v})+\lambda_1(\mathbf{v}))\widetilde{T}} \le e^{2N\widetilde{T}}. \tag{2.36}$$

P r o o f. From Assumption 2.1−(A2), we know that $\{\mathrm{A}(t,\mathbf{v}(t))\}_{t\in[0,\widetilde{T}\wedge\tau_N]}$ is a stable family of infinitesimal generators in $\mathbb{X}$ with stability index $(1,\beta(\mathbf{v}))$. Assumption 2.1−(A2) also implies that $\mathbb{Y}$ is $\mathrm{A}(t,\mathbf{v}(t))$−admissible for every $t \in [0,\widetilde{T}\wedge\tau_N]$ (Theorem 4.5.8, [25], Lemma 7.8.1, [6]). Let $\widetilde{\mathrm{A}}(t,\mathbf{v}(t))$ be the part of $\mathrm{A}(t,\mathbf{v}(t))$ in $\mathbb{Y}$. Then by Theorem 4.5.5, [25], $\widetilde{\mathrm{A}}(t,\mathbf{v}(t))$ is the infinitesimal generator in $\mathbb{Y}$. By using Theorem 5.2.3, [25], $\mathrm{A}(t,\mathbf{v}(t)) + \mathrm{B}(t,\mathbf{v}(t))$ from (2.26) is quasi stable with a stability index $(1,\beta(\mathbf{v})+\lambda_1(\mathbf{v}))$ (Corollary 7.8.2, [6]). Hence, by making use of Theorem 5.2.4, [25], $\widetilde{\mathrm{A}}(t,\mathbf{v}(t))$ is quasi-stable with stability index $(1,\beta(\mathbf{v})+\lambda_1(\mathbf{v}))$. The construction of the random evolution operator in Theorem 2.18 (Theorem I, [9], Theorem 5.3.1, [25]) and the use of stopping time given in (2.25) yield the estimates (2.35) and (2.36). $\square$

Hence the $(\mathscr{F}_t)_{t\in[0,\widetilde{T}\wedge\tau_N]}$−adapted solution of the problem (2.27) in $[0,\widetilde{T}\wedge\tau_N]$ can be written as

$$\mathbf{u}(t) = \mathrm{U}^{\mathbf{v}}(t,0)\mathbf{u}_0$$

with $\mathbf{u} \in \mathrm{C}(0,\widetilde{T}\wedge\tau_N;\mathbb{Y})$ a.s., since by using part (3) of Theorem 2.18 and

$$\sup_{0\le t\le\widetilde{T}\wedge\tau_N}\|\mathbf{u}(t)\|_{\mathbb{Y}} = \sup_{0\le t\le\widetilde{T}\wedge\tau_N}\|\mathrm{U}^{\mathbf{v}}(t,0)\mathbf{u}_0\|_{\mathbb{Y}} = \sup_{0\le t\le\widetilde{T}\wedge\tau_N}\|\mathrm{S}\mathrm{U}^{\mathbf{v}}(t,0)\mathrm{S}^{-1}\mathrm{S}\mathbf{u}_0\|_{\mathbb{X}}$$

$$\leq \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| \mathrm{S} \mathrm{U}^{\mathbf{v}}(t,0) \mathrm{S}^{-1} \right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \mathrm{S} \mathbf{u}_0 \right\|_{\mathbb{X}}$$

$$\leq \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathrm{S}\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \|\mathrm{U}^{\mathbf{v}}(t,0)\|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})} \|\mathrm{S}^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})} \|\mathbf{u}_0\|_{\mathbb{Y}}$$

$$\leq e^{2N\widetilde{T}} \|\mathbf{u}_0\|_{\mathbb{Y}} < \infty, \tag{2.37}$$

where we used the fact that $\|\mathrm{S}\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} = \|\mathrm{S}^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})} = 1$, since S is an isometric isomorphism from $\mathbb{Y}$ to $\mathbb{X}$.

Remark 2.20 Also from (2.37), for $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2 \right] \leq e^{4N\widetilde{T}} \mathbb{E} \left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2 \right] < \infty, \tag{2.38}$$

which implies $\mathbf{u} \in \mathbb{L}^2(\Omega; \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{Y}))$.

2.4 Stochastic Linear Evolution Equation with Random Drift

Let us consider the non-homogenous evolution equation with random drift as

$$\left. \begin{aligned} \mathrm{d}\mathbf{u}(t) + \mathrm{A}(t, \mathbf{v}(t))\mathbf{u}(t)\mathrm{d}t &= \Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(t), \ 0 \leq t \leq \widetilde{T}, \\ \mathbf{u}(0) &= 0, \end{aligned} \right\} \tag{2.39}$$

where $\mathbf{v}(t) := \mathbf{v}(t, \omega)$ belongs to the class $\mathscr{S}$ defined in section 2.2. The random evolution operator $\{\mathrm{U}^{\mathbf{v}}(t,s)\} := \{\mathrm{U}(t, s, (\mathbf{v}(r, \omega))_{s \leq r \leq t})\}$ constructed in Theorem 2.18 is only $\mathscr{F}_t$−measurable and not $\mathscr{F}_s$−measurable. Hence the solvability of (2.39) involves an anticipative stochastic convolution $\int_0^t \mathrm{U}^{\mathbf{v}}(t,s)\Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(s)$ and is not well defined as an Itô stochastic integral. Then by using the integration by parts formula ([26]), we can write down this stochastic integral as

$$\int_0^t \mathrm{U}^{\mathbf{v}}(t,s)\Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(s) = \mathrm{U}^{\mathbf{v}}(t,0) \int_0^t \Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(s) + \int_0^t \mathrm{U}^{\mathbf{v}}(t,s) \mathrm{A}(s, \mathbf{v}(s)) \left(\int_s^t \Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(r) \right) \mathrm{d}s, \tag{2.40}$$

in $\mathbb{Y}$. With the representation (2.40), the stochastic integral is well defined and $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]}$−adapted. The next proposition states that (2.40) satisfies (2.39) in $\mathbb{X}$.

Proposition 2.21 *If $\Phi \in \mathbb{L}^2(\Omega; \mathbb{L}^2(0, \widetilde{T} \wedge \tau_N; \gamma(\mathbb{H}, \mathbb{X})))$ is $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]}$−adapted, then the representation of $\mathbf{u}(t)$ given in (2.40) is adapted and satisfies*

$$\mathbf{u}(t) = - \int_0^t \mathrm{A}(s, \mathbf{v}(s))\mathbf{u}(s)\mathrm{d}s + \int_0^t \Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(s), \tag{2.41}$$

for $0 \leq t \leq \widetilde{T} \wedge \tau_N$.

Proof. See Proposition 4.2, [26] for the case of bounded $\mathrm{A}(\cdot, \cdot)$ and a straightforward extension proves Proposition 2.21. □

Let us now consider the stochastic linear evolution equation with random drift

$$\left. \begin{aligned} \mathrm{d}\mathbf{u}(t) + \mathrm{A}(t, \mathbf{v}(t))\mathbf{u}(t)\mathrm{d}t &= \Phi \mathrm{d}\mathrm{W}_{\mathbb{H}}(t), \ 0 \leq t \leq \widetilde{T}, \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \right\} \tag{2.42}$$

$\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., and prove the existence and uniqueness of a *pathwise mild solution*.

Theorem 2.22 *Assume that the condition* (A1)−(A4) (i) *of Assumption 2.1 are satisfied. Let τ_N be the stopping time defined in (2.25) and $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Then there exists a unique pathwise mild solution of the problem (2.42).*

P r o o f. Let τ_N be the stopping time defined in (2.25) and let $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. We use the notation $A^{\mathbf{v}}(\cdot) = A(\cdot, \mathbf{v}(\cdot))$ and $B^{\mathbf{v}}(\cdot) = B(\cdot, \mathbf{v}(\cdot))$ in the rest of the proof. The $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N)}$−adapted stochastic process

$$\mathbf{u}(t) = U^{\mathbf{v}}(t,0)\mathbf{u}_0 + U^{\mathbf{v}}(t,0)\int_0^t \Phi dW_{\mathbb{H}}(s) + \int_0^t U^{\mathbf{v}}(t,s)A(s,\mathbf{v}(s))\left(\int_s^t \Phi dW_{\mathbb{H}}(r)\right)ds,$$

$$(2.43)$$

in $\mathbb{Y}$, where $U^{\mathbf{v}}(t,s)$ is the evolution system provided by Theorem 2.18, is the unique pathwise mild solution of the problem (2.42). In order to prove $\mathbf{u} \in C(0, \widetilde{T} \wedge \tau_N; \mathbb{Y})$, a.s., we first estimate $\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}$ as

$$\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}$$

$$= \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| U^{\mathbf{v}}(t,0)\mathbf{u}_0 + U^{\mathbf{v}}(t,0)\int_0^t \Phi dW_{\mathbb{H}}(s) + \int_0^t U^{\mathbf{v}}(t,s)A^{\mathbf{v}}(s)\left(\int_s^t \Phi dW_{\mathbb{H}}(r)\right)ds \right\|_{\mathbb{Y}}$$

$$\leq \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|SU^{\mathbf{v}}(t,0)\mathbf{u}_0\|_{\mathbb{X}} + \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| SU^{\mathbf{v}}(t,0)\int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}}$$

$$+ \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| S\int_0^t U^{\mathbf{v}}(t,s)A^{\mathbf{v}}(s)\left(\int_s^t \Phi dW_{\mathbb{H}}(r)\right)ds \right\|_{\mathbb{X}}. \tag{2.44}$$

The first term from the right hand side of the inequality (2.44) can be estimated using (2.36) as (see (2.37))

$$\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|U^{\mathbf{v}}(t,0)\mathbf{u}_0\|_{\mathbb{Y}} \leq e^{2N\widetilde{T}}\|\mathbf{u}_0\|_{\mathbb{Y}}. \tag{2.45}$$

For the second term from the right hand side of the inequality (2.44), we use (2.36) and (2.25) to get

$$\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| SU^{\mathbf{v}}(t,0)\int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}}$$

$$= \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| SU^{\mathbf{v}}(t,0)S^{-1}S\int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}}$$

$$\leq \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left(\|SU^{\mathbf{v}}(t,0)S^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| S\int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \right)$$

$$\leq \|S\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|U^{\mathbf{v}}(t,0)\|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})}\|S^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})} \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| S\int_0^t \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}}$$

$$\leq e^{2N\widetilde{T}} \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| \int_0^t S\Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \leq Ne^{2N\widetilde{T}}. \tag{2.46}$$

The third term in the right hand side of the inequality (2.44) can be simplified using Assumption 2.1-(A2), Hölder's inequality, (2.4), (2.5), (2.36), and (2.25) as follows:

$$\sup_{t \in [0, \widetilde{T} \wedge \tau_N]} \left\| \int_0^t SU^{\mathbf{v}}(t,s)A^{\mathbf{v}}(s)\left(\int_s^t \Phi dW_{\mathbb{H}}(r)\right)ds \right\|_{\mathbb{X}}$$

$$= \sup_{t \in [0, \widetilde{T} \wedge \tau_N]} \left\| \int_0^t SU^{\mathbf{v}}(t,s)A^{\mathbf{v}}(s)S^{-1}\left(\int_s^t S\Phi dW_{\mathbb{H}}(r)\right)ds \right\|_{\mathbb{X}}$$

$$= \sup_{t \in [0, \widetilde{T} \wedge \tau_N]} \left\| \int_0^t SU^{\mathbf{v}}(t,s)\left[S^{-1}A^{\mathbf{v}}(s) + S^{-1}B^{\mathbf{v}}(s)\right]\left(\int_s^t S\Phi dW_{\mathbb{H}}(r)\right)ds \right\|_{\mathbb{X}}$$

$$\leq \sup_{t \in [0, \widetilde{T} \wedge \tau_N]} \left\| \int_0^t \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \mathrm{A}^{\mathbf{v}}(s) \left(\int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right) \mathrm{d}s \right\|_{\mathbb{X}}$$

$$+ \sup_{t \in [0, \widetilde{T} \wedge \tau_N]} \left\| \int_0^t \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \mathrm{B}^{\mathbf{v}}(s) \left(\int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right) \mathrm{d}s \right\|_{\mathbb{X}}$$

$$\leq \int_0^{\widetilde{T} \wedge \tau_N} \left\| \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \mathrm{A}^{\mathbf{v}}(s) \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$+ \int_0^{\widetilde{T} \wedge \tau_N} \left\| \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \mathrm{B}^{\mathbf{v}}(s) \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$\leq \int_0^{\widetilde{T} \wedge \tau_N} \| \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \mathrm{A}^{\mathbf{v}}(s) \mathrm{S}^{-1} \int_s^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$+ \int_0^{\widetilde{T} \wedge \tau_N} \| \mathrm{SU}^{\mathbf{v}}(t,s) \mathrm{S}^{-1} \|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \mathrm{B}^{\mathbf{v}}(s) \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$\leq \sup_{s,t \in [0, \widetilde{T} \wedge \tau_N]} \| \mathrm{U}^{\mathbf{v}}(t,s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})} \int_0^{\widetilde{T} \wedge \tau_N} \| \mathrm{A}^{\mathbf{v}}(s) \mathrm{S}^{-1} \|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \int_s^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$+ \sup_{s,t \in [0, \widetilde{T} \wedge \tau_N]} \| \mathrm{U}^{\mathbf{v}}(t,s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})} \int_0^{\widetilde{T} \wedge \tau_N} \| \mathrm{B}^{\mathbf{v}}(s) \|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(r) \right\|_{\mathbb{X}} \mathrm{d}s$$

$$\leq e^{2N\widetilde{T}} \left[\sup_{s \in [0, \widetilde{T} \wedge \tau_N]} \| \mathrm{A}^{\mathbf{v}}(s) \|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \| \mathrm{S}^{-1} \|_{\mathcal{L}(\mathbb{X},\mathbb{Y})} \int_0^{\widetilde{T} \wedge \tau_N} \left\| \int_s^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \mathrm{d}t \right.$$

$$\left. + \sup_{s \in [0, \widetilde{T} \wedge \tau_N]} \| \mathrm{B}^{\mathbf{v}}(s) \|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \int_0^{\widetilde{T} \wedge \tau_N} \left\| \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \mathrm{d}t \right]$$

$$\leq N e^{2N\widetilde{T}} \left[\int_0^{\widetilde{T} \wedge \tau_N} \left\| \int_s^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \mathrm{d}t + \int_0^{\widetilde{T} \wedge \tau_N} \left\| \int_s^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \mathrm{d}t \right]$$

$$\leq 2N^2 \widetilde{T} e^{2N\widetilde{T}}. \tag{2.47}$$

Finally, a substitution of (2.45), (2.46) and (2.47) in (2.44) yields

$$\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \| \mathbf{u}(t) \|_{\mathbb{Y}} \leq e^{2N\widetilde{T}} \left\{ \| \mathbf{u}_0 \|_{\mathbb{Y}} + N \left(1 + 2N\widetilde{T} \right) \right\} < \infty. \tag{2.48}$$

By using (2.48), part (3) of Theorem 2.18, part (A4)-(i) of Assumption 2.1 and the pathwise continuity of the stochastic integral $\int_0^t \Phi \mathrm{dW}_{\mathbb{H}}(s)$, it can be easily seen that $\mathbf{u}(t)$ given in (2.43) is pathwise continuous in $\mathbb{Y}$, and is adapted to $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]}$, by Proposition 2.21. Thus $\mathbf{u}(t)$ given in (2.43) is $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]}$−adapted and $\mathbf{u} \in \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{Y})$ a.s. A similar calculation of (2.48) yields

$$\mathbb{E} \left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \| \mathbf{u}(t) \|_{\mathbb{Y}}^2 \right] \leq 3 e^{4N\widetilde{T}} \left\{ \mathbb{E} \left[\| \mathbf{u}_0 \|_{\mathbb{Y}}^2 \right] + N^2 \left(1 + 4N^2 \widetilde{T} \right) \right\} < \infty, \tag{2.49}$$

and hence $\mathbf{u} \in \mathbb{L}^2(\Omega; \mathrm{C}(0, \widetilde{T} \wedge \tau_N, \mathbb{Y}))$. The uniqueness of $\mathbf{u}$ is a consequence of the representation (2.43). $\square$

2.5 Existence and Uniqueness of the Pathwise Mild Solution of (2.1)

Let us first prove that the unique pathwise mild solution of the problem (2.42) lies in the class $\mathscr{S}$ (section 2.2). From Theorem 2.22, it is clear that the solution (2.43) is $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N]}$−adapted and hence the condition (2.22)

is satisfied. Let us now verify the condition (2.20). Let us set $\widetilde{\mathbf{u}} = \mathbf{u} - \mathbf{y}_0$, where $\mathbf{y}_0$ is the center of the open ball $\mathbb{W} \subset \mathbb{Y}$. Then $\widetilde{\mathbf{u}}$ satisfies the equation

$$
\left.\begin{array}{r}
\mathrm{d}\widetilde{\mathbf{u}}(t) + \mathrm{A}^{\mathbf{v}}(t)\widetilde{\mathbf{u}}(t)\mathrm{d}t = -\mathrm{A}^{\mathbf{v}}(t)\mathbf{y}_0\mathrm{d}t + \Phi\mathrm{dW}_{\mathbb{H}}(t), \ 0 \le t \le \widetilde{T} \wedge \tau_N, \\
\widetilde{\mathbf{u}}(0) = \mathbf{u}_0 - \mathbf{y}_0,
\end{array}\right\}
\tag{2.50}
$$

$\mathbf{u}_0 - \mathbf{y}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. By using Theorem 2.22, the $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N)}$−adapted unique pathwise mild solution of (2.50) in $0 \le t \le \widetilde{T} \wedge \tau_N$ can be written as

$$
\widetilde{\mathbf{u}}(t) = \mathrm{U}^{\mathbf{v}}(t, 0)\widetilde{\mathbf{u}}_0 - \int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\mathrm{d}s + \mathrm{U}^{\mathbf{v}}(t, 0)\int_0^t \Phi\mathrm{dW}_{\mathbb{H}}(s)
$$
$$
+ \int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t \Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s.
\tag{2.51}
$$

Hence, from (2.51), we get

$$
\mathbf{u}(t) - \mathbf{y}_0 = \mathrm{U}^{\mathbf{v}}(t, 0)(\mathbf{u}_0 - \mathbf{y}_0) - \int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\mathrm{d}s + \mathrm{U}^{\mathbf{v}}(t, 0)\int_0^t \Phi\mathrm{dW}_{\mathbb{H}}(s)
$$
$$
+ \int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t \Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s,
\tag{2.52}
$$

for $t \in [0, \widetilde{T} \wedge \tau_N]$. Now we prove that $\|\mathbf{u}(t) - \mathbf{y}_0\|_{\mathbb{Y}} \le R$ a.s. so that $\mathbf{u}(t) \in \mathbb{W}$, a.s.

Proposition 2.23 *Let τ_N be the stopping time defined in (2.25) and let $\mathbf{u}_0 - \mathbf{y}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Let the Assumption 2.1 be satisfied and if $\mathbf{u}(t)$ satisfies (2.52), then $\mathbf{u}(\cdot) \in \mathbb{W}$, a.s., and satisfies the condition (2.20).*

P r o o f. In order to prove $\mathbf{u} \in \mathbb{W}$, a.s., we first estimate $\|\mathbf{u}(t) - \mathbf{y}_0\|_{\mathbb{Y}}$ as

$$
\|\mathbf{u}(t) - \mathbf{y}_0\|_{\mathbb{Y}} \le \|\mathrm{U}^{\mathbf{v}}(t, 0)(\mathbf{u}_0 - \mathbf{y}_0)\|_{\mathbb{Y}} + \left\|\int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\mathrm{d}s\right\|_{\mathbb{Y}}
$$
$$
+ \left\|\mathrm{U}^{\mathbf{v}}(t, 0)\int_0^t \Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{Y}} + \left\|\int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t \Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s\right\|_{\mathbb{Y}}.
\tag{2.53}
$$

The first term in the the right hand side of the inequality (2.53) can be evaluated by using (2.36) as

$$
\|\mathrm{U}^{\mathbf{v}}(t, 0)(\mathbf{u}_0 - \mathbf{y}_0)\|_{\mathbb{Y}} \le \|\mathrm{U}^{\mathbf{v}}(t, 0)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{Y})}\|\mathbf{u}_0 - \mathbf{y}_0\|_{\mathbb{Y}} \le e^{2N\widetilde{T}}\|\mathbf{u}_0 - \mathbf{y}_0\|_{\mathbb{Y}}.
\tag{2.54}
$$

For the second term in the right hand side of the inequality (2.53), we use (2.36) and (2.7) to get

$$
\left\|\int_0^t \mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\mathrm{d}s\right\|_{\mathbb{Y}} \le \int_0^t \|\mathrm{U}^{\mathbf{v}}(t, s)\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\|_{\mathbb{Y}}\mathrm{d}s
$$
$$
\le \int_0^t \|\mathrm{U}^{\mathbf{v}}(t, s)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{Y})}\|\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\|_{\mathbb{Y}}\mathrm{d}s
$$
$$
\le \sup_{s, t \in \widetilde{\triangle}} \|\mathrm{U}^{\mathbf{v}}(t, s)\|_{\mathcal{L}(\mathbb{Y}, \mathbb{Y})}\int_0^{\widetilde{T} \wedge \tau_N} \|\mathrm{A}^{\mathbf{v}}(s)\mathbf{y}_0\|_{\mathbb{Y}}\mathrm{d}s \le \widetilde{T}Ne^{2N\widetilde{T}}.
\tag{2.55}
$$

Let us apply (2.36) to the third term in the right hand side of the inequality (2.53) to find (see (2.46))

$$
\left\|\mathrm{U}^{\mathbf{v}}(t, 0)\int_0^t \Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{Y}} \le Ne^{2N\widetilde{T}}.
\tag{2.56}
$$

By using (2.47), the final term in the right hand side of the inequality (2.53) can be estimated as

$$\left\| \int_0^t \mathrm{U}^{\mathbf{v}}(t,s) \mathrm{A}^{\mathbf{v}}(s) \left(\int_s^t \Phi \mathrm{dW}_{\mathbb{H}}(r) \right) \mathrm{d}s \right\|_{\mathbb{Y}} \leq 2N^2 \widetilde{T} e^{2N\widetilde{T}}. \tag{2.57}$$

Now we substitute (2.54), (2.55), (2.56) and (2.57) in (2.53) to obtain

$$\|\mathbf{u}(t) - \mathbf{y}_0\|_{\mathbb{Y}} \leq e^{2N\widetilde{T}} \left\{ \|\mathbf{u}_0 - \mathbf{y}_0\|_{\mathbb{Y}} + N \left(\widetilde{T} \left[1 + 2N \right] + 1 \right) \right\}, \tag{2.58}$$

where N is from definition of the stopping time (2.25). For $\mathbf{u}(t)$ to be in $\mathbb{W}$, a.s., the right hand side of the inequality (2.58) should be less than or equal to R. Since $\|\mathbf{u}_0 - \mathbf{y}_0\|_{\mathbb{Y}} < R$, a.s., this is possible if $\widetilde{T} > 0$ is chosen sufficiently small. Thus for sufficiently small $\widetilde{T}$, $\mathbf{u} \in \mathbb{W}$, a.s., and hence the condition (2.20) is satisfied. $\qquad\square$

Using the fact that $\|\mathbf{u}(t)\|_{\mathbb{X}} \leq \|\mathbf{u}(t)\|_{\mathbb{Y}}$, we have

$$\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{X}} \leq \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t) - \mathbf{y}_0\|_{\mathbb{Y}} + \|\mathbf{y}_0\|_{\mathbb{Y}} \leq R + \|\mathbf{y}_0\|_{\mathbb{Y}}, \text{ a.s.} \tag{2.59}$$

The condition (2.22) can be proved using Proposition 2.23, (2.59), strong continuity of $\mathrm{U}^{\mathbf{v}}(\cdot, \cdot)$ on $\widetilde{\triangle}$ to $\mathcal{L}(\mathbb{X}, \mathbb{X})$ (see Theorem 2.18, part (1)), continuity of $t \mapsto \mathrm{A}^{\mathbf{v}}(t)$ in the $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ (see Assumption 2.1-(A4) (i)), and the pathwise continuity of the stochastic integral $\int_0^t \Phi \mathrm{dW}_{\mathbb{H}}(s)$. Hence, $\mathbf{u}(\cdot, \cdot)$ is continuous from $[0, \widetilde{T} \wedge \tau_N] \times \Omega$ to $\mathbb{X}$.

By choosing $\widetilde{T}$ sufficiently small, the map $\mathbf{v} \mapsto \mathbf{u} = \Psi(\mathbf{v})$ sends $\mathscr{S}$ to $\mathscr{S}$. Let us now make $\mathscr{S}$ into a complete metric space by the distance function

$$\Lambda(\mathbf{v}, \mathbf{w}) = \mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbb{X}} \right] = \mathbb{E}\left[\mathrm{d}(\mathbf{v}, \mathbf{w}) \right], \tag{2.60}$$

where $\mathrm{d}(\mathbf{v}, \mathbf{w}) = \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbb{X}}$. Since a closed ball in $\mathbb{Y}$ is a closed ball in $\mathbb{X}$ (Lemma 7.3, [10]), the function space $\mathscr{S}$ is a complete metric space. Now we show that the map $\Psi(\cdot) : \mathscr{S} \to \mathscr{S}$ is a strict contraction map if we choose $\widetilde{T}$ sufficiently small.

Theorem 2.24 *Let τ_N be the stopping time defined in (2.25). Let the Assumption 2.1 be satisfied and let $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Then the map $\Psi(\cdot) : \mathscr{S} \to \mathscr{S}$ is a strict contraction map.*

P r o o f. Let $\Psi(\mathbf{v}_1) = \mathbf{u}_1$ and $\Psi(\mathbf{v}_2) = \mathbf{u}_2$ satisfy

$$\mathrm{d}\mathbf{u}_1(t) + \mathrm{A}^{\mathbf{v}_1}(t)\mathbf{u}_1(t)\mathrm{d}t = \Phi \mathrm{dW}_{\mathbb{H}}(t), \ \mathbf{u}_1(0) = \mathbf{u}_0, \tag{2.61}$$

and

$$\mathrm{d}\mathbf{u}_2(t) + \mathrm{A}^{\mathbf{v}_2}(t)\mathbf{u}_2(t)\mathrm{d}t = \Phi \mathrm{dW}_{\mathbb{H}}(t), \ \mathbf{u}_2(0) = \mathbf{u}_0, \tag{2.62}$$

respectively. Let us denote $\mathbf{z}(t) = \mathbf{u}_1(t) - \mathbf{u}_2(t)$ and take the difference between the equations (2.61) and (2.62) to obtain

$$\left. \begin{array}{r} \mathrm{d}\mathbf{z}(t) + \mathrm{A}^{\mathbf{v}_2}(t)\mathbf{z}(t)\mathrm{d}t = -(\mathrm{A}^{\mathbf{v}_1}(t) - \mathrm{A}^{\mathbf{v}_2}(t))\mathbf{u}_1(t)\mathrm{d}t, \\ \mathbf{z}(0) = 0. \end{array} \right\} \tag{2.63}$$

Note that (2.63) is a stochastic linear evolution equation in $\mathbf{z}(t)$ with random drift $\mathrm{A}^{\mathbf{v}_2}(t) = \mathrm{A}(t, \mathbf{v}_2(t))$. The unique pathwise mild solution of (2.63) is given by

$$\mathbf{z}(t) = -\int_0^t \mathrm{U}^{\mathbf{v}_2}(t,s) \left[\mathrm{A}^{\mathbf{v}_1}(s) - \mathrm{A}^{\mathbf{v}_2}(s) \right] \mathbf{u}_1(s)\mathrm{d}s, \tag{2.64}$$

for $t \in [0, \widetilde{T} \wedge \tau_N]$. Hence by using (2.35), (2.6), and Theorem 2.22, we have

$$
\begin{aligned}
\mathrm{d}(\Psi(\mathbf{v}_1), \Psi(\mathbf{v}_2)) &= \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\Psi(\mathbf{v}_1) - \Psi(\mathbf{v}_2)\|_{\mathbb{X}} \\
&= \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\| \int_0^t \mathrm{U}^{\mathbf{v}_2}(t,s) \left[\mathrm{A}^{\mathbf{v}_1}(s) - \mathrm{A}^{\mathbf{v}_2}(s)\right] \mathbf{u}_1(s) \mathrm{d}s \right\|_{\mathbb{X}} \\
&\leq \int_0^{\widetilde{T} \wedge \tau_N} \left\| \mathrm{U}^{\mathbf{v}_2}(t,s) \left[\mathrm{A}^{\mathbf{v}_1}(s) - \mathrm{A}^{\mathbf{v}_2}(s)\right] \mathbf{u}_1(s) \right\|_{\mathbb{X}} \mathrm{d}s \\
&\leq \int_0^{\widetilde{T} \wedge \tau_N} \left\| \mathrm{U}^{\mathbf{v}_2}(t,s) \right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\| \left[\mathrm{A}^{\mathbf{v}_1}(s) - \mathrm{A}^{\mathbf{v}_2}(s)\right] \mathbf{u}_1(s) \right\|_{\mathbb{X}} \mathrm{d}s \\
&\leq \sup_{s,t \in \widetilde{\triangle}} \left\| \mathrm{U}^{\mathbf{v}_2}(t,s) \right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \int_0^{\widetilde{T} \wedge \tau_N} \left\| \mathrm{A}^{\mathbf{v}_1}(s) - \mathrm{A}^{\mathbf{v}_2}(s) \right\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \|\mathbf{u}_1(s)\|_{\mathbb{Y}} \mathrm{d}s \\
&\leq N e^{N\widetilde{T}} \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}_1(s)\|_{\mathbb{Y}} \int_0^{\widetilde{T} \wedge \tau_N} \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_{\mathbb{X}} \mathrm{d}s \\
&\leq \widetilde{T} e^{3N\widetilde{T}} N \left\{ \|\mathbf{u}_0\|_{\mathbb{Y}} + N \left(1 + 2N\widetilde{T}\right) \right\} \sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_{\mathbb{X}} \\
&\leq \widetilde{T} e^{3N\widetilde{T}} N \left\{ \|\mathbf{u}_0\|_{\mathbb{Y}} + N \left(1 + 2N\widetilde{T}\right) \right\} \mathrm{d}(\mathbf{v}_1, \mathbf{v}_2),
\end{aligned}
\tag{2.65}
$$

Taking expectation on both sides of (2.65), we find

$$
\Lambda(\Psi(\mathbf{v}_1), \Psi(\mathbf{v}_2)) \leq \widetilde{T} e^{3N\widetilde{T}} N \left\{ R + N \left(1 + 2N\widetilde{T}\right) \right\} \Lambda(\mathbf{v}_1, \mathbf{v}_2),
\tag{2.66}
$$

since $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Now by choosing $\widetilde{T}$ sufficiently small, we get $\Psi(\cdot)$ is a contraction map. $\qquad \square$

Theorem 2.25 *Let τ_N be the stopping time defined by*

$$
\begin{aligned}
\tau_N := \inf_{t \geq 0} \Bigg\{ & t : \beta(\mathbf{u}(t)) \vee \lambda_1(\mathbf{u}(t)) \vee \lambda_2(\mathbf{u}(t)) \vee \lambda_3(\mathbf{u}(t)) \vee \mu(\mathbf{u}(t)) \vee \left\| \int_0^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \\
& \vee \left\| \int_0^t \mathrm{S}^2 \Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \geq N \Bigg\},
\end{aligned}
\tag{2.67}
$$

for $N \in \mathbb{N}$. Let the Assumption 2.1 be satisfied and let $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Then there exists a unique local pathwise mild solution $(\mathbf{u}(t))_{t \in [0, \widetilde{T} \wedge \tau_N)}$ of the problem (2.1) in $\mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{W})$ a.s.

P r o o f. From Theorem 2.24, by choosing $\widetilde{T}$ sufficiently small, we obtain the map $\Psi(\cdot)$ as a contraction map. By an application of the contraction mapping theorem, it follows that $\Psi(\cdot)$ has a unique fixed point. Hence for the stopping time τ_N defined by (2.67) there exists an $(\mathscr{F}_t)_{t \in [0, \widetilde{T} \wedge \tau_N)}$−adpated unique pathwise mild solution of the problem

$$
\left.
\begin{aligned}
\mathrm{d}\mathbf{u}(t) + \mathrm{A}(t, \mathbf{u}(t))\mathbf{u}(t)\mathrm{d}t &= \Phi \mathrm{dW}_{\mathbb{H}}(t), \\
\mathbf{u}(0) &= \mathbf{u}_0,
\end{aligned}
\right\}
\tag{2.68}
$$

$\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s., which is given by the stochastic process

$$
\mathbf{u}(t) = \mathrm{U}^{\mathbf{u}}(t,0)\mathbf{u}_0 + \mathrm{U}^{\mathbf{u}}(t,0) \int_0^t \Phi \mathrm{dW}_{\mathbb{H}}(t) + \int_0^t \mathrm{U}^{\mathbf{u}}(t,s) \mathrm{A}(s, \mathbf{u}(s)) \left(\int_s^t \Phi \mathrm{dW}_{\mathbb{H}}(r) \right) \mathrm{d}s.
\tag{2.69}
$$

From Theorem 2.22 and Proposition 2.23, we have $\mathbf{u} \in \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{W})$ a.s., and the estimate

$$\mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right] \leq 3e^{4N\widetilde{T}}\left\{\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right] + N^2\left(1 + 4N^2\widetilde{T}\right)\right\}, \tag{2.70}$$

implies in $\mathbf{u} \in \mathbb{L}^2(\Omega; \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{W}))$. $\square$

Theorem 2.26 *Let $0 < \delta < 1$ be given. Then, we have*

$$\mathbb{P}\{\tau_N > \delta\} \geq 1 - C\delta^2\left\{\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right] + C\left(\delta\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2 + 8\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2 + 8\|\mathrm{S}^2\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2\right)\right\}, \tag{2.71}$$

for some positive constant C independent of δ.

P r o o f. Let $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. For the given $0 < \delta < 1$, there exists a positive integer N such that

$$\frac{1}{N+1} \leq \delta < \frac{1}{N}.$$

Then, $(\mathbf{u}, \widetilde{T} \wedge \tau_N)$ is a local mild solution of (2.68) for the stopping time given by

$$\tau_N := \inf_{t \geq 0}\left\{t : \beta(\mathbf{u}(t)) \vee \lambda_1(\mathbf{u}(t)) \vee \lambda_2(\mathbf{u}(t)) \vee \lambda_3(\mathbf{u}(t)) \vee \mu(\mathbf{u}(t)) \vee \left\|\int_0^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}\right.$$
$$\left. \vee \left\|\int_0^t \mathrm{S}^2\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}} \geq N\right\}.$$

Also it can be easily seen that

$$\mathbb{P}\{\tau_N > \delta\} \geq \mathbb{P}\left\{\sup_{0 \leq t \leq \delta}\left(\beta(\mathbf{u}(t \wedge \tau_N)) \vee \lambda_1(\mathbf{u}(t \wedge \tau_N)) \vee \lambda_2(\mathbf{u}(t \wedge \tau_N)) \vee \lambda_3(\mathbf{u}(t \wedge \tau_N))\right.\right.$$
$$\left.\left. \vee \mu(\mathbf{u}(t \wedge \tau_N)) \vee \left\|\int_0^{t \wedge \tau_N} \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}} \vee \left\|\int_0^{t \wedge \tau_N} \mathrm{S}^2\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}\right) < N\right\}$$
$$\geq \mathbb{P}\left\{\sup_{0 \leq t \leq \delta} \|\mathbf{u}(t \wedge \tau_N)\|_{\mathbb{Y}} < \mathcal{K}N\right\}, \tag{2.72}$$

where $\mathcal{K}$ is a positive constant defined by

$$\mathcal{K} = \sup\left\{C \in \mathbb{R}^+ \,\middle|\, C\left(\beta(\mathbf{u}(t)) \vee \lambda_1(\mathbf{u}(t)) \vee \lambda_2(\mathbf{u}(t)) \vee \lambda_3(\mathbf{u}(t)) \vee \mu(\mathbf{u}(t))\right.\right.$$
$$\left.\left. \vee \left\|\int_0^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}} \vee \left\|\int_0^t \mathrm{S}^2\Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}\right) \leq \|\mathbf{u}(t)\|_{\mathbb{Y}}, \text{ for all } \mathbf{u} \in \mathbb{Y}\right\}. \tag{2.73}$$

Now, we consider $\mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right]$ and use (2.44) to get

$$\mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right]$$
$$\leq 3\left\{\mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathrm{SU}^{\mathbf{v}}(t,0)\mathbf{u}_0\|_{\mathbb{X}}^2\right] + \mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \left\|\mathrm{SU}^{\mathbf{v}}(t,0)\int_0^t \Phi \mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\right]\right.$$

$$+\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{T}\wedge\tau_N}\left\|\mathrm{S}\int_0^t\mathrm{U}^{\mathbf{v}}(t,s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t\Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2\right]\right\}.\tag{2.74}$$

Using (2.45), the first term from the right hand side of the inequality (2.74) can be estimated as

$$\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{T}\wedge\tau_N}\|\mathrm{U}^{\mathbf{v}}(t,0)\mathbf{u}_0\|_{\mathbb{Y}}^2\right]\leq e^{4N\widetilde{T}}\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right].\tag{2.75}$$

The second term from the right hand side of the inequality (2.74) can be simplified using (2.24), (2.46) as

$$\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{T}\wedge\tau_N}\left\|\mathrm{SU}^{\mathbf{v}}(t,0)\int_0^t\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\right]\leq e^{4N\widetilde{T}}\mathbb{E}\left[\sup_{0\leq t\leq\widetilde{T}\wedge\tau_N}\left\|\int_0^t\mathrm{S}\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\right]$$
$$\leq e^{4N\widetilde{T}}\widetilde{T}\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2.\tag{2.76}$$

The third term from the right hand side of the inequality (2.74) can be evaluated using (2.47) as

$$\mathbb{E}\left[\sup_{t\in[0,\widetilde{T}\wedge\tau_N]}\left\|\int_0^t\mathrm{SU}^{\mathbf{v}}(t,s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t\Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2\right]$$
$$\leq 2N^2e^{4N\widetilde{T}}\left\{\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_s^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]+\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_s^t\mathrm{S}\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]\right\}.\tag{2.77}$$

Let us consider the term $\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_s^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]$ from (2.77), and use Hölder's inequality and (2.24) to obtain

$$\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_s^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]$$
$$=\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_0^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)-\int_0^s\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]$$
$$\leq 2\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_0^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]+2\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_0^s\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]$$
$$\leq 2\widetilde{T}\mathbb{E}\left[\sup_{t\in[0,\widetilde{T}\wedge\tau_N]}\left\|\int_0^t\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\right]+2\widetilde{T}\mathbb{E}\left[\left\|\int_0^s\mathrm{S}^2\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\right]$$
$$\leq 2C\widetilde{T}\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\|\mathrm{S}^2\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2\mathrm{d}t\right]+2C\widetilde{T}\mathbb{E}\left[\int_0^{\widetilde{T}}\|\mathrm{S}^2\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2\mathrm{d}t\right]$$
$$\leq 4C\widetilde{T}^2\|\mathrm{S}^2\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2,\tag{2.78}$$

where C is a positive constant independent of $\widetilde{T}$. Similarly one can prove that

$$\mathbb{E}\left[\int_0^{\widetilde{T}\wedge\tau_N}\left\|\int_s^t\mathrm{S}\Phi\mathrm{dW}_{\mathbb{H}}(s)\right\|_{\mathbb{X}}^2\mathrm{d}t\right]\leq 4C\widetilde{T}^2\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2.\tag{2.79}$$

Substituting (2.78) and (2.79) in (2.77), we get

$$\mathbb{E}\left[\sup_{t\in[0,\widetilde{T}\wedge\tau_N]}\left\|\int_0^t\mathrm{SU}^{\mathbf{v}}(t,s)\mathrm{A}^{\mathbf{v}}(s)\left(\int_s^t\Phi\mathrm{dW}_{\mathbb{H}}(r)\right)\mathrm{d}s\right\|_{\mathbb{X}}^2\right]$$

$$\leq 8C\widetilde{T}^2 N^2 e^{4N\widetilde{T}} \left[\|\mathrm{S}^2\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + \|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} \right]. \tag{2.80}$$

Finally, a substitution of (2.75), (2.76) and (2.80) in (2.74) yields

$$\mathbb{E}\left[\sup_{0 \leq t \leq \widetilde{T} \wedge \tau_N} \|\mathbf{u}(t)\|^2_{\mathbb{Y}} \right]$$
$$\leq e^{4N\widetilde{T}} \left\{ \mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{Y}} \right] + C\widetilde{T}\left[\left(1 + 8\widetilde{T}N^2\right) \|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\widetilde{T}N^2\|\mathrm{S}^2\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} \right] \right\}. \tag{2.81}$$

From (2.81), it can be easily seen that

$$\mathbb{E}\left(\sup_{0 \leq t \leq \delta} \|\mathbf{u}(t \wedge \tau_N)\|^2_{\mathbb{Y}} \right)$$
$$\leq e^{4N\delta} \left\{ \mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{Y}} \right] + C\left(\delta\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\delta^2 N^2\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\delta^2 N^2\|\mathrm{S}^2\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} \right) \right\}, \tag{2.82}$$

for all $t \in [0, \widetilde{T}]$. Then by using Markov's inequality and (2.82), we obtain

$$\mathbb{P}\left\{ \tau_N > \delta \right\} \geq \mathbb{P}\left\{ \sup_{0 \leq t \leq \delta} \|\mathbf{u}(t \wedge \tau_N)\|_{\mathbb{Y}} < \mathcal{K}N \right\}$$
$$= \mathbb{P}\left\{ \sup_{0 \leq t \leq \delta} \|\mathbf{u}(t \wedge \tau_N)\|^2_{\mathbb{Y}} < \mathcal{K}^2 N^2 \right\}$$
$$\geq 1 - \frac{1}{\mathcal{K}^2 N^2} \mathbb{E}\left(\sup_{0 \leq t \leq \delta} \|\mathbf{u}(t \wedge \tau_N)\|^2_{\mathbb{Y}} \right)$$
$$\geq 1 - \frac{e^{4N\delta}}{\mathcal{K}^2 N^2} \left\{ \mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{Y}} \right] + C\left(\delta\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\delta^2 N^2\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\delta^2 N^2\|\mathrm{S}^2\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} \right) \right\}$$
$$\geq 1 - C\delta^2 \left\{ \mathbb{E}\left[\|\mathbf{u}_0\|^2_{\mathbb{Y}} \right] + C\left(\delta\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\|\mathrm{S}\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} + 8\|\mathrm{S}^2\Phi\|^2_{\gamma(\mathbb{H},\mathbb{X})} \right) \right\}, \tag{2.83}$$

where C is a positive constant independent of $\mathbf{u}(t)$ and δ. $\qquad\square$

Similar ideas for proving the positivity of stopping time can be found in [14, 16, 15, 18].

A characterization of the maximal pathwise mild solution for the stochastic quasilinear evolution equation of hyperbolic type (2.1) is given in the next theorem (Theorem 2.27).

Theorem 2.27 *Let the Assumption 2.1 be satisfied and let $\mathbf{u}_0 \in \mathbb{W} \subset \mathbb{Y}$, a.s. Then there exists a unique maximal pathwise mild solution $(\mathbf{u}(t))_{[0,\tau_\infty)}$ in $\mathrm{C}(0, \tau_\infty; \mathbb{W})$, a.s. of (2.68).*

P r o o f. Let us denote by $\mathcal{L}$, the set of all stopping times such that $\tau \in \mathcal{L}$ if and only if there exists a process $\mathbf{u}(\cdot)$ such that $(\mathbf{u}, \tau)$ is a local mild solution the problem (2.68). It can be easily seen that

$$\tau_1, \tau_2 \in \mathcal{L} \Rightarrow \tau_1 \vee \tau_2, \tau_1 \wedge \tau_2 \in \mathcal{L}. \tag{2.84}$$

For each $k \in \mathbb{N}$, let us take $\tau_k \in \mathcal{L}$ such that $(\mathbf{u}_k, \tau_k)$ be the unique local mild solution of (2.68). Then for each τ_k, the process $\mathbf{u}_k(\cdot)$ having continuous paths such that $(\mathbf{u}_k, \tau_k)$ is a local mild solution of (2.68) with

$$\tau_k = \inf_{t \geq 0} \left\{ t : \beta(\mathbf{u}_k(t)) \vee \lambda_1(\mathbf{u}_k(t)) \vee \lambda_2(\mathbf{u}_k(t)) \vee \lambda_3(\mathbf{u}_k(t)) \vee \mu(\mathbf{u}_k(t)) \right.$$
$$\left. \vee \left\| \int_0^t \mathrm{S}\Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \vee \left\| \int_0^t \mathrm{S}^2\Phi \mathrm{dW}_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \geq k \right\} \wedge T, k \in \mathbb{N},$$

for some $T > 0$. Let us now show that for $n > k$, $\tau_n > \tau_k$, a.s. For $n > k$, let us define a sequence of stopping times $\tau_{k,n}$ such that

$$\tau_{k,n} = \inf_{t \geq 0} \left\{ t : \beta(\mathbf{u}_n(t)) \vee \lambda_1(\mathbf{u}_n(t)) \vee \lambda_2(\mathbf{u}_n(t)) \vee \lambda_3(\mathbf{u}_n(t)) \vee \mu(\mathbf{u}_n(t)) \right.$$

$$\vee \left\| \int_0^t S\Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \vee \left\| \int_0^t S^2 \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \geq k \right\} \wedge T, k, n \in \mathbb{N}.$$

Since $(\mathbf{u}_n, \tau_n)$ is also a local mild solution, where

$$\tau_n = \inf_{t \geq 0} \left\{ t : \beta(\mathbf{u}_n(t)) \vee \lambda_1(\mathbf{u}_n(t)) \vee \lambda_2(\mathbf{u}_n(t)) \vee \lambda_3(\mathbf{u}_n(t)) \vee \mu(\mathbf{u}_n(t)) \right.$$

$$\left. \vee \left\| \int_0^t S\Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \vee \left\| \int_0^t S^2 \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \geq n \right\} \wedge T, n \in \mathbb{N},$$

it is clear from the definition of τ_n that $\tau_{k,n} \leq \tau_n$, a.s., for $n > k$. Thus $(\mathbf{u}_n, \tau_{k,n})$ is also a local mild solution of (2.68). If $\tau_k < \tau_{k,n}$, a.s., then from above, we have $\tau_k < \tau_{k,n} \leq \tau_n$, a.s., and hence we are done. Let us now assume that $\tau_k > \tau_{k,n}$, a.s. Since $(\mathbf{u}_k, \tau_k)$ and $(\mathbf{u}_n, \tau_{n,k})$ are both local mild solutions of (2.68), by the uniqueness of local mild solution, we have $\mathbf{u}_k(t) = \mathbf{u}_n(t)$, a.s., for all $t \in [0, \tau_k \wedge \tau_{k,n}) = [0, \tau_{k,n})$. Thus, $\tau_{k,n}$ is the first exit time for $\mathbf{u}_k(\cdot)$ at k with $\tau_{k,n} < \tau_k$, a.s., which is a contradiction. Hence $\tau_k < \tau_n$, a.s., for all $k < n$. Thus $\{\tau_k : k \in \mathbb{N}\}$ is an increasing sequence in $\mathcal{L}$ and hence it has a limit in $\mathcal{L}$. Let us denote the limit by $\tau_\infty := \lim_{k \to \infty} \tau_k$. By letting $k \to \infty$, let $\{\mathbf{u}(t), 0 \leq t < \tau_\infty\}$ be the stochastic process defined by

$$\mathbf{u}(t) = \mathbf{u}_k(t), \quad t \in [\tau_{k-1}, \tau_k), \quad k \geq 1, \tag{2.85}$$

where $\tau_0 = 0$. By making use of uniqueness results, we have $\mathbf{u}(t \wedge \tau_k) = \mathbf{u}_k(t \wedge \tau_k)$ for any $t > 0$. As $k \to \infty$, we are thus justified to define a process $(\mathbf{u}, \tau_\infty)$ such that $(\mathbf{u}, \tau_\infty)$ is a local mild solution of (2.68) on the set $\{\omega : \tau_\infty(\omega) < T\}$ and hence we have

$$\lim_{t \uparrow \tau_\infty} \left[\sup_{0 \leq s \leq t} \|\mathbf{u}(s)\|_{\mathbb{Y}} \right] \geq \lim_{k \uparrow \infty} \left[\sup_{0 \leq s \leq \tau_k} \|\mathbf{u}(s)\|_{\mathbb{Y}} \right] = \lim_{k \uparrow \infty} \left[\sup_{0 \leq s \leq \tau_k} \|\mathbf{u}_k(s)\|_{\mathbb{Y}} \right]$$

$$\geq \mathcal{K} \lim_{k \uparrow \infty} \left[\sup_{0 \leq s \leq \tau_k} \left(\beta(\mathbf{u}_k(t)) \vee \lambda_1(\mathbf{u}_k(t)) \vee \lambda_2(\mathbf{u}_k(t)) \vee \lambda_3(\mathbf{u}_k(t)) \vee \mu(\mathbf{u}_k(t)) \right. \right.$$

$$\left. \left. \vee \left\| \int_0^t S\Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \vee \left\| \int_0^t S^2 \Phi dW_{\mathbb{H}}(s) \right\|_{\mathbb{X}} \right) \right] = \infty, \tag{2.86}$$

where $\mathcal{K}$ is defined in (2.73). Thus $\tau_\infty(\omega)$ is an explosion time of $\mathbf{u}(t) \in C(0, \tau_\infty; \mathbb{W})$, a.s. $\square$

Similar ideas of proving maximal local solutions can be found in [5, 2, 16, 15]. Let us now give an example for which the abstract theory we discussed above is applicable.

Example 2.28 Let us apply the abstract theory to the Euler and Navier-Stokes equations for incompressible fluids. We write the combined stochastic Euler and Navier-Stokes equation as

$$\left. \begin{array}{r} d\mathbf{u}(t) - \nu \mathscr{P} \Delta \mathbf{u}(t) dt + \mathscr{P}(\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t) dt = \Phi dW_{\mathbb{H}}(t), \ t \in (0, T], \\ \mathbf{u}(0) = \mathbf{u}_0, \end{array} \right\} \tag{2.87}$$

where $\mathbf{u}(t) := (u_1(t, x, \omega), \cdots, u_n(t, x, \omega))$, $(t, x, \omega) \in [0, T] \times \mathbb{R}^n \times \Omega$, is the velocity field, $\nu \geq 0$ is the coefficient of kinematic viscosity, and $\mathscr{P}$ is the Helmholtz-Hodge projection operator (see [16]). The operator $\mathscr{P}$ annihilates gradients and maps into divergence free vector fields. $\mathscr{P}$ is a bounded operator on $\mathbb{L}^p$, $1 < p < \infty$, into itself, and commutes with translation. Hence it is also bounded on $\mathbb{L}^p_s := J^{-s}\mathbb{L}^p(\mathbb{R}^n)$, where $J := (I-\Delta)^{1/2}$, for any real s. Let us now choose the basic spaces $\mathbb{H}$, $\mathbb{X}$ and $\mathbb{Y}$, and the isomorphism S as

$$\mathbb{H} = \mathscr{P}\mathbb{L}^2, \mathbb{X} = \mathscr{P}\mathbb{L}^p_{s-2}, \mathbb{Y} = \mathscr{P}\mathbb{L}^p_s \subset \mathbb{W}^{1,\infty}, S = J^2 = I - \Delta, 1 < p < \infty, s > 1 + \frac{n}{p}. \tag{2.88}$$

Note that $\mathbb{X}$ and $\mathbb{Y}$ consisting of divergence free vector fields are closed subspaces of vector-valued $\mathbb{L}^p_{s-2}$ and $\mathbb{L}^p_s$ respectively, and inherits their norms $\|\cdot\|_{s-2,p}$ and $\|\cdot\|_{s,p}$. Thus $\mathbb{X}$ and $\mathbb{Y}$ are UMD Banach spaces of type 2

(see Theorem 4.5.2, [1], Corollary A.6, [3]) with $\|\mathbf{u}\|_{s,p} = \|\mathrm{S}\mathbf{u}\|_{s-2,p}$ and hence verifying the condition (A1) of Assumption 2.1.

For each $\mathbf{w} \in \mathbb{Y}$, we define the operator $\mathrm{A}(\mathbf{w})$ by

$$\mathrm{A}(\mathbf{w})\mathbf{z} = -\nu\mathscr{P}\Delta\mathbf{z} + \mathscr{P}(\mathbf{w}\cdot\nabla)\mathbf{z} = -\nu\mathscr{P}\Delta\mathbf{z} + \mathscr{P}\nabla\cdot(\mathbf{w}\otimes\mathbf{z}). \tag{2.89}$$

The operator $\mathrm{A}(\mathbf{w})$ is quasi$-m-$accretive with $\mathcal{S}_{\#}$ as a core, where $\mathcal{S}_{\#}$ denotes the subset of the Schwartz space $\mathcal{S}$ consisting of divergence free vector fields (see Proposition 3.3, [13]), and hence it verifies the condition (A2) of Assumption 2.1.

Using Proposition 3.3, [13], we obtain the operator $\mathcal{C}(\mathbf{w}) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ with $\|\mathcal{C}(\mathbf{w})\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})} \le C\|\mathbf{w}\|_{\mathbb{Y}}$ such that

$$[\mathrm{J}^2, \overline{\mathrm{A}}(\mathbf{w})]\psi = \mathcal{C}(\mathbf{w})\psi, \text{ for all } \psi \in \mathcal{S}_{\#}, \ [\mathrm{J}^2, \overline{\mathrm{A}}(\mathbf{w})]\psi = \mathrm{J}^2(\overline{\mathrm{A}}(\mathbf{w})\psi) - \overline{\mathrm{A}}(\mathbf{w})\mathrm{J}^2\psi, \tag{2.90}$$

where $\overline{\mathrm{A}}(\mathbf{w}) = \mathscr{P}(\mathbf{w}\cdot\nabla)$ and $[\cdot,\cdot]$ is the commutator. Thus (2.90) implies that $\mathrm{J}^2\mathrm{A}(\mathbf{w})\psi = \mathrm{A}(\mathbf{w})\mathrm{J}^2\psi + \mathcal{C}(\mathbf{w})\psi$. Let us take $\phi = \mathrm{J}^2\psi$ and an application of $\mathrm{S}^{-1} = \mathrm{J}^{-2}$ yields

$$\mathrm{A}(\mathbf{w})\mathrm{S}^{-1}\phi = \mathrm{S}^{-1}\mathrm{A}(\mathbf{w})\phi + \mathrm{S}^{-1}\mathrm{B}(\mathbf{w})\phi, \text{ for all } \phi \in \mathcal{S}_{\#}, \tag{2.91}$$

where $\mathrm{B}(\mathbf{w}) = \mathcal{C}(\mathbf{w})\mathrm{J}^{-2} \in \mathcal{L}(\mathbb{X},\mathbb{X})$ with

$$\|\mathrm{B}(\mathbf{w})\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \le C\|\mathbf{w}\|_{\mathbb{Y}}. \tag{2.92}$$

Since, $\mathcal{S}_{\#}$ is the core of $\mathrm{A}(\mathbf{w})$ in $\mathbb{X}$, (2.91) verifies the condition (A3) of Assumption 2.1.

By using the algebra property of $\mathbb{L}_s^p$ norm for $s > n/p$ and Hölder's inequality, for all $\mathbf{v} \in \mathbb{Y}$, we have

$$\|\mathrm{A}(\mathbf{w})\mathbf{v}\|_{\mathbb{X}} \le \|\mathscr{P}\Delta\mathbf{v}\|_{s-2,p} + \|\mathscr{P}\nabla\cdot(\mathbf{w}\otimes\mathbf{v})\|_{s-2,p} \le (1 + \|\mathbf{w}\|_{s-1,p})\|\mathbf{v}\|_{\mathbb{Y}}, \tag{2.93}$$

and

$$\begin{aligned}\|(\mathrm{A}(\mathbf{w}_1) - \mathrm{A}(\mathbf{w}_2))\mathbf{v}\|_{\mathbb{X}} &\le \|\mathscr{P}((\mathbf{w}_1 - \mathbf{w}_2)\cdot\nabla)\mathbf{v}\|_{s-2,p} \le \|\mathbf{w}_1 - \mathbf{w}_2\|_{s-2,p}\|\nabla\mathbf{v}\|_{\mathbb{L}^\infty} \\ &\le C\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbb{X}}\|\mathbf{v}\|_{\mathbb{Y}},\end{aligned} \tag{2.94}$$

for $s > n/p + 1$, verifying the condition (A4) of Assumption 2.1.

By taking the center of the open ball $\mathbb{W} \subset \mathbb{Y}$ as the origin, the condition (A5) of Assumption 2.1 follows easily. Thus, we can apply our abstract theory to the problem (2.87) for $\nu \ge 0$. Hence, for $\mathbf{u}_0 \in \mathbb{Y}$, a.s., there exists a unique local pathwise mild solution to the problem (2.87) in $\mathbb{L}^2(\Omega; \mathrm{C}(0, \widetilde{T} \wedge \tau_N; \mathbb{Y}))$, where τ_N is the stopping time given by $\tau_N := \inf_{t \ge 0}\left\{t : \|\mathbf{u}(t)\|_{\mathbb{Y}} \ge N\right\}$. The $\mathbb{L}^p-$theory for stochastic Navier-Stokes equations perturbed by Lévy noise is established in [17] and this method establishes the $\mathbb{L}_s^p-$theory in the case of Gaussian noise.

Remark 2.29 If we consider the problem

$$\left.\begin{aligned}\mathrm{d}\mathbf{u}(t) + \mathrm{A}(t)\mathbf{u}(t)\mathrm{d}t &= \Phi\mathrm{d}\mathrm{W}_{\mathbb{H}}(t), \ 0 \le t \le T, \\ \mathbf{u}(0) &= \mathbf{u}_0,\end{aligned}\right\} \tag{2.95}$$

where $\{\mathrm{A}(t)\}_{0 \le t \le T}$ is a deterministic stable family of generators in $\mathrm{G}(\mathbb{X})$ with the stability index M and β, then one can establish a global pathwise mild solution for this problem. The deterministic evolution operator $\mathrm{U}(t,s) \in \mathcal{L}(\mathbb{X},\mathbb{X})$ defined on the triangle $\triangle := 0 \le s \le t \le T$ can be constructed same as in Theorem I, [10] under the assumption (i)-(iii), page 29, [10], satisfying

$$\sup_{s,t\in\triangle}\|\mathrm{U}(t,s)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \le Me^{\beta T} \text{ and } \sup_{s,t\in\triangle}\|\mathrm{U}(t,s)\|_{\mathcal{L}(\mathbb{Y},\mathbb{Y})} \le \|\mathrm{S}\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})}\|\mathrm{S}^{-1}\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})}Me^{\beta T + MV},$$

where $V = \displaystyle\int_0^{T^-}\|\mathrm{B}(t)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\mathrm{d}t$, with $\mathrm{S}\mathrm{A}(t)\mathrm{S}^{-1} = \mathrm{A}(t) + \mathrm{B}(t), \ \mathrm{B}(t) \in \mathcal{L}(\mathbb{X},\mathbb{X}), 0 \le t \le T$.

Then the unique global mild solution $\mathbf{u}(t)$ of (2.95) is given by

$$\mathbf{u}(t) = \mathrm{U}(t,0)\mathbf{u}_0 + \int_0^t \mathrm{U}(t,s)\Phi\mathrm{d}\mathrm{W}_{\mathbb{H}}(s), \tag{2.96}$$

and

(i) if $\mathbf{u}_0 \in \mathbb{L}^2(\Omega;\mathbb{X})$, then $\mathbf{u} \in \mathbb{L}^2(\Omega;\mathrm{C}(0,T;\mathbb{X}))$,

(ii) if $\mathbf{u}_0 \in \mathbb{L}^2(\Omega;\mathbb{Y})$, then $\mathbf{u} \in \mathbb{L}^2(\Omega;\mathrm{C}(0,T;\mathbb{Y}))$.

Also, we have

(i) $\mathbb{E}\left[\displaystyle\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|_{\mathbb{X}}^2\right] \leq 2M^2 e^{2\beta T}\left[\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{X}}^2\right] + CT\|\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2\right],$

(ii) $\mathbb{E}\left[\displaystyle\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|_{\mathbb{Y}}^2\right] \leq 2M^2 e^{2(\beta T + MV)}\left[\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbb{Y}}^2\right] + CT\|\mathrm{S}\|_{\mathcal{L}(\mathbb{Y},\mathbb{X})}^4\|\mathrm{S}^{-1}\|_{\mathcal{L}(\mathbb{X},\mathbb{Y})}^4\|\mathrm{S}\Phi\|_{\gamma(\mathbb{H},\mathbb{X})}^2\right].$

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