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In this work we construct a Markov family of martingale solutions for 3D stochastic Navier–Stokes equations (SNSE) perturbed by Lévy noise with periodic boundary conditions. Using the Kolmogorov equations of integrodifferential type associated with the SNSE perturbed by Lévy noise, we construct a transition semigroup and establish the existence of a unique invariant measure. We also show that it is ergodic and strongly mixing.



## 1 Introduction

The ergodic properties of infinite dimensional systems have been intensively studied over the past three decades. The ergodic principle lies at the basis of statistical approach to the theory fluid dynamics. It states that as the averaging interval becomes infinitely large, the time average of an observable defined over a phase space converge to the corresponding ensemble average. Ergodicity results for the two and three dimensional stochastic Navier–Stokes equations (SNSE) on various domains have been established in the literature. The existence and uniqueness of invariant measures for the 2D and 3D SNSE with degenerate and non-degenerate Gaussian noise have been studied by using various methods in [22, 9, 23, 34, 8, 39, 27, 13, 40, 41, 14]. Ergodicity results for the 2D SNSE driven by Lévy noise with non-degenerate Gaussian part is established in [16] and the 2D stochastic magnetohydrodynamic(MHD) equations with Lévy noise is established in [32]. The paper [17] studies the Markov selection of martingale solutions for the 3D SNSE with Lévy noise. The authors in [44, 36] proved the existence of an ergodic control which is optimal in the class of all stationary measures for the SNSE with Gaussian and Lévy noise respectively for a suitable class of cost functions. By deriving rigorous estimates for solutions of the Kolmogorov equations associated with the 3D SNSE with additive and multiplicative Gaussian noise, the ergodicity results of these models have been studied in [8] and [13] respectively.

The study of this model with general jump random noise is motivated by (1) the engineering scenario where the flow field is often subjected to structural and environmental disturbances; and also (2) intermittency phenomena observed in turbulence signifying space-time concentrated abrupt fluctuations in velocity and in particular vorticity field can be studied by introducing jump noise forcing to the Navier–Stokes equations and understanding its impact on the dynamics. A phenomenological study of fully developed turbulence and intermittency is carried out in [3] where it is proposed that experimental observations of these physical characteristics could be modeled by stochastic Navier–Stokes equations with Lévy noise, which is the sum of Gaussian and compensated Poisson processes.

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Using the methodology developed in [8] and [13], we first construct a Markov family of martingale solutions for the 3D SNSE subject to Lévy noise. It is then used to obtain the existence of a unique invariant measure, which is ergodic and strongly mixing. These results are established by proving suitable estimates for the Kolmogorov equations of integrodifferential type associated to the SNSE with Lévy noise. The mild form of the Galerkin approximated Kolmogorov equations are obtained from the Feynman–Kac formula. Rigorous estimates for this semigroup are established by obtaining the differentiability of the Feynman–Kac semigroup and Bismut–Elworthy–Li type formula derived for the SNSE with Lévy noise. Several crucial higher order weighted estimates used in this context have been proved for the SNSE as well as the associated Kolmogorov equations using stochastic convolution estimates derived for compensated Poisson integral (see Lemma 11). This extends these kinds of estimates derived for the SNSE with Gaussian noise in [8] and [13]. The limiting solutions of the approximated Kolmogorov equation and that of the SNSE are combined to arrive at the required result. Moreover, in this work we only consider the SNSE in periodic domains. This restriction arises due to the bilinear estimates of Lemma 3 which hold true for periodic boundary conditions (see, Remark 4).

The construction of the paper is as follows. In section 2, we give a mathematical formulation of the problem, define the necessary functional spaces for this paper and state the main theorems. In section 3, we derive a-priori estimates needed to establish the main results. A Markov family of martingale solutions is constructed in section 4. The existence and uniqueness of invariant measures and hence the ergodicity of the 3D SNSE perturbed by Lévy noise is established in section 5. Comparison lemma, Bismut–Elworthy–Li formula and the differentiability of the Feynman–Kac semigroup for such systems are given in Appendices A, B and C.

## 2 Mathematical formulation

Let $\mathcal{O} = [0, \mathrm{L}] \times [0, \mathrm{L}] \times [0, \mathrm{L}]$, and we define the spaces

$$\mathbb{H} = \left\{ \mathrm{X} \in \mathbb{L}^2(\mathcal{O}; \mathbb{R}^3),\ \mathrm{div}\ \mathrm{X} = 0,\ \int_{\mathcal{O}} \mathrm{X}(\xi) d\xi = 0, \mathrm{X} \cdot \mathbf{n} \text{ is periodic} \right\},$$

$$\mathbb{V} = \left\{ \mathrm{X} \in \mathbb{H}^1(\mathcal{O}; \mathbb{R}^3),\ \mathrm{div}\ \mathrm{X} = 0,\ \int_{\mathcal{O}} \mathrm{X}(\xi) d\xi = 0, \mathrm{X} \cdot \mathbf{n} \text{ is periodic} \right\},$$

where $\mathbf{n}$ is the unit outward normal, and for an integer $k \geq 1$, $\mathbb{H}^k(\mathcal{O}; \mathbb{R}^3)$ is the space of $\mathbb{R}^3$-valued functions X that are in $\mathbb{H}^k_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ and such that $\mathrm{X}(\xi + \mathrm{L}\mathbf{e}_i) = \mathrm{X}(\xi)$ for every $\xi \in \mathbb{R}^3$ and $i = 1, 2, 3$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. We denote by $(\cdot, \cdot)$ and $\|\cdot\|$, the usual $\mathbb{L}^2$-inner product and norm in $\mathbb{H}$ with $\|\mathrm{X}\|^2 := \int_{\mathcal{O}} |\mathrm{X}(\xi)|^2 \mathrm{d}\xi$. Due to the zero mean condition, we also have the *Poincaré inequality*, $\|\mathrm{X}\| \leq \frac{1}{\lambda} \|\nabla \mathrm{X}\|$, where $\lambda$ is defined to be the smallest constant for which this inequality holds (see [25]). From the Poincaré inequality, we may endow $\mathbb{V}$ with the norm $\|\mathrm{X}\|_{\mathbb{V}}^2 := \int_{\mathcal{O}} |\nabla \mathrm{X}(\xi)|^2 \mathrm{d}\xi$.

Let $\mathrm{X}(t, \xi) = (\mathrm{X}_1(t, \xi), \mathrm{X}_2(t, \xi), \mathrm{X}_3(t, \xi))$ denotes the *velocity field* and the scalar valued function $p = p(t, \xi)$ denotes the *pressure field*. Let $T$ be an arbitrary but fixed positive number. For $t \in [0, T]$, let us consider the stochastic Navier–Stokes equation perturbed by Lévy noise as follows:

$$\begin{cases} \mathrm{d}\mathrm{X}(t, \xi) = [\nu \Delta \mathrm{X}(t, \xi) - (\mathrm{X}(t, \xi) \cdot \nabla) \mathrm{X}(t, \xi) - \nabla p(t, \xi)] \mathrm{d}t \\ \qquad + \sqrt{\mathrm{Q}} \mathrm{d}\mathrm{W}(t, \xi) + \displaystyle\int_{\mathrm{Z}} \Psi(t-, \xi, z) \widetilde{\pi}(\mathrm{d}t, \mathrm{d}z),\ (t, \xi) \in (0, T) \times \mathcal{O}, \\ \mathrm{div}\ \mathrm{X}(t, \xi) = 0,\ (t, \xi) \in (0, T) \times \mathcal{O}, \\ \mathrm{X}(0, \xi) = x(\xi),\ \xi \in \mathcal{O}. \end{cases} \tag{2.1}$$

Here $\nu > 0$ is the *coefficient of kinematic viscosity* and is scaled to unity in the rest of the paper. The characterization of noise coefficients are given in the next subsection.

Let $\mathrm{P}_{\mathbb{H}}$ be the orthogonal projection of $\mathbb{L}^2(\mathcal{O}; \mathbb{R}^3)$ onto $\mathbb{H}$. Let us define the Stokes operator $\mathrm{A} : \mathrm{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ by

$$\mathrm{A}\mathrm{X} = -\mathrm{P}_{\mathbb{H}} \Delta \mathrm{X} \text{ with } \mathrm{D}(\mathrm{A}) = \mathbb{H}^2(\mathcal{O}; \mathbb{R}^3) \cap \mathbb{V}.$$

The operator A is a selfadjoint, positive (unbounded) operator in $\mathbb{H}$ with a compact resolvent. There is a complete orthonormal system $\{e_i\}_{i \in \mathbb{N}} \subset \mathbb{H}$ made of eigenfunctions of A, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ such that $\mathrm{A}e_i = \lambda_i e_i$. We may take the Poincaré constant $\lambda$ in the Poincaré inequality equal to $\lambda_1$. We also have $(\mathrm{AX},\mathrm{X}) = \|\mathrm{X}\|^2_{\mathbb{V}}$ for every $\mathrm{X}\in\mathrm{D(A)}$, and in particular $(\mathrm{AX},\mathrm{X})\geq\lambda_1\|\mathrm{X}\|^2$. The spaces $\mathbb{V}$ and $\mathrm{D(A)}$ are densely and compactly embedded in $\mathbb{H}$. The fractional powers $\mathrm{A}^\alpha$ of A, $\alpha\geq 0$, are defined by $\mathrm{A}^\alpha\mathrm{X} = \sum\limits_{j=1}^{\infty}\lambda_j^\alpha(\mathrm{X},e_j)e_j$ with domain $\mathrm{D}(\mathrm{A}^\alpha) = \Big\{\mathrm{X}\in\mathbb{H} : \|\mathrm{X}\|_{\mathrm{D}(\mathrm{A}^\alpha)} < +\infty\Big\}$, where $\|\mathrm{X}\|^2_{\mathrm{D}(\mathrm{A}^\alpha)} = \sum\limits_{j=1}^{\infty}\lambda_j^{2\alpha}|(\mathrm{X},e_j)|^2 = \|\mathrm{A}^\alpha\mathrm{X}\|^2$. The space $\mathrm{D}(\mathrm{A}^\alpha)$ is a Hilbert space with the inner product $(\mathrm{X},\mathrm{Y})_{\mathrm{D}(\mathrm{A}^\alpha)} = (\mathrm{A}^\alpha\mathrm{X},\mathrm{A}^\alpha\mathrm{Y})$, for all $\mathrm{X},\mathrm{Y}\in\mathrm{D}(\mathrm{A}^\alpha)$. It is well known that $\mathbb{V}$ coincides with $\mathrm{D}(\mathrm{A}^{1/2})$ and we endow $\mathbb{V}$ with the norm $\|\mathrm{X}\|_{\mathbb{V}} = \|\mathrm{A}^{1/2}\mathrm{X}\|$. The space $\mathrm{D}(\mathrm{A}^\alpha)$ is a closed subspace of the Sobolev space $\mathbb{H}^{2\alpha}(\mathcal{O};\mathbb{R}^3)$ and $\|\cdot\|_{\mathrm{D}(\mathrm{A}^\alpha)} = \|\mathrm{A}^\alpha\cdot\|$ is equivalent to the usual $\mathbb{H}^{2\alpha}(\mathcal{O};\mathbb{R}^3)$ norm (see [45]). For $\alpha_1<\alpha_2$, the embedding of $\mathbb{H}^{\alpha_2}\subset\mathbb{H}^{\alpha_1}$ is compact.

Let $\mathbb{V}'$ be the dual of $\mathbb{V}$. By the proper identifications, we also have $\mathbb{V}\subset\mathbb{H}\equiv\mathbb{H}'\subset\mathbb{V}'$ with continuous, dense injections and the scalar product $(\cdot,\cdot)$ extends to the duality pairing $\langle\cdot,\cdot\rangle_{\mathbb{V}'\times\mathbb{V}}$ between $\mathbb{V}$ and $\mathbb{V}'$. We denote by $\mathrm{D}(\mathrm{A}^{-\alpha})$, the dual space of $\mathrm{D}(\mathrm{A}^\alpha)$ and we perform identifications as above to get the dense continuous inclusions, for $\alpha>1/2$,

$$\mathrm{D}(\mathrm{A}^\alpha)\subset\mathbb{V}\subset\mathbb{H}\equiv\mathbb{H}'\subset\mathbb{V}'\subset\mathrm{D}\big(\mathrm{A}^{-\alpha}\big).$$

For negative powers, we have $(\mathrm{X},\mathrm{Y})_{\mathrm{D}(\mathrm{A}^{-\alpha})} = (\mathrm{A}^{-\alpha}\mathrm{X},\mathrm{A}^{-\alpha}\mathrm{Y})$.

Let us define $\mathrm{B}:\mathrm{D(B)}\subset\mathbb{V}\times\mathbb{V}\to\mathbb{V}'$ by $\mathrm{B}(\mathrm{X},\mathrm{Y}) = \mathrm{P}_{\mathbb{H}}(\mathrm{X}\cdot\nabla)\mathrm{Y}$, with $\mathrm{B}(\mathrm{X}) = \mathrm{B}(\mathrm{X},\mathrm{X})$. Moreover, for any $\mathrm{X},\mathrm{Y},\mathrm{Z}\in\mathbb{V}$ and integration by parts yields

$$\langle\mathrm{B}(\mathrm{X},\mathrm{Y}),\mathrm{Z}\rangle_{\mathbb{V}'\times\mathbb{V}} = \sum_{i,j=1}^{3}\int_{\mathcal{O}}\mathrm{X}_i\frac{\partial\mathrm{Y}_j}{\partial\xi_i}\mathrm{Z}_j\mathrm{d}\xi = -\langle\mathrm{B}(\mathrm{X},\mathrm{Z}),\mathrm{Y}\rangle_{\mathbb{V}'\times\mathbb{V}},$$

and $\langle\mathrm{B}(\mathrm{X},\mathrm{Y}),\mathrm{Y}\rangle_{\mathbb{V}'\times\mathbb{V}} = 0$. By applying the projection operator $\mathrm{P}_{\mathbb{H}}$ on (2.1), we get

$$\begin{cases}\mathrm{dX}(t,x) = -[\mathrm{AX}(t,x)+\mathrm{B}(\mathrm{X}(t,x))]\mathrm{d}t+\sqrt{\mathrm{Q}}\mathrm{dW}(t)+\displaystyle\int_{\mathrm{Z}}\Psi(t-,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{X}(0,x) = x\in\mathbb{H}.\end{cases}\tag{2.2}$$

### 2.1 Lévy noise and assumptions

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a given complete probability space equipped with an increasing family of sub-sigma fields $\{\mathscr{F}_t\}_{0\leq t\leq T}$ of $\mathscr{F}$ satisfying usual conditions.

Let $\mathbf{L}(\mathbb{H})$ be the space of all bounded linear operators on $\mathbb{H}$. Let $\mathrm{Q}\in\mathbf{L}(\mathbb{H})$ be a positive, symmetric and trace class operator on $\mathbb{H}$ with $\mathrm{Ker\,Q} = \{0\}$. Thus there exists an orthonormal basis $\{f_k\}_{k=1}^\infty$ of $\mathbb{H}$ such that $\mathrm{Q}f_k = \varrho_k f_k, k\in\mathbb{N}$. Here $\varrho_k$ is the eigenvalue corresponding to $\{f_k\}$ which is real and positive satisfying

$$\mathrm{Tr(Q)} = \sum_{k=1}^{\infty}\varrho_k < +\infty \ \text{ and } \ \mathrm{Q}^{1/2}v = \sum_{k=1}^{\infty}\sqrt{\varrho_k}(v,f_k)f_k,\ v\in\mathbb{H}.$$

The stochastic process $\{\mathrm{W}(t):0\leq t\leq T\}$ is an $\mathbb{H}$-valued cylindrical Wiener process on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ if and only if for arbitrary $t$, the process $\mathrm{W}(t)$ can be expressed as $\mathrm{W}(t) = \sum\limits_{k=1}^{\infty}\beta_k(t)f_k$, where $\beta_k(t), k\in\mathbb{N}$ are independent, one dimensional Brownian motions on the space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$.

Let $(\mathcal{Z},|\cdot|)$ be a separable Banach space and $(\mathbf{L}_t)_{t\geq 0}$ be an $\mathcal{Z}$-valued Lévy process. For every $\omega\in\Omega$, $\mathbf{L}_t(\omega)$ has at most countable number of jumps in an interval and the jump $\Delta\mathbf{L}_t(\omega):[0,T]\to\mathcal{Z}$ is defined by $\Delta\mathbf{L}_t(\omega) := \mathbf{L}_t(\omega)-\mathbf{L}_{t-}(\omega)$ at $t\geq 0$. For $\mathrm{Z}:=\mathcal{Z}\backslash\{0\}$, we define

$$\pi([0,T],\Gamma) = \#\{t\in[0,T]:\Delta\mathbf{L}_t(\omega)\in\Gamma\},\ \text{where } \Gamma\in\mathscr{B}(\mathrm{Z}),\ \omega\in\Omega.$$

The measure $\pi(\cdot,\cdot)$ is the Poisson random measure(or jump measure) with respect to $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{0\leq t\leq T},\mathbb{P})$ associated with the Lévy process $(\mathbf{L}_t)_{t\geq 0}$. Here $\mathscr{B}(\mathrm{Z})$ is the Borel $\sigma$-field, $\pi([0,T],\Gamma)$ is the Poisson random measure defined on $(([0,T]\times \mathrm{Z}),\mathscr{B}([0,T]\times \mathrm{Z}))$, and $\lambda(\cdot)=\mathbb{E}(\pi(1,\cdot))$ is the $\sigma$-finite measure defined on $(\mathrm{Z},\mathscr{B}(\mathrm{Z}))$. The intensity measure $\lambda(\cdot)$ on Z satisfies the conditions $\lambda(\{0\})=0$ and

$$\int_{\mathrm{Z}}(1\wedge |z|^{p})\lambda(\mathrm{d}z)<+\infty,\ \ p\geq 2. \tag{2.3}$$

The compensated Poisson random measure is defined by $\widetilde{\pi}(\mathrm{d}t,\Gamma)=\pi(\mathrm{d}t,\Gamma)-\mathrm{d}t\lambda(\Gamma)$, where $\mathrm{d}t\lambda(\Gamma)$ is the compensator of the Lévy process $(\mathbf{L}_t)_{t\geq 0}$ and $\mathrm{d}t$ is the Lebesgue measure. We assume that the processes W and $\widetilde{\pi}$ are mutually independent, which is crucial in obtaining the Bismut–Elworthy–Li formula (see Appendix B).

The co-variance operator Q is sufficiently smooth and nondegenerate with

$$\mathrm{Tr}\big[\mathrm{A}^{1+g}\mathrm{Q}\big]<+\infty,\ \text{for some } g>0, \tag{2.4}$$

and

$$\|\mathrm{Q}^{-1/2}x\|\leq C_r\|\mathrm{A}^r x\|,\ \text{for all } x\in \mathrm{D}(\mathrm{A}^r),\ \text{for some } r\in(1,3/2) \text{ and } C_r>0. \tag{2.5}$$

**Remark 1.** The operator $\mathrm{Q}=\mathrm{A}^{-\alpha}$ with suitable $\alpha>0$ satisfy the conditions (2.4) and (2.5). Indeed, since the asymptotic behavior of the eigenvalues in periodic domain is given by $\lambda_k\sim\lambda_1 k^{2/3}$ (Theorem 4.11, [5], page 54, [25]),

$$\mathrm{Tr}(\mathrm{A}^{1+g-\alpha})=\sum_{k=1}^{\infty}\lambda_k^{1+g-\alpha}\sim\lambda_1^{1+g-\alpha}\sum_{k=1}^{\infty}k^{2/3(1+g-\alpha)}<+\infty,\quad \text{provided } \ g<\alpha-5/2.$$

For any $\alpha<2r$ and $x\in\mathrm{D}(\mathrm{A}^r)$, one can obtain

$$\|\mathrm{A}^{\alpha/2}x\|^2=\sum_{k=1}^{\infty}\lambda_k^{\alpha}|(x,e_k)|^2\leq\sup_k[\lambda_k^{\alpha-2r}]\sum_{k=1}^{\infty}\lambda_k^{2r}|(x,e_k)|^2\leq\lambda_1^{\alpha-2r}\|\mathrm{A}^r x\|^2.$$

For any $g>0$ and $r\in(1,3/2)$, we see that $\alpha\in(5/2,3)$ satisfies the requirements.

Now we state the main assumptions of the jump noise coefficient and other assumptions are stated in the relevant sections. The jump noise coefficient $\Psi:[0,T]\times\mathcal{O}\times\mathrm{Z}\to\mathrm{D}(\mathrm{A}^{\delta/2})$, where $\delta\in(1/2,1+g]$, satisfies

$$(i)\int_0^T\int_{\mathrm{Z}}\|\mathrm{A}\Psi(s,z)\|^p\lambda(\mathrm{d}z)\mathrm{d}s\leq C(T)<+\infty, \tag{2.6}$$

$$(ii)\int_0^T\left(\int_{\mathrm{Z}}\|\mathrm{A}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\right)^{p/2}\mathrm{d}s\leq C(T)<+\infty, \tag{2.7}$$

$$(iii)\int_0^T\int_{\mathrm{Z}}\|\mathrm{A}^{\delta/2}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s\leq C(T)<+\infty, \tag{2.8}$$

for $p\geq 2$. Also we fix measurable subsets $\mathrm{Z}_m$ of Z such that $\lambda(\mathrm{Z}_m)<+\infty$ and $\mathrm{Z}_m\uparrow\mathrm{Z}$ as $m\to\infty$.

**Remark 2.** It is worth noting that from the conditions of the Lévy measure $\lambda(\cdot)$ (see (2.3)), the boundedness assumptions of the integrals given in (2.6)-(2.8) can be validated under reasonable growth condition of $\Psi$ with respect to $z$. Moreover, we are forced to assume that the jump noise coefficient to be in the domain of the Stokes operator rather than square integrable coefficients in $\mathbb{H}$ which is usually assumed to prove the solvability of SNSE with Lévy noise.

For the right continuous martingale, $\mathrm{M}_t:=\int_0^t\int_{\mathrm{Z}}\Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)$, the *Meyer process* and *quadratic variation process* are given by (see [35, 42])

$$\lhd\mathrm{M}\rhd_t=\int_0^t\|\Psi(s,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s\ \text{ and } [\![\mathrm{M}]\!]_t=\int_0^t\|\Psi(s,z)\|^2\pi(\mathrm{d}z,\mathrm{d}s), \tag{2.9}$$

so that $[\![\mathrm{M}]\!]_t - \lhd\mathrm{M}\rhd_t$ is a martingale and $\mathbb{E}(\lhd\mathrm{M}\rhd_t) = \mathbb{E}([\![\mathrm{M}]\!]_t)$.

The Itô stochastic integral $\mathrm{M}_t$ has a càdlàg modification and satisfies the following Itô isometry (Remark 3.5.3, [31]):

$$\mathbb{E}\left[\left\|\int_0^t \int_{\mathrm{Z}} \Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\right\|^2\right] = \int_0^t \int_{\mathrm{Z}} \|\Psi(s,z)\|^2 \lambda(\mathrm{d}z)\mathrm{d}s, \text{ for all } t \in (0,T]. \tag{2.10}$$

### 2.2 Estimates on the nonlinear term

For any real $\alpha \leq \beta \leq \gamma$, we have the following interpolation inequality and Agmon estimate [8]:

$$\|\mathrm{A}^\beta x\| \leq C\|\mathrm{A}^\alpha x\|^{(\gamma-\beta)/(\gamma-\alpha)}\|\mathrm{A}^\gamma x\|^{(\beta-\alpha)/(\gamma-\alpha)}, \ x \in \mathrm{D}(\mathrm{A}^\gamma), \tag{2.11}$$

$$\|x\|_{\mathbb{L}^\infty} \leq C\|\mathrm{A}^{1/2}x\|^{1/2}\|\mathrm{A}x\|^{1/2}, \ x \in \mathrm{D}(\mathrm{A}). \tag{2.12}$$

Now, we have the following estimates on the bilinear operator $\mathrm{B}(x,y)$:

**Lemma 3** (Lemma 2.1, [8], Lemma 2.3, [24])**.** *Whenever the right hand side makes sense, we have*

*(i)* $\left|\left(\mathrm{B}(x,y), \mathrm{A}^{1/2}z\right)\right| \leq C\|\mathrm{A}x\|\|\mathrm{A}y\|\|z\|$,

*(ii)* $\left|\left(\mathrm{B}(x), \mathrm{A}^\delta x\right)\right| \leq C\|\mathrm{A}^{\delta/2}x\|^{1/2+\delta}\|\mathrm{A}^{(1+\delta)/2}x\|^{5/2-\delta}$, *for* $\delta > 1/2$.

**Remark 4.** Note that the estimates in Lemma 3 can be obtained when the domain is periodic (see [24]). In the case of bounded domains $\mathcal{O}$ with zero Dirichlet boundary conditions, when we apply the Helmholtz-Hodge projection $\mathrm{P}_{\mathbb{H}}$, the range of the nonlinear operator $\mathrm{B}(x,y) := \mathrm{P}_{\mathbb{H}}(x\cdot\nabla)y$ belongs to

$$\mathbb{H} := \left\{\mathbf{u} \in \mathbb{L}^2(\Omega) : \text{div } \mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}\big|_{\partial\Omega} = 0\right\}.$$

In the estimation of $(i)$, we need to use

$$\left(\mathrm{B}(x,y), \mathrm{A}^{1/2}z\right) = \left(\mathrm{A}^{1/2}\mathrm{B}(x,y), z\right), \tag{2.13}$$

and it demands that the operator $\mathrm{B} \in \mathrm{D}(\mathrm{A}^{1/2})$. In the case of bounded domains, we have (see page 163, [20])

$$\mathrm{D}(\mathrm{A}^\alpha) \approx \begin{cases} \mathbb{H}\cap\mathbb{H}^{2\alpha}, & \text{for } 0 < \alpha < 1/4, \\ \mathbb{H}\cap\mathbb{H}_0^{2\alpha}, & \text{for } 1/4 < \alpha < 1. \end{cases}$$

For instance, in order to define (2.13) when $\alpha = 1/2$, $\mathrm{B}(x,y)$ has to vanish on the boundary, but we only have $\mathrm{B}(x,y)\cdot\mathbf{n} = 0$ on the boundary.

### 2.3 Main results

Our aim in this paper is to establish the theorems (Theorem 7 and Theorem 8) given below.

**Definition 5.** [13] Let $(\Omega_x, \mathscr{F}_x, \mathbb{P}_x)_{x\in\mathrm{D}(\mathrm{A})}$ be a family of probability spaces and $(\mathrm{X}(\cdot,x))_{x\in\mathrm{D}(\mathrm{A})}$ be a family of random processes on $(\Omega_x, \mathscr{F}_x, \mathbb{P}_x)_{x\in\mathrm{D}(\mathrm{A})}$. We denote by $(\mathscr{F}_x^t)_{t\geq 0}$, the filtration generated by $\mathrm{X}(\cdot,x)$ and by $\mathcal{P}_x$, the law of $\mathrm{X}(t,x)$ under $\mathbb{P}_x$. The family $(\Omega_x, \mathscr{F}_x, \mathbb{P}_x, \mathrm{X}(\cdot,x))_{x\in\mathrm{D}(\mathrm{A})}$ is a *Markov family* if the following conditions hold:

(i) For any $x \in \mathrm{D}(\mathrm{A})$, $t \geq 0$, we have

$$\mathbb{P}_x\left\{\mathrm{X}(t,x) \in \mathrm{D}(\mathrm{A})\right\} = 1,$$

(ii) the map $x \mapsto \mathcal{P}_x$ is measurable and for any $x \in \mathrm{D}(\mathrm{A})$, $t_0,\ldots,t_n \geq 0$, $A_0,\ldots,A_n$, the Borel subsets of $\mathrm{D}(\mathrm{A})$, we have

$$\mathbb{P}_x\left\{\mathrm{X}(t+s) \in \mathcal{A}\big|\mathscr{F}_x^t\right\} = \mathcal{P}_{\mathrm{X}(t,x)}(\mathcal{A}),$$

for all $s \geq 0$, where $\mathcal{A} = \left\{y \in (\mathbb{H})^{\mathbb{R}^+}\big|y(t_0) \in A_0,\ldots,y(t_n) \in A_n\right\}$.

The *Markov transition semigroup* $(\mathrm{P}_t)_{t\geq 0}$ associated to the family is then defined by

$$\mathrm{P}_t\phi(x) = \mathbb{E}[\phi(\mathrm{X}(t,x))],\ x\in \mathrm{D}(\mathrm{A}), t\geq 0,$$

for $\phi\in \mathrm{B}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$, where $\mathrm{B}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ is the space of all *Borel bounded mappings* from $\mathrm{D}(\mathrm{A})$ into $\mathbb{R}$.

Following theorem of existence of martingale solutions to (2.2) is proved in Theorem 2.1, [42].

**Theorem 6.** *For any $x\in\mathbb{H}$ and $T>0$, there exists a martingale solution of the problem (2.2) with trajectories in $\mathbb{D}([0,T];\mathrm{D}(\mathrm{A}^{-1/2}))$ and $\mathrm{L}^\infty(0,T;\mathbb{H})\cap \mathrm{L}^2(0,T;\mathrm{D}(\mathrm{A}^{1/2}))$, where $\mathbb{D}([0,T];\mathrm{D}(\mathrm{A}^{-1/2}))$ is the space of all Càdlàg paths from $[0,T]$ to $\mathrm{D}(\mathrm{A}^{-1/2})$.*

Now we state the first main theorem concerning the Markov family of martingale solutions as in Definition 5 and it is proved in section 4.

**Theorem 7.** *There exists a* Markov family of martingale solutions $(\Omega_x,\mathscr{F}_x,\mathbb{P}_x,\mathrm{X}(\cdot,x))_{x\in\mathrm{D}(\mathrm{A})}$ *of the stochastic Navier–Stokes equations (2.2). Moreover, the transition semigroup $(\mathrm{P}_t)_{t\geq 0}$ is Markovian and stochastically continuous.*

The second main result is the following theorem proved in section 5.

**Theorem 8.** *There exists a Markov process $\mathrm{X}(\cdot,\mu)$ on a probability space $(\Omega_\mu,\mathscr{F}_\mu,\mathbb{P}_\mu)$ which is a martingale stationary solution of the stochastic Navier–Stokes equation (2.2). The law $\mu$ of $\mathrm{X}(\cdot,\mu)$ is the* unique invariant measure *on $\mathrm{D}(\mathrm{A})$ of the transition semigroup $(\mathrm{P}_t)_{t\geq 0}$. Furthermore*

(i) *the transition semigroup $(\mathrm{P}_t)_{t\geq 0}$ is* strong Feller, irreducible, *and the invariant measure $\mu$ is* ergodic *and* strongly mixing,

(ii) *the law $\mathcal{P}_\mu$ of $\mathrm{X}(\cdot,\mu)$ is given by*

$$\mathcal{P}_\mu(\mathcal{A}) = \int_{\mathrm{D}(\mathrm{A})} \mathcal{P}_x(\mathcal{A})\mu(\mathrm{d}x),$$

*for $\mathcal{A} = \left\{ y\in(\mathbb{H})^{\mathbb{R}^+} \middle| y(t_0)\in A_0,\dots,y(t_n)\in A_n\right\}$ with $t_0,\dots,t_n\geq 0$ and Borel subsets $A_0,\dots,A_n$ of* $\mathrm{D}(\mathrm{A})$.

### 2.4 Functional spaces

Let $\phi:\mathrm{D}(\mathrm{A})\to\mathbb{R}$. For any $x,h\in\mathrm{D}(\mathrm{A})$, we set

$$\langle \mathrm{D}_x\phi(x),h\rangle_{\mathrm{D}(\mathrm{A}^{-1})\times\mathrm{D}(\mathrm{A})} = \lim_{s\to 0}\frac{\phi(x+sh)-\phi(x)}{s},$$

provided the limit exists and is in $\mathbb{R}$. We define the following function spaces:

- $\mathrm{C}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ is the space of all *continuous and bounded* mappings from $\mathrm{D}(\mathrm{A})$ into $\mathbb{R}$ endowed with the norm
$$\|\phi\|_0 := \sup_{x\in\mathrm{D}(\mathrm{A})}|\phi(x)| < +\infty,\ \phi\in\mathrm{C}_b(\mathrm{D}(\mathrm{A});\mathbb{R}).$$
- For any $k\in\mathbb{N}$, we define $\mathrm{C}_k(\mathrm{D}(\mathrm{A});\mathbb{R})$ as the space of all continuous mappings from $\mathrm{D}(\mathrm{A})$ into $\mathbb{R}$ such that
$$\|\phi\|_k := \sup_{x\in\mathrm{D}(\mathrm{A})}\frac{|\phi(x)|}{(1+\|\mathrm{A}x\|^2)^{k/2}} < +\infty.$$
- For any $k\in\mathbb{N}$, we define
$$\mathscr{E} = \left\{\phi\in\mathrm{C}_b(\mathrm{D}(\mathrm{A});\mathbb{R}) \middle| \sup_{x_1,x_2\in\mathrm{D}(\mathrm{A})}\frac{|\phi(x_2)-\phi(x_1)|}{\|\mathrm{A}(x_2-x_1)\|(1+\|\mathrm{A}x_1\|^2+\|\mathrm{A}x_2\|^2)} < +\infty\right\},$$
with the norm
$$\|\phi\|_{\mathscr{E}} = \|\phi\|_0 + \sup_{x_1,x_2\in\mathrm{D}(\mathrm{A})}\frac{|\phi(x_2)-\phi(x_1)|}{\|\mathrm{A}(x_2-x_1)\|(1+\|\mathrm{A}x_1\|^2+\|\mathrm{A}x_2\|^2)}.$$

### 2.5 Galerkin approximations

Let $\{e_1, \dots, e_m\}$ be the first $m$ eigenvectors of A and we define the projector $\mathrm{P}_m$ as the projector of $\mathbb{H}$ onto the space spanned by these $m$ vectors. We set $\mathrm{B}_m(x) = \mathrm{P}_m \mathrm{B}(\mathrm{P}_m x)$, for $x \in \mathbb{H}$, $\mathrm{Q}_m = \mathrm{P}_m \mathrm{Q}, \mathrm{W}_m = \mathrm{P}_m \mathrm{W}$, $\Psi_m(t, z) = \mathrm{P}_m \Psi(t, \xi, z)$ and $\mathrm{Z}_m = \mathrm{P}_m \mathrm{Z}$. Then, we write the following finite dimensional system:

$$\begin{cases} \mathrm{dX}_m(t,x) = -[\mathrm{AX}_m(t,x) + \mathrm{B}_m(\mathrm{X}_m(t,x))]\mathrm{d}t + \sqrt{\mathrm{Q}_m}\,\mathrm{dW}_m(t) + \displaystyle\int_{\mathrm{Z}_m} \Psi_m(t-,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{X}_m(0) = \mathrm{P}_m x = x_m. \end{cases} \tag{2.14}$$

Let $(\mathrm{P}^m_t)_{t\geq 0}$ be the Markov transition semigroup associated to the system (2.14). Then

$$u_m(t,x) := \mathrm{P}^m_t \phi(x) = \mathbb{E}[\phi(\mathrm{X}_m(t,x))], \tag{2.15}$$

formally solves the following Kolmogorov equation:

$$\begin{cases} \dfrac{\partial u_m(t,x)}{\partial t} = \mathscr{L}^m_x u_m(t,x), \\ u_m(0,x) = \phi(x), \text{ for } x \in \mathrm{P}_m\mathbb{H}, \end{cases} \tag{2.16}$$

where the integro-differential operator

$$\begin{aligned} \mathscr{L}^m_x \phi(x) = &-(\mathrm{A}x + \mathrm{B}_m(x), \mathrm{D}_x\phi(x)) + \frac{1}{2}\mathrm{Tr}\big[\mathrm{Q}_m \mathrm{D}^2_x \phi(x)\big] \\ &+ \int_{\mathrm{Z}_m} [\phi(x + \Psi_m(z)) - \phi(x) - (\mathrm{D}_x\phi(x), \Psi_m(z))]\lambda(\mathrm{d}z). \end{aligned} \tag{2.17}$$

We can extend the definition of $u_m(t,x)$ to any $x \in \mathbb{H}$ by setting $u_m(t,x) = u_m(t, \mathrm{P}_m x)$. If $\phi$ is a $\mathrm{C}^1(\mathrm{P}_m\mathbb{H};\mathbb{R})$ function, then for any $h \in \mathrm{P}_m\mathbb{H}$, we have

$$(\mathrm{D}_x u_m(t,x), h) = \mathbb{E}\big[\big(\mathrm{D}_x\phi(\mathrm{X}_m(t,x)), \eta^h_m(t,x)\big)\big], \tag{2.18}$$

where $\eta^h_m := \mathrm{D}_x\mathrm{X}_m(t,x)h$ is the solution of the linear equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\eta^h_m(t,x) = -\big[\mathrm{A}\eta^h_m(t,x) + \mathrm{B}_m(\mathrm{X}_m(t,x), \eta^h_m(t,x)) + \mathrm{B}_m(\eta^h_m(t,x), \mathrm{X}_m(t,x))\big], \\ \eta^h_m(0,x) = \mathrm{P}_m h. \end{cases} \tag{2.19}$$

Moreover, since Ker $\mathrm{Q} = \{0\}$, the differential of $u_m$ exists even if when $\phi$ is only continuous due to the *Bismut–Elworthy–Li formula* (see Appendix B):

$$(\mathrm{D}_x u_m(t,x), h) = \frac{1}{t}\mathbb{E}\left[\phi(\mathrm{X}_m(t,x)) \int_0^t \Big(\mathrm{Q}_m^{-1/2}\eta^h_m(s,x), \mathrm{dW}_m(s)\Big)\right]. \tag{2.20}$$

It is difficult to get any estimate on the differential of $u_m$ as we are not able to prove an estimate of $\eta^h_m(t,x)$ uniform in $m$ (see Lemma 3.2, [8] or Lemma 15 below). Thus, we introduce an auxiliary *Kolmogorov equation*:

$$\begin{cases} \dfrac{\partial v_m(t,x)}{\partial t} = \mathscr{L}^m_x v_m(t,x) - K\|\mathrm{A}x\|^2 v_m(t,x), \\ v_m(0,x) = \phi(x), \text{ for } x \in \mathrm{P}_m\mathbb{H}, \end{cases} \tag{2.21}$$

where $K > 0$ is a fixed constant will be chosen later appropriately. The formal solution of (2.21) can be written by the *Feynman–Kac formula*:

$$v_m(t,x) := \mathrm{S}^m_t \phi(x) = \mathbb{E}\left[e^{-K\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}_m(t,x))\right]. \tag{2.22}$$

Thus the function $u_m(\cdot,\cdot)$ can be expressed in terms of the function $v_m(\cdot,\cdot)$ by the variation of constants formula:

$$u_m(t,\cdot) = \mathrm{S}_t^m\phi + K\int_0^t \mathrm{S}_{t-s}^m\big(\|\mathrm{A}\cdot\|^2 u_m(s,\cdot)\big)\mathrm{d}s. \tag{2.23}$$

Now, since the covariance operator Q is non-degenerate, we know that for any $\phi\in \mathrm{C}_b(\mathrm{P}_m\mathbb{H};\mathbb{R})$, $\mathrm{S}_t^m\phi$ is differentiable in any direction $h\in \mathrm{P}_m\mathbb{H}$, and we have (see Appendix C)

$$\begin{aligned}
&(\mathrm{D}_x\mathrm{S}_t^m\phi(x),h)\\
&=\frac{1}{t}\mathbb{E}\bigg[e^{-K\int_0^t\|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}_m(t,x))\int_0^t\Big(\mathrm{Q}_m^{-1/2}\eta_m^h(s,x),\mathrm{dW}_m(s)\Big)\bigg]\\
&\quad+2K\mathbb{E}\bigg[e^{-K\int_0^t\|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}_m(t,x))\int_0^t\Big(1-\frac{s}{t}\Big)\big(\mathrm{AX}_m(s,x),\mathrm{A}\eta_m^h(s,x)\big)\mathrm{d}s\bigg].
\end{aligned}\tag{2.24}$$

Our aim is to prove estimates for the derivatives of $u_m(t,\cdot)$ through the corresponding estimates for $v_m(t,\cdot)$.

### 2.6 The linear stochastic differential equations

Let us consider the linear stochastic differential equation with Gaussian noise as

$$\begin{cases}\mathrm{dG}(t) = -\mathrm{AG}(t)\mathrm{d}t+\sqrt{\mathrm{Q}}\mathrm{dW}(t),\\ \mathrm{G}(0)=0.\end{cases}\tag{2.25}$$

The unique strong solution of (2.25) can be defined by the variation of constant formula as follows:

$$\mathrm{G}(t)=\int_0^t e^{-(t-s)\mathrm{A}}\sqrt{\mathrm{Q}}\mathrm{dW}(s).\tag{2.26}$$

The following estimate of $\mathrm{G}(\cdot)$ is useful in the sequel.

**Lemma 9** (Proposition 34, [14])**.** *For any $T\geq 0$, $\varepsilon<g/2$ and any $k\geq 1$, there exists a constant $C(\varepsilon,k,T)$ such that $\mathrm{G}(\cdot)$ has continuous paths with values in $\mathrm{D}\big(\mathrm{A}^{1+\varepsilon}\big)$ and*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}\|\mathrm{A}^{1+\varepsilon}\mathrm{G}(t)\|^{2k}\bigg]\leq C(\varepsilon,k,T)\big[\mathrm{Tr}(\mathrm{A}^{1+g}\mathrm{Q})\big]^k.\tag{2.27}$$

*Moreover, for any $\beta<\min\{g/2-\varepsilon,1/2\}$, there exists a constant $C(\varepsilon,\beta,k,T)$ such that for $t_1,t_2\in[0,T]$, we have*

$$\mathbb{E}\bigg[\sup_{t_1,t_2\in[0,T]}\|\mathrm{A}^{1+\varepsilon}(\mathrm{G}(t_1)-\mathrm{G}(t_2))\|^{2k}\bigg]\leq C(\varepsilon,\beta,k,T)|t_1-t_2|^{2\beta k}\big[\mathrm{Tr}(\mathrm{A}^{1+g}\mathrm{Q})\big]^k.\tag{2.28}$$

We now consider the linear stochastic differential equation with jump noise as

$$\begin{cases}\mathrm{dJ}(t)=-\mathrm{AJ}(t)\mathrm{d}t+\displaystyle\int_{\mathrm{Z}}\Psi(t-,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{J}(0)=0.\end{cases}\tag{2.29}$$

The unique strong solution to the system (2.29) with càdlàg trajectories can be defined by the variation of constant formula as follows (see Lemma 3.2, [4]):

$$\mathrm{J}(t)=\int_0^t\int_{\mathrm{Z}}e^{-(t-s)\mathrm{A}}\Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z).\tag{2.30}$$

The next two lemmas give maximal inequalities for stochastic convolutions driven by compensated Poisson random measures in Hilbert spaces.

**Lemma 10** (Theorem 2, [28])**.** *For* $0 < p < 2$*, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^p\right] \leq C_p\left(\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2}. \tag{2.31}$$

**Lemma 11** (Maximal inequalities for stochastic convolutions)**.** *For* $p \geq 2$*, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^p\right] \leq C_p\left[\left(\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2} + \int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^p\lambda(\mathrm{d}z)\mathrm{d}t\right]. \tag{2.32}$$

P r o o f. Let us apply Itô's formula (Proposition 2, [35], Theorem 3.7.2, [31]) to the process $\|\mathrm{J}(t)\|^p$, for $p \geq 2$ to obtain[1]

$$\begin{aligned}\|\mathrm{J}(t)\|^p &= -p\int_0^t\|\mathrm{J}(s)\|^{p-2}\|\mathrm{A}^{1/2}\mathrm{J}(s)\|^2\mathrm{d}s + p\int_0^t\int_{\mathrm{Z}}\|\mathrm{J}(s-)\|^{p-2}(\mathrm{J}(s-),\Psi(s-,z))\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\\ &\quad+\int_0^t\int_{\mathrm{Z}}\left[\|\mathrm{J}(s)+\Psi(s,z)\|^p - \|\mathrm{J}(s)\|^p - p\|\mathrm{J}(s)\|^{p-2}(\mathrm{J}(s),\Psi(s,z))\right]\pi(\mathrm{d}s,\mathrm{d}z). \end{aligned}\tag{2.33}$$

Let us take supremum over time and then take expectation in (2.33) to get

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^p\right]\\ &\leq p\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathrm{Z}}\|\mathrm{J}(s-)\|^{p-2}(\mathrm{J}(s-),\Psi(s-,z))\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\right|\right]\\ &\quad+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathrm{Z}}\left[\|\mathrm{J}(s)+\Psi(s,z)\|^p - \|\mathrm{J}(s)\|^p - p\|\mathrm{J}(s)\|^{p-2}(\mathrm{J}(s),\Psi(s,z))\right]\pi(\mathrm{d}s,\mathrm{d}z)\right|\right]\\ &=: I_1 + I_2.\end{aligned}\tag{2.34}$$

We now use the Burkholder–Davis–Gundy inequality (Theorem 1, [28]), Cauchy-Schwarz inequality, Hölder's inequality and Young's inequality to $I_1$ to estimate

$$\begin{aligned}I_1 &\leq C\mathbb{E}\left(\int_0^T\int_{\mathrm{Z}}\|\mathrm{J}(t)\|^{2(p-1)}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{1/2}\\ &\leq C\mathbb{E}\left[\left(\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^{2(p-1)}\right)^{1/2}\left(\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{1/2}\right]\\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^p\right] + C\mathbb{E}\left(\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2}.\end{aligned}\tag{2.35}$$

Now by using Taylor's formula[2], for $0 < \theta < 1$, we also have

$$\|\mathrm{J}(s)+\Psi(s,z)\|^p - \|\mathrm{J}(s)\|^p - p\|\mathrm{J}(s)\|^{p-2}(\mathrm{J}(s),\Psi(s,z))$$

[1] For $\phi(x) = \|x\|^p$, we have $(\mathrm{D}_x\phi(x),h) = p\|x\|^{p-2}(x,h)$ and $\mathrm{D}_x^2\phi(x)(h,h) = p(p-2)\|x\|^{p-4}|(x,h)|^2 + p\|x\|^{p-2}(h,h)$.

[2] If $f:\mathbb{H}\to\mathbb{R}$ is Fréchet differentiable, then

$$f(x+h) = f(x) + (\mathrm{D}_x f(x),h) + \frac{1}{2}\mathrm{D}_x^2 f(x+\theta h)(h,h), \text{ for } 0<\theta<1 \text{ and } x,h\in\mathbb{H}.$$

$$= \frac{p}{2}\Big[\|\mathrm{J}(s)+\theta\Psi(s,z)\|^{p-2}\|\Psi(s,z)\|^2 + (p-2)\|\mathrm{J}(s)+\theta\Psi(s,z)\|^{p-4}|(\mathrm{J}(s)+\theta\Psi(s,z),\Psi(s,z))|^2\Big]$$
$$\leq \frac{p(p-1)}{2}\|\mathrm{J}(s)+\theta\Psi(s,z)\|^{p-2}\|\Psi(s,z)\|^2$$
$$\leq \begin{cases} \|\Psi(s,z)\|^p, & \text{for } p=2,\\ 2^{p-4}p(p-1)\big[\|\mathrm{J}(s)\|^{p-2}\|\Psi(s,z)\|^2+\|\Psi(s,z)\|^p\big], & \text{for } p>2. \end{cases} \tag{2.36}$$

We now take $I_2$ from (2.34), then use (2.36), (2.9), Hölder's inequality and Young's inequality to obtain

$$\begin{aligned} I_2 &\leq C_p\mathbb{E}\left[\int_0^T\int_{\mathrm{Z}}\big[\|\mathrm{J}(t)\|^{p-2}\|\Psi(t,z)\|^2+\|\Psi(t,z)\|^p\big]\pi(\mathrm{d}t,\mathrm{d}z)\right]\\ &= C_p\mathbb{E}\left[\int_0^T\int_{\mathrm{Z}}\big[\|\mathrm{J}(t)\|^{p-2}\|\Psi(t,z)\|^2+\|\Psi(t,z)\|^p\big]\lambda(\mathrm{d}z)\mathrm{d}t\right]\\ &\leq C_p\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^{p-2}\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right]+C_p\mathbb{E}\left[\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^p\lambda(\mathrm{d}z)\mathrm{d}t\right]\\ &\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{J}(t)\|^p\right]+C_p\mathbb{E}\left[\left(\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2}+\int_0^T\int_{\mathrm{Z}}\|\Psi(t,z)\|^p\lambda(\mathrm{d}z)\mathrm{d}t\right]. \end{aligned} \tag{2.37}$$

Combine (2.35) and (2.37), and substitute it in (2.34) to obtain (2.32). □

**Remark 12.** The proof of Lemma 11 makes use of the Burkholder–Davis–Gundy type inequality derived in Theorem 1, [28]. However, the authors in [33] establish a similar inequality as in Lemma 11 using Bichteler-Jacod inequality for Poisson integrals (see Lemma 3.1) and Sz.-Nagy's theorem on unitary dilations in Hilbert spaces (see Proposition 3.3).

**Lemma 13.** *The process*

$$\mathrm{F}(t) := \mathrm{A}\mathrm{J}(t) = \int_0^t\int_{\mathrm{Z}} e^{-(t-s)\mathrm{A}}\mathrm{A}\Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z), \tag{2.38}$$

*has càdlàg trajectories and is stochastically continuous. Moreover, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathrm{F}(t)\|^p\right] \tag{2.39}$$
$$\leq C_p\begin{cases}\left(\int_0^T\int_{\mathrm{Z}}\|\mathrm{A}\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2}, & \textit{for } 0<p<2,\\ \left(\int_0^T\int_{\mathrm{Z}}\|\mathrm{A}\Psi(t,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}t\right)^{p/2}+\int_0^T\int_{\mathrm{Z}}\|\mathrm{A}\Psi(t,z)\|^p\lambda(\mathrm{d}z)\mathrm{d}t, & \textit{for } p\geq 2.\end{cases}$$

P r o o f. We use the Assumption (2.1) (ii) and Lemma 3.2, [4] to conclude that the process defined in (2.38) is the unique strong solution of

$$\begin{cases}\mathrm{d}\mathrm{F}(t) = -\mathrm{A}\mathrm{F}(t)\mathrm{d}t+\int_{\mathrm{Z}}\mathrm{A}\Psi(t-,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{F}(0)=0,\end{cases} \tag{2.40}$$

and has càdlàg trajectories. Using (2.10), for $0\leq t_1<t_2\leq T$, we have

$$\mathbb{E}\big[\|\mathrm{F}(t_2)-\mathrm{F}(t_1)\|^2\big] \leq 2\mathbb{E}\left[\left\|\int_{t_1}^{t_2}\int_{\mathrm{Z}} e^{-(t_2-s)\mathrm{A}}\mathrm{A}\Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\right\|^2\right]$$

$$
\begin{aligned}
&+ 2\mathbb{E}\left[\left\|\int_0^{t_1}\int_{\mathrm{Z}}\left(e^{-(t_2-s)\mathrm{A}} - e^{-(t_1-s)\mathrm{A}}\right)\mathrm{A}\Psi(s-,z)\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\right\|^2\right] \\
\leq\ & 2\int_0^T\int_{\mathrm{Z}}\mathbb{1}_{(t_1,t_2]}(s)\|\mathrm{A}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s \\
&+ 2\int_0^T\int_{\mathrm{Z}}\mathbb{1}_{(0,t_1]}(s)\left\|\left(e^{-(t_2-s)\mathrm{A}} - e^{-(t_1-s)\mathrm{A}}\right)\mathrm{A}\Psi(s,z)\right\|^2\lambda(\mathrm{d}z)\mathrm{d}s,
\end{aligned} \tag{2.41}
$$

where $\mathbb{1}_{(t_1,t_2]}(\cdot)$ is the characteristic function on the interval $(t_1, t_2]$. An application of the Lebesgue dominated convergence theorem implies that both terms in (2.41) converges to 0 as $t_2 \searrow t_1$ or $t_1 \nearrow t_2$, and the process $\mathrm{F}(\cdot)$ is stochastically continuous. The estimate (2.39) can be obtained in a similar way as in Lemma 10 and Lemma 11. □

## 3 A-priori estimates

In this section, we derive higher order weighted estimates for the SNSE (2.2) using various convolution estimates obtained for Gaussian and jump noise integrals in the previous section.

**Lemma 14.** *There exist $C, \widetilde{C} > 0$ such that for any $m \in \mathbb{N}$, $t \in [0, T]$ and any $x \in \mathrm{D}(\mathrm{A})$, we have*

$$
\begin{aligned}
e^{-\widetilde{C}\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\|\mathrm{AX}_m(t,x)\|^2 &\leq 3\|\mathrm{A}x\|^2 + 6\left(\sup_{s\in[0,T]}\|\mathrm{AG}_m(s)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AJ}_m(s)\|^2\right) \\
&\leq C\big(1 + \|\mathrm{A}x\|^2\big), \quad \textit{a.s.},
\end{aligned} \tag{3.1}
$$

*where* $\mathrm{G}_m := \mathrm{P}_m\mathrm{G}$, $\mathrm{J}_m := \mathrm{P}_m\mathrm{J}$.

Proof. If $\mathrm{Y}_m = \mathrm{X}_m - \mathrm{G}_m - \mathrm{J}_m$, then $\mathrm{Y}_m$ is the solution of the equation

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Y}_m(t,x) = -[\mathrm{AY}_m(t,x) + \mathrm{B}_m(\mathrm{Y}_m(t,x) + \mathrm{G}_m(t) + \mathrm{J}_m(t))], \\
\mathrm{Y}_m(0,x) = x.
\end{cases} \tag{3.2}
$$

Let us now multiply both sides of (3.2) by $\mathrm{A}^2\mathrm{Y}_m(t,x)$, and use Lemma 3 and Young's inequality to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{AY}_m(t,x)\|^2 + \|\mathrm{A}^{3/2}\mathrm{Y}_m(t,x)\|^2 \\
&\leq \widetilde{C}\|\mathrm{AX}_m(t,x)\|^2\big(\|\mathrm{AY}_m(t,x)\|^2 + \|\mathrm{AG}_m(t)\|^2 + \|\mathrm{AJ}_m(t)\|^2\big).
\end{aligned} \tag{3.3}
$$

Hence, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathrm{AY}_m(t,x)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AG}_m(s)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AJ}_m(s)\|^2\right) + \|\mathrm{A}^{3/2}\mathrm{Y}_m(t,x)\|^2 \\
&\leq \widetilde{C}\|\mathrm{AX}_m(t,x)\|^2\left(\|\mathrm{AY}_m(t,x)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AG}_m(s)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AJ}_m(s)\|^2\right).
\end{aligned} \tag{3.4}
$$

The comparison lemma (see Lemma 33) yields

$$
\begin{aligned}
&e^{-\widetilde{C}\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\|\mathrm{AY}_m(t,x)\|^2 + \int_0^t e^{-\widetilde{C}\int_0^s \|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{A}^{3/2}\mathrm{Y}_m(s,x)\|^2\mathrm{d}s \\
&\leq \|\mathrm{A}x\|^2 + \sup_{s\in[0,T]}\|\mathrm{AG}_m(s)\|^2 + \sup_{s\in[0,T]}\|\mathrm{AJ}_m(s)\|^2.
\end{aligned} \tag{3.5}
$$

Since $\mathrm{X}_m = \mathrm{Y}_m + \mathrm{G}_m + \mathrm{J}_m$, by using (3.5), we obtain

$$
e^{-\widetilde{C}\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\|\mathrm{AX}_m(t,x)\|^2
$$

$$
\begin{aligned}
&\leq 3e^{-\widetilde{C}\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\left[\|\mathrm{AY}_m(t,x)\|^2+\|\mathrm{AG}_m(t)\|^2+\|\mathrm{AJ}_m(t)\|^2\right]\\
&\leq 3\|\mathrm{A}x\|^2+6\left(\sup_{s\in[0,T]}\|\mathrm{AG}_m(s)\|^2+\sup_{s\in[0,T]}\|\mathrm{AJ}_m(s)\|^2\right). \qquad (3.6)
\end{aligned}
$$

The last two terms on the right hand side of the inequality (3.6) is finite, a.s., by using (2.27) and (2.39). □

**Lemma 15.** *For any $\gamma\in(0,1]$, there exists $C>0$ such that for any $m\in\mathbb{N}$, $t\in[0,T]$ and any $h\in\mathrm{D}(\mathrm{A})$, we have*

$$
e^{-C\int_0^t \|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\|\mathrm{A}^\gamma\eta_m^h(t,x)\|^2+\int_0^t e^{-C\int_0^s \|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{A}^{\gamma+1/2}\eta_m^h(s,x)\|^2\mathrm{d}s\leq\|\mathrm{A}^\gamma h\|^2.
$$

P r o o f. Let us multiply both sides of (2.19) by $\mathrm{A}^{2\gamma}\eta_m^h(t,x)$ and use Lemma 3 to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{A}^\gamma\eta_m^h(t,x)\|^2+\|\mathrm{A}^{\gamma+1/2}\eta_m^h(t,x)\|^2\\
&\quad=-\big(\mathrm{B}_m(\mathrm{X}_m(t,x),\eta_m^h(t,x))+\mathrm{B}_m(\eta_m^h(t,x),\mathrm{X}_m(t,x)),\mathrm{A}^{2\gamma}\eta_m^h(t,x)\big)\\
&\quad\leq C\|\mathrm{AX}_m(t,x)\|\|\mathrm{A}\eta_m^h(t,x)\|\|\mathrm{A}^{2\gamma-1/2}\eta_m^h(t,x)\|\\
&\quad\leq C\|\mathrm{AX}_m(t,x)\|\|\mathrm{A}^\gamma\eta_m^h(t,x)\|\|\mathrm{A}^{\gamma+1/2}\eta_m^h(t,x)\|\\
&\quad\leq C\|\mathrm{AX}_m(t,x)\|^2\|\mathrm{A}^\gamma\eta_m^h(t,x)\|^2+\frac{1}{2}\|\mathrm{A}^{\gamma+1/2}\eta_m^h(t,x)\|^2, \qquad (3.7)
\end{aligned}
$$

where we also applied the interpolation inequality (2.11) for the last but previous estimate and Young's inequality for the final estimate. Hence from (3.7), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{A}^\gamma\eta_m^h(t,x)\|^2+\|\mathrm{A}^{\gamma+1/2}\eta_m^h(t,x)\|^2\leq C\|\mathrm{AX}_m(t,x)\|^2\|\mathrm{A}^\gamma\eta_m^h(t,x)\|^2, \qquad (3.8)
$$

and using comparison lemma, we get the required estimate. □

The following estimate on the regularity of the semigroup can be proved by similar arguments as in Lemma 3.4, [13] or Lemma 4.1, [8].

**Lemma 16.** *Suppose $r-1/2<\gamma\leq 1$, where $r\in(1,3/2)$ is defined in (2.5) and $k=1,2,\ldots$. If $K$ is sufficiently large, then there exists $C(\gamma,k)>0$ such that for any $\phi\in\mathrm{C}_k(\mathrm{D}(\mathrm{A});\mathbb{R})$, we have*

$$
\|\mathrm{A}^{-\gamma}\mathrm{D}_x\mathrm{S}_t^m\phi\|_k\leq C\Big(1+t^{-1/2-(r-\gamma)}\Big)\|\phi\|_k,\ t>0, \qquad (3.9)
$$

*for all $m\in\mathbb{N}$.*

Now we can obtain uniform estimates on the Galerkin approximated Kolmogorov equation (2.16).

**Proposition 17.** *If $\phi\in\mathrm{C}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$, then $u_m(t)\in\mathrm{C}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ and, for any $r-1/2<\gamma\leq 1$, $\mathrm{A}^{-\gamma}\mathrm{D}_x u_m\in\mathrm{C}_2(\mathrm{D}(\mathrm{A});\mathbb{R})$ for all $t>0$, $m\in\mathbb{N}$. Moreover, we have*

$$
\|u_m(t)\|_0\leq\|\phi\|_0, \qquad (3.10)
$$

*and*

$$
\|\mathrm{A}^{-\gamma}\mathrm{D}_x u_m(t)\|_2\leq C\Big(1+t^{-1/2-(r-\gamma)}\Big)\|\phi\|_0,\ t>0. \qquad (3.11)
$$

P r o o f. The estimate (3.10) follows from Markov property and (2.15). Using (2.23) and Lemma 16, it follows that

$$
\begin{aligned}
&\|\mathrm{A}^{-\gamma}\mathrm{D}_x u_m(t)\|_2\\
&\leq C\Big(1+t^{-1/2-(r-\gamma)}\Big)\|\phi\|_2+\int_0^t C\Big(1+(t-s)^{-1/2-(r-\gamma)}\Big)\|\|\mathrm{A}x\|^2u_m(s)\|_2\mathrm{d}s. \qquad (3.12)
\end{aligned}
$$

Note that $\|\phi\|_2 \le \|\phi\|_0$ and $\|\|\mathrm{A}x\|^2 u_m(s)\|_2 \le \|u_m(s)\|_0 \le \|\phi\|_0$ and hence from (3.12), we obtain

$$\|\mathrm{A}^{-\gamma}\mathrm{D}_x u_m(t)\|_2 \le C\Big(1 + t^{-1/2-(r-\gamma)}\Big)\|\phi\|_0 + C\bigg(t + \frac{2}{1-2(r-\gamma)}t^{1/2-(r-\gamma)}\bigg)\|\phi\|_0, \quad (3.13)$$

and (3.11) follows. □

**Proposition 18.** *Let $\phi \in \mathscr{E}$, then for any $\beta < \min\{g/2, 1/2\}$, there exists $C(\beta)$ such that for any $0 < t_1 < t_2 \le T$, $m \in \mathbb{N}$ and $x \in \mathrm{D}(\mathrm{A})$, we have*

$$\begin{aligned}
&|u_m(t_1,x) - u_m(t_2,x)| \\
&\le C\|\phi\|_{\mathscr{E}}\big(1+\|\mathrm{A}x\|^2\big)^3\Bigg(\|\mathrm{A}\big(e^{-t_1\mathrm{A}} - e^{-t_2\mathrm{A}}\big)x\| + |t_1-t_2|^{\beta} \\
&\quad + |t_1-t_2|^{1/4}\bigg(\int_0^T\bigg(\int_{\mathrm{Z}}\|\mathrm{A}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\bigg)^2\mathrm{d}s\bigg)^{1/4} \\
&\quad + \bigg[\int_0^T\int_{\mathrm{Z}}\mathbb{1}_{(0,t_1]}(s)\Big\|\Big(e^{-(t_2-s)\mathrm{A}} - e^{-(t_1-s)\mathrm{A}}\Big)\mathrm{A}\Psi(s,z)\Big\|^2\lambda(\mathrm{d}z)\mathrm{d}s\bigg]^{1/2}\Bigg). \qquad (3.14)
\end{aligned}$$

P r o o f. From (2.23), for $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
u_m(t_1,x) - u_m(t_2,x) &= \big(\mathrm{S}^m_{t_1} - \mathrm{S}^m_{t_2}\big)\phi(x) + K\int_0^{t_1}\big(\mathrm{S}^m_{t_1-s} - \mathrm{S}^m_{t_2-s}\big)\big(\|\mathrm{A}x\|^2 u_m(s,x)\big)\mathrm{d}s \\
&\quad - K\int_{t_1}^{t_2}\mathrm{S}^m_{t_2-s}\big(\|\mathrm{A}x\|^2 u_m(s,x)\big)\mathrm{d}s \\
&:= I_3 + I_4 + I_5. \qquad (3.15)
\end{aligned}$$

Using a decomposition and fundamental theorem of calculus, we estimate $|I_3|$ as

$$\begin{aligned}
|I_3| &= \Big|\mathbb{E}\Big[e^{-K\int_0^{t_1}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}_m(t_1,x))\Big] - \mathbb{E}\Big[e^{-K\int_0^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}_m(t_2,x))\Big]\Big| \\
&= \Big|\mathbb{E}\Big[\Big(e^{-K\int_0^{t_1}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s} - e^{-K\int_0^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}\Big)\phi(\mathrm{X}_m(t_1,x))\Big] \\
&\quad + \mathbb{E}\Big[e^{-K\int_0^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}(\phi(\mathrm{X}_m(t_1,x)) - \phi(\mathrm{X}_m(t_2,x)))\Big]\Big| \\
&\le K\|\phi\|_0\mathbb{E}\bigg[\int_{t_1}^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2 e^{-K\int_0^s\|\mathrm{A}\mathrm{X}_m(r,x)\|^2\mathrm{d}r}\mathrm{d}s\bigg] \\
&\quad + C\|\phi\|_{\mathscr{E}}\mathbb{E}\bigg[e^{-K\int_0^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}\bigg(1 + \sup_{t\in[0,T]}\|\mathrm{A}\mathrm{X}_m(t)\|^2\bigg)\|\mathrm{A}(\mathrm{X}_m(t_1) - \mathrm{X}_m(t_2))\|\bigg] \\
&\le C\|\phi\|_{\mathscr{E}}\big(1+\|\mathrm{A}x\|^2\big)|t_1-t_2| + I_6, \qquad (3.16)
\end{aligned}$$

where we used Lemma 14, and $I_6$ stands for the final term from the inequality (3.16). We estimate $I_6$ as follows. We know that

$$\begin{aligned}
\mathrm{X}_m(t_1) - \mathrm{X}_m(t_2) &= \big(e^{-t_1\mathrm{A}} - e^{-t_2\mathrm{A}}\big)x + (\mathrm{G}_m(t_1) - \mathrm{G}_m(t_2)) + (\mathrm{J}_m(t_1) - \mathrm{J}_m(t_2)) \\
&\quad - \bigg(\int_0^{t_1}e^{-(t_1-s)\mathrm{A}}\mathrm{B}_m(\mathrm{X}_m(s))\mathrm{d}s - \int_0^{t_2}e^{-(t_2-s)\mathrm{A}}\mathrm{B}_m(\mathrm{X}_m(s))\mathrm{d}s\bigg). \qquad (3.17)
\end{aligned}$$

Using Hölder's inequality and (2.28), we obtain

$$\mathbb{E}\bigg[e^{-K\int_0^{t_2}\|\mathrm{A}\mathrm{X}_m(s,x)\|^2\mathrm{d}s}\bigg(1 + \sup_{t\in[0,T]}\|\mathrm{A}\mathrm{X}_m(t)\|^2\bigg)\|\mathrm{A}(\mathrm{G}_m(t_1) - \mathrm{G}_m(t_2))\|\bigg]$$

$$\leq \left\{\mathbb{E}\left[e^{-K\int_0^{t_2}\|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\left(1+\sup_{t\in[0,T]}\|\mathrm{AX}_m(t)\|^2\right)\right]^2\right\}^{1/2}\left\{\mathbb{E}\left[\|\mathrm{A}(\mathrm{G}_m(t_1)-\mathrm{G}_m(t_2))\|^2\right]\right\}^{1/2}$$
$$\leq C\big(1+\|\mathrm{A}x\|^2\big)|t_1-t_2|^{\beta}, \tag{3.18}$$

provided $K$ is sufficiently large. We now use Hölder's inequality and (2.41) to get

$$\mathbb{E}\left[e^{-K\int_0^{t_2}\|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\left(1+\sup_{t\in[0,T]}\|\mathrm{AX}_m(t)\|^2\right)\|\mathrm{A}(\mathrm{J}_m(t_1)-\mathrm{J}_m(t_2))\|\right]$$
$$\leq \left\{\mathbb{E}\left[e^{-K\int_0^{t_2}\|\mathrm{AX}_m(s,x)\|^2\mathrm{d}s}\left(1+\sup_{t\in[0,T]}\|\mathrm{AX}_m(t)\|^2\right)\right]^2\right\}^{1/2}\left\{\mathbb{E}\left[\|\mathrm{A}(\mathrm{J}_m(t_1)-\mathrm{J}_m(t_2))\|^2\right]\right\}^{1/2}$$
$$\leq C\big(1+\|\mathrm{A}x\|^2\big)\left\{|t_1-t_2|^{1/4}\left(\int_0^T\left(\int_{\mathrm{Z}_m}\|\mathrm{A}\Psi_m(s,z)\|^2\lambda(\mathrm{d}z)\right)^2\mathrm{d}s\right)^{1/4}\right.$$
$$\left.+\left[\int_0^T\int_{\mathrm{Z}_m}\mathbb{1}_{(0,t_1]}(s)\left\|\left(e^{-(t_2-s)\mathrm{A}}-e^{-(t_1-s)\mathrm{A}}\right)\mathrm{A}\Psi_m(s,z)\right\|^2\lambda(\mathrm{d}z)\mathrm{d}s\right]^{1/2}\right\}. \tag{3.19}$$

Using Lemma 3, and properties of the analytic semigroup $\left(e^{-t\mathrm{A}}\right)_{t\geq 0}$ (see Appendix A, [9]), for any $\lambda\in(0,1/2)$, we have

$$\left\|\mathrm{A}\left(\int_0^{t_1}e^{-(t_1-s)\mathrm{A}}\mathrm{B}_m(\mathrm{X}_m(s))\mathrm{d}s-\int_0^{t_2}e^{-(t_2-s)\mathrm{A}}\mathrm{B}_m(\mathrm{X}_m(s))\mathrm{d}s\right)\right\|$$
$$\leq\int_{t_1}^{t_2}\|\mathrm{A}^{1/2}e^{-(t_2-s)\mathrm{A}}\|_{\mathbf{L}(\mathbb{H})}\|\mathrm{A}^{1/2}\mathrm{B}_m(\mathrm{X}_m(s))\|\mathrm{d}s$$
$$+\int_0^{t_1}\left\|\mathrm{A}^{1/2}\left(e^{-(t_1-s)\mathrm{A}}-e^{-(t_2-s)\mathrm{A}}\right)\right\|_{\mathbf{L}(\mathbb{H})}\|\mathrm{A}^{1/2}\mathrm{B}_m(\mathrm{X}_m(s))\|\mathrm{d}s$$
$$\leq\int_{t_1}^{t_2}(t_2-s)^{-1/2}\|\mathrm{AX}_m(s)\|^2\mathrm{d}s+\int_0^{t_1}(t_1-s)^{-(1/2+\lambda)}|t_1-t_2|^{\lambda}\|\mathrm{AX}_m(s)\|^2\mathrm{d}s$$
$$\leq\left[2+\upbeta\left(1,\frac{1}{2}-\lambda\right)\right]T^{1/2-\lambda}\sup_{s\in[0,t_2]}\|\mathrm{AX}_m(s,x)\|^2|t_1-t_2|^{\lambda}, \tag{3.20}$$

where $\upbeta(x,y)=2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\mathrm{d}\theta$ for $x,y>0$ is the beta function. Substituting (3.17)-(3.20) in (3.16), we find

$$|I_3|\leq C\|\phi\|_{\mathscr{E}}\big(1+\|\mathrm{A}x\|^2\big)^2\mathscr{K}, \tag{3.21}$$

where

$$\mathscr{K}:=\|\mathrm{A}\big(e^{-t_1\mathrm{A}}-e^{-t_2\mathrm{A}}\big)x\|+|t_1-t_2|+|t_1-t_2|^{\beta}+|t_1-t_2|^{\lambda}$$
$$+|t_1-t_2|^{1/4}\left(\int_0^T\left(\int_{\mathrm{Z}_m}\|\mathrm{A}\Psi_m(s,z)\|^2\lambda(\mathrm{d}z)\right)^2\mathrm{d}s\right)^{1/4}$$
$$+\left[\int_0^T\int_{\mathrm{Z}_m}\mathbb{1}_{(0,t_1]}(s)\left\|\left(e^{-(t_2-s)\mathrm{A}}-e^{-(t_1-s)\mathrm{A}}\right)\mathrm{A}\Psi_m(s,z)\right\|^2\lambda(\mathrm{d}z)\mathrm{d}s\right]^{1/2}. \tag{3.22}$$

Using the Markov property, we can estimate $I_5$ as

$$|I_5|=K\left|\int_{t_1}^{t_2}\mathrm{S}^m_{t_2-s}\big(\|\mathrm{A}x\|^2u_m(s,x)\big)\mathrm{d}s\right|$$

$$
\begin{aligned}
&= K\left|\int_{t_1}^{t_2}\mathbb{E}\Big[e^{-K\int_0^{t_2-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_2-s,x)\|^2u_m(s,\mathrm{X}_m(t_2-s,x))\mathrm{d}s\Big]\right|\\
&= K\int_{t_1}^{t_2}\mathbb{E}\Big[e^{-K\int_0^{t_2-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_2-s,x)\|^2|\mathrm{P}_{t_2-s}(\mathrm{P}_s\phi)(x)|\mathrm{d}s\Big]\\
&\leq C\|\phi\|_0\int_{t_1}^{t_2}\mathbb{E}\Big[e^{-K\int_0^{t_2-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_2-s,x)\|^2\mathrm{d}s\Big]\\
&\leq C\|\phi\|_0\big(1+\|\mathrm{A}x\|^2\big)|t_1-t_2|. \qquad (3.23)
\end{aligned}
$$

Once again an application of the Markov property yields

$$
\begin{aligned}
I_4 &= \int_0^{t_1}\big(\mathrm{S}^m_{t_1-s}-\mathrm{S}^m_{t_2-s}\big)\big(\|\mathrm{A}x\|^2u_m(s,x)\big)\mathrm{d}s\\
&= \int_0^{t_1}\mathbb{E}\Big[e^{-K\int_0^{t_1-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_1-s,x)\|^2u_m(s,\mathrm{X}_m(t_1-s,x))\mathrm{d}s\Big]\\
&\quad-\int_0^{t_1}\mathbb{E}\Big[e^{-K\int_0^{t_2-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_2-s,x)\|^2u_m(s,\mathrm{X}_m(t_2-s,x))\mathrm{d}s\Big]\\
&= \int_0^{t_1}\mathbb{E}\Big[e^{-K\int_0^{t_1-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_1-s,x)\|^2\\
&\qquad\times\big(u_m(s,\mathrm{X}_m(t_1-s,x))-u_m(s,\mathrm{X}_m(t_2-s,x))\big)\mathrm{d}s\Big]\\
&\quad+\int_0^{t_1}\mathbb{E}\Big[\Big(e^{-K\int_0^{t_1-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_1-s)\|^2-e^{-K\int_0^{t_2-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_2-s)\|^2\Big)\\
&\qquad\times u_m(s,\mathrm{X}_m(t_2-s,x))\mathrm{d}s\Big]\\
&\leq I_7+C\|\phi\|_0\big(1+\|\mathrm{A}x\|^2\big), \qquad (3.24)
\end{aligned}
$$

where $I_7$ is the integral of the first term from the right hand side of the inequality (3.24). By applying the fundamental theorem of calculus, we get

$$
\begin{aligned}
|I_7| &\leq \int_0^{t_1}\|\mathrm{A}^{-1}\mathrm{D}_xu_m(s)\|_2\mathbb{E}\Big[e^{-K\int_0^{t_1-s}\|\mathrm{AX}_m(r,x)\|^2\mathrm{d}r}\|\mathrm{AX}_m(t_1-s,x)\|^2\left(1+\sup_{t\in[0,T]}\|\mathrm{AX}_m(t)\|^2\right)\\
&\qquad\times\|\mathrm{A}(\mathrm{X}_m(t_1,x)-\mathrm{X}_m(t_2,x))\|\Big]\mathrm{d}s.
\end{aligned}
$$

Now proceeding as in the estimate of $I_3$, we arrive at

$$
|I_7| \leq C\big(1+\|\mathrm{A}x\|^2\big)^3\mathscr{K}\int_0^{t_1}\|\mathrm{A}^{-1}\mathrm{D}_xu_m(s)\|_2\mathrm{d}s, \qquad (3.25)
$$

where $\mathscr{K}$ is defined in (3.22). An application of Proposition 17 yields

$$
\int_0^{t_1}\|\mathrm{A}^{-1}\mathrm{D}_xu_m(s)\|_2\mathrm{d}s \leq C\|\phi\|_0\int_0^{t_1}\left(1+s^{-1/2-(r-\gamma)}\right)\mathrm{d}s = C\|\phi\|_0\left(t_1+\frac{2t_1^{1/2-(r-\gamma)}}{1-2(r-\gamma)}\right) < +\infty,
$$

since $\gamma > r-1/2$. Thus, we have

$$
|I_4| \leq C\|\phi\|_{\mathscr{E}}\big(1+\|\mathrm{A}x\|^2\big)^3\mathscr{K}. \qquad (3.26)
$$

By combining (3.21), (3.23) and (3.26), one can obtain (3.14). □

**Remark 19.** It is clear from the upper bound of the estimate (3.9) that there is a singularity near $t=0$. When the Gaussian noise is non-degenerate, as in this paper, we are able to obtain the estimate of the integral $I_4$ to prove Proposition 18. However, when the Gaussian noise is degenerate, one may need to work with some interpolation techniques to handle the singularity as it is done in [2].

## 4 Markov family of martingale solutions

In this section, we first prove the convergence of the approximated solutions $(u_m)_{m\in\mathbb{N}}$ of the Kolmogorov equations (2.16) by using the a-priori estimates established in section 3. Making use of these convergence arguments, we construct the transition semigroup associated with the stochastic process $\mathrm{X}(\cdot,\cdot)$. Let us set

$$K_R = \Big\{x \in \mathrm{D}(\mathrm{A}) : \|\mathrm{A}x\| \le R\Big\},$$

and $K_R$ is endowed with the topology of $\mathbb{H}$. Since the embedding of $\mathrm{D}(\mathrm{A}^\gamma)$ in $\mathbb{H}$ is compact, $K_R$ is a compact subset of $\mathrm{D}(\mathrm{A}^\gamma)$, for $0 < \gamma \le 1$.

**Lemma 20.** *Suppose the Assumptions 2.1 and 2.1 hold, and let $\phi \in \mathscr{E}$. Then, there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ of $(u_m)$ and a function u bounded on $[0,T]\times \mathrm{D}(\mathrm{A})$ such that*

(i) $u \in \mathrm{C}_b((0,T]\times\mathrm{D}(\mathrm{A}))$ *and for any $\delta > 0$, $R > 0$, we have*

$$\lim_{k\to\infty} u_{m_k}(t,x) = u(t,x) \text{ uniformly on } [\delta,T]\times K_R. \tag{4.1}$$

(ii) *For any $x \in \mathrm{D}(\mathrm{A})$, $u(\cdot,x)$ is continuous on $[0,T]$.*

(iii) *For any $r - 1/2 < \gamma \le 1$, $\delta > 0$, $R \ge 0$, $\beta < \min\{g/2, 1/2\}$, $\tau \in (0,1/4)$, there exists a constant $C(\gamma,\beta,\delta,R,T,\phi) > 0$ such that for any $x,y \in K_R$, $t,s > \delta$, we have*

$$|u(t,x)-u(s,y)| \le C\Big[\|\mathrm{A}^\gamma(x-y)\| + |t-s|^\beta + |t-s|^\tau M^{1/4}\Big], \tag{4.2}$$

*where*

$$M = \int_0^T\bigg(\int_{\mathrm{Z}}\|\mathrm{A}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\bigg)^2\mathrm{d}s.$$

(iv) *For any $t \in [0,T]$, $u(t,\cdot) \in \mathscr{E}$.*

(v) $u(0,\cdot) = \phi$.

*Proof.* Let $R > 0$, $\delta > 0$, $s < t$ and $s,t \in [\delta,T]$, $x,y \in K_R$. By using Proposition 18, for $\beta < \min\{g/2,1/2\}$, we get

$$\begin{aligned}
|u_m(t,x)-u_m(s,y)| &\le |u_m(t,x)-u_m(t,y)| + |u_m(t,y)-u_m(s,y)| \\
&\le \sup_{z\in K_R}\|\mathrm{A}^{-\gamma}\mathrm{D}_z u_m(t,z)\|\|\mathrm{A}^\gamma(x-y)\| + |u_m(t,y)-u_m(s,y)| \\
&\le \sup_{z\in K_R}\big(1+\|\mathrm{A}z\|^2\big)\|\mathrm{A}^{-\gamma}\mathrm{D}_z u_m(t)\|_2\|\mathrm{A}^\gamma(x-y)\| \\
&\quad + C\|\phi\|_{\mathscr{E}}\big(1+\|\mathrm{A}x\|^2\big)^3\Bigg(\|\mathrm{A}\big(e^{-t\mathrm{A}}-e^{-s\mathrm{A}}\big)y\| + |t-s|^\beta \\
&\quad + |t-s|^{1/4}\bigg(\int_0^T\bigg(\int_{\mathrm{Z}}\|\mathrm{A}\Psi(s,z)\|^2\lambda(\mathrm{d}z)\bigg)^2\mathrm{d}s\bigg)^{1/4} \\
&\quad + \bigg[\int_0^T\int_{\mathrm{Z}}\mathbb{1}_{(\delta,s]}(r)\Big\|\Big(e^{-(t-r)\mathrm{A}}-e^{-(s-r)\mathrm{A}}\Big)\mathrm{A}\Psi(r,z)\Big\|^2\lambda(\mathrm{d}z)\mathrm{d}r\bigg]^{1/2}\Bigg).
\end{aligned} \tag{4.3}$$

We denote the final integral in (4.3) as $I_8$ and estimate it as follows:

$$I_8 \le \bigg(\int_\delta^s\bigg\|\int_{s-r}^{t-r}\mathrm{A}e^{-\rho\mathrm{A}}\mathrm{d}\rho\bigg\|^2_{\mathbf{L}(\mathbb{H})}\int_{\mathrm{Z}}\|\mathrm{A}\Psi(r,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}r\bigg)^{1/2}.$$

Let us choose $\tau \in (0, 1/4)$ and use the semigroup property to have

$$\left\|\int_{s-r}^{t-r} \mathrm{A}e^{-\rho\mathrm{A}}\mathrm{d}\rho\right\|_{\mathbf{L}(\mathbb{H})}^2 \leq \left(\int_{s-r}^{t-r} \frac{1}{\rho}d\rho\right)^2 \leq (s-r)^{-2\tau}\left(\int_{s-r}^{t-r} \rho^{\tau-1}d\rho\right)^2 \leq \frac{1}{\tau^2}(s-r)^{-2\tau}(t-s)^{2\tau}.$$

By Hölder's inequality, we arrive at

$$\begin{aligned} I_8 &\leq \frac{1}{\tau}(t-s)^\tau\left[\int_0^s (s-r)^{-4\tau}\mathrm{d}r\right]^{1/4}\left[\int_0^T\left(\int_{\mathrm{Z}}\|\mathrm{A}\Psi(r,z)\|^2\lambda(\mathrm{d}z)\right)^2\mathrm{d}r\right]^{1/4} \\ &\leq \frac{1}{\tau}(t-s)^\tau T^{1/4-\tau}(\beta(1,1-4\tau))^{1/4}M^{1/4} < +\infty. \end{aligned} \tag{4.4}$$

Using this estimate and Proposition 17 in (4.3), one can get

$$|u_m(t,x) - u_m(s,y)| \leq C(\delta,T,R)\|\phi\|_{\mathscr{E}}\left[\|\mathrm{A}^\gamma(x-y)\| + |t-s|^\beta + |t-s|^\tau M^{1/4}\right], \tag{4.5}$$

where $M$ is defined in (4.2). Hence the uniformly bounded sequence $(u_m)_{m\in\mathbb{N}}$ is also equicontinuous, and by an application of the Arzelà-Ascoli theorem and a diagonal extraction argument, we can construct a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoting it as $(u_m)_{m\in\mathbb{N}}$, such that

$$u_m(t,x) \to u(t,x), \text{ as } m \to \infty,$$

uniformly in $[\delta, T] \times K_R$, for any $\delta > 0$, $R > 0$. This proves (i). By taking limit $m \to \infty$ in (4.5), we get (4.2), which proves (iii). Let us define $u(0,\cdot) = \phi(\cdot)$. Setting $t_1 = 0$ and $t_2 = t$ with $x \in K_R$ in Proposition 18, and taking limit $m \to \infty$, we obtain

$$|u(t,x) - \phi(x)| \leq C(\phi, R)\Big(\|\mathrm{A}(e^{-t\mathrm{A}} - \mathrm{I})x\| + t^\beta + t^\tau M^{1/4}\Big).$$

Since the semigroup $\left(e^{-t\mathrm{A}}\right)_{t\geq 0}$ is strongly continuous, we get $u(\cdot, x)$ is continuous on $[0,T]$ for any $x \in \mathrm{D}(\mathrm{A})$, which establishes (ii). Now, for any fixed $t \in [0,T]$ and $x, x_1, x_2 \in K_R$ by using Proposition 17, we have

$$\begin{aligned} \|u(t,\cdot)\|_{\mathscr{E}} &= \|u(t,\cdot)\|_0 + \sup_{x_1,x_2\in\mathrm{D}(\mathrm{A})} \frac{|u(t,x_1) - u(t,x_2)|}{\|\mathrm{A}(x_1-x_2)\|(1+\|\mathrm{A}x_1\|^2 + \|\mathrm{A}x_2\|^2)} \\ &\leq \|\phi\|_0 + \sup_{x,x_1,x_2\in\mathrm{D}(\mathrm{A})} \frac{\left(1+\|\mathrm{A}x\|^2\right)\|\mathrm{A}^{-1}\mathrm{D}_x u_m(t)\|_2}{(1+\|\mathrm{A}x_1\|^2 + \|\mathrm{A}x_2\|^2)} < +\infty. \end{aligned} \tag{4.6}$$

This proves (iv). □

The existence of a martingale solution of (2.2) is known. This fact is helpful in constructing a transition semigroup $\mathrm{P}_t\phi$, for $\phi \in \mathrm{B}_b(\mathrm{D}(\mathrm{A}); \mathbb{R})$. In order to prove Theorem 7, we further prove some regularity estimates on $\mathrm{X}_m(t,x)$.

**Lemma 21.** *For any $\delta \in (1/2, 1+g]$, there exists a constant $C(\delta) > 0$ such that for any $x \in \mathbb{H}$, $m \in \mathbb{N}$, and $t \in [0,T]$:*

$$\begin{aligned} &(i)\ \mathbb{E}_x\left[\|\mathrm{X}_m(t,x)\|^2\right] + \mathbb{E}_x\left[\int_0^t \|\mathrm{A}^{1/2}\mathrm{X}_m(s,x)\|^2\mathrm{d}s\right] \\ &\quad \leq \|x\|^2 + t\mathrm{Tr}\mathrm{Q} + \int_0^t\int_{\mathrm{Z}}\|\Psi(s,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s, \end{aligned} \tag{4.7}$$

$$(ii)\ \mathbb{E}_x\left[\int_0^T \frac{\|\mathrm{A}^{(\delta+1)/2}\mathrm{X}_m(s,x)\|^2}{\left(1+\|\mathrm{A}^{\delta/2}\mathrm{X}_m(s,x)\|^2\right)^{\gamma_\delta}}\mathrm{d}s\right] \leq C(\delta)\left(1+\|x\|^2\mathbb{1}_{\delta\leq 1}\right), \tag{4.8}$$

*with $\gamma_\delta = \frac{2}{2\delta-1}$ if $\delta \leq 1$ and $\gamma_\delta = \frac{2\delta+1}{2\delta-1}$ if $\delta > 1$.*

P r o o f. (i) The proof of (4.7) is standard and it follows by applying Itô's formula to the process $\|X_m(t,x)\|^2$ (for example, see Theorem 3.1, [42]).

(ii) In order to prove (4.8), we apply Itô's formula to the process

$$F_\delta(X_m(t,x)) = -\frac{1}{\left(1+\|A^{\delta/2}X_m(t,x)\|^2\right)^{\gamma_\delta-1}}.$$

Noting that the expectation of the stochastic integral is zero, one can obtain[3]

$$\begin{aligned}
&\frac{1}{\left(1+\|A^{\delta/2}P_m x\|^2\right)^{\gamma_\delta-1}} + 2(\gamma_\delta-1)\mathbb{E}_x\left[\int_0^t \frac{\|A^{(\delta+1)/2}X_m(s,x)\|^2}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta}}ds\right]\\
&= -2(\gamma_\delta-1)\mathbb{E}_x\left[\int_0^t \frac{\left(B_m(X_m(s,x)), A^\delta X_m(s,x)\right)}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta}}ds\right]\\
&\quad + (\gamma_\delta-1)\mathbb{E}_x\left[\int_0^t \frac{1}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta}}ds\right]\mathrm{Tr}\left[Q_m A^\delta\right]\\
&\quad - 2\gamma_\delta(\gamma_\delta-1)\mathbb{E}_x\left[\int_0^t \frac{\|Q_m^{1/2}A^\delta X_m(s,x)\|^2}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta+1}}ds\right]\\
&\quad + \mathbb{E}_x\Bigg[\int_0^t\int_{Z_m}\Bigg(-\frac{1}{\left(1+\|A^{\delta/2}(X_m(s,x)+\Psi_m(s,z))\|^2\right)^{\gamma_\delta-1}} + \frac{1}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta-1}}\\
&\qquad -2(\gamma_\delta-1)\frac{\left(A^{\delta/2}X_m(s,x), A^{\delta/2}\Psi_m(s,z)\right)}{\left(1+\|A^{\delta/2}X_m(s,x)\|^2\right)^{\gamma_\delta}}\Bigg)\lambda(dz)ds\Bigg]\\
&\quad + \mathbb{E}_x\left[\frac{1}{\left(1+\|A^{\delta/2}X_m(t,x)\|^2\right)^{\gamma_\delta-1}}\right] := \sum_{i=1}^5 J_i. \qquad (4.9)
\end{aligned}$$

For the nonlinear term in $J_1$, let us use Lemma 3- (ii) and Young's inequality to obtain

$$\begin{aligned}
\left|\left(B_m(X_m), A^\delta X_m\right)\right| &\le C\|A^{\delta/2}X_m\|^{1/2+\delta}\|A^{(\delta+1)/2}X_m\|^{5/2-\delta}\\
&\le \frac{1}{2}\|A^{(\delta+1)/2}X_m\|^2 + C\|A^{\delta/2}X_m\|^{2(2\delta+1)/(2\delta-1)}. \qquad (4.10)
\end{aligned}$$

By Assumption 2.1, the integral $J_2$ is bounded. For integral $J_3$, we note by Assumption 2.1 with some $r \in (1, 3/2)$ that

$$\begin{aligned}
\|Q_m^{1/2}A^\delta X_m\|^2 &\le C_r\|A^r Q_m A^\delta X_m\|^2 \le C_r\|A^r Q_m A^{\delta/2}\|^2\|A^{\delta/2}X_m\|^2\\
&\le C_r \mathrm{Tr}\left(A^{r+\delta/2}Q\right)^2\|A^{\delta/2}X_m\|^2,
\end{aligned}$$

and hence this integral is bounded. Using Taylor's formula, we estimate $J_4$ as[4]

$$J_4 = (\gamma_\delta-1)\mathbb{E}_x\left[\int_0^t\int_{Z_m}\left(\frac{\|A^{\delta/2}\Psi_m(s,z)\|^2}{\left(1+\|A^{\delta/2}(X_m(s,x)+\theta\Psi_m(s,z))\|^2\right)^{\gamma_\delta}}\right)\lambda(dz)ds\right]$$

[3] $D_x F_\delta = 2(\gamma_\delta-1)\frac{A^\delta x}{(1+\|A^{\delta/2}x\|^2)^{\gamma_\delta}}$ and $D_x^2 F_\delta = 2(\gamma_\delta-1)\frac{A^\delta}{(1+\|A^{\delta/2}x\|^2)^{\gamma_\delta}} - 4\gamma_\delta(\gamma_\delta-1)\frac{A^\delta x\otimes A^\delta x}{(1+\|A^{\delta/2}x\|^2)^{\gamma_\delta+1}}$.

[4] Using $A^\delta x = \sum_{j=1}^\infty (x,e_j)\lambda_j^\delta e_j$, we have

$$\begin{aligned}
\left(A^\delta x\otimes A^\delta x\right)(h,h) &= \sum_{j,k=1}^\infty (x,e_j)(x,e_k)\lambda_j^\delta\lambda_k^\delta(e_j\otimes e_k)(h,h) = \sum_{j,k=1}^\infty (x,e_j)(x,e_k)\lambda_j^\delta\lambda_k^\delta(e_j,h)(e_k,h)\\
&= \left(\sum_{j=1}^\infty (x,e_j)\lambda_j^\delta(e_j,h)\right)^2 = \left|\left(A^\delta x, h\right)\right|^2 = \left|\left(A^{\delta/2}x, A^{\delta/2}h\right)\right|^2.
\end{aligned}$$

$$- 2\gamma_\delta(\gamma_\delta - 1)\mathbb{E}_x\left[\int_0^t \int_{\mathrm{Z}_m} \left(\frac{\left|\left(\mathrm{A}^{\delta/2}\Psi_m(s,z), \mathrm{A}^{\delta/2}(\mathrm{X}_m(s,x) + \theta\Psi_m(s,z))\right)\right|^2}{\left(1 + \|\mathrm{A}^{\delta/2}(\mathrm{X}_m(s,x) + \theta\Psi_m(s,z))\|^2\right)^{\gamma_\delta+1}}\right)\lambda(\mathrm{d}z)\mathrm{d}s\right]$$
$$\leq (\gamma_\delta - 1)\int_0^t \int_{\mathrm{Z}_m} \|\mathrm{A}^{\delta/2}\Psi_m(s,z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s < +\infty, \tag{4.11}$$

for $0 < \theta < 1$. The integral $J_5$ is also bounded. Thus, we have

$$\mathbb{E}_x\left[\int_0^T \frac{\|\mathrm{A}^{(\delta+1)/2}\mathrm{X}_m(s,x)\|^2}{\left(1 + \|\mathrm{A}^{\delta/2}\mathrm{X}_m(s,x)\|^2\right)^{\gamma_\delta}}\mathrm{d}s\right] \leq C\mathbb{E}_x\left[\int_0^T \frac{\|\mathrm{A}^{\delta/2}\mathrm{X}_m(s,x)\|^{2(2\delta+1)/(2\delta-1)}}{\left(1 + \|\mathrm{A}^{\delta/2}\mathrm{X}_m(s,x)\|^2\right)^{\gamma_\delta}}\mathrm{d}s\right] + C(\delta). \tag{4.12}$$

Hence, if $\delta > 1$, by setting $\gamma_\delta = (2\delta+1)/(2\delta-1)$, we see that the left hand side of the inequality (4.12) is clearly bounded. If $\delta \leq 1$, let us set $\gamma_\delta = (2\delta+1)/(2\delta-1) - 1 = 2/(2\delta-1)$ and use the fact that $\mathrm{D}(\mathrm{A}^{1/2}) \subset \mathrm{D}(\mathrm{A}^{\delta/2})$ is compact, the integral on the right hand side of (4.12) is again bounded due to the energy estimate (4.7). □

The following estimate can be proved as in Corollary 46, [14]:

**Corollary 22.** *There exists a constant $C > 0$ such that for any $x \in \mathbb{H}$, $m \in \mathbb{N}$, we have*

$$(i)\ \mathbb{E}_x\left[\int_0^T \|\mathrm{A}\mathrm{X}_m(s,x)\|^{2/3}\mathrm{d}s\right] \leq C\left(1 + \|x\|^2\right),$$
$$(ii)\ \mathbb{E}_x\left[\int_0^T \|\mathrm{A}^{1+\frac{\widetilde{g}}{2}}\mathrm{X}_m(s,x)\|^{\frac{1}{2+\widetilde{g}}}\mathrm{d}s\right] \leq C\left(1 + \|x\|^2\right),$$

*where* $\widetilde{g} = \min\{g, 1\}$.

We can use Lemma 21-(i) to prove that the family of laws $(\mathcal{L}(\mathrm{X}_m(\cdot,x)))_{m\in\mathbb{N}}$ is tight in $\mathrm{L}^2(0,T;\mathrm{D}(\mathrm{A}^{s/2}))$ for $s < 1$ and in $\mathbb{D}\big([0,T];\mathrm{D}(\mathrm{A}^{-1/2})\big)$ (see Proposition 3.1, [42]). Thus, by the Prokhorov theorem, it has a weakly convergent subsequence $(\mathcal{L}(\mathrm{X}_{m_k}(\cdot,x)))_{k\in\mathbb{N}}$ converging to say $\mu_x$. By the Skorokhod representation theorem, there exists a stochastic process $\mathrm{X}(\cdot,x)$ on a probability space $(\Omega_x, \mathscr{F}_x, \mathbb{P}_x)$ which belongs to $\mathrm{L}^2(0,T;\mathrm{D}(\mathrm{A}^{s/2}))$ for $s < 1$ and in $\mathbb{D}\big([0,T];\mathrm{D}(\mathrm{A}^{-1/2})\big)$, satisfying (2.2) and such that for any $x \in \mathrm{D}(\mathrm{A})$

$$\mathrm{X}_{m_k}(\cdot,x) \to \mathrm{X}(\cdot,x),\ \mathbb{P}_x - \text{a.s., in } \mathrm{L}^2(0,T;\mathrm{D}(\mathrm{A}^{s/2})) \cap \mathbb{D}([0,T];\mathrm{D}(\mathrm{A}^{-1/2})). \tag{4.13}$$

By the estimate (see, Theorem 3.1, [42])

$$\mathbb{E}_x\left[\sup_{0\leq t\leq T}\|\mathrm{X}_m(t,x)\|^p\right] + \mathbb{E}_x\left[\int_0^T \|\mathrm{X}_m(s,x)\|^{p-2}\|\mathrm{A}^{1/2}\mathrm{X}_m(s,x)\|^2\mathrm{d}s\right] \leq C(1 + \|x\|^p), \tag{4.14}$$

for $p \geq 2$ and uniform integrability, the convergence also holds in $\mathbb{L}^p\left(\Omega_x;\mathbb{D}\big([0,T];\mathrm{D}\big(\mathrm{A}^{-1/2}\big)\big)\right)$. Moreover, for $\widetilde{g} = \min\{g, 1\}$, we have

$$\|\mathrm{A}x\| \leq \|\mathrm{A}^{1+\frac{\widetilde{g}}{2}}x\|^{\frac{3}{3+\widetilde{g}}}\|\mathrm{A}^{-1/2}x\|^{\frac{\widetilde{g}}{3+\widetilde{g}}},\ x \in \mathrm{D}\left(\mathrm{A}^{1+\frac{\widetilde{g}}{2}}\right).$$

Let $\gamma := \frac{(3+\widetilde{g})}{6(2+\widetilde{g})}$, then by the above inequality and Corollary 22, we have

$$\mathbb{E}_x\left(\int_0^T \|\mathrm{A}\mathrm{X}_m(t)\|^\gamma \mathrm{d}t\right)$$
$$\leq \frac{1}{2}\mathbb{E}_x\left(\sup_{t\in[0,T]}\|\mathrm{A}^{-1/2}\mathrm{X}_m(t)\|^{\frac{\widetilde{g}}{3(1+\widetilde{g})}}\right) + \frac{T}{2}\mathbb{E}_x\left(\int_0^T \|\mathrm{A}^{1+\widetilde{g}/2}\mathrm{X}_m(t)\|^{\frac{1}{2+\widetilde{g}}}\mathrm{d}t\right) < +\infty.$$

Hence, we deduce that

$$\mathbb{E}_x\left[\int_0^T \|\mathrm{A}(\mathrm{X}_{m_k}(s,x)-\mathrm{X}(s,x))\|^\gamma \mathrm{d}s\right] \to 0, \text{ as } k\to\infty.$$

Thus, by taking an another subsequence, for any $x \in \mathrm{D}(\mathrm{A})$, we have

$$\mathrm{X}_{m_k}(t,x) \to \mathrm{X}(t,x) \text{ in } \mathrm{D}(\mathrm{A}), \mathrm{d}\mathbb{P}_x \times \mathrm{d}t, \text{ a.s.} \tag{4.15}$$

Since the sequence $(m_k)_{k\in\mathbb{N}}$ in (4.13) may depend on $x$, one cannot directly build a Markov family of solutions $(\mathrm{X}(t,x), \Omega_x, \mathscr{F}_x, \mathbb{P}_x)$. The following Lemma states that the sequence $(m_k)_{k\in\mathbb{N}}$ in Lemma 20 can be chosen independent of $\phi$. This Lemma can be proven by using the convergence arguments of Lemma 7.5, [8] or Lemma 4.4, [13]:

**Lemma 23.** *There exists a sequence $(m_k)_{k\in\mathbb{N}}$ such that for any $\phi \in \mathscr{E}$, we have*

$$u^\phi_{m_k}(t,x) \to u^\phi(t,x) \text{ uniformly in } [\delta,T]\times K_R, \text{ for any } \delta>0, R>0.$$

*If $\phi \in \mathrm{C}_b\big(\mathrm{D}\big(\mathrm{A}^{-1/2}\big);\mathbb{R}\big)$, then we have $u^\phi(t,x) = \mathbb{E}_x[\phi(\mathrm{X}(t,x))]$, $x\in\mathrm{D}(\mathrm{A})$, $t\in[0,T]$.*

**Remark 24.** By Lemma 20-(ii) and (iv), there is only one limit and the whole sequence $u^\phi_{m_k}(t,x)$ converges to $u^\phi(t,x)$.

Now we are ready to construct the transition semigroup and complete the proof of Theorem 7.

Proof of Theorem 7. Using Lemma 23, we fix the sequence $(m_k)_{k\in\mathbb{N}}$ and define for $\phi\in\mathscr{E}$:

$$\mathrm{P}_t\phi(x) = u^\phi(t,x), \ t\in[0,T], x\in \mathrm{D}(A).$$

Our next aim is to extend the definition of $\mathrm{P}_t\phi$ to all $\phi\in\mathrm{B}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$. By Proposition 17, we have $\|\mathrm{P}_t\phi\|_0 \leq \|\phi\|_0$, for all $t\in[0,T]$. Since $\mathrm{C}^{1,1}_b(\mathrm{D}(A);\mathbb{R})$[5] is a subspace of $\mathscr{E}$ and is dense in $\mathrm{UC}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$, the space of all uniformly continuous and bounded functions on $\mathrm{D}(\mathrm{A})$ (see [30]), we can extend $(\mathrm{P}_t)_{t\geq 0}$ to $\mathrm{UC}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$. However, by the existence theorem due to Getoor (Proposition 4.1, [26] and also see Lemma 3.9, [21]), there exists a measure $\nu^t_x$, $x\in\mathrm{D}(\mathrm{A})$, $t\in[0,T]$ such that

$$\mathrm{P}_t\phi(x) = \left\langle \nu^t_x, \phi\right\rangle, \ \phi\in\mathrm{UC}_b(\mathrm{D}(\mathrm{A});\mathbb{R}),$$

where $\langle\cdot,\cdot\rangle$ denote the duality product between bounded Borel functions and probability measures. Hence it can be easily seen that $(\mathrm{P}_t)_{t\geq 0}$ can be extended to $\mathrm{B}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ by this formula. In particular, $\mathrm{P}^*_t\delta_x = \nu^t_x$ defines a probability measure on $\mathrm{D}(\mathrm{A})$.

Moreover, for any $x\in\mathrm{D}(\mathrm{A})$, by extracting a subsequence $(m^x_k)_{k\in\mathbb{N}}$ of $(m_k)_{k\in\mathbb{N}}$ such that (4.13) holds, one can complete the existence of a martingale solution $(\Omega_x, \mathscr{F}_x, \mathbb{P}_x, \mathrm{X}(\cdot,x))$. Using the relation

$$u_{m^x_k}(t,x) = \mathrm{P}^{m^x_k}_t\phi(x) = \mathbb{E}_x\left[\phi\big(\mathrm{X}_{m^x_k}(t,x)\big)\right], \tag{4.16}$$

and Lemma 23, we arrive at

$$\mathrm{P}_t\phi(x) = \mathbb{E}_x[\phi(\mathrm{X}(t,x))], \ x\in\mathrm{D}(\mathrm{A}), \ t\in[0,T], \tag{4.17}$$

provided $\phi\in\mathrm{C}_b\big(\mathrm{D}\big(\mathrm{A}^{-1/2}\big);\mathbb{R}\big)\cap\mathscr{E}$. By a density argument one can show that (4.17) is true for all uniformly continuous functions $\phi$ on $\mathrm{D}(\mathrm{A}^{-1/2})$. Thus $\mathrm{P}^*_t\delta_x$, which can be seen as a probability measure on $\mathrm{D}(\mathrm{A}^{-1/2})$, is the law of $\mathrm{X}(t,x)$. Since $\mathrm{P}^*_t\delta_x$ is a probability measure on $\mathrm{D}(\mathrm{A})$, we have

$$\mathbb{P}_x\Big\{\mathrm{X}(t,x)\in\mathrm{D}(\mathrm{A})\Big\} = \mu^t_x(\mathrm{D}(\mathrm{A})) = 1.$$

[5] $\mathrm{C}^{1,1}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ is the space of all functions $\phi\in\mathrm{C}^1_b(\mathrm{D}(\mathrm{A});\mathbb{R})$ such that $\mathrm{D}_x\phi$ is Lipschitz continuous with norm $\|\phi\|_{1,1} := \sup\limits_{x\in\mathrm{D}(\mathrm{A})}|\phi(x)| + \sup\limits_{x\neq y, x,y\in\mathrm{D}(\mathrm{A})}\left[\frac{|\phi(x)-\phi(y)|}{\|\mathrm{A}(x-y)\|} + \frac{\|\mathrm{A}^{-1}(\mathrm{D}_x\phi(x)-\mathrm{D}_y\phi(y))\|}{\|\mathrm{A}(x-y)\|}\right]$.

Moreover the definition of $\mathrm{P}_t\phi$ in (4.17) remains true for $\phi \in \mathrm{B}_b(\mathrm{D}(\mathrm{A});\mathbb{R})$.

The second part of the Definition 5 follows by similar arguments in Theorem 2.4, [13] or Theorem 43, [14]. The proof of showing that $(\mathrm{P}_t)_{t\geq 0}$ is a one parameter family of semigroups follows from Lemma 20 and by the similar arguments given in Theorem 7.1, [8]. One can easily verify that the semigroup $(\mathrm{P}_t)_{t\geq 0}$ is also Markovian in the sense Defenition 5.1, [6]. The stochastic continuity of the semigroup $(\mathrm{P}_t)_{t\geq 0}$ is the direct consequence of Lemma 20-(ii) and Proposition 2.1.1, [9]. This completes the proof of Theorem 7. □

## 5 Invariant measures and ergodicity

In this section, we prove that the probability measure $\mu$ is a unique invariant measure on $\mathrm{D}(\mathrm{A})$. Let us recall that a measure $\mu$ defined on $\mathbb{H}$ is an *invariant measure* if

$$\int_{\mathbb{H}} \phi(x) d\mu(x) = \int_{\mathbb{H}} \mathrm{P}_t\phi(x) d\mu(x), \quad \text{for all } \ t \geq 0, \ \phi \in \mathrm{C}_b(\mathbb{H};\mathbb{R}), \tag{5.1}$$

where $\mathrm{P}_t$ is the Markov semigroup of the process $\mathrm{X}(t,x)$. In other words, the measure $\mu$ is invariant if $\mathrm{P}_t^*\mu = \mu$ for all $t \geq 0$, where $\mathrm{P}_t^*$ is the dual semigroup of $\mathrm{P}_t$.

Since Q is nondegenerate and the jump noise coefficient is independent of the state, $(\mathrm{P}_t^{m_k})_{t\geq 0}$ has a unique invariant measure $\mu_{m_k}$ (see [1, 37]). By Dynkin's formula, we have

$$\mathbb{E}[\phi(\mathrm{X}_{m_k}(t,x))] = \mathbb{E}[\phi(x)] + \mathbb{E}\bigg[\int_0^t \mathscr{L}_x^{m_k}\phi(\mathrm{X}_{m_k}(s,x))\mathrm{d}s\bigg], \text{ for all } \phi \in \mathrm{D}(\mathscr{L}_x^{m_k}), \tag{5.2}$$

where $\mathscr{L}_x^{m_k}$ is defined in (2.17). Thus, from (5.2), we obtain

$$\big\langle (\mathrm{P}_t^{m_k})^*\mu_{m_k}, \phi\big\rangle = \big\langle (\mathrm{P}_0^{m_k})^*\mu_{m_k}, \phi\big\rangle + \int_0^t \big\langle (\mathrm{P}_s^{m_k})^*\mu_{m_k}, \mathscr{L}_x^{m_k}\phi\big\rangle \mathrm{d}s. \tag{5.3}$$

Since $\mu_{m_k}$ is an invariant measure, $(\mathrm{P}_t^{m_k})^*\mu_{m_k} = \mu_{m_k}$, for all $t \geq 0$, and hence we have

$$\langle \mu_{m_k}, \mathscr{L}_x^{m_k}\phi\rangle = 0, \text{ for all } \phi \in \mathrm{D}(\mathscr{L}_x^{m_k}). \tag{5.4}$$

The jump noise coefficient $\Psi(\cdot,\cdot)$ satisfies the following uniform bound:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{Z}} \|\mathrm{A}^k\Psi(t,z)\|^2 \lambda(\mathrm{d}z)\mathrm{d}t = C < +\infty, \text{ for } k = 0, \frac{1}{2}, 1+\frac{g}{2} \text{ and } g > 0. \tag{5.5}$$

**Lemma 25.** *Suppose Assumption* (5) *holds true. There exists a constant* $C > 0$ *such that for any* $k \in \mathbb{N}$ :

$$\int_{\mathbb{H}} \Big[\|\mathrm{A}^{1/2}x\|^2 + \|\mathrm{A}x\|^{2/3} + \|\mathrm{A}^{1+\frac{g}{2}}x\|^{(1+2g)/(10+8g)}\Big]\mathrm{d}\mu_{m_k}(x) \leq C. \tag{5.6}$$

Proof. Applying (2.17) for $\phi(x) = \|x\|^2$, we have

$$\mathscr{L}_x^{m_k}\|x\|^2 = \mathrm{Tr}[\mathrm{Q}_{m_k}] + \int_{\mathrm{Z}_{m_k}} \|\Psi_{m_k}(z)\|^2\lambda(\mathrm{d}z) - 2\|\mathrm{A}^{1/2}x\|^2,$$

and by the invariance of $\mu_{m_k}$, we also have

$$0 = \int_{\mathbb{H}} \mathscr{L}_x^{m_k}\|x\|^2\mathrm{d}\mu_{m_k}(x) = \mathrm{Tr}[\mathrm{Q}_{m_k}] + \int_{\mathrm{Z}_{m_k}} \|\Psi_{m_k}(z)\|^2\lambda(\mathrm{d}z) - 2\int_{\mathbb{H}} \|\mathrm{A}^{1/2}x\|^2\mathrm{d}\mu_{m_k}(x). \tag{5.7}$$

Integrating (5.7) with respect to time in $[0,T]$, dividing by $T$ and using Assumption 5 gives an estimate for the first term in the left hand side of (5.6). Let us now take $\phi(x) = \frac{1}{(1+\|\mathrm{A}^{1/2}x\|^2)}$, then we have

$$\mathrm{D}_x\phi = \frac{-2\mathrm{A}x}{\big(1+\|\mathrm{A}^{1/2}x\|^2\big)^2} \text{ and } \mathrm{D}_x^2\phi = \frac{-2\mathrm{A}}{\big(1+\|\mathrm{A}^{1/2}x\|^2\big)^2} + \frac{8\mathrm{A}x\otimes\mathrm{A}x}{\big(1+\|\mathrm{A}^{1/2}x\|^2\big)^3}. \tag{5.8}$$

Therefore, we get

$$
\begin{aligned}
&\mathscr{L}_x^{m_k}\phi \\
&= -\frac{1}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\mathrm{Tr}[\mathrm{Q}_{m_k}\mathrm{A}] + \frac{4}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^3}\|\mathrm{Q}_{m_k}^{1/2}\mathrm{A}x\|^2 \\
&\quad + \int_{\mathrm{Z}_{m_k}}\left[\frac{1}{1+\|\mathrm{A}^{1/2}(x+\Psi_{m_k}(z))\|^2} - \frac{1}{1+\|\mathrm{A}^{1/2}x\|^2} + 2\frac{\left(\mathrm{A}^{1/2}x, \mathrm{A}^{1/2}\Psi_{m_k}(z)\right)}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\right]\lambda(\mathrm{d}z) \\
&\quad + \frac{2}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\left(\|\mathrm{A}x\|^2 + (\mathrm{B}_{m_k}(x), \mathrm{A}x)\right). \qquad (5.9)
\end{aligned}
$$

Lemma 3-(ii) gives

$$
|(\mathrm{B}_{m_k}(x), \mathrm{A}x)| \le C\|\mathrm{A}^{1/2}x\|^{3/2}\|\mathrm{A}x\|^{3/2} \le C\|\mathrm{A}^{1/2}x\|^6 + \frac{1}{2}\|\mathrm{A}x\|^2.
$$

Using Taylor's formula, we obtain

$$
\begin{aligned}
&\int_{\mathrm{Z}_{m_k}}\left[\frac{1}{1+\|\mathrm{A}^{1/2}(x+\Psi_{m_k}(z))\|^2} - \frac{1}{1+\|\mathrm{A}^{1/2}x\|^2} + 2\frac{\left(\mathrm{A}^{1/2}x, \mathrm{A}^{1/2}\Psi_{m_k}(z)\right)}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\right]\lambda(\mathrm{d}z) \\
&= -\int_{\mathrm{Z}_{m_k}}\frac{\|\mathrm{A}^{1/2}\Psi_{m_k}(z)\|^2}{\left(1+\|\mathrm{A}^{1/2}(x+\theta\Psi_{m_k}(z))\|^2\right)^2}\lambda(\mathrm{d}z) \\
&\quad + 4\int_{\mathrm{Z}_{m_k}}\frac{\left|\left(\mathrm{A}^{1/2}\Psi_{m_k}(z), \mathrm{A}^{1/2}(x+\theta\Psi_{m_k}(z))\right)\right|^2}{\left(1+\|\mathrm{A}^{1/2}(x+\theta\Psi_{m_k}(z))\|^2\right)^3}\lambda(\mathrm{d}z) \\
&\ge -\int_{\mathrm{Z}_{m_k}}\frac{\|\mathrm{A}^{1/2}\Psi_{m_k}(z)\|^2}{\left(1+\|\mathrm{A}^{1/2}(x+\theta\Psi_{m_k}(z))\|^2\right)^2}\lambda(\mathrm{d}z) \ge -\int_{\mathrm{Z}_{m_k}}\|\mathrm{A}^{1/2}\Psi_{m_k}(z)\|^2\lambda(\mathrm{d}z),
\end{aligned}
$$

for $0<\theta<1$. Assumptions 2.1 and 2.1 imply

$$
\begin{aligned}
\mathscr{L}_x^{m_k}\phi &\ge \frac{\|\mathrm{A}x\|^2}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2} - C\frac{\|\mathrm{A}^{1/2}x\|^6}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2} - C - \int_{\mathrm{Z}_{m_k}}\|\mathrm{A}^{1/2}\Psi_{m_k}(z)\|^2\lambda(\mathrm{d}z) \\
&\ge \frac{\|\mathrm{A}x\|^2}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2} - C\|\mathrm{A}^{1/2}x\|^2 - C - \int_{\mathrm{Z}_{m_k}}\|\mathrm{A}^{1/2}\Psi_{m_k}(z)\|^2\lambda(\mathrm{d}z). \qquad (5.10)
\end{aligned}
$$

The integral with Lévy measure in (5.10) is again bounded by Assumption 5. We integrate (5.10) over $\mathbb{H}$ to get

$$
0 = \int_{\mathbb{H}}\mathscr{L}_x^{m_k}\phi\,\mathrm{d}\mu_{m_k}(x) \ge \int_{\mathbb{H}}\frac{\|\mathrm{A}x\|^2}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\mathrm{d}\mu_{m_k}(x) - C\int_{\mathbb{H}}\|\mathrm{A}^{1/2}x\|^2\mathrm{d}\mu_{m_k}(x) - C. \qquad (5.11)
$$

We use Hölder's inequality, (5.7) and (5.11) to obtain

$$
\begin{aligned}
&\int_{\mathbb{H}}\|\mathrm{A}x\|^{2/3}\mathrm{d}\mu_{m_k}(x) \\
&\le \left(\int_{\mathbb{H}}\frac{\|\mathrm{A}x\|^2}{\left(1+\|\mathrm{A}^{1/2}x\|^2\right)^2}\mathrm{d}\mu_{m_k}(x)\right)^{1/3}\left(\int_{\mathbb{H}}\left(1+\|\mathrm{A}^{1/2}x\|^2\right)\mathrm{d}\mu_{m_k}(x)\right)^{2/3} \le C. \qquad (5.12)
\end{aligned}
$$

Let us now take[6]

$$
\phi(x) = \frac{1}{\left(1+\|\mathrm{A}^{(1+g)/2}x\|^2\right)^{2/(1+2g)}}.
$$

[6]For $k(x) = \left(1+\|\mathrm{A}^{(1+g)/2}x\|^2\right)$, we have

$$
\mathrm{D}_x\phi = \frac{-4}{(1+2g)}\frac{\mathrm{A}^{1+g}x}{k(x)^{\frac{3+2g}{1+2g}}}, \mathrm{D}_x^2\phi = \frac{-4}{(1+2g)}\frac{\mathrm{A}^{1+g}}{k(x)^{\frac{3+2g}{1+2g}}} + \frac{8(3+2g)}{(1+2g)^2}\frac{\mathrm{A}^{1+g}x\otimes\mathrm{A}^{1+g}x}{k(x)^{\frac{4(1+g)}{1+2g}}},
$$

Thus, we get

$$\begin{aligned}\mathscr{L}^x_{m_k}\phi = & \frac{-2}{(1+2g)}\frac{1}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{\frac{3+2g}{1+2g}}}\operatorname{Tr}\big[\mathrm{Q}_{m_k}\mathrm{A}^{1+g}\big]\\ & + \frac{4(3+2g)}{(1+2g)^2}\frac{1}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{\frac{4(1+g)}{1+2g}}}\|\mathrm{Q}^{1/2}_{m_k}\mathrm{A}^{1+g}\|^2\\ & + \int_{\mathrm{Z}_{m_k}}\Bigg[\frac{1}{\big(1+\|\mathrm{A}^{(1+g)/2}(x+\Psi_{m_k}(z))\|^2\big)^{2/(1+2g)}} - \frac{1}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{2/(1+2g)}}\\ & \quad + \frac{4}{(1+2g)}\frac{\big(\mathrm{A}^{(1+g)/2}x, \mathrm{A}^{(1+g)/2}\Psi_{m_k}(z)\big)}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{\frac{3+2g}{1+2g}}}\Bigg]\lambda(\mathrm{d}z)\\ & + \frac{4}{(1+2g)}\frac{1}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{\frac{3+2g}{1+2g}}}\Big(\|\mathrm{A}^{1+g/2}x\|^2 + \big(\mathrm{B}_{m_k}(x), \mathrm{A}^{1+g}x\big)\Big).\end{aligned}$$

Let us use Lemma 3-(ii) with $\delta = 1+g$ to obtain

$$\big|\big(\mathrm{B}_{m_k}(x), \mathrm{A}^{1+g}x\big)\big| \le \frac{1}{2}\|\mathrm{A}^{1+g/2}x\|^2 + C\|\mathrm{A}^{(1+g)/2}x\|^{2(3+2g)/(1+2g)}.$$

Proceeding as before and using the assumptions on noise coefficient, we get

$$\mathscr{L}^x_{m_k}\phi \ge \frac{2}{1+2g}\frac{\|\mathrm{A}^{1+g/2}x\|^2}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{\frac{3+2g}{1+2g}}} - C.$$

Let us integrate the above inequality to find

$$\int_{\mathbb{H}}\frac{\|\mathrm{A}^{1+g/2}x\|^2}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{(3+2g)/(1+2g)}}\mathrm{d}\mu_{m_k}(x) \le C. \tag{5.13}$$

Now, by using Hölder's inequality, (5.12) and (5.13), we obtain

$$\begin{aligned}&\int_{\mathbb{H}}\|\mathrm{A}^{1+g/2}x\|^{(1+2g)/(10+8g)}\mathrm{d}\mu_{m_k}(x)\\ &\le \left(\int_{\mathbb{H}}\frac{\|\mathrm{A}^{1+g/2}x\|^2}{\big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\big)^{(3+2g)/(1+2g)}}\mathrm{d}\mu_{m_k}(x)\right)^{(1+2g)/(20+16g)}\\ &\quad\times\left(\int_{\mathbb{H}}\Big(1+\|\mathrm{A}^{(1+g)/2}x\|^2\Big)^{1/3}\mathrm{d}\mu_{m_k}(x)\right)^{(19+14g)/(20+16g)} \le C.\end{aligned} \tag{5.14}$$

Combining (5.7), (5.12) and (5.14), we finally get (5.6). □

By Lemma 25, it follows that the sequence $(\mu_{m_k})_{k\in\mathbb{N}}$ is tight on $\mathrm{D}(\mathrm{A})$ and there exists a subsequence, denoted by $(\mu_{m_k})_{k\in\mathbb{N}}$ for simplicity, and a measure $\mu$ on $\mathrm{D}(\mathrm{A})$ such that $\mu_{m_k}$ converges weakly to $\mu$. Furthermore, $\mu\big(\mathrm{D}\big(\mathrm{A}^{1+\frac{g}{2}}\big)\big) = 1$.

Let us take $\phi \in \mathscr{E}$. By the invariance of $\mu_{m_k}$, we have

$$\int_{\mathbb{H}}\mathrm{P}^{m_k}_t\phi(x)\mathrm{d}\mu_{m_k}(x) = \int_{\mathbb{H}}\phi(x)\mathrm{d}\mu_{m_k}(x), \tag{5.15}$$

for any $t \ge 0$. Note that

$$\left|\int_{\mathbb{H}}\mathrm{P}^{m_k}_t\phi(x)\mathrm{d}\mu_{m_k}(x) - \int_{\mathbb{H}}\mathrm{P}_t\phi(x)\mathrm{d}\mu(x)\right|$$

$$\leq \left|\int_{\mathbb{H}}[\mathrm{P}_t^{m_k}\phi(x)-\mathrm{P}_t\phi(x)]\mathrm{d}\mu_{m_k}(x)\right|+\left|\int_{\mathbb{H}}\mathrm{P}_t\phi(x)[\mathrm{d}\mu_{m_k}(x)-\mathrm{d}\mu(x)]\right|.$$

By the weak convergence of $\mu_{m_k}$, the second integral on the right hand side goes to 0 as $k \to \infty$. The first integral can be further estimated as

$$\left|\int_{\mathbb{H}}[\mathrm{P}_t^{m_k}\phi(x)-\mathrm{P}_t\phi(x)]\mathrm{d}\mu_{m_k}(x)\right|$$
$$\leq \sup_{\|\mathrm{A}x\|\leq R}|\mathrm{P}_t^{m_k}\phi(x)-\mathrm{P}_t\phi(x)|+2\|\phi\|_0R^{-2/3}\int_{\mathbb{H}}\|\mathrm{A}x\|^{2/3}\mathrm{d}\mu_{m_k}(x).$$

By Lemma 20, the approximations $\mathrm{P}_t^{m_k}\phi$ converges to $\mathrm{P}_t\phi$ uniformly on $K_R$ and the second integral in the right hand side of the above inequality is also bounded by Lemma 25. Since $R$ is arbitrary and by the weak convergence of $\mu_{m_k}$, taking limit $k \to \infty$ in (5.15), one can get (5.1). Hence $\mu$ is an invariant measure.

Let $\mu$ be an invariant measure for $(\mathrm{P}_t)_{t\geq 0}$. We say that the measure $\mu$ is an *ergodic measure,* if for all $\varphi \in \mathbb{L}^2(\mathbb{H};\mu)$, we have (see, page 74, [6])

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T(\mathrm{P}_t\phi)(x)\mathrm{d}t=\int_{\mathbb{H}}\phi(x)\mathrm{d}\mu(x)\ \text{ in } \mathbb{L}^2(\mathbb{H};\mu).$$

The invariant measure $\mu$ for $(\mathrm{P}_t)_{t\geq 0}$ is called *strongly mixing* if for all $\varphi \in \mathbb{L}^2(\mathbb{H};\mu)$, we have

$$\lim_{t\to+\infty}\mathrm{P}_t\phi(x)=\int_{\mathbb{H}}\phi(x)\mathrm{d}\mu(x) \text{ in } \mathbb{L}^2(\mathbb{H};\mu).$$

Due to the classical result of Khasminskii and Doob (see Theorem 4.2.1, [9] or Theorem 22, [14]), ergodicity and strongly mixing properties of the measure $\mu$ are the direct consequence of strong Feller property and irreducibility of the transition semigroup $(\mathrm{P}_t)_{t\geq 0}$.

**Definition 26.** A transition semigroup $(\mathrm{P}_t)_{t\geq 0}$ is *strong Feller* on $\mathrm{D(A)}$ if for any $\phi \in \mathrm{B}_b(\mathrm{D(A)};\mathbb{R})$ and $t>0$, one can get $\mathrm{P}_t\phi \in \mathrm{C}_b(\mathrm{D(A)};\mathbb{R})$.

Let $\mathrm{B}_{\mathrm{D(A)}}(y,\varepsilon)$ denote the ball in $\mathrm{D(A)}$ of center $y$ and radius $\varepsilon$. The semigroup $(\mathrm{P}_t)_{t\geq 0}$ on $\mathrm{D(A)}$ is *irreducible*, if for all $t>0$, all $x, y \in \mathrm{D(A)}$, and all $\varepsilon>0$, we have

$$\mathrm{P}_t(x,\mathrm{B}_{\mathrm{D(A)}}(y,\varepsilon))=\mathbb{P}\Big\{\mathrm{X}(t,x)\in \mathrm{B}_{\mathrm{D(A)}}(y,\varepsilon)\Big\}>0.$$

**Proposition 27.** *The transition semigroup* $(\mathrm{P}_t)_{t\geq 0}$ *is strong Feller on* $\mathrm{C}_b(\mathrm{D(A)};\mathbb{R})$.

Proof. By Proposition 17, it is clear that $\|\mathrm{P}_t\phi\|_0 \leq \|\phi\|_0$, for $\phi \in \mathrm{C}_b(\mathrm{D(A)};\mathbb{R})$, and in view of Theorem 7, it is true for any $\phi \in \mathrm{B}_b(\mathrm{D(A)};\mathbb{R})$, and the strong Feller property holds. □

The irreducibility of the transition semigroup is proved in the next proposition.

**Proposition 28.** *Suppose that Assumptions 2.1 and 2.1 hold true. Then the transition semigroup* $(\mathrm{P}_t)_{t\geq 0}$ *corresponding to the system (2.2) is irreducible on* $\mathrm{D(A)}$.

We first prove Proposition 28 when $\lambda(\mathrm{Z})<+\infty$. Since we are assuming that the Lévy measure $\lambda(\mathrm{Z})<+\infty$, we can define the jump times of $\pi(\mathrm{d}t,\mathrm{d}z)$ as $0<\sigma_1(\omega)<\sigma_2(\omega)<\dots$. The jump integral is

$$\int_0^t\int_{\mathrm{Z}}\Psi(t-,\xi,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z)=\int_0^t\int_{\mathrm{Z}}\Psi(t-,\xi,z)\pi(\mathrm{d}t,\mathrm{d}z)-\int_0^t\int_{\mathrm{Z}}\Psi(t,\xi,z)\lambda(\mathrm{d}z)\mathrm{d}t.$$

Since the jump occurs only at $\sigma_1$, the first integral is zero on $[0,\sigma_1)$. Hence the equation (2.2) is equivalent to the following:

$$\begin{cases}\mathrm{dX}(t,x)=-[\mathrm{AX}(t,x)+\mathrm{B}(\mathrm{X}(t,x))]\mathrm{d}t+\sqrt{\mathrm{Q}}\mathrm{dW}(t)-\displaystyle\int_{\mathrm{Z}}\Psi(t,z)\lambda(\mathrm{d}z)\mathrm{d}t,\\ \mathrm{X}(0,x)=x,\end{cases}\tag{5.16}$$

on $[0, \sigma_1)$. By Theorem 2.1 [42], there exists a martingale solution of the problem (5.16) in $[0, \sigma_1(\omega))$. We can recursively obtain a martingale solution $(\Omega_x, \mathbb{P}_x, \mathscr{F}_x, \mathrm{X}(t,x))$ of the system (5.16) for the interval $[0,T]$ (see [16, 21]).

Now, the proof of Proposition 28 is completed by proving Lemmas given below. The irredcibility of the semigroup $(\mathrm{P}_t)_{t\geq 0}$ is closely related to the controllability of the Navier–Stokes equation (5.16) with the noise replaced by a right hand side forcing/control term. This technique has been developed for the Navier–Stokes equations with Gaussian noise in [23, 8, 14]. More precisely, we consider a control system with $\lambda(\mathrm{Z}) < +\infty$ :

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{dt} = -[\mathrm{A}y(t) + \mathrm{B}(y(t))] - \displaystyle\int_{\mathrm{Z}} \Psi(t,z)\lambda(\mathrm{d}z) + \mathrm{U}(t), \\ y(0) = x, \end{cases} \tag{5.17}$$

where $\mathrm{U}(t)$ is the control function. Then, we have

**Lemma 29.** *Let $T > 0$, $x \in \mathrm{D}(\mathrm{A})$ and $y_T \in \mathrm{D}\big(\mathrm{A}^{3/2}\big)$ be given, and assume that the Lévy measure satisfies*

$$\sup_{t\in[0,T]} \int_{\mathrm{Z}} \|\mathrm{A}^{1/2}\Psi(t,z)\|\lambda(\mathrm{d}z) \leq C(T,\Psi). \tag{5.18}$$

*Then, there exists a control $\mathrm{U} \in \mathrm{L}^\infty\big(0,T;\mathrm{D}\big(\mathrm{A}^{1/2}\big)\big)$ and $y \in \mathrm{C}([0,T];\mathrm{D}(\mathrm{A})) \cap \mathrm{L}^2\big(0,T;\mathrm{D}\big(\mathrm{A}^{3/2}\big)\big)$ satisfying (5.17) such that $y(T) = y_T$.*

Proof. Let us first set $\mathrm{U} = 0$. Now we show that there exists a time $T^*$ such that $0 < T^* < T$ and $y \in \mathrm{C}([0,T^*];\mathrm{D}(\mathrm{A}))$ satisfying (5.17). Let $M > 0$ and consider

$$\mathrm{S} = \Big\{ v \in \mathrm{C}([0,T];\mathrm{D}(\mathrm{A})) : \|\mathrm{A}v(t)\| \leq M \text{ for all } t \in [0,T] \Big\}.$$

Let us take any $v \in \mathrm{S}$ and define $y = \mathrm{F}(v)$ by

$$y(t) = e^{-t\mathrm{A}}x - \int_0^t e^{-(t-s)\mathrm{A}}\mathrm{B}(v(s))\mathrm{d}s - \int_0^t \int_{\mathrm{Z}} e^{-(t-s)\mathrm{A}}\Psi(s,z)\lambda(\mathrm{d}z)\mathrm{d}s. \tag{5.19}$$

Note that by the properties of the semigroup $\big(e^{-t\mathrm{A}}\big)_{t\geq 0}$, Lemma 3-(i) and Assumption 2.1, we have

$$\begin{aligned} \|\mathrm{A}y(t)\| &\leq \|e^{-t\mathrm{A}}\mathrm{A}x\| + \int_0^t \|\mathrm{A}^{1/2}e^{-(t-s)\mathrm{A}}\|_{\mathbf{L}(\mathbb{H})}\|\mathrm{A}^{1/2}\mathrm{B}(v(s))\|\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathrm{Z}} \|e^{-(t-s)\mathrm{A}}\|_{\mathbf{L}(\mathbb{H})}\|\mathrm{A}\Psi(s,z)\|\lambda(\mathrm{d}z)\mathrm{d}s \\ &\leq \|\mathrm{A}x\| + C\int_0^t (t-s)^{-1/2}\|\mathrm{A}v(s)\|^2\mathrm{d}s + \int_0^t \int_{\mathrm{Z}} \|\mathrm{A}\Psi(s,z)\|\lambda(dz)\mathrm{d}s \\ &\leq \|\mathrm{A}x\| + 2CM^2t^{1/2} + (\lambda(\mathrm{Z})TC(T,\Psi))^{1/2}. \end{aligned} \tag{5.20}$$

Thus, $\|\mathrm{A}y(t)\| \leq M$ for all $t \in [0,T]$ provided,

$$\|\mathrm{A}x\| + 2CM^2T^{1/2} + (\lambda(\mathrm{Z})TC(T,\Psi))^{1/2} \leq M.$$

For any $M > \|\mathrm{A}x\|$, there exists $0 < T^* < T$ such that the above inequality holds. Moreover, for any $v_1, v_2 \in \mathrm{S}$, we have

$$\|\mathrm{A}(\mathrm{F}(v_1) - \mathrm{F}(v_2))\| \leq 4CMt^{1/2}\|\mathrm{A}(v_1 - v_2)\|. \tag{5.21}$$

Choose $T^* \in (0,T)$ such that $4CMT^{*^{1/2}} < 1$, so that F is a strict contraction on S. Therefore by a fixed point argument, we get a unique solution $y \in \mathrm{C}([0,T^*];\mathrm{D}(\mathrm{A}))$ to the problem (5.17) with $\mathrm{U} = 0$.

It can also be established that $y \in \mathrm{L}^2\big(0,T^*;\mathrm{D}\big(\mathrm{A}^{3/2}\big)\big)$, so that $y(t) \in \mathrm{D}\big(\mathrm{A}^{3/2}\big)$, a.e., and one can change $T^*$ so that $y(T^*) \in \mathrm{D}\big(\mathrm{A}^{3/2}\big)$. Next we set $\mathrm{U} = 0$ on $[0,T^*]$ and define $y$ on $[T^*,T]$ as follows:

$$\begin{cases} y(t) = \dfrac{T-t}{T-T^*}y(T^*) + \dfrac{t-T^*}{T-T^*}y_T\,,\ t\in[T^*,T],\ \text{and set} \\ \mathrm{U}(t) = \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} + \mathrm{A}y(t) + \mathrm{B}(y(t)) + \displaystyle\int_{\mathrm{Z}} \Psi(t,z)\lambda(\mathrm{d}z),\ t\in[T^*,T]. \end{cases} \tag{5.22}$$

Using Lemma 2.4-(i), and Assumption (5.18), one can verify that U and $y$ have the properties described in the Lemma. □

**Remark 30.** Note that Lemma 29 also works for bounded or even for exterior domains. Using the continuous embedding and algebra property of $\mathbb{H}^\alpha$ norm for $\alpha > 3/2$, one can obtain that for any $0 < \varepsilon < 1/4$ (see, [29]):

$$\|\mathrm{A}^{1/4-\varepsilon}\mathrm{P}_{\mathbb{H}}(\mathbf{u}\cdot\nabla\mathbf{u})\| \leq C\|\mathrm{A}\mathbf{u}\|^2. \tag{5.23}$$

Hence, we have

$$\begin{aligned} \left\|\int_0^t \mathrm{A}e^{-(t-s)\mathrm{A}}\mathrm{B}(v(s))\mathrm{d}s\right\| &\leq \int_0^t \|\mathrm{A}^{3/4+\varepsilon}e^{-(t-s)\mathrm{A}}\|_{\mathbf{L}(\mathbb{H})}\|\mathrm{A}^{1/4-\varepsilon}\mathrm{B}(v(s))\|\mathrm{d}s \\ &\leq C\int_0^t (t-s)^{-(3/4+\varepsilon)}\|\mathrm{A}v(s)\|^2\mathrm{d}s \leq \frac{4C}{1-4\varepsilon}M^2t^{1/4-\varepsilon} < +\infty. \end{aligned}$$

Using this non-linear estimate in (5.20), the controllability Lemma 29 can be proved for bounded/exterior domains as well.

Let us define

$$\widetilde{w}(t) = \int_0^t e^{-(t-s)\mathrm{A}}\mathrm{U}(s)\mathrm{d}s,\ t\in[0,T].$$

Since $\mathrm{U} \in \mathrm{L}^\infty\big(0,T;\mathrm{D}\big(\mathrm{A}^{1/2}\big)\big)$, the properties of the analytic semigroup leads to $\widetilde{w} \in \mathrm{C}([0,T];\mathrm{D}(\mathrm{A}^\sigma))$ for any $\sigma < 3/2$. Now, let $\widetilde{y}_m := \mathrm{P}_m(y-\widetilde{w})$. Then $\widetilde{y}_m$ satisfies the equation

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{y}_m}{\mathrm{d}t} = -\mathrm{A}\widetilde{y}_m - \mathrm{B}_m(\widetilde{y}_m + \widetilde{w}) + g_m - \displaystyle\int_{\mathrm{Z}_m} \Psi_m(t,z)\lambda(\mathrm{d}z), \\ \widetilde{y}_m(0) = \mathrm{P}_m x, \end{cases}$$

where

$$g_m = -\mathrm{P}_m\mathrm{B}(y) + \mathrm{B}_m(\widetilde{y}_m + \widetilde{w}) = \mathrm{P}_m(-\mathrm{B}(y) + \mathrm{B}(\mathrm{P}_m y)).$$

Moreover, for any $w \in \mathrm{L}^\infty(0,T;\mathrm{D}(\mathrm{A}))$, consider

$$\begin{cases} \dfrac{\mathrm{d}\widehat{y}_m}{\mathrm{d}t} = -\mathrm{A}\widehat{y}_m - \mathrm{B}_m(\widehat{y}_m + w) - \displaystyle\int_{\mathrm{Z}_m} \Psi_m(t,z)\lambda(\mathrm{d}z), \\ y_m(0) = \mathrm{P}_m x. \end{cases}$$

Using Lemma 29, one can prove the following Lemma by the arguments similar to that of Lemma 7.7, [8] or Lemma 51, [14] .

**Lemma 31.** *There exists a constant $C > 0$ such that $\|\widetilde{y}_m - \widehat{y}_m\|_{\mathrm{L}^\infty(0,T;\mathrm{D}(\mathrm{A}))} \leq e^{CKT}K_1$, where*

$$K := \big(\|y\|_{\mathrm{L}^\infty(0,T;\mathrm{D}(\mathrm{A}))} + \|\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D}(\mathrm{A}))} + 1\big)^4$$

*and*

$$K_1 := \|w - \widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D}(\mathrm{A}))} + \|g_m\|_{\mathrm{L}^4\left(0,T;\mathrm{D}\left(\mathrm{A}^{1/2}\right)\right)},$$

*provided* $\|w-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq 1$ *and* $e^{CKT}K_1 \leq \frac{1}{2}\left[\|y\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})}+\|\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})}\right]$. *Moreover, we have* $\lim\limits_{m\to\infty} g_m = 0$ *in* $\mathrm{L}^4\big(0,T;\mathrm{D}(\mathrm{A}^{1/2})\big)$.

*Consequently, let* $x_0 \in \mathrm{D(A)}$ *and* $\varepsilon > 0$. *Then, for any* $w \in \mathrm{C}([0,T];\mathrm{D(A)})$ *such that* $\|w-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq \eta$, *where* $\eta = \min\big\{1, \tilde{\varepsilon}e^{-CKT}\big\}$ *and* $\widetilde{\varepsilon} > 0$ *depending on* $\varepsilon$, *we also have*

$$\|\mathrm{A}(\widehat{y}_m(T)+w_m(T)-x_0)\| \leq \varepsilon. \tag{5.24}$$

Let $\mathrm{P}_t^\Psi(x,\cdot)$ be the transition probability corresponding to the system (2.2) with finite Lévy measure $\lambda(\mathrm{Z}) < +\infty$. The irreducibility of this case is proved in the following lemma by appropriately choosing the function $\phi$ :

**Lemma 32.** *Let* $x_0 \in \mathrm{D(A)}$, $\varepsilon > 0$ *and* $\phi \in \mathscr{E}$ *be such that*

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon), \\ 0 & \text{if } x \notin \mathrm{B}_{\mathrm{D(A)}}(x_0,2\varepsilon), \end{cases} \tag{5.25}$$

*and* $0 \leq \phi(x) \leq 1,$ *if* $x \in \mathrm{B}_{\mathrm{D(A)}}(x_0,2\varepsilon)\backslash \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon)$. *Then, for any* $t > 0$ *and* $x \in \mathrm{D(A)}$, *we have* $\mathrm{P}_t\phi(x) > 0$, *where* $\mathrm{P}_t$ *is the transition semigroup corresponding to the system (5.16)* .

*Moreover, for any* $t > 0$ *and* $x \in \mathrm{D(A)}$, *the transition probability* $\mathrm{P}_t^\Psi(\cdot,\cdot)$ *is also irreducible.*

P r o o f. Let us first show the irreducibility of the transition semigroup associated with the system (5.16). Let $\mathrm{X}_m(t,x)$ be the solution of the finite dimensional approximations of (5.16) on $[0,T]$ and $\mathrm{G}(t)$ be the solution of (2.25). Let $x_0 \in \mathrm{D(A)}$ and $\varepsilon > 0$. Then, by using (5.24), there exists $m_0 \in \mathbb{N}$ and $\eta > 0$ such that, for $m_k \geq m_0$, we have

$$\begin{aligned}\mathbb{P}\Big\{\mathrm{X}_{m_k}(T,x) \in \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon)\Big\} &= \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}_{m_k}(T,x)-x_0)\| \leq \varepsilon\Big\} \\ &\geq \mathbb{P}\Big\{\|\mathrm{G}-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq \eta\Big\}.\end{aligned}$$

We know that Ker $\mathrm{Q} = \{0\}$ and hence we have $\mathbb{P}\Big\{\|\mathrm{G}-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq \eta\Big\} > 0$. Also, for $\Gamma_{m_k} := \big\{\omega \in \Omega : \mathrm{X}_{m_k}(T,x) \in \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon)\big\}$, we have

$$\begin{aligned}\mathrm{P}_T^{m_k}\phi(x) = \mathbb{E}[\phi(\mathrm{X}_{m_k}(T,x))] &= \int_{\Gamma_{m_k}} \phi(\mathrm{X}_{m_k}(T,x))\mathrm{d}\mathbb{P}(\omega) + \int_{\Gamma_{m_k}^c} \phi(\mathrm{X}_{m_k}(T,x))\mathrm{d}\mathbb{P}(\omega) \\ &\geq \mathbb{P}\Big\{\mathrm{X}_{m_k}(T,x) \in \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon)\Big\} \geq \mathbb{P}\Big\{\|\mathrm{G}-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq \eta\Big\}.\end{aligned}$$

Since $\mathrm{P}_T^{m_k}\phi(x) \to \mathrm{P}_T\phi(x)$ as $m_k \to \infty$, we arrive at

$$\mathrm{P}_T\phi(x) \geq \mathbb{P}\Big\{\|\mathrm{G}-\widetilde{w}\|_{\mathrm{L}^\infty(0,T;\mathrm{D(A)})} \leq \eta\Big\} > 0.$$

But, by the definition of $\phi \in \mathscr{E}$, we know that

$$\begin{aligned}0 < \mathrm{P}_T\phi(x) &= \int_\Gamma \phi(\mathrm{X}(T,x))\mathrm{d}\mathbb{P}(\omega) + \int_{\Gamma^c} \phi(\mathrm{X}(T,x))\mathrm{d}\mathbb{P}(\omega) \\ &= \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(T,x)-x_0)\| < \varepsilon\Big\} + \int_{\{\omega\in\Omega:\varepsilon\leq\|\mathrm{A}(\mathrm{X}(T,x)-x_0)\|<2\varepsilon\}} \phi(\mathrm{X}(T,x))\mathrm{d}\mathbb{P}(\omega) \\ &\leq \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(T,x)-x_0)\| < \varepsilon\Big\} + \mathbb{P}\Big\{\varepsilon \leq \|\mathrm{A}(\mathrm{X}(T,x)-x_0)\| < 2\varepsilon\Big\} \\ &= \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(T,x)-x_0)\| < 2\varepsilon\Big\},\end{aligned} \tag{5.26}$$

where $\Gamma := \big\{\omega \in \Omega : \mathrm{X}(T,x) \in \mathrm{B}_{\mathrm{D(A)}}(x_0,\varepsilon)\big\}$.

Next we complete the proof of Lemma 32 by showing that $\mathrm{P}_t^\Psi(x,\cdot)$ is irreducible. Let $\{\sigma_k\}_{k\geq 1}$ be the interarrival times of the Poisson process $\pi$ associated with $\{z_{\sigma_k} : k \geq 1\} \subset \mathrm{Z}$. Then $\{\sigma_k, z_{\sigma_k}\}$ is independent and

$$\mathbb{P}\Big\{z_{\sigma_k} \in \mathrm{V}, \sigma_k > t\Big\} = e^{-\lambda(\mathrm{Z})t}\lambda(\mathrm{V}), \text{ for all } t > 0, \mathrm{V} \in \mathscr{B}(\mathrm{Z}).$$

Let $(\Omega_x, \mathbb{P}_x, \mathscr{F}_x, \mathrm{X}_0(t,x))$ be a martingale solution of the system (2.1) on $[0, \sigma_1)$. Since $\{\sigma_k, z_{\sigma_k}\}$ is independent of $(\Omega_x, \mathbb{P}_x, \mathscr{F}_x, \mathrm{X}_0(t,x))$, we have $\mathrm{X}(t,x) = \mathrm{X}_0(t,x)$ for $0 \leq t < \sigma_1$, and $\mathrm{X}(\sigma_1, x) = \mathrm{X}(\sigma_1-, x) + \Psi(\sigma_1-, \triangle z_{\sigma_1})$.

Let $\mathrm{P}_t^0(x,\cdot)$ be the transition probability of the solution $\mathrm{X}_0(t,x)$. Then, the relation between $\mathrm{P}_t^0(x,\cdot)$ and $\mathrm{P}_t^\Psi(x,\cdot)$ can be derived as follows (see, Theorem 14, [43] or [15, 16])

$$\mathrm{P}_t^\Psi(x,\mathrm{V}) = e^{-t\lambda(\mathrm{Z})}\mathrm{P}_t^0(x,\mathrm{V}) + \int_0^t\int_{\mathbb{H}}\int_{\mathrm{Z}} e^{-s\lambda(\mathrm{Z})}\mathrm{P}_{t-s}^\Psi(y+\Psi(s,z),\mathrm{V})\mathrm{P}_s^0(x,\mathrm{d}y)\lambda(\mathrm{d}z)\mathrm{d}s. \quad (5.27)$$

Since $\mathrm{P}_t^0(x,\cdot)$ is irreducible by the first part of this Lemma and by the above relationship, we have that $\mathrm{P}_t^\Psi(x,\cdot)$ is also irreducible. □

Using all the above Lemmas we complete the proof of Proposition 28.

Proof of Proposition 28. Let $(\Omega_x, \mathbb{P}_x, \mathscr{F}_x, \mathrm{X}_m(t,x))$ be the martingale solution of the equation

$$\begin{cases} \mathrm{dX}_m(t,x) = -[\mathrm{AX}_m(t,x) + \mathrm{B}_m(\mathrm{X}_m(t,x))]\mathrm{d}t + \sqrt{\mathrm{Q}_m}\mathrm{dW}(t) + \displaystyle\int_{\mathrm{Z}_m}\Psi_m(t,z)\widetilde{\pi}(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{X}_m(0,x) = \mathrm{P}_m x. \end{cases} \quad (5.28)$$

By Lemma 32, we know that $\mathrm{X}_m(t,x)$ is irreducible, then for any $y \in \mathrm{D(A)}$ and $\varepsilon > 0$, we have

$$\mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}_m(t,x) - y)\| < \varepsilon\Big\} = \delta > 0. \quad (5.29)$$

Since we know that $(\Omega_x, \mathbb{P}_x, \mathscr{F}_x, \mathrm{X}(t,x))$ is a martingale solution of the problem (2.1), by using (4.15) (if necessary along a subsequence of $\mathrm{X}_m$), we also have

$$\mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(t,x) - \mathrm{X}_m(t,x))\| \geq \varepsilon\Big\} \leq \frac{\delta}{2}. \quad (5.30)$$

Using (5.29) and (5.30), we obtain

$$\begin{aligned} &\mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(t,x) - y)\| \geq 2\varepsilon\Big\} \\ &\leq \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}(t,x) - \mathrm{X}_m(t,x))\| \geq \varepsilon\Big\} + \mathbb{P}\Big\{\|\mathrm{A}(\mathrm{X}_m(t,x) - y)\| \geq \varepsilon\Big\} \leq \frac{\delta}{2} + 1 - \delta < 1, \end{aligned} \quad (5.31)$$

and hence $\mathrm{X}(t,x)$ is irreducible on $\mathrm{D(A)}$. □

Now let us complete the proof of Theorem 8.

Proof of Theorem 8. Let us use Propositions 27 and 28 to see that the the transition semigroup $(\mathrm{P}_t)_{t\geq 0}$ is strong Feller and irreducible on $\mathrm{D(A)}$. Hence by Doob's theorem, $\mu$ is the unique invariant measure on $\mathrm{D(A)}$ and therefore it is ergodic and strongly mixing. The Theorem 8, part (ii) can be established in a similar way as in [13, 14]. □

**Conclusions and future works:** The ergodicity of 3D stochastic Navier–Stokes equations perturbed by Lévy noise has been established. Since we do not have uniqueness of solutions for the 3D SNSE with Lévy noise, we consider the Kolmogorov equation involving an integro-differential operator with Lévy mesure associated with this SNSE. The existence of martingale solutions for the 3D SNSE with Lévy noise and the solution to the associated Kolmogorov equation help us to construct a transition semigroup and prove the uniqueness of invariant measures. The classical result of Khasminskii and Doob gives the uniqueness of invariant measure as a direct consequence of the strong Feller property and irreducibility of the transition semigroup associated with the SNSE with Lévy noise. Moreover, such an invariant measure is ergodic and strongly mixing. By assuming that the Gaussian and jump noises are independent, we obtain the BEL formula and thereby we proved the strong Feller property. The irreducibility has been established by proving the controllability of the NSE perturbed by an integral with Lévy measure and a distributed control.

One can extend this work for various hydrodynamic models and different structure of the noise coefficients. In particular, this paper can be extended to the case where the noise coefficients are multiplicative in nature. As the BEL formula is established for general stochastic differential equations with $\alpha-$stable noise (see [46]), one can extend our ideas for SNSE with $\alpha-$stable noise. In fact, the exponential ergodicity of stochastic Burgers equation driven by $\alpha-$stable noise has already been done in [18]. Since the abstract functional setting for a class of nonlinear stochastic hydrodynamic models perturbed by Lévy noise, namely *3D magnetohydrodynamic(MHD) equation*, *3D Leray $\alpha$-model for Navier–Stokes equation*, *Shell models of turbulence* are same as that of *3D Navier–Stokes equation*, the methods used in this paper can be extended to these models as well.

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## A Comparison lemma

**Lemma 33** (Comparison Lemma)**.** *Let* $\mathrm{I}$ *denote an interval of the real line of the form* $[a,\infty)$ *or* $[a,b]$ *or* $[a,b)$ *with* $a<b$. *Let* $f$, $g$ *and* $k$ *be non-negative continuous functions defined on* $\mathrm{I}$. *If* $f$ *is differentiable in the interior* $\mathrm{I}^o$ *of* $\mathrm{I}$ *and satisfies the differential inequality*

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t}+g(t)\leq Ck(t)f(t),\ t\in\mathrm{I}^o, \tag{A.1}$$

*where* $C>0$ *is a constant. Then we have*

$$e^{-C\int_a^t k(s)\mathrm{d}s}f(t)+\int_a^t e^{-C\int_a^s k(r)\mathrm{d}r}g(s)\mathrm{d}s\leq f(a),\quad \textit{for all}\ \ t\in\mathrm{I}^o. \tag{A.2}$$

Proof. Using Leibniz rule of differentiation under the integral sign, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-C\int_a^t k(s)\mathrm{d}s}f(t)\Big)=e^{-C\int_a^t k(s)\mathrm{d}s}\frac{\mathrm{d}f(t)}{\mathrm{d}t}-Ck(t)e^{-C\int_a^t k(s)\mathrm{d}s}f(t).$$

Thus by using (A.1), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-C\int_a^t k(s)\mathrm{d}s}f(t)\Big)+e^{-C\int_a^t k(s)\mathrm{d}s}g(t)\\&=e^{-C\int_a^t k(s)\mathrm{d}s}\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}+g(t)\right)-e^{-C\int_a^t k(s)\mathrm{d}s}Ck(t)f(t)\leq 0.\end{aligned}$$

Integrating the above inequality from $a$ to $t$, one can get (A.2). □

## B Bismut–Elworthy–Li formula(BEL formula)

The BEL formula for Gaussian case has been derived in [19, 11]. This formula is also obtained for some general stochastic evolution equations with Lévy noise in finite and infinite dimensions in [38, 12, 33]. For the sake of readers point of view, we give a formal derivation in the case of SNSE with Lévy noise.

Let us consider the following Kolmogorov equation:

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}=\mathscr{L}_x u(t,x),\\ u(0,x)=\phi(x),\end{cases} \tag{B.1}$$

where

$$\begin{aligned}\mathscr{L}_x\phi(x)&=-(\mathrm{A}x+\mathrm{B}(x),\mathrm{D}_x\phi(x))+\frac{1}{2}\mathrm{Tr}\big[\mathrm{Q}\mathrm{D}_x^2\phi(x)\big]\\&\quad+\int_{\mathrm{Z}}[\phi(x+\Psi(z))-\phi(x)-(\mathrm{D}_x\phi(x),\Psi(z))]\lambda(\mathrm{d}z).\end{aligned}$$

For $\mathrm{F}\in\mathrm{C}^{1,2}([0,T]\times\mathrm{D}(\mathscr{L}_x);\mathbb{R})$, Itô's formula yields (see [35, 16])

$$\begin{aligned}&\mathrm{F}(t,\mathrm{X}(t,x))\\&=\mathrm{F}(0,x)+\int_0^t\left[\frac{\partial}{\partial s}\mathrm{F}(s,\mathrm{X}(s,x))+\mathscr{L}_x\mathrm{F}(s,\mathrm{X}(s,x))\right]\mathrm{d}s+\int_0^t\Big(\mathrm{D}_x\mathrm{F}(s,\mathrm{X}(s,x)),\mathrm{Q}^{1/2}\mathrm{d}\mathrm{W}(s)\Big)\\&\quad+\int_0^t\int_{\mathrm{Z}}[\mathrm{F}(s-,\mathrm{X}(s-,x)+\Psi(s-,z))-\mathrm{F}(s,\mathrm{X}(s-,x))]\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{B.2}$$

For $\phi \in \mathrm{C}_b^2(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$, we can prove that $u(t,x) = \mathbb{E}[\phi(\mathrm{X}(t,x)]$ is a solution of (B.1) by using Itô's formula. Let us take $\mathrm{F}(s,x) = u(t-s,x)$ for $s \in [0,t]$, then $\mathrm{F} \in \mathrm{C}^{1,2}([0,T]\times \mathrm{D}(\mathscr{L}_x);\mathbb{R})$. Using (B.2), we have

$$\begin{aligned}\phi(\mathrm{X}(t,x)) &= u(t,x) + \int_0^t \left[-\frac{\partial}{\partial s}u(t-s,\mathrm{X}(s,x)) + \mathscr{L}_x u(t-s,\mathrm{X}(s,x))\right]\mathrm{d}s \\ &\quad + \int_0^t \Big(\mathrm{D}_x[u(t-s,\mathrm{X}(s,x))], \mathrm{Q}^{1/2}\mathrm{dW}(s)\Big) \\ &\quad + \int_0^t\int_{\mathrm{Z}}[u(t-s,\mathrm{X}(s-,x)+\Psi(s-,z)) - u(t-s,\mathrm{X}(s-,x))]\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z) \\ &= u(t,x) + \int_0^t \Big(\mathrm{D}_x[u(t-s,\mathrm{X}(s,x))], \mathrm{Q}^{1/2}\mathrm{dW}(s)\Big) \\ &\quad + \int_0^t\int_{\mathrm{Z}}[u(t-s,\mathrm{X}(s-,x)+\Psi(s-,z)) - u(t-s,\mathrm{X}(s-,x))]\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{B.3}$$

Let us take expectation on both sides of (B.3) and note that the final two terms on the right hand side of (B.3) are martingales, we obtain $u(t,x) = \mathbb{E}[\phi(\mathrm{X}(t,x)]$. Now we multiply both sides of (B.3) by $\mathrm{G}(t) = \int\limits_0^t \big(\mathrm{Q}^{-1/2}\eta^h(s,x),\mathrm{dW}(s)\big)$, where $\eta^h(t,x) := \mathrm{D}_x\mathrm{X}(s,x)h$ is the solution of the equation:

$$\left\{\begin{aligned} &\frac{\partial}{\partial t}\eta^h(t,x) = -\big[\mathrm{A}\eta^h(t,x) + \mathrm{B}(\mathrm{X}(t,x),\eta^h(t,x)) + \mathrm{B}(\eta^h(t,x),\mathrm{X}(t,x))\big], \\ &\eta^h(0,x) = h. \end{aligned}\right. \tag{B.4}$$

Then taking expectation and using the Markov property of semigroup, we get the Bismut–Elworthy–Li formula for the stochastic Navier–Stokes equations perturbed by Lévy noise as follows:

$$(\mathrm{D}_x u(t,x),h) = \frac{1}{t}\mathbb{E}\left[\phi(\mathrm{X}(t,x))\int_0^t\Big(\mathrm{Q}^{-1/2}\eta^h(s,x),\mathrm{dW}(s)\Big)\right]. \tag{B.5}$$

In order to get (B.5), we used Itô's product rule to obtain $\mathbb{E}[\mathrm{G}(t)\mathrm{J}(t)] = 0$, where $\mathrm{J}(t)$ is the final term from the right hand side of the equality (B.3). This follows from the fact that the stochastic integrals $\mathrm{G}(t)$ and $\mathrm{J}(t)$ are uncorrelated, since $\mathrm{W}(\cdot)$ and $\widetilde{\pi}(\cdot,\cdot)$ are independent. Since $\mathrm{C}_b^2(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$ is dense in $\mathrm{C}_b(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$, given any $\phi \in \mathrm{C}_b(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$, there exists a sequence $\phi_n \subset \mathrm{C}_b^2(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$ convergent to $\phi$ in $\mathrm{C}_b(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$. It can be shown that (B.5) is true for any $\phi \in \mathrm{C}_b(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$ in a similar way as in Proposition 9.22, [7].

**Remark 34.** Since the Gaussian noise and jump noise are assumed to be independent, the BEL formula for the SNSE with Lévy noise looks the same as that of the formula for SNSE with Gaussian noise. This BEL formula helps to complete the strong Feller property of the transition semigroup associated with the SNSE with Lévy noise. Therefore, one may have to carefully look at the case where these two noises really have to be dependent.

## C Differentiability of the Feynman–Kac semigroup

Following the ideas developed in [11] for Gaussian case, we derive the differentiability of the Feynman–Kac semigroup associated with SNSE perturbed by Lévy noise.

Let us now consider the auxiliary Kolmogorov equation:

$$\left\{\begin{aligned} &\frac{\partial v(t,x)}{\partial t} = \mathscr{L}_x v(t,x) - K\|\mathrm{A}x\|^2 v(t,x), \\ &v(0,x) = \phi(x), \end{aligned}\right. \tag{C.1}$$

where $K > 0$ is a fixed constant. The equation (C.1) has a unique solution given by the Feynman–Kac formula:

$$v(t,x) := \mathrm{S}_t\phi(x) = \mathbb{E}\Big[e^{-K\int_0^t \|\mathrm{AX}(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}(t,x))\Big]. \tag{C.2}$$

Let us apply Itô's formula to the process

$$\mathrm{F}(s,x)=e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}v(t-s,\mathrm{X}(s,x)),\ s\in[0,t],$$

to obtain

$$\begin{aligned}
&e^{-K\int_0^t\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\phi(\mathrm{X}(t,x))\\
&=v(t,x)+\int_0^t e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\bigg[-\frac{\partial}{\partial s}v(t-s,\mathrm{X}(s,x))-K\|\mathrm{AX}(s,x)\|^2v(t-s,\mathrm{X}(s,x))\\
&\qquad+\mathscr{L}_x v(t-s,\mathrm{X}(s,x))\bigg]\mathrm{d}s\\
&\quad+\int_0^t e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\Big(\mathrm{D}_x[v(t-s,\mathrm{X}(s,x))],\mathrm{Q}^{1/2}\mathrm{dW}(s)\Big)\\
&\quad+\int_0^t\int_{\mathrm{Z}} e^{-K\int_0^{s-}\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}[v(t-s,\mathrm{X}(s-,x)+\Psi(s-,z))-v(t-s,\mathrm{X}(s-,x))]\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z)\\
&=v(t,x)+\int_0^t e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\Big(\mathrm{D}_x[v(t-s,\mathrm{X}(s,x))],\mathrm{Q}^{1/2}\mathrm{dW}(s)\Big)\\
&\quad+\int_0^t\int_{\mathrm{Z}} e^{-K\int_0^{s-}\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}[v(t-s,\mathrm{X}(s-,x)+\Psi(s-,z))-v(t-s,\mathrm{X}(s-,x))]\widetilde{\pi}(\mathrm{d}s,\mathrm{d}z).
\end{aligned}\tag{C.3}$$

Let us take expectation in (C.3) to get (C.2). Now we multiply both sides of (C.3) by $\int_0^t\big(\mathrm{Q}^{-1/2}\eta^h(s,x),\mathrm{dW}(s)\big)$ and then taking expectation to obtain

$$\begin{aligned}
\mathscr{H}:&=\mathbb{E}\bigg[e^{-K\int_0^t\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\phi(\mathrm{X}(t,x))\int_0^t\Big(\mathrm{Q}^{-1/2}\eta^h(s,x),\mathrm{dW}(s)\Big)\bigg]\\
&=\mathbb{E}\bigg[\int_0^t e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\big(\mathrm{D}_x v(t-s,\mathrm{X}(s,x)),\eta^h(s,x)\big)\mathrm{d}s\bigg].
\end{aligned}\tag{C.4}$$

On the other hand, we have

$$\begin{aligned}
&\Big(\mathrm{D}_x\Big[e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}v(t-s,\mathrm{X}(s,x))\Big],h\Big)\\
&=e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\bigg[-2K\bigg(\int_0^s\big(\mathrm{AX}(r,x),\mathrm{A}\eta^h(r,x)\big)\mathrm{d}r\bigg)v(t-s,\mathrm{X}(s,x))\\
&\qquad+\big(\mathrm{D}_x v(t-s,\mathrm{X}(s,x)),\eta^h(s,x)\big)\bigg].
\end{aligned}\tag{C.5}$$

Thus from (C.5), we obtain

$$\begin{aligned}
\mathscr{H}&=\mathbb{E}\bigg[\int_0^t\Big(\mathrm{D}_x\Big[e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}v(t-s,\mathrm{X}(s,x))\Big],h\Big)\mathrm{d}s\bigg]\\
&\quad+2K\mathbb{E}\bigg[\int_0^t e^{-K\int_0^s\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\bigg(\int_0^s\big(\mathrm{AX}(r,x),\mathrm{A}\eta^h(r,x)\big)\mathrm{d}r\bigg)v(t-s,\mathrm{X}(s,x))\mathrm{d}s\bigg].
\end{aligned}\tag{C.6}$$

We use the Markov property to both expressions in the right hand side of (C.6) to find

$$\begin{aligned}
\mathscr{H}&=\bigg(\mathrm{D}_x\int_0^t\mathrm{S}_s(\mathrm{S}_{t-s}\phi)(x)\mathrm{d}s,h\bigg)\\
&\quad+2K\mathbb{E}\bigg[\phi(\mathrm{X}(t,x))e^{-K\int_0^t\|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\int_0^t\int_0^s\big(\mathrm{AX}(r,x),\mathrm{A}\eta^h(r,x)\big)\mathrm{d}r\mathrm{d}s\bigg]
\end{aligned}\tag{C.7}$$

$$= t(\mathrm{D}_x\mathrm{S}_t\phi(x),h) + 2K\mathbb{E}\Big[\phi(\mathrm{X}(t,x))e^{-K\int_0^t \|\mathrm{AX}(r,x)\|^2\mathrm{d}r}\int_0^t (t-r)\big(\mathrm{AX}(r,x),\mathrm{A}\eta^h(r,x)\big)\mathrm{d}r\Big].$$

Combining (C.4) and (C.7), for all $\phi \in \mathrm{C}_b^2(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$, we get the differentiability of the Feynman–Kac semigroup associated to the system (C.1) as

$$(\mathrm{D}_x\mathrm{S}_t\phi(x),h) = \frac{1}{t}\mathbb{E}\Big[e^{-K\int_0^t \|\mathrm{AX}(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}(t,x))\int_0^t \Big(\mathrm{Q}^{-1/2}\eta^h(s,x),\mathrm{dW}(s)\Big)\Big] \\ + 2K\mathbb{E}\Big[e^{-K\int_0^t \|\mathrm{AX}(s,x)\|^2\mathrm{d}s}\phi(\mathrm{X}(t,x))\int_0^t \Big(1-\frac{s}{t}\Big)\big(\mathrm{AX}(s,x),\mathrm{A}\eta^h(s,x)\big)\mathrm{d}s\Big]. \tag{C.8}$$

For $\phi \in \mathrm{C}_b(\mathrm{D}(\mathscr{L}_x);\mathbb{R})$ also, the result (C.8) follows and the details can be obtained from page 187, [11].