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Chapter

$\mathcal{I}$–Convergence of Arithmetical Functions

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Abstract

Let $n > 1$ be an integer with its canonical representation, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Put $H(n) = \max\{\alpha_1, \dots, \alpha_k\}$, $h(n) = \min\{\alpha_1, \dots, \alpha_k\}$, $\omega(n) = k$, $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, $f(n) = \prod_{d|n} d$ and $f^*(n) = \frac{f(n)}{n}$. Many authors deal with the statistical convergence of these arithmetical functions. For instance, the notion of normal order is defined by means of statistical convergence. The statistical convergence is equivalent with $\mathcal{I}_d$–convergence, where $\mathcal{I}_d$ is the ideal of all subsets of positive integers having the asymptotic density zero. In this part, we will study $\mathcal{I}$–convergence of the well-known arithmetical functions, where $\mathcal{I} = \mathcal{I}_c^{(q)} = \left\{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-q} < +\infty\right\}$ is an admissible ideal on $\mathbb{N}$ such that for $q \in (0, 1\rangle$ we have $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_d$, thus $\mathcal{I}_c^{(q)}$–convergence is stronger than the statistical convergence ($\mathcal{I}_d$–convergence).

Keywords: sequences, $\mathcal{I}$–convergence, arithmetical functions, normal order, binomial coefficients

1. Introduction

The notion of statistical convergence was introduced independently by Fast and Schoenberg in [1, 2], and the notion of $\mathcal{I}$–convergence introduced by Kostyrko et al. in the paper [3] coresponds to the natural generalization of statistical convergence (see also [4] where $\mathcal{I}$–convergence is defined by means of filter – the dual notion to ideal). These notions have been developed in several directions in [5–18] and have been used in various parts of mathematics, in particular in number theory and ergodic theory, for example [15, 19–28] also in Economic Theory [29, 30] and Political Science [31]. Many authors deal with average and normal order of the well-known arithmetical functions (see [20, 21, 23, 24, 26, 28, 32, 33] and the monograph [34] for basic properties of the well-known arithmetical functions). In what follows, we shall strengthen these results from the standpoint of $\mathcal{I}$–convergence of sequences, mainly by $\mathcal{I}_c^{(q)}$–convergence and $\mathcal{I}_u$–convergence. On connection with that we can obtain a good information about behaviour and properties of the well-known arithmetical functions by investigating $\mathcal{I}$–convergence of these functions or some sequences connected with these functions. Specifically in [28] by means of $\mathcal{I}_d$–convergence, there is recalled the result that normal order of $\Omega(n)$ or $\omega(n)$ respectively is $\log\log n$. We managed to completely determine for which $q \in (0, 1\rangle$ the sequences $\frac{\Omega(n)}{\log\log n}$ and $\frac{\omega(n)}{\log\log n}$ are $\mathcal{I}_c^{(q)}$–convergent. As consequence of our results, we have that the above sequences are $\mathcal{I}_d$–convergent to 1, what is equivalent that normal order of $\Omega(n)$ or $\omega(n)$ respectively is $\log\log n$. Further in [26], there is proved that the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)$ is $\mathcal{I}_d$–convergent to 0 (see also [21]). We shall extend this result by means of $\mathcal{I}_u$–convergence of the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)$. So we can get a better view of the structure of the set $B(\varepsilon) = \left\{n \in \mathbb{N} : \log p \cdot \frac{a_p(n)}{\log n} < \varepsilon\right\}$, $\varepsilon > 0$. We also want to investigate the $\mathcal{I}_c^{(q)}$–convergence of further arithmetical functions.

2. Basic notions

Let $\mathbb{N}$ be the set of positive integers. Let $A \subseteq \mathbb{N}$. If $m, n \in \mathbb{N}$, $m \leq n$, we denote by $A(m,n)$ the cardinality of the set $A \cap [m,n]$. $A(1,n)$ is abbreviated by $A(n)$. We recall the concept of asymthotic, logarithmic and uniform density of the set $A \subseteq \mathbb{N}$ (see [35–38]).

Definition 1.1. *Let* $A \subseteq \mathbb{N}$.

a. Put $d_n(A) = \frac{A(n)}{n} = \frac{1}{n}\sum_{k=1}^{n}\chi_A(k)$, where χ_A is the characteristic function of the set A. Then the numbers $\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$ and $\overline{d}(A) = \limsup_{n\to\infty} d_n(A)$ are called the *lower* and *upper asymptotic density* of the set A, respectively. If there exists $\lim_{n\to\infty} d_n(A)$, then $d(A) = \underline{d}(A) = \overline{d}(A)$ is said to be the *asymptotic density* of A.

b. Put $\delta_n(A) = \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_A(k)}{k}$, where $s_n = \sum_{k=1}^{n}\frac{1}{k}$. Then the numbers $\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n(A)$ and $\overline{\delta}(A) = \limsup_{n\to\infty} \delta_n(A)$ are called the *lower* and *upper logarithmic density* of A, respectively. Similarly, if there exists $\lim_{n\to\infty} \delta_n(A)$, then $\delta(A) = \underline{\delta}(A) = \overline{\delta}(A)$ is said to be the *logarithmic density* of A. Since $s_n = \log n + \gamma + O\left(\frac{1}{k}\right)$ for $n \to \infty$ and γ is the Euler constant, s_n can be replaced by $\log n$ in the definition of $\delta_n(A)$.

c. Put $\alpha_s = \min_{n \geq 0} A(n+1, n+s)$ and $\alpha^s = \max_{n \geq 0} A(n+1, n+s)$. The following limits $\underline{u}(A) = \lim_{s\to\infty}\frac{\alpha_s}{s}$, $\overline{u}(A) = \lim_{s\to\infty}\frac{\alpha^s}{s}$ exist (see [17, 37, 39, 40]) and they are called *lower* and *upper uniform density* of the set A, respectively. If $\overline{u}(A) = \underline{u}(A)$, then we denote it by $u(A)$ and it is called the *uniform density* of A. It is clear that for each $A \subseteq \mathbb{N}$ we have

$$\underline{u}(A) \leq \underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A) \leq \overline{u}(A). \tag{1}$$

Further densities can be found in papers [11, 12].

Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the canonical representation of the integer $n \in \mathbb{N}$. Recall some arithmetical functions, which belong to our interest.

1. $\omega(n)$ – the number of distinct prime factors of n $(\omega(n) = k)$,

2. $\Omega(n)$ – the number of prime factors of n counted with multiplicities $(\Omega(n) = \alpha_1 + \cdots + \alpha_k)$,

3. $d(n)$ – the number of divisors of n $\left(d(n) = \sum_{d|n}1\right)$,

4. define *h(n) and H(n)*, put $h(1) = 1$, $H(1) = 1$ and for $n > 1$ denote

$$h(n) = \min_{1 \le j \le k} \alpha_j, \quad H(n) = \max_{1 \le j \le k} \alpha_j,$$

5. $f(n) = \prod_{d|n} d$, $f^*(n) = \frac{f(n)}{n}$, where $n = 1, 2, \dots,$

6. $a_p(n)$ is defined as follows: $a_p(1) = 0$ and if $n > 0$, then $a_p(n)$ is a unique integer $j \ge 0$ satisfying $p^j | n$, but $p^{j+1} \nmid n$ i.e., $p^{a_p(n)} \| n$,

7. $\gamma(n)$ and $\tau(n)$ – were introduced in connection with representation of natural numbers of the form $n = a^b$, where a, b are positive integers. Let

$$n = a_1^{b_1} = a_2^{b_2} = \dots = a_{\gamma(n)}^{b_{\gamma(n)}}$$

be all such representations of given natural number n, where $a_i, b_i \in \mathbb{N}$. Denote by

$$\tau(n) = b_1 + b_2 + \dots + b_{\gamma(n)}, \quad (n > 1).$$

It is clear that $\gamma(n) \ge 1$, because for any $n > 1$ there exist representation in the form n^1.

3. Ideals

A lot of mathematical disciplines use the term small (large) set from different point of view. For instance a final set, a set having the measure zero and nowhere dense set is a small set from point of view of cardinality, measure (probability) and topology, respectively. The notion of ideal $\mathcal{I} \subseteq 2^X$ is the unifying principle how to express that a subset of $X \ne \emptyset$ is small. We say a set $A \subseteq X$ is a small set if $A \in \mathcal{I}$. Recall the notion of an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$. $\mathcal{I}$ is said to be an *ideal* in $\mathbb{N}$, if $\mathcal{I}$ is additive (if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$) and hereditary (if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$). An ideal $\mathcal{I}$ is said to be *non-trivial ideal* if $\mathcal{I} \ne \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible ideal* if it contains all finite subsets of $\mathbb{N}$. The dual notion to the ideal is the notion filter. A non-empty family of sets $\mathcal{F} \subset 2^{\mathbb{N}}$ is a *filter* if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$ (for definitions see e.g. [4, 41, 42]). Let $\mathcal{I}$ be a proper ideal in $\mathbb{N}$ (i.e. $\mathbb{N} \notin \mathcal{I}$). Then a family of sets $\mathcal{F}(\mathcal{I}) = \{B \subseteq \mathbb{N} : \textit{there exists } A \in \mathcal{I} \textit{ such that } B = \mathbb{N} \setminus A\}$ is a filter in $\mathbb{N}$, so called the *associated filter* with the ideal $\mathcal{I}$.

The following example shows the most commonly used admissible ideals in different areas of mathematics.

Example 1.2.

a. The class of all finite subsets of $\mathbb{N}$ forms an admissible ideal usually denoted by $\mathcal{I}_f$.

b. Let ϱ be a density function on $\mathbb{N}$, the set $\mathcal{I}_\varrho = \{A \subseteq \mathbb{N} : \varrho(A) = 0\}$ is an admissible ideal. We will use namely the ideals $\mathcal{I}_d$, $\mathcal{I}_\delta$ and $\mathcal{I}_u$ related to asymptotic, logarithmic and uniform density, respectively.

c. A wide class of ideals $\mathcal{I}$ can be obtained by means of regular non negative matrixes $\mathbf{T} = (t_{n,k})_{n,k \in \mathbb{N}}$ (see [43]). For $A \subset \mathbb{N}$, we put $d_{\mathbf{T}}^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k} \chi_A(k)$

for $n \in \mathbb{N}$. If $\lim_{n\to\infty} d_{\mathbf{T}}^{(n)}(A) = d_{\mathbf{T}}(A)$ exists, then $d_{\mathbf{T}}(A)$ is called $\mathbf{T}$–*density* of A (see [3, 44]). Put $\mathcal{I}_{d_\mathbf{T}} = \{A \subset \mathbb{N} : d_{\mathbf{T}}(A) = 0\}$. Then $\mathcal{I}_{d_\mathbf{T}}$ is a non-trivial ideal and $\mathcal{I}_{d_\mathbf{T}}$ contains both $\mathcal{I}_d$ and $\mathcal{I}_\delta$ ideals as a special case. Indeed $\mathcal{I}_d$ can be obtained by choosing $t_{n,k} = \frac{1}{n}$ for $k \le n$, $t_{n,k} = 0$ for $k > n$ and $\mathcal{I}_\delta$ by choosing $t_{n,k} = \frac{\frac{1}{k}}{s_n}$ for $k \le n$, $t_{n,k} = 0$ for $k > n$ where $s_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}$.

For the matrix $\mathbf{T} = (t_{n,k})_{n,k\in\mathbb{N}}$, where $t_{n,k} = \frac{\varphi(k)}{n}$ for $k \le n$, $k|n$ and $t_{n,k} = 0$ otherwise we obtain $\mathcal{I}_\varphi$ ideal of Schoenberg (see [2]), where φ is Euler function.

Another special case of $\mathcal{I}_{d_\mathbf{T}}$ is the following. Take an arbitrary divergent series $\sum_{n=1}^{\infty} c_n$, where $c_n > 0$ for $n \in \mathbb{N}$ and put $t_{n,k} = \frac{c_k}{S_n}$ for $k \le n$, where $S_n = \sum_{i=1}^{n} c_i$, and $t_{n,k} = 0$ for $k > n$.

d. Let μ be a finitely additive normed measure on a field $\mathcal{S} \subseteq 2^{\mathbb{N}}$. Suppose that $\mathcal{S}$ contains all singletons $\{n\}$, $n \in \mathbb{N}$. Then the family $\mathcal{I}_\mu = \{A \subseteq \mathbb{N} : \mu(A) = 0\}$ is an admissible ideal. In the case if μ is the Buck measure density (see [13, 45]), $\mathcal{I}_\mu$ is an admissible ideal and $\mathcal{I}_\mu \subsetneq \mathcal{I}_d$.

e. Suppose that $\mu_n : 2^{\mathbb{N}} \to [0, 1]$ is a finitely additive normed measure for $n \in \mathbb{N}$. If for $A \subseteq \mathbb{N}$ there exists $\mu(A) = \lim_{n\to\infty} \mu_n(A)$, then the set A is said to be *measurable* and $\mu(A)$ is called the *measure* of A. Obviously μ is a finitely additive measure on some field $\mathcal{S} \subseteq 2^{\mathbb{N}}$. The family $\mathcal{I}_\mu = \{A \subseteq \mathbb{N} : \mu(A) = 0\}$ is a non-trivial ideal. For μ_n we can take for instance d_n, δ_n or $d_{\mathbf{T}}^{(n)}$.

f. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$ be a decomposition on $\mathbb{N}$ (i.e. $D_k \cap D_l = \emptyset$ for $k \ne l$). Assume that D_j ($j = 1, 2, \dots$) are infinite sets (e.g. we can choose $D_j = \{2^{j-1} \cdot (2s - 1) : s \in \mathbb{N}\}$ for $j = 1, 2, \dots$). Denote $\mathcal{I}_{\mathbb{N}}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of D_j. Then $\mathcal{I}_{\mathbb{N}}$ is an admissible ideal.

g. For an $q \in (0, 1\rangle$ the set $\mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-q} < +\infty\}$ is an admissible ideal (see [23]). The ideal $\mathcal{I}_c^{(1)} = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < +\infty\}$ is usually denoted by $\mathcal{I}_c$. It is easy to see, that for any $q_1, q_2 \in (0, 1)$, $q_1 < q_2$ we have

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)} \subsetneq \mathcal{I}_c \subsetneq \mathcal{I}_d \subsetneq \mathcal{I}_\delta. \tag{2}$$

The fact $\mathcal{I}_c \subsetneq \mathcal{I}_d$ in Eq. (2) follows from the following result. Let $A \subseteq \mathbb{N}$ and $\sum_{a\in A} \frac{1}{a} < \infty$ then $d(A) = 0$ (see [46]) thus if $A \in \mathcal{I}_c$ then $A \in \mathcal{I}_d$. The opposite is not true, consider the set of primes $\mathbb{P}$, for which we have $d(\mathbb{P}) = 0$ but $\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty$ thus $\mathbb{P} \in \mathcal{I}_d$ but $\mathbb{P} \notin \mathcal{I}_c$ ($\mathcal{I}_c \ne \mathcal{I}_d$).

The fact that for any $q_1, q_2 \in (0, 1\rangle$, $q_1 < q_2$ we have $\mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)}$ in Eq. (2) is clear. For showing that $\mathcal{I}_c^{(q_1)} \ne \mathcal{I}_c^{(q_2)}$ it suffices to find a set $H = \{h_1 < h_2 < \dots < h_k < \dots\} \subset \mathbb{N}$ such that $\sum_{k=1}^{\infty} h_k^{-q_1} = +\infty$ and $\sum_{k=1}^{\infty} h_k^{-q_2} < +\infty$. Put $h_k = \left[k^{\frac{1}{q_1}}\right]$. Since $h_1 < h_2 < \dots < h_k < \dots$ and $h_k^{q_1} \le k$ we have $\sum_{k=1}^{\infty} h_k^{-q_1} \ge \sum_{k=1}^{\infty} k^{-1} = +\infty$. On the other side $h_k > k^{\frac{1}{q_1}} - 1 \ge \frac{1}{2} k^{\frac{1}{q_1}}$ for $k \ge 2$, so we obtain $\sum_{k=1}^{\infty} h_k^{-q_2} \le 2^{q_2} \sum_{k=1}^{\infty} k^{-\frac{q_2}{q_1}} < +\infty$ since $\frac{q_2}{q_1} > 1$.

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4. $\mathcal{I}$– and $\mathcal{I}^*$–convergence

The notion of statistical convergence was introduced in [1, 2] and the notion of $\mathcal{I}$–convergence introduced in [3] corresponds to the natural generalization of the notion of statistical convergence.

Let us recall notions of statistical convergence, $\mathcal{I}$– and $\mathcal{I}^*$–convergence of sequence of real numbers (see [3]).

Definition 1.3. We say that a sequence $(x_n)_{n=1}^{\infty}$ is *statistically convergent* to a number $L \in \mathbb{R}$ and we write $\lim \operatorname{stat} x_n = L$, provided that for each $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$.

Definition 1.4.

i. We say that a sequence $(x_n)_{n=1}^{\infty}$ is *$\mathcal{I}$–convergent* to a number $L \in \mathbb{R}$ and we write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.

ii. Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is said to be *$\mathcal{I}^*$–convergent* to $L \in \mathbb{R}$, if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \backslash H = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ we have $\lim_{k \to \infty} x_{m_k} = L$, where the limit is in the usual sense.

In the definition of usual convergence the set $A(\varepsilon)$ is finite, it means that it is small from point of view of cardinality, $A(\varepsilon) \in \mathcal{I}_f$. Similarly in the definition of statistical convergence the set $A(\varepsilon)$ has asymptotic density zero, it is small from point of view of density, $A(\varepsilon) \in \mathcal{I}_d$. The natural generalization of these notions is the following, let $\mathcal{I}$ be an admissible ideal (e.g. anyone from Example 1.2) then for each $\varepsilon > 0$ we ask whether the set $A(\varepsilon)$ belongs in the ideal $\mathcal{I}$. In this way we obtain the notion of the $\mathcal{I}$–convergence. For the following use, we note that the concept of $\mathcal{I}$–convergence can be extended for such sequences that are not defined for all $n \in \mathbb{N}$, but only for "almost" all $n \in \mathbb{N}$. This means that instead of a sequence $(x_n)_{n=1}^{\infty}$ we have $(x_s)_{s \in S}$, where s runs over all positive integers belonging to $S \subseteq \mathbb{N}$ and $S \in \mathcal{F}(\mathcal{I})$.

Remember that $\mathcal{I}$–convergence in $\mathbb{R}$ has many properties similar to properties of the usual convergence. All notions which are used next we considered in real numbers $\mathbb{R}$. The following theorem can be easily proved.

Theorem 1.5 (Theorem 2.1 from [9]).

i. If $\mathcal{I} - \lim x_n = L$ and $\mathcal{I} - \lim y_n = K$, then $\mathcal{I} - \lim (x_n \pm y_n) = L \pm K$.

ii. If $\mathcal{I} - \lim x_n = L$ and $\mathcal{I} - \lim y_n = K$, then $\mathcal{I} - \lim (x_n \cdot y_n) = L \cdot K$.

The following properties are the most familiar axioms of convergence (see [47]).

(S) Every constant sequence $(x, x, \ldots, x, \ldots)$ converges to x.

(H) The limit of any convergent sequence is uniquely determined.

(F) If a sequence $(x_n)_{n=1}^{\infty}$ has the limit L, then each of its subsequences has the same limit.

(U) If each subsequence of the sequence $(x_n)_{n=1}^{\infty}$ has a subsequence which converges to L, then $(x_n)_{n=1}^{\infty}$ converges to L.

A natural question arises which above axioms are satisfied for the concept of $\mathcal{I}$–convergence.

Theorem 1.6 (see [14] and Proposition 3.1 from [3], where the concept of $\mathcal{I}$–convergence has been investigated in a metric space) *Let* $\mathcal{I} \subset 2^{\mathbb{N}}$ *be an admissible ideal.*

i. $\mathcal{I}$–convergence satisfies (S), (H) and (U).

ii. If $\mathcal{I}$ contains an infinite set, then $\mathcal{I}$–convergence does not satisfy (F).

Theorem 1.7 (see [3]) *Let* $\mathcal{I}$ *be an admissible ideal in* $\mathbb{N}$. *If* $\mathcal{I}^* - \lim x_n = L$ *then* $\mathcal{I} - \lim x_n = L$.

The following example shows that the converse of Theorem 1.7 is not true.

Example 1.8. Let $\mathcal{I} = \mathcal{I}_{\mathbb{N}}$ be an ideal from Example 1.2 f). Define $(x_n)_{n=1}^{\infty}$ as follows: For $n \in D_j$ we put $x_n = \frac{1}{j}$ for $j = 1, 2, \ldots$. Then obviously $\mathcal{I} - \lim x_n = 0$. But we show that $\mathcal{I}^* - \lim x_n = 0$ does not hold.

If $H \in \mathcal{I}$ then directly from the definition of $\mathcal{I}$ there exists $p \in \mathbb{N}$ such that $H \subseteq D_1 \cup D_2 \cup \cdots \cup D_p$. But then $D_{p+1} \subseteq \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ and so we have $x_{m_k} = \frac{1}{p+1}$ for infinitely many indices $k \in \mathbb{N}$. Therefore $\lim_{k \to \infty} x_{m_k} = 0$ cannot be true.

In [3] was formulated a necessary and sufficient condition for an admissible ideal $\mathcal{I}$ under which $\mathcal{I}$– and $\mathcal{I}^*$–convergence to be equivalent. Recall this condition (AP) that is similar to the condition (APO) in [7, 35].

Definition 1.9 (see also [40]) An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that symmetric difference $A_j \Delta B_j$ is finite for $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Remark. Observe that each B_j from the previous Definition belong to $\mathcal{I}$.

Theorem 1.10 (see [14]) *From* $\mathcal{I} - \lim x_n = L$ *the statement* $\mathcal{I}^* - \lim x_n = L$ *follows if and only if* $\mathcal{I}$ *satisfies the condition (AP).*

In [44] it is proved that $\mathcal{I}_{d_{\mathbf{T}}}$– and $\mathcal{I}^*_{d_{\mathbf{T}}}$–convergence are equivalent in $\mathbb{R}$ provided that $\mathbf{T} = (t_{n,k})_{n,k \in \mathbb{N}}$ from Example 1.2 c) is a non-negative triangular matrix with $\sum_{k=1}^{n} t_{n,k} = 1$ for $n \in \mathbb{N}$. From this we get that $\mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_\varphi$–convergence coinside with $\mathcal{I}^*_d, \mathcal{I}^*_\delta, \mathcal{I}^*_\varphi$–convergence, respectively. On the other hand for further ideals from Example 1.2 e.g. $\mathcal{I}_u, \mathcal{I}_{\mathbb{N}}$ and $\mathcal{I}_\mu$, respectively, we have that they do not fulfill the assertion that their $\mathcal{I}$–convergence coincides with $\mathcal{I}^*$–convergence. Since these ideals do not fulfill condition (AP) (see [13, 38, 40]).

The following Theorem shows that also for all ideals $\mathcal{I}_c^{(q)}$ for $q \in (0, 1\rangle$ the concepts $\mathcal{I}$– and $\mathcal{I}^*$–convergence coincide.

Theorem 1.11 (see, [20, 23]) *For any* $q \in (0, 1\rangle$ *the notions* $\mathcal{I}_c^{(q)}$– *and* $\mathcal{I}_c^{(q)*}$–*convergence are equivalent.*

Proof. It suffices to prove that for any $\mathcal{I}_c^{(q)}$, $q \in (0, 1\rangle$ and any sequence $(x_n)_{n=1}^{\infty}$ of real numbers such that $\mathcal{I}_c^{(q)} - \lim x_n = L$ for $q \in (0, 1\rangle$ there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus M \in \mathcal{I}_c^{(q)}$ and $\lim_{k \to \infty} x_{m_k} = L$.

For any positive integer k let $\varepsilon_k = \frac{1}{2^k}$ and $A_k = \left\{ n \in \mathbb{N} : |x_n - L| \geq \frac{1}{2^k} \right\}$.

As $\mathcal{I}_c^{(q)} - \lim x_n = L$, we have $A_k \in \mathcal{I}_c^{(q)}$, i.e.

$$\sum_{a \in A_k} a^{-q} < \infty.$$

Therefore there exists an infinite sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of integers such that for every $k = 1, 2, \ldots$

$$\sum_{\substack{a > n_k \\ a \in A_k}} a^{-q} < \frac{1}{2^k}.$$

Let $H = \cup_{k=1}^{\infty}[(n_k, n_{k+1}\rangle \cap A_k]$. Then

$$\sum_{a \in H} a^{-q} \le \sum_{\substack{a > n_1 \\ a \in A_1}} a^{-q} + \sum_{\substack{a > n_2 \\ a \in A_2}} a^{-q} + \cdots + \sum_{\substack{a > n_k \\ a \in A_k}} a^{-q} + \cdots$$
$$< \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \cdots < +\infty.$$

Thus $H \in \mathcal{I}_c^{(q)}$. Put $M = \mathbb{N} \backslash H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$. Now it suffices to prove that $\lim_{k\to\infty} x_{m_k} = L$. Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $m_k > n_{k_0}$. Then m_k belongs to some interval $\left(n_j, n_{j+1}\right\rangle$ where $j \ge k_0$ and does not belong to A_j ($j \ge k_0$). Hence m_k belongs to $\mathbb{N} \backslash A_j$, and then $|x_{m_k} - L| < \varepsilon$ for every $m_k > n_{k_0}$, thus $\lim_{k\to\infty} x_{m_k} = L$.

Corollary 1.12 *Ideals* $\mathcal{I}_c^{(q)}$ *for* $q \in (0, 1\rangle$ *have the property (AP).*

It is easy to prove the following lemma.

Lemma 1.13 (see [3]). *If* $\mathcal{I}_1 \subseteq \mathcal{I}_2$ *then the statement* $\mathcal{I}_1 - \lim x_n = L$ *implies* $\mathcal{I}_2 - \lim x_n = L$.

5. $\mathcal{I}$–convergence of arithmetical functions

We can obtain a good information about behaviour and properties of the well-known arithmetical functions by investigating $\mathcal{I}$–convergence of these functions or some sequences connected with these functions. Recall the concept of normal order.

Definition 1.14. The sequence $(x_n)_{n=1}^{\infty}$ has *the normal order* $\left(y_n\right)_{n=1}^{\infty}$ if for every $\varepsilon > 0$ and almost all (almost all in the sense of asymptotic density) values n we have $(1-\varepsilon)y_n < x_n < (1+\varepsilon)y_n$.

Schinzel and Šalát in [28] pointed out that one of equivalent definitions to have the normal order is as follows. The sequence $(x_n)_{n=1}^{\infty}$ has the normal order $\left(y_n\right)_{n=1}^{\infty}$ if and only if $\mathcal{I}_d - \lim \frac{x_n}{y_n} = 1$. The results concerning the normal order will be formulated using the concept of statistical convergence, which coincides with $\mathcal{I}_d$–convergence. For equivalent definitions of the normal order and more examples concerning this notion see [34, 38, 48].

In the papers [21, 27, 28] and in the monograph [38] there are studied various kinds of convergence of arithmetical functions which were mentioned at the beginning. The following equalities were proved in the paper [28] by using the concept of the normal order.

$$\mathcal{I}_d - \lim \frac{\omega(n)}{\log\, \log n} = \mathcal{I}_d - \lim \frac{\Omega(n)}{\log\, \log n} = 1$$

and

$$\mathcal{I}_d - \lim \frac{h(n)}{\log n} = \mathcal{I}_d - \lim \frac{H(n)}{\log n} = 0.$$

Similarly for the functions $f(n)$ and $f^*(n)$. In [27] it is proved the following equality:

$$\mathcal{I}_d - \lim \frac{\log \log f(n)}{\log \log n} = \mathcal{I}_d - \lim \frac{\log \log f^*(n)}{\log \log n} = 1 + \log 2.$$

Let us recall one more result from [26], there was proved that the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is $\mathcal{I}_d$–convergent to 0. Moreover the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is $\mathcal{I}_c^{(q)}$–convergent to 0 for $q = 1$ and it is not $\mathcal{I}_c^{(q)}$–convergent for all $q \in (0, 1)$, as it was shown in [21]. In [19] it was proved that this sequence is also $\mathcal{I}_u$–convergent.

The following theorem shows that the assertions using the notion $\mathcal{I}_u$ instead of $\mathcal{I}_c^{(q)}$, $q \in (0, 1\rangle$ need to use a different technique for their proofs. First of all we recall a new kind of convergence so called the uniformly strong p–Cesàro convergence. This convergence is an analog of the notion of strong almost convergence (see [6]).

Definition 1.15. A sequence $(x_n)_{n=1}^{\infty}$ is said to be uniformly strong p–Cesàro convergent $(0 < p < \infty)$ to a number L if $\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} |x_i - L|^p = 0$ uniformly in k.

The following Theorem shows a connection between uniformly strong p–Cesàro convergence and $\mathcal{I}_u$–convergence.

Theorem 1.16 (see [6]). *If $(x_n)_{n=1}^{\infty}$ is a bounded sequence, then $(x_n)_{n=1}^{\infty}$ is $\mathcal{I}_u$–convergent to L if and only if $(x_n)_{n=1}^{\infty}$ is uniformly strong p–Cesàro convergent to L for some p, $0 < p < \infty$.*

The sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is $\mathcal{I}_u$–convergent to zero i.e. for arbitrary $\varepsilon > 0$ the set $A(\varepsilon) = \left\{n \in \mathbb{N} : \log p \cdot \frac{a_p(n)}{\log n} \ge \varepsilon > 0\right\}$ has uniform density equal to zero.

Theorem 1.17 (see [19]). *We have $\mathcal{I}_u - \lim \log p \cdot \frac{a_p(n)}{\log n} = 0$.*

Proof. The sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is bounded. Using Theorem 1.16, it is sufficient to show that the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is uniformly strong p–Cesàro convergent to 0 for $p = 1$. For the reason that all members of $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ are positive, we shall prove that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=k+1}^{k+N} \frac{a_p(n)}{\log n} = 0$, uniformly in k. $a_p(n) = \alpha$ if $p^{\alpha} \| n$. Let $\alpha_0 = \left[\frac{\log N}{\log p}\right]$. This immediately implies that $p^{\alpha_0} \le N < p^{\alpha_0+1}$. Then for all $n \in (k, k+N]$ we have $a_p(n) = \alpha < \alpha_0$ with the possible exception of one $n_1 \in (k, k+N]$ for which we could have $a_p(n_1) = \alpha_1 > \alpha_0$. Assume that there exist two such numbers $n_1, n_2 \in (k, k+N]$ for which $a_p(n_1) = \alpha_1 > \alpha_0$ and $a_p(n_2) = \alpha_2 > \alpha_0$, then $n_1 = m_1 p^{\alpha_1}$, $n_2 = m_2 p^{\alpha_2}$ hence $p^{\alpha_0+1} | n_1 - n_2$. We have $p^{\alpha_0+1} < |n_1 - n_2| \le N$, what is a contradiction with $p^{\alpha_0+1} > N$. When we omit such an n_1 from the sum, the error is less than $\frac{1}{N} \frac{a_p(n_1)}{\log n_1} \le \frac{1}{N} \frac{\alpha_1}{\alpha_1} \log p$. Using the Hölder's inequality we get

$$\frac{1}{N} \sum_{n=k+1}^{k+N} \frac{a_p(n)}{\log n} \le \sqrt{\frac{1}{N} \sum_{n=k+1}^{k+N} (a_p(n))^2} \sqrt{\frac{1}{N} \sum_{n=k+1}^{k+N} \frac{1}{(\log n)^2}}. \tag{3}$$

We are going to estimate the first factor of Eq. (3)

$$\frac{1}{N}\sum_{n=k+1}^{k+N}\left(a_p(n)\right)^2 \cong \frac{1}{N}\sum_{\alpha=0}^{\alpha_0}\alpha^2 \sum_{\substack{n=k+1\\ a_p(n)=\alpha}}^{k+N} 1 =$$

$$\frac{1}{N}\sum_{\alpha=0}^{\alpha_0}\alpha^2\left(\left[\frac{k+N}{p^\alpha}\right]-\left[\frac{k}{p^\alpha}\right]-\left(\left[\frac{k+N}{p^{\alpha+1}}\right]-\left[\frac{k}{p^{\alpha+1}}\right]\right)\right) =$$

$$\frac{1}{N}\sum_{\alpha=0}^{\alpha_0}\alpha^2\left(\frac{N}{p^\alpha}-\frac{N}{p^{\alpha+1}}+O(1)\right) = \frac{1}{N}N\left(1-\frac{1}{p}\right)\sum_{\alpha=0}^{\alpha_0}\alpha^2\frac{1}{p^\alpha}+\frac{O(1)}{N}\sum_{\alpha=0}^{\alpha_0}\alpha^2.$$

Formula $\sum_{\alpha=0}^{\alpha_0}\alpha^2 = P(\alpha_0)$, where $P(x) = \frac{x(x+1)(2x+1)}{6}$ and simple estimations give $\sum_{\alpha=0}^{\alpha_0}\alpha^2\frac{1}{p^\alpha} \le \sum_{\alpha=0}^{\infty}\frac{\alpha^2}{p^\alpha} < \infty$.

So we get

$$\frac{1}{N}\sum_{n=k+1}^{k+N}\left(a_p(n)\right)^2 = O(1). \tag{4}$$

Estimate the second factor Eq. (3)

$$\frac{1}{N}\sum_{n=k+1}^{k+N}\frac{1}{(\log n)^2} \le \frac{1}{N}\sum_{n=2}^{N+1}\frac{1}{(\log n)^2} \to 0, \quad \text{since} \quad \frac{1}{(\log n)^2} \to 0. \tag{5}$$

Let $N \to \infty$, from Eqs. (4) and (5) we obtain

$$\frac{1}{N}\sum_{n=k+1}^{k+N}\frac{a_p(n)}{\log n} \le c\sqrt{\frac{1}{N}\sum_{n=2}^{N+1}\frac{1}{(\log n)^2}} \to 0,$$

uniformly in k.

Remark. It is known that $\mathcal{I}_u \subsetneq \mathcal{I}_d$ (see e.g. [5, 6]) but the ideals $\mathcal{I}_c$ and $\mathcal{I}_u$ are not disjoint, and moreover $\mathcal{I}_u \nsubseteq \mathcal{I}_c$ and $\mathcal{I}_c \nsubseteq \mathcal{I}_u$. For example the set of all prime numbers belongs to $\mathcal{I}_u$ but not belongs to $\mathcal{I}_c$. On the other hand there exists the set $B = \cup_{k=1}^{\infty}B_k$, where $B_k = \{k^3+1, k^3+2, \dots, k^3+k\}$ which not belongs to $\mathcal{I}_u$ but it belongs to $\mathcal{I}_c$.

Under the fact that $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_d$ for all $q \in \langle 0, 1\rangle$ and Lemma 1.13 it is useful to investigate $\mathcal{I}_c^{(q)}$–convergence of special sequences described in the introduction. Under the Lemma 1.13 it is clear that if there exists the $\mathcal{I}_c^{(q)}$–limit of some sequence for any $q \in \langle 0, 1\rangle$, then it is equal to the $\mathcal{I}_d$–limit of the same sequence. There are no other options.

Consider the sequences $\left(\frac{h(n)}{\log n}\right)_{n=2}^{\infty}$ and $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$. In [45] it was proved that these sequences are dense on $\left(0, \frac{1}{\log 2}\right)$ and moreover they both are statistically convergent to zero. The same result we have for $\mathcal{I}_c^{(q)}$–convergence, but only for the sequence $\left(\frac{h(n)}{\log n}\right)_{n=2}^{\infty}$ for all $q \in \langle 0, 1\rangle$.

Theorem 1.18 (see [20]). *We have*

$$\mathcal{I}_c^{(q)} - \lim \frac{h(n)}{\log n} = 0, \quad \textit{for all} \quad q \in \langle 0, 1\rangle.$$

Proof. Let $k \in \mathbb{N}$ and $k \geq 2$. It is easy to see that the following equality holds

$$1 + \sum_{\substack{n \in \mathbb{N} \\ h(n) \geq k}} n^{-q} = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^{kq}} + \frac{1}{p^{(k+1)q}} + \cdots \right), \tag{6}$$

where $\mathbb{P}$ denotes the set of all primes.

The right-hand side of the equality Eq. (6) equals

$$\prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^{kq}} \cdot \frac{1}{1 - \frac{1}{p^q}}\right) = \prod_{p \in \mathbb{P}} \left(1 + \frac{1}{p^{(k-1)q} \cdot (p^q - 1)}\right).$$

Then for $q > \frac{1}{k}$, the product on the right-hand side of the previous equality converges. Thus, the series on the left-hand side of Eq. (6) converges.

Let $\varepsilon > 0$. Put $A(\varepsilon) = \left\{n \in \mathbb{N} : \frac{h(n)}{\log n} \geq \varepsilon > 0\right\}$. There exists an $n_0^{(k)} \in \mathbb{N}$ for all $k \geq 2$ such that for all $n > n_0^{(k)}$ and $n \in A(\varepsilon)$ we have $h(n) \geq \varepsilon \cdot \log n > k$ (it is sufficient to put $n_0^{(k)} = \left[e^{\frac{k}{\varepsilon}}\right]$).

From this $A(\varepsilon) \cap \left\{n_0^{(k)} + 1, n_0^{(k)} + 2, \dots \right\} \subseteq \{n \in \mathbb{N} : h(n) \geq k\}$ for all $k \geq 2$, $k \in \mathbb{N}$.

Therefore $\sum_{n \in A(\varepsilon)} n^{-q} < +\infty$ for all $k \geq 2$ and $\mathcal{I}_c^{(q)} - \lim \frac{h(n)}{\log n} = 0$ since the series Eq. (6) converges for all $q > \frac{1}{k}$. If $k \to \infty$ for sufficient large then $\mathcal{I}_c^{(q)} - \lim \frac{h(n)}{\log n} = 0$ for all $q \in (0, 1\rangle$.

Corollary 1.19. We have

$$\mathcal{I}_c^{(q)*} - \lim \frac{h(n)}{\log n} = 0 \ \ for \ \ all \ \ q \in (0, 1\rangle.$$

For the sequence $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ we get the result of different character.

Theorem 1.20 (see [20]). *The sequence* $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*–convergent for every* $q \in (0, 1)$.

Proof. In the paper [21] is proved, that the sequence $\left(\log p \cdot \frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is not $\mathcal{I}_c^{(q)}$–convergent for any $q \in (0, 1)$. The sequence $\left(\frac{a_p(n)}{\log n}\right)_{n=2}^{\infty}$ is also not $\mathcal{I}_c^{(q)}$–convergent to zero. The inequality $H(n) \geq a_p(n)$ holds for all $n = 1, 2, \dots$ and for any prime number p. Then we have $\frac{H(n)}{\log n} \geq \frac{a_p(n)}{\log n}$ for all $n = 2, 3, \dots$. This implies that the sequence $\left(\frac{H(n)}{\log n}\right)_{n=2}^{\infty}$ is also not $\mathcal{I}_c^{(q)}$–convergent to zero for every $q \in (0, 1)$.

Theorem 1.21 (see [20]). *For* $q = 1$, *we obtain*

$$\mathcal{I}_c - \lim \frac{H(n)}{\log n} = 0.$$

Proof. We will show that

$$A(\varepsilon) = \left\{n \in \mathbb{N} : \frac{H(n)}{\log n} \geq \varepsilon\right\} \in \mathcal{I}_c$$

for any $\varepsilon > 0$.

Every non-negative integer n can be represented as $n = ab^2$, where a is a square-free number. Hence $H(a) = 1$ and

$$H(n) \in \{H(b^2), H(b^2) + 1\}.$$

If $n \in A(\varepsilon)$ then from $H(n) \geq \varepsilon \cdot \log n$ we have

$$\log(ab^2) \leq \frac{H(b^2)+1}{\varepsilon} \text{ and so } \log a \leq \frac{H(b^2)+1}{\varepsilon}.$$

Therefore

$$A(\varepsilon) \subseteq B = \left\{ n \in \mathbb{N} : n = ab^2, \ \ \log a \leq \frac{H(b^2)+1}{\varepsilon}, \ \ b \in \mathbb{N} \right\}.$$

It is enough to prove that $\sum_{n \in B} n^{-1} < +\infty$. We have

$$\sum_{n \in B} \frac{1}{n} = \sum_{b=1}^{\infty} \frac{1}{b^2} \sum_{\log a \leq \frac{H(b^2)+1}{\varepsilon}} \frac{1}{a}.$$

We use the inequality $S_k = \sum_{j=1}^{k} \frac{1}{j} \leq 1 + \log k$ for the harmonic series. Then we have the following inequality

$$\sum_{n \in B} \frac{1}{n} \leq \sum_{b=1}^{\infty} \frac{1}{b^2} \left(\frac{H(b^2)+1}{\varepsilon} + 1 \right). \tag{7}$$

Because the $\sum \frac{1}{b^2} = \frac{\pi^2}{6} < +\infty$, it is enough to prove that the

$$\sum_{b=1}^{\infty} \frac{H(b^2)}{b^2} < +\infty. \tag{8}$$

For any $n \in \mathbb{N}$ we have $n = p_1^{a_1} \cdots p_k^{a_k} \geq 2^{H(n)}$ and from this $H(n) \leq \frac{\log n}{\log 2}$. Therefore

$$\sum_{b=1}^{\infty} \frac{H(b^2)}{b^2} \leq \frac{2}{\log 2} \sum_{b=1}^{\infty} \frac{\log b}{b^2} < +\infty.$$

We have shown that the sum in Eq. (8) is finite and therefore the sum in Eq. (7) is also finite.

Moreover $B \in \mathcal{I}_c$ and because $A(\varepsilon) \subseteq B$ we have $A(\varepsilon) \in \mathcal{I}_c$.

The situation for sequences $\left(\frac{\omega(n)}{\log \log n}\right)_{n=2}^{\infty}$, $\left(\frac{\Omega(n)}{\log \log n}\right)_{n=2}^{\infty}$ is following.

Theorem 1.22 (see [20]). *The sequences* $\left(\frac{\omega(n)}{\log \log n}\right)_{n=2}^{\infty}$ *and* $\left(\frac{\Omega(n)}{\log \log n}\right)_{n=2}^{\infty}$ *are not* $\mathcal{I}_c^{(q)}$*–convergent for all* $q \in (0, 1\rangle$.

Proof. We prove this assertion only for $\left(\frac{\omega(n)}{\log \log n}\right)_{n=2}^{\infty}$. The proof for the sequence $\left(\frac{\Omega(n)}{\log \log n}\right)_{n=2}^{\infty}$ is analogous. Let $q = 1$. On the basis of the Theorem 2.2 of [28] and Lemma 1.13 we can assume that $\mathcal{I}_c - \lim \frac{\omega(n)}{\log \log n} = 1$. Take $\varepsilon \in \left(0, \frac{1}{2}\right)$ and consider the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \left| \frac{\omega(n)}{\log \, \log n} - 1 \right| \geq \varepsilon \right\}.$$

Put $n = p$, where p is a prime number, then $\omega(p) = 1$ and $\left|\frac{1}{\log \, \log p} - 1\right| \geq \varepsilon$ holds for all prime numbers $p > p_0$. Therefore the set A_ε contains all prime numbers greater than p_0. For these p we have: $\sum_{p > p_0} \frac{1}{p} = +\infty$ and so $A(\varepsilon) \notin \mathcal{I}_c$. From this $\mathcal{I}_c - \lim \frac{\omega(n)}{\log \, \log n} \neq 1$. Under the inclusion $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_c^{(1)} \equiv \mathcal{I}_c$ and according to Lemma 1.13 we have $\mathcal{I}_c^{(q)} - \lim \frac{\omega(n)}{\log \, \log n} \neq 1$ for $q \in (0, 1\rangle$. This complete the proof.

Further possibility where the results can be strengthened by the way that the statistical convergence in them is replaced by $\mathcal{I}_c^{(q)}$–convergence is the concept of the famous *Pascal's triangle*. The n-th row of the Pascal's triangle consists of the numbers $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1}, \binom{n}{n}$. Their sum equals to $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$. Let $\Gamma(t)$ denote the number of times the positive integer t, $t > 2$ occurs in the Pascal's triangle. That is, $\Gamma(t)$ is the number of binomial coefficient $\binom{n}{k}$ satisfying $\binom{n}{k} = t$. From this point of view Γ is the function which maps the set $\mathbb{N}$ in the set $\mathbb{N} \cup \{\aleph_0\}$ $(\Gamma(1) = \aleph_0)$. Let us observe that for every $t \in \mathbb{N}$, $\Gamma(t) \geq 1$.

In [32] it is proved that the average and normal order of the function Γ is 2. Since the normal order is 2, we have

$$\mathcal{I}_d - \lim \Gamma(t) = 2$$

(see [28]). We are going to show two results which strengthen the result of [32] and their proofs are outlined in [24].

Theorem 1.23 (see [24]). $\mathcal{I}_c - \lim \Gamma(t) = 2$.

Proof. The values of the function Γ are positive integers for $t \neq 1$. Thus for $\varepsilon > 0$ the set $A_\varepsilon = \{t \in \mathbb{N} : |\Gamma(t) - 2| \geq \varepsilon\}$ is a subset of the set $H = \{1\} \cup \{2\} \cup M$, where $M = \{t \in \mathbb{N} : \Gamma(t) > 2\}$. Note that $\Gamma(2) = 1$. Therefore is suffices to show that $\sum_{n \in H} \frac{1}{n} < +\infty$. Evidently this is equivalent with

$$\sum_{n \in M} \frac{1}{n} < +\infty. \tag{9}$$

We shall prove Eq. (9). Firstly, we write the left-hand site of Eq. (9) in the form

$$\sum_{n \in M} \frac{1}{n} = \frac{M(1)}{1} + \frac{M(2) - M(1)}{2} + \dots + \frac{M(k) - M(k-1)}{k} + \cdots$$

$$= \frac{M(1)}{1 \cdot 2} + \frac{M(2)}{2 \cdot 3} + \dots + \frac{M(k)}{k \cdot (k+1)} + \cdots.$$

In [32] it is shown that $M(x) = O(\sqrt{x})$. Therefore there exists such $c_1 > 0$ that for every $k \in \mathbb{N}$, $M(k) \leq c_1 \sqrt{k}$ holds. But then

$$\frac{M(k)}{k \cdot (k+1)} \leq \frac{c_1}{k^{\frac{3}{2}}} \quad (k = 1, 2, \dots).$$

According these inequalities by comparison test of the convergence of the series in Eq. (9) follows.

Now we shall use the concept of $\mathcal{I}_c^{(q)}$–convergence.

Theorem 1.24 (see [24]). *For every* $q > \frac{1}{2}$, $\mathcal{I}_c^{(q)} - \lim \Gamma(t) = 2$ *holds and* $\mathcal{I}_c^{(q)} - \lim \Gamma(t) = 2$ *does not hold for any* q, $0 < q \le \frac{1}{2}$.

Proof. Let $0 < q < 1$ and let M have the same meaning as in the proof of Theorem 1.23. Let us examine the series $\sum_{n \in M} \frac{1}{n^q}$. We write it in the form

$$\begin{aligned}\sum_{n \in M} \frac{1}{n^q} &= \frac{M(1)}{1^q} + \frac{M(2) - M(1)}{2^q} + \dots + \frac{M(k) - M(k-1)}{k^q} + \dots \\ &= M(1)\left(\frac{1}{1^q} - \frac{1}{2^q}\right) + \dots + M(k)\left(\frac{1}{k^q} - \frac{1}{(k+1)^q}\right) + \dots \\ &= \sum_{k=1}^{\infty} M(k)\left(\frac{1}{k^q} - \frac{1}{(k+1)^q}\right).\end{aligned} \tag{10}$$

In virtue of Lagrange's mean value theorem we have

$$(k+1)^q - k^q = q z_k^{q-1}, \quad k < z_k < k+1 \quad (k = 1, 2, \dots).$$

Therefore the series Eq. (10) can be written in the form

$$\sum_{k \in M} \frac{1}{k^q} = \sum_{k=1}^{\infty} \frac{M(k) q z_k^{q-1}}{k^q (k+1)^q} = \sum_{k=1}^{\infty} \frac{q M(k)}{k^q (k+1)^q z_k^{1-q}}. \tag{11}$$

But $z_k^{1-q} > k^{1-q}$, $(k+1)^q > k^q$ and so

$$\sum_{n \in M} \frac{1}{n^q} \le q \sum_{k=1}^{\infty} \frac{M(k)}{k^{1+q}}.$$

We have already seen, that $M(k) \le c_1 \sqrt{k}$, $(k = 1, 2, \dots)$ (in the proof of Theorem 1.23). Consider that every binomial coefficient $t = \binom{n}{2}$, $n \ge 4$ occurs in Pascal's triangle at least four times $\left(\text{as } \binom{n}{2}, \binom{n}{n-2}, \binom{t}{1}, \binom{t}{t-1}\right)$. Therefore every number of this form belongs to M. Consequently for $x > 4$, $x \in \mathbb{N}$ the number $M(x)$ is greater then or equal to the number $V(x)$ of all numbers of the form $\binom{n}{2}$, $n \ge 4$ not exceeding x. But $V(x) \ge s - 3$, where s is the integer satisfying

$$\binom{s}{2} \le x < \binom{s+1}{2}.$$

From this we get

$$x < \frac{s(s+1)}{2}, \quad s > \sqrt{2x} - 1.$$

So we obtain

$$M(x) \ge V(x) \ge s - 3 > \sqrt{2x} - 4, \qquad \frac{M(x)}{\sqrt{x}} > \sqrt{2} - \frac{4}{x} \ge \underbrace{\sqrt{2} - 1}_{c_2} > 0.$$

Now it is clear that if $x \geq \binom{4}{2}$ then

$$c_2\sqrt{x} \leq M(x) \leq c_1\sqrt{x}, \quad c_2 = \sqrt{2} - 1 > 0. \tag{12}$$

Therefore by Eq. (11) we get

$$\sum_{n \in M} \frac{1}{n^q} = q \sum_{k=1}^{\infty} \frac{M(k)}{k^q (k+1)^q z_k^{1-q}}.$$

But $z_k < k+1$, hence $z_k^{1-q} < (k+1)^{1-q}$ and so

$$\sum_{n \in M} \frac{1}{n^q} \geq q \sum_{k=1}^{\infty} \frac{M(k)}{(k+1)^{1+q}}.$$

From this owing to Eq. (12) we obtain

$$\sum_{n \in M} \frac{1}{n^q} \geq q \sum_{k=1}^{\infty} \frac{c_2\sqrt{k}}{(k+1)^{1+q}} \geq q \frac{c_2}{2} \sum_{k=1}^{\infty} \frac{1}{(k+1)^{\frac{1}{2}+q}} = +\infty \quad \text{if} \quad 0 < q \leq \frac{1}{2}.$$

Thus $\sum_{n \in M} \frac{1}{n^q} = +\infty$, and so $\sum_{n \in A_\varepsilon} \frac{1}{n^q} = +\infty$ for every $\varepsilon > 0$.

Similar results we can prove for functions $f(n)$ and $f^*(n)$.

Theorem 1.25 (see [20, 27]). *The sequence* $\left(\frac{\log \log f(n)}{\log \log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*–convergent for all* $q \in (0, 1\rangle$.

Proof. According to Theorem 2.1 of [27] suppose that the

$$\mathcal{I}_c^{(q)} - \lim \frac{\log \log f(n)}{\log \log n} = 1 + \log 2,$$

where $q \in (0, 1\rangle$. Let $\varepsilon \in (0, \log 2)$ and define the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \left| \frac{\log \log f(n)}{\log \log n} - (1 + \log 2) \right| \geq \varepsilon \right\}.$$

Put $n = p$, where p is a prime number, then $f(p) = p$ and $\frac{\log \log p}{\log \log p} = 1$. Therefore the set $A(\varepsilon)$ contains all prime numbers. Next we have:

$$\sum_{n \in A(\varepsilon)} n^{-q} \geq \sum_{j=1}^{\infty} p_j^{-q} \geq \sum_{j=1}^{\infty} p_j^{-1} = +\infty, \quad q \in (0, 1\rangle.$$

Hence $A(\varepsilon) \notin \mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)} - \lim \frac{\log \log f(n)}{\log \log n} \neq 1 + \log 2$ for all $q \in (0, 1\rangle$.

Theorem 1.26 (see [20.2]). *The sequence* $\left(\frac{\log \log f^*(n)}{\log \log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*–convergent for all* $q \in (0, 1\rangle$.

Proof. According to Theorem 2.2 of [27] again suppose that the

$$\mathcal{I}_c^{(q)} - \lim \frac{\log \log f^*(n)}{\log \log n} = 1 + \log 2,$$

where $q \in \langle 0,1\rangle$. The proof is going similar as in the previous Theorem. Put $n = p_i p_j$, $i \neq j$, where p_i, p_j are distinct prime numbers. Then $f^*(n) = f^*\left(p_i p_j\right) = \frac{f\left(p_i p_j\right)}{p_i p_j} = \frac{p_i p_j \left(p_i p_j\right)}{p_i p_j} = p_i p_j$, $i \neq j$. Hence $\frac{\log\log f^*\left(p_i p_j\right)}{\log\log p_i p_j} = 1$. Let $\varepsilon \in (0, \log 2)$ and define the set

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \left| \frac{\log\log f^*(n)}{\log\log n} - (1 + \log 2) \right| \geq \varepsilon \right\}.$$

This set contains all numbers of the type $p_i p_j$, $i \neq j$. For $q \in \langle 0,1\rangle$ we have:

$$\sum_{n \in A(\varepsilon)} n^{-q} \geq \sum_{\substack{j=1 \\ p_j \neq 2}}^{\infty} \frac{1}{2p_j}, \qquad (p_i = 2).$$

Since the series $\sum_{j=1}^{\infty} \frac{1}{2p_j}$ diverges, we have $A(\varepsilon) \notin \mathcal{I}_c^{(q)}$ for all $q \in \langle 0,1\rangle$. Therefore $\mathcal{I}_c^{(q)} - \lim \frac{\log\log f^*(n)}{\log\log n} \neq 1 + \log 2$ and the proof is complete.

There exists a relationship between functions $f(n)$ and $d(n)$ (where $d(n)$ is the number of divisors of n). The following equality holds: $\log f(n) = \frac{d(n)}{2} \cdot \log n$, $(n > e)$ (see [34]). From this we have

$$\log\log f(n) = \log\frac{1}{2} + \log d(n) + \log\log n, \quad n > e^e.$$

Therefore

$$\frac{\log\log f(n)}{\log\log n} = 1 + \frac{\log d(n)}{\log\log n} + \frac{\log\frac{1}{2}}{\log\log n}, \quad n > e^e.$$

From Theorem 1.25 we have the following statement.

Corollary 1.27. *The sequence* $\left(\frac{\log d(n)}{\log\log n}\right)_{n=2}^{\infty}$ *is not* $\mathcal{I}_c^{(q)}$*–convergent for all* $q \in \langle 0,1\rangle$.

The following results concerning the functions $\gamma(n)$ and $\tau(n)$.

In [33, Theorem 3, 5] there are proofs of the following results:

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n} = 1, \qquad \sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n} = 1 + \frac{\pi^2}{6}.$$

In connection with these results we have investigated the convergence of series for any $\alpha \in (0,1)$,

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha}, \qquad \sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^\alpha},$$

that we need for $\mathcal{I}_c^{(q)}$–convergence of functions $\gamma(n)$ and $\tau(n)$. The following results are outlined in [21].

Theorem 1.28. *The series*

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha}$$

diverges for $0 < \alpha \leq \frac{1}{2}$ and converges for $\alpha > \frac{1}{2}$.

Proof.

a. Let $0 < \alpha \le \frac{1}{2}$. Put $K = \{k^2 : k \in \mathbb{N}, k > 1\}$. A simple estimation gives

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^{\alpha}} \ge \sum_{n \in K} \frac{\gamma(n) - 1}{n^{\alpha}}.$$

Clearly $\gamma(n) \ge 2$ for $n \in K$. Therefore

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^{\alpha}} \ge \sum_{n \in K} \frac{1}{n^{\alpha}} = \sum_{k=2}^{\infty} \frac{1}{k^{2\alpha}} \ge \sum_{k=2}^{\infty} \frac{1}{k} = +\infty. \tag{13}$$

b. Let $\alpha > \frac{1}{2}$. We will use the formula

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^{\alpha}} = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{k^{\alpha s}} = \sum_{k=2}^{\infty} \frac{1}{k^{\alpha}(k^{\alpha} - 1)}. \tag{14}$$

For a sufficiently large number k $(k > k_0)$ we have $\frac{k^{\alpha}}{k^{\alpha} - 1} < 2$. We can estimate the series on the right-hand side of Eq. (14) with

$$\sum_{k=2}^{\infty} \frac{1}{k^{\alpha}(k^{\alpha} - 1)} < \sum_{k=2}^{k_0} \frac{1}{k^{\alpha}(k^{\alpha} - 1)} + 2 \sum_{k > k_0} \frac{1}{k^{2\alpha}}.$$

Since $2\alpha > 1$ we get

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^{\alpha}} < +\infty.$$

Corollary 1.29. *The sequence $\gamma(n)$ is*

i. $\mathcal{I}_c$–convergent to 1,

ii. $\mathcal{I}_c^{(q)}$–divergent for $q \in \langle 0, \frac{1}{2} \rangle$ and $\mathcal{I}_c^{(q)}$–convergent to 1 for $q \in \left(\frac{1}{2}, 1\right)$.

Proof.

i. Let $\varepsilon > 0$. The set of numbers $\{n \in \mathbb{N}, n > 1 : |\gamma(n) - 1| \ge \varepsilon\}$ is a subset of $H = \{n = t_s : n \in \mathbb{N}, t > 1, s > 1\}$ and $\sum_{a \in H} \frac{1}{a} < +\infty$. From the definition of $\mathcal{I}_c$–convergence Cor. 1.29 i. (Cor. is the abbreviation for Corollary) follows.

ii. Let $\varepsilon > 0$ and denote $A_{\varepsilon} = \{n \in \mathbb{N} : |\gamma(n) - 1| \ge \varepsilon\}$. When $0 < q \le \frac{1}{2}$ then for the numbers $n \in K$, $K = \{k^2 : k \in N, k > 1\}$ considering Eq. (13) for $q = \alpha$ holds

$$\sum_{n \in A_{\varepsilon}} \frac{1}{n^{\alpha}} \ge \sum_{n \in K} \frac{1}{n^{\alpha}} \ge +\infty.$$

Therefore $\gamma(n)$ is $\mathcal{I}_c^{(q)}$–divergent. If $\frac{1}{2} < q < 1$, $q = \alpha$ then $A_{\varepsilon} \subset H$ and

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha}} \le \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{k^{\alpha s}}.$$

The convergence of the series on the right-hand side we proved previously in Theorem 1.28. Therefore $\gamma(n)$ is $\mathcal{I}_c^{(q)}$–convergent to 1 if $q \in \left(\frac{1}{2}, 1\right)$.

Remark. We have $\lim \text{stat}\, \gamma(n) = 1$.

Theorem 1.30 The series

$$\sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^{\alpha}}$$

diverges for $0 < \alpha \le \frac{1}{2}$ and converges for $\alpha > \frac{1}{2}$.

Proof. Let $0 < \alpha < 1$. We write the given series in the form

$$\sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^{\alpha}} = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{s}{k^{\alpha s}}, \tag{15}$$

We shall try to use a similar method to Mycielski's proof of the convergence of $\sum_{n=2}^{\infty} \frac{\tau(n)-1}{n^{\alpha}}$ to explain the equality Eq. (15). Since $\frac{s}{k^{\alpha s}} = -\frac{k}{\alpha} \frac{d}{dt} \left(\frac{1}{t^{\alpha s}}\right)_{t=k}$ and $\sum_{s=2}^{\infty} \frac{1}{t^{\alpha s}} = \frac{1}{t^{\alpha}(t^{\alpha}-1)}$ the right-hand side of Eq. (15) is equal to

$$\sum_{s=2}^{\infty} \frac{2k^{\alpha} - 1}{k^{\alpha}(k^{\alpha} - 1)^2} = \sum_{s=2}^{\infty} a_k$$

For the k-th term of $\sum a_k$ we have

$$a_k = \frac{2 - \frac{1}{k^{\alpha}}}{\left(1 - \frac{1}{k^{\alpha}}\right)^2} \cdot \frac{1}{k^{2\alpha}}.$$

Denote by $b_k = \frac{1}{k^{2\alpha}}$ and consider that $\lim_{k\to\infty} \frac{a_k}{b_k} = 2$. Hence the series $\sum_{s=2}^{\infty} a_k$ converges (diverges) if and only if the series $\sum_{s=2}^{\infty} b_k$ converges (diverges). Since $\sum b_k$ is convergent (divergent) for any $\alpha > \frac{1}{2}$ $\left(0 < \alpha \le \frac{1}{2}\right)$ so does the series $\sum a_k$ and therefore the series $\sum \frac{\tau(n)-1}{n^{\alpha}}$.

Corollary 1.31. *The sequence $\tau(n)$ is*

i. *$\mathcal{I}_c$–convergent to 1,*

ii. *$\mathcal{I}_c^{(q)}$–divergent for $q \in \left(0, \frac{1}{2}\right\rangle$ and $\mathcal{I}_c^{(q)}$–convergent to 1 for $q \in \left(\frac{1}{2}, 1\right)$.*

Proof. Similar to the proof of Corollary 1.29.

Remark. We have $\lim \text{stat}\, \tau(n) = 1$.

6. Conclusions

It turns out that the study of $\mathcal{I}$–convergence of arithmetical functions or some sequences related to these arithmetical functions for different kinds of ideals $\mathcal{I}$ (see [18]) gives a deeper insight into the behaviour and properties of these arithmetical functions.

On the other hand Algebraic number theory has many deep applications in cryptology. Many basic algorithms, which are widely used, have its security due to ANT. The theory of arithmetic functions has many connections to the classical ciphers, and to the general theory as well.

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