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Circuits and Cycles in Graphs and Matroids

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in
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ABSTRACT

Circuits and Cycles in Graphs and Matroids

Yang Wu

This dissertation mainly focuses on characterizing cycles and circuits in graphs, line graphs and matroids. We obtain the following advances.

1. Results in graphs and line graphs.

For a connected graph G not isomorphic to a path, a cycle or a $K_{1,3}$, let $\text{pc}(G)$ denote the smallest integer n such that the n th iterated line graph $L^n(G)$ is panconnected. A path P is a **divalent** path of G if the internal vertices of P are of degree 2 in G . If every edge of P is a cut edge of G , then P is a **bridge divalent path** of G ; if the two ends of P are of degree s and t , respectively, then P is called a **divalent** (s, t) -**path**. Let $\ell(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length 2 and in a } K_3\}$. We prove the following.

- (i) If G is a connected triangular graph, then $L(G)$ is panconnected if and only if G is essentially 3-edge-connected.
- (ii) $\text{pc}(G) \leq \ell(G) + 2$. Furthermore, if $\ell(G) \geq 2$, then $\text{pc}(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$ -path of length $\ell(G)$.

For a graph G , the **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G has a spanning $(k; u, v)$ -trail-system, for any integer k with $1 \leq k \leq s$, and for any $u, v \in V(G)$ with $u \neq v$. Thus $\mu'(G) \geq 2$ implies that G is supereulerian, and so graphs with higher supereulerian width are natural generalizations of supereulerian graphs. Settling an open problem of Bauer, Catlin in [J. Graph Theory 12 (1988), 29-45] proved that if a simple graph G on $n \geq 17$ vertices satisfy $\delta(G) \geq \frac{n}{4} - 1$, then $\mu'(G) \geq 2$. In this paper, we show that for any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite graph family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for a simple graph G with $n = |V(G)|$, if for any $u, v \in V(G)$ with $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq an + b$, then either $\mu'(G) \geq s + 1$ or G is contractible to a member in \mathcal{F} . When $a = \frac{1}{4}, b = -\frac{3}{2}$, we show that if n is sufficiently large, $K_{3,3}$ is the only obstacle for a 3-edge-connected graph G to satisfy $\mu'(G) \geq 3$.

An hourglass is a graph obtained from K_5 by deleting the edges in a cycle of length 4, and an hourglass-free graph is one that has no induced subgraph isomorphic to an hourglass. Kriesell in [J. Combin. Theory Ser. B, 82 (2001), 306-315] proved that every 4-connected hourglass-free line graph is Hamilton-connected, and Kaiser, Ryjáček and Vrána in [Discrete Mathematics, 321 (2014) 1-11] extended it by showing that every 4-connected hourglass-free line graph is 1-Hamilton-connected. We characterize all essentially 4-edge-connected graphs whose line graph is hourglass-free. Consequently we prove that for any integer s and for any hourglass-free line

graph $L(G)$, each of the following holds.

- (i) If $s \geq 2$, then $L(G)$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$;
- (ii) If $s \geq 1$, then $L(G)$ is s -Hamilton-connected if and only if $\kappa(L(G)) \geq s + 3$.

For integers $s_1, s_2, s_3 > 0$, let N_{s_1, s_2, s_3} denote the graph obtained by identifying each vertex of a K_3 with an end vertex of three disjoint paths P_{s_1+1} , P_{s_2+1} , P_{s_3+1} of length s_1, s_2 and s_3 , respectively. We prove the following results.

- (i) Let $\mathcal{N}_1 = \{N_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1 + s_2 + s_3 \leq 6\}$. Then for any $N \in \mathcal{N}_1$, every N -free line graph $L(G)$ with $|V(L(G))| \geq s + 3$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.
- (ii) Let $\mathcal{N}_2 = \{N_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1 + s_2 + s_3 \leq 4\}$. Then for any $N \in \mathcal{N}_2$, every N -free line graph $L(G)$ with $|V(L(G))| \geq s + 3$ is s -Hamilton-connected if and only if $\kappa(L(G)) \geq s + 3$.

2. Results in matroids.

A matroid M with a distinguished element $e_0 \in E(M)$ is a rooted matroid with e_0 being the root. We present a characterization of all connected binary rooted matroids whose root lies in at most three circuits, and a characterization of all connected binary rooted matroids whose root lies in all but at most three circuits. While there exist infinitely many such matroids, the number of serial reductions of such matroids is finite. In particular, we find two finite families of binary matroids \mathcal{M}_1 and \mathcal{M}_2 and prove the following.

- (i) For some $e_0 \in E(M)$, M has at most three circuits containing e_0 if and only if the serial reduction of M is isomorphic to a member in \mathcal{M}_1 .
- (ii) If for some $e_0 \in E(M)$, M has at most three circuits not containing e_0 if and only if the serial reduction of M is isomorphic to a member in \mathcal{M}_2 .

These characterizations will be applied to show that every connected binary matroid M with at least four circuits has a 1-hamiltonian circuit graph.

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DEDICATION

To

my father Song-Chu Wu, my mother Jie Liu

and

my wife Yu-Lin Xiao

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Chapter 1

Introduction

1.1 Notations and Terminology

We consider finite loopless graphs and matroids. We follow the notations and terminology in [3] for graphs and [84] for matroids except otherwise defined. Throughout this dissertation, G denotes an graph with vertex set $V(G)$ and edge set $E(G)$; M denotes a matroid with ground set $E(M)$ and circuit set $\mathcal{C}(M)$; $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of a graph G , respectively. $\alpha(G)$, $\kappa(G)$ and $\kappa'(G)$ represent the stability number (also called the independence number), the connectivity and the edge connectivity of the graph G , respectively. We use $c(G)$ and $g(G)$ to denote the **circumference** and the **girth** of G , which are the length of a longest cycle in G and the length of a shortest cycle of G , respectively. A graph is trivial if it has no edges.

The **line graph** of G , denoted $L(G)$, has vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges share at least one common vertex in G .

A trail with initial vertex u and terminal vertex v will be referred as a (u, v) -trail. We use $O(G)$ to denote the set of all odd degree vertices in G . A graph G is **Eulerian** if it is connected and $O(G) = \emptyset$, and is **supereulerian** if G has a Eulerian subgraph H with $V(H) = V(G)$.

Following [84], a matroid M is connected if for any pair of distinct elements $e, e' \in E(M)$, there exists a circuit $C \in \mathcal{C}(M)$ with $e, e' \in C$. Throughout this paper, for any edge subset $X \subseteq E(G)$ of a graph G , X denotes an edge subset as well as the subgraph $G[X]$ induced by the edge subset X . Following matroid terminology, if G is a graph and $M = M(G)$ is the cycle matroid of M , any edge subset Z (as well as the subgraph $G[Z]$ induced by Z) will be called a **circuit** if $Z \in \mathcal{C}(M(G))$. Let $h > 0$ be an integer. If $Z \in \mathcal{C}(M)$ with $|Z| = h$, we often call Z an h -circuit of M .

1.2 Background on Index Problems for Line Graphs

For a connected graph G , the n -th iterated line graph $L^n(G)$ is defined recursively by $L^0(G) = G$ and $L^n(G) = L(L^{n-1}(G))$. Since the iterated line graph of a path will eventually diminish, and since the line graph of a cycle remains unchanged, in the discussions of iterated line graph problems, it is generally assumed that graphs under considerations are connected but not isomorphic to paths, cycles or $K_{1,3}$. For this reason, we let \mathcal{G} denote the family of all connected graphs that are neither a path or a cycle, nor isomorphic to $K_{1,3}$.

The hamiltonian index (to be defined below) of a graph was first introduced in [19] by Chartrand. Other hamiltonian like indices were given by Clark and Wormald in [27]. More generally, we have the following definition.

Definition 1.2.1. ([48]) *For a property \mathcal{P} and a connected nonempty graph $G \in \mathcal{G}$, the \mathcal{P} -index of G , denoted $\mathcal{P}(G)$, is defined by*

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} & \text{if at least one such integer } k \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

When \mathcal{P} represents the properties of being hamiltonian, edge-hamiltonian, pancyclic, vertex-pancyclic, edge-pancyclic, hamiltonian-connected, the corresponding indices are denoted (as in [27]) by $h(G)$, $eh(G)$, $p(G)$, $vp(G)$, $ep(G)$, $hc(G)$, respectively. In particular, $h(G)$ is called the **hamiltonian index** of G . Clark and Wormald [27] showed that if $G \in \mathcal{G}$, then the indices $h(G)$, $eh(G)$, $p(G)$, $vp(G)$, $ep(G)$, $hc(G)$ exist as finite numbers. In [20] and [48], it is shown that if G has any one of these properties mentioned above, then $L(G)$ also has the same property. In [86], Ryjáček, Woeginger and Xiong proved that determining the value of $h(G)$ is a difficult problem.

There have been many studies to investigate upper bounds of the hamiltonian index, hamiltonian-connected index and (vertex) pancyclic index. Interested readers may refer to [14, 20, 22, 24, 33, 44, 45, 75, 89, 90, 91, 99, 101, 105] for further details. A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G . Define

$$\ell(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length 2 and in a } K_3\}. \quad (1.1)$$

Let P be a divalent path of G . If every edge of P is a cut edge of G , then P is a **bridge divalent path** of G ; Moreover, if the two ends of P are of degree s and t , respectively, then P is called a **divalent (s, t) -path**. Sharp upper bounds of the hamiltonian index, hamiltonian-connected index, s -hamiltonian index and pan-cyclic index have been obtained in terms of $\ell(G)$, see [24, 44, 45, 75, 90, 91, 105], among others. A graph G on $n \geq 3$ vertices is **panconnected** if for every pair of vertices u and v in G and for each s with $d(u, v) \leq s \leq n - 1$, G always has a (u, v) -path of length s . Let \mathcal{P} denote the property of being panconnected and following [27], let

$\text{pc}(G)$ denote the panconnected index of a graph $G \in \mathcal{G}$. There has been little study on $\text{pc}(G)$. This observation motivates the current study.

1.3 Background on Supereulerian Width Problem

The study of supereulerian graphs was first raised by Boesch, Suffel and Tindel in [6]. Pulleyblank [85] showed that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete.

Motivated by the Menger Theorem, a generalization of supereulerian graphs has been considered in the literature (see [66], for example). For a graph G and an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$ -trail-system of G is a subgraph H consisting of s edge-disjoint (u, v) -trails. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G has a spanning $(k; u, v)$ -trail-system, for any integer k with $1 \leq k \leq s$. For any $u, v \in V(G)$ with $u \neq v$, Luo et al in [74] defined graphs with $\mu'(G) \geq 1$ as **Eulerian-connected** graphs. They also investigated, for a given integer $r > 0$, the minimum value $\psi(r)$ such that if G is a $\psi(r)$ -edge-connected graph, then for any $X \subseteq E(G)$ with $|X| \leq r$, $\mu'(G - X) \geq 2$. An open problem on $\psi(r)$ is raised in [74] and is settled in [103]. By definition, $\mu'(G) \geq 2$ implies that G is supereulerian. Supereulerian graphs have been intensively studied, as seen in surveys [12, 21, 47], among others.

The concept of $\mu'(G)$ is formally introduced in [66], as a natural generalization of supereulerian graphs. Related studies can be found in [23] and [101]. One of the main problems in the study on the supereulerian width of graphs is to determine $\mu'(G)$ for a given graph G . As shown in [66], every collapsible graph (to be defined in Section 2) has supereulerian width at least 2. Settling an open problem of Bauer ([4, 5]), Catlin prove Theorem 1.4.3(i) below, which was recently extended by Li et al. in [66].

Theorem 1.3.1. *Let G be a simple graph on n vertices.*

- (i) *(Catlin, Theorem 9(ii) of [12]) If $n \geq 17$ and $\delta(G) \geq \frac{n}{4} - 1$, then $\mu'(G) \geq 2$.*
- (ii) *(Li et al, Theorem 5.3(i) of [66]) For any positive integers p and s with $p \geq 2$, there exists an integer $N = N(s, p)$ and a finite family \mathcal{F}_0 of graphs with supereulerian width at most s such that if $\delta(G) \geq \frac{n}{p} - 1$, then either $\mu'(G) \geq s + 1$, or G is contractible to a member in \mathcal{F}_0 .*

These motivate the current research.

1.4 Background on s -hamiltonian and s -hamiltonian-connected Problems for Line Graphs

A few most fascinating problems in studying hamiltonian problems of line graphs are presented below. By an ingenious argument of Z. Ryjáček ([87]), Conjecture 1.4.1(i) below is equivalent to a seeming stronger conjecture of Conjecture 1.4.1(ii). In [88], it is shown that all conjectures stated in Conjecture 1.4.1 below are equivalent to each other.

Conjecture 1.4.1. (i) (Thomassen [94]) *Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [76]) *Every 4-connected claw-free graph is hamiltonian.*
(iii) (Kužel and Xiong [42]) *Every 4-connected line graph is Hamilton-connected.*
(iv) (Ryjáček and Vrána [88]) *Every 4-connected claw-free graph is Hamilton-connected.*

Towards Conjecture 1.4.1, Zhan gave a first result in this direction, and the best known result is given by Kaiser and Vrána, as shown below.

Theorem 1.4.2. *Let G be a graph.*

- (i) (Zhan, Theorem 3 in [107]) *If $\kappa(L(G)) \geq 7$, then $L(G)$ is Hamilton-connected.*
- (ii) (Kaiser and Vrána [39]) *Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.*
- (iii) (Kaiser, Ryjáček and Vrána [41]) *Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.*

There have been many researches on hamiltonian properties in 3-connected claw-free graphs forbidding a $N_{k,0,0}$, as seen in the surveys in [9, 30, 34, 35], among others. The following have been proved.

Theorem 1.4.3. *Let Q^* be the graph obtained from the Petersen graph by adding one pendant edge to each vertex. Let G be a 3-connected simple claw-free graph.*

- (i) (Brousek, Ryjáček and Favaron, [10]) *If G is $N_{4,0,0}$ -free, then G is hamiltonian.*
- (ii) ([57]) *If G is $N_{8,0,0}$ -free, then G is hamiltonian. Moreover, the graph Q^* indicates the sharpness of this result.*
- (iii) (Fujisawa, [32], see also Ma et al. [77]) *If G is $N_{9,0,0}$ -free graph, then G is hamiltonian unless G is the line graph of Q^* .*

It is natural to seek necessary and sufficient conditions for hamiltonicity of line graphs. For an integer $s \geq 0$, a graph G of order $n \geq s + 3$ is **s -hamiltonian** (**s -Hamilton-connected**, respectively), if for any $X \subseteq V(G)$ with $|X| \leq s$, $G - X$ is hamiltonian ($G - X$ is Hamilton-connected, respectively). It is well known that if a graph G is s -hamiltonian, then G is $(s + 2)$ -connected, and if G is s -Hamilton-connected, then G is $(s + 3)$ -connected. Broersma and Veldman

in [8] initiated the problem of investigating graphs whose line graph is s -hamiltonian if and only if the connectivity of the line graph is at least $s + 2$. They define, for an integer $k \geq 0$, a graph G to be k -**triangular** if every edge of G lies in at least k triangles of G . The following is obtained.

Theorem 1.4.4. *(Broersma and Veldman, [8]) Let $k \geq s \geq 0$ be integers and let G be a k -triangular simple graph. Then $L(G)$ is s -hamiltonian if and only if $L(G)$ is $(s + 2)$ -connected.*

Broersma and Veldman in [8] proposed an open problem of determining the range of an integer s such that within triangular graphs, $L(G)$ is s -hamiltonian if and only if $L(G)$ is $(s + 2)$ -connected. This problem was first settled by Chen et al. in [25].

Theorem 1.4.5. *Each of the following holds.*

- (i) *(Chen et al. [25]) Let k and s be positive integers such that $0 \leq s \leq \max\{2k, 6k - 16\}$, and let G be a k -triangular simple graph. Then $L(G)$ is s -hamiltonian if and only if $L(G)$ is $(s + 2)$ -connected.*
- (ii) *([55]) Let G be a connected graph and let $s \geq 5$ be an integer. Then $L(G)$ is s -hamiltonian if and only if $L(G)$ is $(s + 2)$ -connected.*

An hourglass is a graph isomorphic to $K_5 - E(C_4)$, where C_4 is a cycle of length 4 in K_5 . The following are proved recently.

Theorem 1.4.6. *Each of the following holds.*

- (i) *(Kaiser, Ryjáček and Vrána [41]) Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*
- (ii) *([58]) For an integer $s \geq 2$, the line graph $L(G)$ of a claw-free graph G is s -hamiltonian if and only if $L(G)$ is $(s + 2)$ -connected.*
- (iii) *([58]) The line graph $L(G)$ of a claw-free graph G is 1-Hamilton-connected if and only if $L(G)$ is 4-connected.*

In view of Conjecture 1.4.1 and motivated by Theorems 1.4.2, 1.4.3, 1.4.5 and 1.4.6, it is conjectured ([55]) that

Conjecture 1.4.7. *Let G be a connected graph and let s be an integer.*

- (i) *([55]) If $s \geq 2$, then $L(G)$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.*
- (ii) *([51]) If $s \geq 1$, then $L(G)$ is s -Hamilton-connected if and only if $\kappa(L(G)) \geq s + 3$.*

Let $P(10, \ell)$ be the graph obtained from $P(10)$, the Petersen graph by attaching $\ell > 0$ pendant edges at every vertex of $P(10)$. It is known that $L(P(10, \ell))$ is 3-connected but not hamiltonian. Hence the values of s in Conjecture 1.4.7 cannot be smaller. This conjecture motivated our investigation on s -hamiltonicity of line graphs.

1.5 Background on Matroid Circuit Problem

The distribution of circuits in a graph or a matroid has been studied by quite a few researchers. Murty [81] initially characterized all connected binary matroids with exactly one circuit length. Lemos, Reid and Wu in [59] extended Murty's result by successfully characterizing all connected binary matroids with at most two circuit lengths. It is indicated in [59] that it is difficult to characterize the matroids having a particular circuit-spectrum set even when the set is small and the matroids belong to an interesting class. Cordovil et al in [28], and B.M. Junior and M. Lemos in [78] constructed all matroids M whose circuit lengths are at most 5, and constructed all 3-connected binary matroids M whose circuit lengths are in $\{3, 4, 5, 6, 7\}$. In [2], Bollobás presented a characterization of all graphs with minimum degree at least 3 that do not have edge disjoint circuits. He indicated that this characterization can be applied to imply a slight extension of an earlier result of Erdős and Pósa [29]. The corresponding characterization of regular matroids without disjoint circuits is obtained in [31]. In this paper, we consider the problem of determining all binary matroids with an element lying in at most 3 circuits, as well as all binary matroids with an element lying in all but at most three circuits. The main results of this paper, to be stated in the next section after some of the terms are defined, are characterizations of such matroids. Li and Liu ([62], [63] and [64]) initiated the investigation of graphical properties of matroid circuit graphs. These motivated us to study matroids circuit problem and circuit graph of matroids.

Chapter 2

Panconnected index of graphs

2.1 Main Results

A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G . Define

$$\ell(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length 2 and in a } K_3\}. \quad (2.1)$$

Let P be a divalent path of G . If every edge of P is a cut edge of G , then P is a **bridge divalent path** of G ; Moreover, if the two ends of P are of degree s and t , respectively, then P is called a **divalent (s, t) -path**. A graph G is **triangular** if G is connected with $E(G) \neq \emptyset$ such that every edge in $E(G)$ lies in a cycle of length at most 3 in G . Let \mathcal{G} denote the family of all connected graphs that are neither a path or a cycle, nor isomorphic to $K_{1,3}$. Our main purpose of this study is to investigate $pc(G)$ for graphs $G \in \mathcal{G}$ and we obtained the following results.

Theorem 2.1.1. *Let G be a graph in \mathcal{G} . Then $pc(G) \leq \ell(G) + 2$. Furthermore, if $\ell(G) \geq 2$, then $pc(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$ -path of length $\ell(G)$.*

Theorem 2.1.2. *Let G be a graph in \mathcal{G} . If every edge of G lies in a cycle of length at most 3, in G , then $L(G)$ is panconnected if and only if G is essentially 3-edge-connected.*

2.2 Sufficient condition for a triangular graph to be panconnected.

It is a well known fact that

$$\text{every panconnected graph is 3-connected.} \quad (2.2)$$

We found the sufficient condition for a triangular graph to be panconnected in this section, by which together with (2.2) Theorem 2.1.2 is proved.

Let G be a graph. For a vertex $v \in V(G)$, define $N_G(v)$ to be the set of all vertices in G adjacent to v , and $E_G(v) = \{e \in E(G) \mid e \text{ is incident with } v \text{ in } G\}$. Following [3], we denote a trail $T = v_0 e_1 v_1 \cdots v_{t-1} e_t v_t$ such that each edge $e_i = v_{i-1} v_i$, for every i with $1 \leq i \leq t$, and such that all edges are distinct. For convenience, we sometimes view that T is associate with a natural orientation in which every edge e_i in the trail is oriented from v_{i-1} to v_i . If $v_0 = v_t$, then T is a closed trail. To emphasize the terminal vertices, T is called a (v_0, v_t) -trail. As the terminal edges of this trail T are e_1 and e_t , we also refer to T as an (e_1, e_t) -trail. The set of internal vertices of T is defined to be $T^\circ = \{v_1, v_2, \dots, v_{t-1}\}$. If T is a trail of G , define

$$\partial_G(T) = \{e \in E(G) : e \text{ is incident with a vertex in } T^\circ\}. \quad (2.3)$$

As in [24, 49, 56], an (e, e') -trail T in G is a dominating trail if $\partial(T) = E(G)$, and is a spanning trail if T is dominating with $V(T) = V(G)$. The theorem below is well known.

Theorem 2.2.1. *Let G be a graph with $|E(G)| \geq 3$.*

- (i) *(Harary and Nash-Williams) $L(G)$ is hamiltonian if and only if G has a dominating closed trail.*
- (ii) *(Proposition 2.2 of [49]) $L(G)$ is hamiltonian-connected if and only if for every pair of distinct edges e, e' in $E(G)$, G has a dominating (e, e') -trail.*

Inspired by Theorem 2.2.1, we obtain the following lemma which is an important tool in the discussion on line graphs.

Lemma 2.2.2. *Let $s > 0$ be an integer, and $e, e' \in E(G)$. Each of the following holds.*

- (i) *There is an (e, e') -path of length s in $L(G)$ if and only if G has an (e, e') -trail T with $|E(T)| \leq s + 1$ and $|\partial_G(T)| \geq s + 1$.*
- (ii) *The distance between e and e' in $L(G)$ is s if and only if G has a shortest (e, e') -path of length $s + 1$.*

Proof. By the definition of line graphs, (ii) follows from (i) and so it suffices to prove Part (i) only. Suppose G has an (e, e') -trail $T = v_0 e_1 v_1 e_2 \dots v_{m-1} e_m v_m$ with $e = e_1$ and $e' = e_m$, satisfying $m = |E(T)| \leq s + 1$ and $|\partial_G(T)| \geq s + 1$. Then $L(T)$ is an (e, e') -path of length $m - 1$ in $L(G)$. For each i with $0 < i < m$, let $X'_i = E_G(v_i) - E(T)$, $X_1 = X'_1$ and for $2 \leq j < m$, let $X_j = X'_j - (\cup_{1 \leq i < j} X'_i)$. Then X_1, X_2, \dots, X_m are pairwise disjoint and $\partial(T) = E(T) \cup (\cup_{i=1}^{m-1} X_i)$. Since $m = |E(T)| \leq s + 1$ and $|\partial_G(T)| \geq s + 1$, we have $\sum_{i=1}^{m-1} |X_i| \geq (s + 1) - m$. Hence there must be an integer m' with $1 \leq m' \leq m - 1$ and a subset $X' \subseteq X_{m'}$ such that $|X_1 \cup X_2 \cup \dots \cup X_{m'-1} \cup X'| = s - m$. Since every $E_G(v_i)$ induces a complete subgraph of $L(G)$, for each i with $1 \leq i \leq m' - 1$, $L(G)[E_G(v_i)]$ has an (e_i, e_{i+1}) -path P_i using exactly

the vertices in $X_i \cup \{e_i, e_{i+1}\}$ in $L(G)$, and $L(G)[E_G(v_{m'})]$ has an $(e_{m'}, e_{m'+1})$ -path $P_{m'}$ using exactly the vertices in $X' \cup \{e_{m'}, e_{m'+1}\}$ in $L(G)$. Let $P_{m'+1}$ be the subpath $e_{m'+1}e_{m'+2}\dots e_m$ of $L(T)$. It follows that $L(G)$ has an (e, e') -path of length s obtained by putting all the paths $P_1, P_2, \dots, P_{m'}, P_{m'+1}$ together.

Conversely, assume that $L(G)$ has an (e, e') -path P of length s . Then $V(P) \subseteq E(G)$. Since P is an (e, e') -path, the edge induced subgraph $G[V(P)]$ of G is connected and contains e and e' . Thus $G[V(P)]$ has a longest (e, e') -trail T . Since T is longest, and since $L(G[V(P)]) = P$, it follows that $E(T) \subseteq V(P) \subseteq \partial(T)$, and so $|E(T)| \leq |V(P)| = s + 1$ and $|\partial_G(T)| \geq |V(P)| = s + 1$. \square

If H is a subgraph of a graph G , the **vertex of attachment of H in G** , is

$$A_G(H) = \{v \in V(H) : v \text{ is adjacent to a vertex in } V(G) - V(H)\}. \quad (2.4)$$

If $X \subseteq E(H)$ and $Y \subseteq E(G) - E(H)$, then we define $H - X + Y = G[(E(H) - X) \cup Y]$. For sets X and Y , the symmetric difference of X and Y is defined as

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Now we are ready to give the sufficient condition for a triangular graph to be panconnected.

Theorem 2.2.3. *If G is an essentially 3-edge-connected triangular graph, then $L(G)$ is panconnected.*

Proof. Since G is triangular, throughout the rest of the proof of this theorem, for each edge $f \in E(G)$, we define C_f to be a shortest cycle containing f . Thus $|E(C_f)| \leq 3$ for any $f \in E(G)$. We argue by contradiction and assume that Theorem 2.2.3 has a counterexample G with $e, e' \in E(G)$ and a positive integer $s < |E(G)| - 1$ such that

$$L(G) \text{ has an } (e, e')\text{-path of every length at most } s \text{ but no } (e, e')\text{-paths of length } s + 1. \quad (2.5)$$

By Lemma 2.2.2, G has an (e, e') -trail

$$T = v_0 e_1 v_1 e_2 \dots v_{m-1} e_m v_m \text{ with } e = e_1 \text{ and } e' = e_m \quad (2.6)$$

with $|E(T)| \leq s + 1$ and $|\partial(T)| \geq s + 1$. Assume that the choice of G satisfies (2.5), and that, subject to $|E(T)| \leq s + 1$ and $|\partial(T)| \geq s + 1$,

$$|E(T)| \text{ is maximized.} \quad (2.7)$$

If $|\partial(T)| \geq s + 2$, then by Lemma 2.2.2, $L(G)$ has an (e, e') -path of length $s + 1$, contradicting (2.5). Hence we must have $|\partial(T)| = s + 1 < |E(G)|$.

Claim 1. For any edge $f = uv \in \partial(T) - E(T)$, we have $u, v \in V(T)$.

By contradiction, assume that there exists an edge $f = uv \in \partial(T) - E(T)$ violating the claim. By (2.3), we may assume that $v \in T^o$ and $u \notin V(T)$. Since G is triangular, there exists a cycle C_f of length at most 3 in G containing f . If $C_f = \{f, f'\}$ is a cycle of length 2, then the trail $G[E(T) \cup C_f]$ violates (2.7). Assume that C_f is a 3-cycle with $V(C_f) = \{u, v, w\}$. As $uv, uw \notin E(T)$, it follows that $G[E(T) \Delta E(C_f)]$ is an (e, e') -trail in G violating (2.7). This justifies Claim 1.

Claim 2. With the notation in (2.6), each of the following holds.

- (i) There exists an edge $f \in E(G) - \partial(T)$ such that f is adjacent to an edge in $\partial(T)$.
- (ii) For any edge $f \in E(G) - \partial(T)$, if f is adjacent to an edge in $\partial(T)$, then f must be adjacent to either v_0 or v_m .
- (iii) $A_G(G[\partial(T)]) \subseteq \{v_0, v_m\}$.
- (iv) Let $H = G[\partial(T)]$. For any $z \in \{v_0, v_m\}$, let $x \in \{e, e'\}$ be the corresponding edge incident with z . If there exists an edge $f \in E_G(z) - \partial(T)$, then x is not in any cycle of length 2 and

$$N_H(z) - V(C_x) = \emptyset. \quad (2.8)$$

Claim 2(i) follows from the assumptions that G is connected and that $|\partial(T)| < |E(G)|$. To justify Claim 2(ii), suppose that an edge $f \in E(G) - \partial(T)$ is adjacent to an edge y in $\partial(T)$. If $y \notin \{e, e'\}$, then by Claim 1, both ends are in T^o , and so f is incident with a vertex in T^o , leading to the contradiction that $f \in \partial(T)$. Hence we may assume that $y = e$. As $v_1 \in T^o$ and as $f \in E(G) - \partial(T)$, f must be incident with v_0 . Similarly, if $y = e'$, then f must be incident with v_m . This shows Claim 2(ii).

Claim 2(iii) follows from (ii) and (2.4). We now show (iv) and assume, by symmetry, that $z = v_0$ and $x = e$, and there exists an edge $f \in E_G(v_0) - \partial(T)$. By (2.3), $v_0 \notin T^o$. If $\{e, e''\}$ is a cycle of G , then by (2.6), $T' = v_1 e v_0 e'' v_1 e_2 \dots v_{m-1} e' v_m$ is also an (e, e') -trail in G with $E(T') \subseteq \partial(T)$ and with $|E(T')| > |E(T)|$, a contradiction to (2.7). Thus e is not in any cycle of length 2. Assume now that there exists a vertex $z' \in N_H(v_0) - V(C_e)$. Since G is triangular, G has a cycle $C_{v_0 z'}$ of length 2 or 3. By Claim 1, $E(C_{v_0 z'}) \subseteq \partial(T)$. As $v_0 \notin T^o$, $E(C_{v_0 z'}) \cap E(T) = \emptyset$ and so $G[E(T) \cup E(C_{v_0 z'})]$ is an (e, e') -trail violating (2.7). Hence we have (2.8).

Claim 3. Suppose that $v_0 \neq v_m$ and that there exists an edge $f \in E_G(v_0) - \partial(T)$. Then each of the following holds.

- (i) $V(C_e) = \{v_0, v_1, w\}$ is a cycle of length 3 and $v_1 \neq v_m$.
- (ii) If $v_m \notin V(C_e)$, then G has an (e, e') -trail T' such that $|E(T')| \leq |\partial(T)|$ and $|\partial(T')| \geq |\partial(T)| + 1$.
- (iii) If C'_e is a cycle of length 3 in G containing e , then $v_0 v_1, v_1 w \in E(C'_e)$.
- (iv) $w = v_m$.
- (v) If $v_1 \neq v_{m-1}$, then either $v_1 v_m \notin E(T)$ or $v_1 v_m$ is not a cut edge of T .
- (vi) The vertex v_1 is a cut vertex of G .

Throughout the justification of Claim 3, we let $H = G[V(T)]$ and assume that $v_0 \neq v_m$, and there exists an edge $f \in E_G(v_0) - \partial(T)$. If $E(C_e) \cap E(T) = \{e\}$, then $G[E(C_e) \cup E(T)]$ is an (e, e') -trail violating (2.7). Hence we may assume that $V(C_e) = \{v_0, v_1, w\}$ and $v_0v_1, v_1w \in E(T)$. By contradiction, assume that $v_1 = v_m$. If $w = v_{m-1}$, then let $L = T[E(T) - \{e, e'\}]$ be a (v_1, v_{m-1}) -subtrail of T . View L as a trail oriented by its direction from v_1 to v_{m-1} . Define L^{-1} to be the (v_{m-1}, v_1) -trail obtained from L by reversing the orientations. Then L^{-1} together with the oriented edges v_1v_0 and v_mv_{m-1} is an (e, e') -trail T_2 with $E(T_2) = E(T)$ and $\partial(T) \cup \{f\} \subseteq \partial(T_2)$. It follows that $|\partial(T_2)| \geq |\partial(T)| + 1 = s + 2$. By Lemma 2.2.2, $L(G)$ has an (e, e') -path of length $s + 1$, which contradicts (2.5).

Hence we assume that $w \neq v_{m-1}$. If v_1w is not a cut edge of $T - \{e, e'\}$, then $T - \{e, e', v_1w\} + \{v_0w\}$ is a (v_0, v_{m-1}) -trail. Thus $T_1 = T - \{v_1w\} + \{v_0w\}$ is an (e, e') -trail with $\partial(T) \subseteq \partial(T_1)$ and $v_0 \in T_1^o$. It follows that $f \in \partial(T_1)$, and so by Lemma 2.2.2, we obtain a contradiction to (2.5).

Thus v_1w is a cut edge of $T - \{e, e'\}$. Let J_1 and J_2 be the two components of $T - \{e, e', v_1w\}$ with $v_1 = v_m \in V(J_1)$ and $w \in V(J_2)$. Since v_1w is a cut edge of $T - \{e, e'\}$, we have $v_{m-1} \in V(J_2)$. Thus $T - \{e, e'\}$ is a (v_1, v_{m-1}) -trail. If $v_0 \in V(C_{e'})$, then $V(C_{e'}) = \{v_0, v_1, v_{m-1}\}$, and so by $v_1 = v_m$, $T - \{e, e'\} + \{v_0v_{m-1}\}$ is a (v_0, v_m) -trail. It follows that $T_2 = T + \{v_0v_{m-1}\}$ is a (e, e') -trail with $\partial(T) \subseteq \partial(T_2)$ and $v_0 \in T_2^o$, and so a contradiction to (2.5) is obtained. Hence $v_0 \notin V(C_{e'})$. By (2.7), $|E(C_{e'})| = 3$ and so $V(C_{e'}) = \{z, v_1, v_{m-1}\}$ for some $z \neq v_0$. It follows that $T_3 = G[E(T) \Delta E(C_{e'}) \Delta (E(C_e) + \{e, e'\})]$ is an (e, e') -trail with $\partial(T) \subseteq \partial(T_3)$ and $v_0 \in T_3^o$, once again a contradiction to (2.5) is obtained. This shows that $v_1 \neq v_m$, and justifies (i).

Assume that $v_m \notin V(C_e)$. If $T - v_1w$ is connected, then $T_4 = G[E(T) \Delta (E(C_e) - \{e\})]$ is an (e, e') -trail with $v_0 \in T_4^o$, and so $E(T_4) \subseteq \partial(T)$ and $\partial(T) \cup \{f\} \subseteq \partial(T_4)$, implying (ii). Hence we may assume that $T - v_1w$ has two components L_1 and L_2 such that $e \in E(L_1)$ and $w \in V(L_2)$. Since T is an (e, e') -trail and $w \neq v_m$, $e' \neq v_1w$ and so $e' \in E(L_2)$. Since G is essentially 3-edge-connected, $\{v_0v_1, v_1w\}$ is not an essential edge cut. By Claim 2(ii), there must be an edge $e'' = z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1)$ and $z_2 \in V(L_2)$. Since G is triangular, there exists a cycle $C_{e''}$ of length 2 or 3 containing e'' . Since any cycle intersects any edge cut with an even number of edges, $C_{e''}$ has two edges incident with both $V(L_1)$ and $V(L_2)$. It follows that $T_5 = G[E(T) \Delta E(C_{e''})]$ is also an (e, e') -trail of G with $E(T_5) \subseteq \partial(T)$ and $\partial(T) = \partial(T_5)$. Since $C_{e''}$ has two edges incident with both $V(L_1)$ and $V(L_2)$, the edge v_1w is not a cut edge of T' . Hence $T_6 = G[E(T_5) \Delta E(C_e - e)]$ is an (e, e') -trail with the first edge v_1v_0 and with $E(T_6) \subseteq \partial(T) \subseteq \partial(T_6)$. However, $v_0 \in (T_6)^o$, and so $f \in \partial(T_6) - \partial(T)$, which is a contradiction to (2.5). This proves Claim 3(ii).

For (iii), suppose next that C'_e is a cycle of length 3 in G containing e . By contradiction, assume that $V(C'_e) = \{v_0, v_1, w'\}$ for some $w' \neq w$. By Claim 3(i), $v_1 \neq v_m$. Hence we may assume that $w \neq v_m$. It follows that by Claim 3(ii), G has an (e, e') -trail T' such that

$|E(T')| \leq |\partial(T)| = s + 1$ and $|\partial(T')| \geq |\partial(T)| + 1 = s + 2$. By Lemma 2.2.2, we have a contradiction to (2.5). This justifies (iii). Claim 3(iv) now follows from Claim 3(ii) and (iii).

Now suppose that $v_1 \neq v_{m-1}$. Since $v_0 \notin T^o$, we have $v_1v_m \notin E(T)$. By contradiction, assume that $v_1v_m \in E(T)$ and v_1v_m is a cut edge of T . To avoid introducing too many new notations, we again assume that $T - v_1v_m$ has two components L_1 and L_2 such that $e \in E(L_1)$ and $v_m \in V(L_2)$. Since T is an (e, e') -trail and $v_{m-1} \neq v_1$, $e' \neq v_1w$ and so $e' \in E(L_2)$. Since G is essentially 3-edge-connected, $\{v_0v_1, v_1v_m\}$ is not an essential edge cut. By Claim 2(ii), there must be an edge $e'' = z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L_1)$ and $z_2 \in V(L_2)$. Since G is triangular, there exists a cycle $C_{e''}$ of length 2 or 3 containing e'' . Since any cycle intersects any edge cut with an even number of edges, $C_{e''}$ has two edges incident with both $V(T_1)$ and $V(T_2)$. It follows that $T_5 = G[E(T)\Delta E(C_{e''})]$ is also an (e, e') -trail of G with $E(T_5) \subseteq \partial(T)$ and $\partial(T) = \partial(T_5)$. Since $C_{e''}$ has two edges incident with both $V(L_1)$ and $V(L_2)$, the edge v_1w is not a cut edge of T' . Hence $T_6 = G[E(T_5)\Delta E(C_e - e)]$ is an (e, e') -trail with the first edge v_1v_0 and with $E(T_6) \subseteq \partial(T) \subseteq \partial(T_6)$. However, $v_0 \in (T_6)^o$, and so $f \in \partial(T_6) - \partial(T)$, which contradicts (2.5). Therefore, if $v_1v_m \in E(T)$, then v_1v_m is not a cut edge of T . This justifies (v).

We argue by contradiction to prove (vi) and assume that v_1 is not a cut vertex of G . By Claim 2(ii), $\{v_0, v_m\}$ is a vertex cut of G such that if J is a component of $G - \{v_0, v_m\}$ containing v_1 , then $G[V(J) \cup \{v_0, v_m\}] = H$.

Suppose first that $v_1 \neq v_{m-1}$ or $v_m \in T^o$. If v_1v_m is not a cut edge of $T - \{e, e'\}$, then $T - \{e, e', v_1v_m\} + \{v_0v_m\}$ is a (v_0, v_m) -trail, and so $T_7 = T - \{v_1v_m\} + \{v_0v_m\}$ is an (e, e') -trail with $E(T_7) \subseteq \partial(T) \cup \{f\} \subseteq \partial(T_7)$. By Lemma 2.2.2, this is a contradiction to (2.5). Hence $T - \{e, e', v_1v_m\}$ has two components L'_1 and L'_2 with $v_1 \in V(L'_1)$ and $v_m \in V(L'_2)$. By Claim 2(iii) and (iv), $N_H(v_0) - V(C_e) = \emptyset$ and $A_G(H) = \{v_0, v_m\}$. Thus either $\{e, v_1v_m\}$ is an edge cut of G , or there is an edge $z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L'_1)$ and $z_2 \in V(L'_2)$. Since G is essentially 3-edge-connected, if $\{e, v_1v_m\}$ is an edge cut of G , then one side of $G - \{e, v_1v_m\}$ is a singleton v_1 . In this case, as $T = v_0ev_1e_2v_2e_3v_3\dots v_{m-1}e'v_m$ with $v_2 = v_m$, we obtain an (e, e') -trail $T_8 = v_1ev_0e'_2v_2e_3v_3\dots v_{m-1}e'v_m$ where $e'_2 = v_0v_m \in E(C_r)$. Since $E(T_8) \subseteq \partial(T) \cup \{f\} \subseteq \partial(T_8)$, this leads to a contradiction to (2.5). Assume then that there is an edge $z_1z_2 \in \partial(T) - E(T)$ with $z_1 \in V(L'_1)$ and $z_2 \in V(L'_2)$. Since G is triangular, G has a cycle $C_{z_1z_2}$ of length 2 or 3 containing z_1z_2 , and so $T_9 = G[E(T)\Delta E(C_e)\Delta(C_{z_1z_2})]$ is an (e, e') -trail violating (2.7). As in either case, a contradiction is always obtained, we conclude that both $v_1 = v_{m-1}$ and $v_m \notin T^o$. Since v_1 is not a cut vertex of G , we must have $N_G(v_1) - V(C_e) = \emptyset$. It follows that we must have $s = 2$, and $T_{10} = v_1ev_0e''v_me'v_{m-1}$, having $f \in \partial(T_{10})$, which leads to a contradiction to (2.5). This proves (vi).

We continue our proof of Theorem 2.2.3. If $v_0 \neq v_m$, then by Claim 3(v), v_1 is a cut vertex of G . By Claim 2(iv), $\{v_0v_1, v_1v_m\}$ is an essential edge cut of G , contradicting the assumption

that G is essentially 3-edge-connected. Therefore, we must have $v_0 = v_m$. By Claim 2(iii), v_0 is a cut vertex of G and $V(C_e) = \{v_0, v_1, v_{m-1}\}$. By Claim 2(iv) and by the existence of f and $v_1 v_{m-1}$, $\{v_0 v_1, v_0 v_{m-1}\}$ is an essential edge-cut of G , which is a contradiction to the assumption that G is essentially 3-edge-connected. This final contradiction indicates that (2.5) does not hold, which proves Theorem 2.2.3. \square

2.3 Panconnected index for $G \in \mathcal{G}$

We start with some former results and lemmas. Recall that if $G \in \mathcal{G}$, then $\ell(G)$ is defined in (2.1).

Lemma 2.3.1. (Zhang et al, Lemma 3.2 [106]) *If $G \in \mathcal{G}$, then $L^{\ell(G)}(G)$ is triangular.*

Lemma 2.3.2. (Zhang et al, Proposition 2.3 [105]) *Let G be a simple connected triangular graph. Each of the following holds.*

- (i) *The line graph $L(G)$ is triangular.*
- (ii) *If G is k -connected, then $L(G)$ is $(k+1)$ -connected.*
- (iii) *If G is essentially k -edge-connected, then $L(G)$ is essentially $(k+1)$ -edge-connected.*

From the definition of line graphs, we make the following observations.

Observation 1. *Let $G \in \mathcal{G}$ be a graph, let $\mathcal{H}(G)$ denote the collection of all edge-induced subgraphs of G and let $\mathcal{L}(G)$ denote the collection of all induced subgraphs of $L(G)$.*

(i) *For any $H \in \mathcal{H}(G)$, by the definition of line graphs, $L(H) = L(G[E(H)])$ is an induced subgraph of $L(G)$, and so $L(H) \in \mathcal{L}(G)$. Conversely, if $\Gamma \in \mathcal{L}(G)$, then $H = G[V(\Gamma)] \in \mathcal{H}(G)$. Hence there exists a bijection between $\mathcal{H}(G)$ and $\mathcal{L}(G)$. We also use $L : \mathcal{H}(G) \mapsto \mathcal{L}(G)$ to denote this bijection, and L^{-1} denotes the inverse mapping of L . By the definition of iterated line graphs, for any integer $s > 1$, L^s is an operator mapping subgraphs in $\mathcal{H}(G)$ into subgraphs in $L^s(G)$; and L^{-s} pulls back induced subgraphs in $L^s(G)$ to subgraphs in $\mathcal{H}(G)$.*

(ii) *In particular, if $e \in E(G)$, we define $v_e = L(e)$. Thus $v_e \in V(L(G))$ is a cut vertex of $L(G)$ if and only if $\{e\}$ is an essential edge-cut of G ; if $v_{e_1} v_{e_2} \in E(L(G))$ is an edge which is not lying in a K_3 of $L(G)$, then $L^{-1}(v_{e_1} v_{e_2}) = G[\{e_1, e_2\}]$ is a divalent path of G .*

(iii) *By (i), we conclude that if P is a divalent path of length $h > 0$, then for any integer k with $0 \leq k < h$, $L^k(P)$ is a divalent path of length $h - k$ in $L^k(G)$; and $L^h(P)$ is a vertex of $L^h(G)$.*

(iv) *By (ii), we observe that for integers $s \geq t \geq 2$, if v is a cut vertex of $L^s(G)$, then $\{L^{-1}(v)\}$ is an essential edge cut of $L^{s-1}(G)$; and $L^{-2}(v)$ is a bridge divalent path of length 2 in $L^{s-2}(G)$. Inductively, if $s - t \geq 0$, then $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$.*

(v) *Similarly, if e is an edge which is not in a complete subgraph of order at least 3 in $L(G)$, $L^{-1}(e)$ is a divalent path of length 2 in G . For integers $s \geq t \geq 2$, if e is an edge which is not*

in a complete subgraph of order at least 3 in $L^s(G)$, then $\{L^{-1}(e)\}$ is a divalent path of length 2 in $L^{s-1}(G)$. Inductively, if $s - t \geq 0$, then $L^{-t}(e)$ is a divalent path of length $t + 1$ in $L^{s-t}(G)$.

We are now ready to prove the main results, restated below as Theorem 2.3.4. We observe that if G is a triangular graph, then G is connected and every edge of G lies in a cycle. Hence

$$\text{every triangular graph is 2-edge-connected.} \quad (2.9)$$

Lemma 2.3.3. *Let $G \in \mathcal{G}$ be a graph with $\ell = \ell(G) \geq 2$. Each of the following holds.*

- (i) *If G has a bridge divalent $(3, t)$ -path of length ℓ for some integer $t \geq 3$, then $pc(G) = \ell(G) + 2$.*
- (ii) *If G does not have any bridge divalent $(3, t)$ -path of length ℓ , then $pc(G) \leq \ell(G) + 1$.*

Proof. (i). Suppose first that $P = v_0 e_1 v_1 \cdots v_{\ell-1} e_\ell v_\ell$ is a bridge divalent $(3, t)$ -path for some integer $t \geq 3$ with $d_G(v_0) = 3$. Let e'_1, e'_2, e_1 be the three edges of G incident with v_0 . By the definition of line graphs, the neighbors of vertex e_1 in $L(G)$ are the vertices $\{e'_1, e'_2, e_2\}$, and so $L(P)$ is bridge divalent $(3, t')$ -path in $L(G)$ for some integer $t' \geq 3$. Inductively, we conclude that $L^{\ell-1}(P)$ is a cut edge $z_1 z_2$ of $L^{\ell-1}(G)$ such that $d_{L^{\ell-1}(G)}(z_1) = 3$ (say) and $d_{L^{\ell-1}(G)}(z_2) \geq 3$. Thus $\{z_1 z_2\}$ is an essential edge cut of $L^{\ell-1}(G)$. By Observation 1(i), the cut edge $z_1 z_2$ in $L^{\ell-1}(G)$ is a cut vertex v of $L^\ell(G)$. Since $d_{L^{\ell-1}(G)}(z_1) = 3$, the three edges in $N_{L^{\ell-1}(G)}(z_1)$ form a 3-cycle C of $L^\ell(G)$ containing the cut vertex v . Since v is a cut vertex of $L^\ell(G)$, the two edges in C incident with v form an essential edge cut of $L^\ell(G)$. By Observation 1, $L^{s+1}(G)$ is not 3-connected. By (2.2), $L^{\ell+1}(G)$ is not panconnected. Hence by (2.2), $pc(G) \geq \ell(G) + 2$. On the other hand, by (2.9) and by Lemmas 2.3.1 and 2.3.2, $L^{\ell+1}$ is triangular and essentially 3-edge-connected, and so by Theorem 2.2.3, $pc(G) = \ell(G) + 2$. This proves (i).

(ii). Assume that G does not have any bridge divalent $(3, t)$ -path of length ℓ . Let $P = v_0 e_1 v_1 \cdots v_{\ell-1} e_\ell v_\ell$ be a divalent (s, t) -path of length $\ell(G)$. Since P is a maximal divalent path of G , $s \neq 2$ and $t \neq 2$. Let $Q(G)$ be the collection of all divalent paths of G of length ℓ . We have the following cases.

Case 1. Every bridge divalent path of G of length ℓ is either an (s, t) -path with $s \geq t \geq 4$, or with $s \geq 3$ and $t = 1$.

By Observation 1(iii), for every $Q \in Q(G)$, $L^\ell(Q)$ is a vertex of $L^\ell(G)$. Moreover, if Q is a bridge divalent path, then $L^\ell(Q)$ is a cut vertex of $L^\ell(G)$. Since every bridge divalent path of G is either an (s, t) -path with $s \geq t \geq 4$, or with $s \geq 3$ and $t = 1$, $L^\ell(G)$ does not have any essential edge cut of size 2. By (2.9), $L^\ell(G)$ is essentially 3-edge-connected. By Lemma 2.3.1, $L^\ell(G)$ is triangular. By Theorem 2.2.3, $L(L^\ell(G))$ is panconnected, and so $pc(G) \leq \ell + 1$. Hence (ii) holds for Case 1.

Case 2. G does not have a bridge divalent path of length ℓ .

By Lemma 2.3.1, $L^\ell(G)$ is triangular. By (2.9), $L^\ell(G)$ is 2-edge-connected. If $L^\ell(G)$ has an essential edge cut X of size 2, then since $L^\ell(G)$ is triangular, X must be in a cycle of size 3, and

so the vertex incident with both edges in X must be a cut vertex v of $L^\ell(G)$. By Observation 1(vi), $L^{-\ell}(v)$ is a bridge divalent path of length ℓ of G , contradicting the assumption that G does not have a bridge divalent path of length ℓ . This contradiction implies that $L^\ell(G)$ is essentially 3-edge-connected. By Theorem 2.2.3, $L^\ell(G)$ is panconnected, and so $\text{pc}(G) \leq \ell + 1$. Hence (ii) holds for Case 2 as well. \square

Theorem 2.3.4. *For a graph $G \in \mathcal{G}$, $\text{pc}(G) \leq \ell(G) + 2$. Furthermore, if $\ell(G) \geq 2$, then $\text{pc}(G) = \ell(G) + 2$ if and only if G has a bridge divalent $(3, t)$ -path of length $\ell(G)$, for some integer $t \geq 3$.*

Proof. Let $G \in \mathcal{G}$ and $\ell = \ell(G)$. By Lemma 2.3.1, $L^\ell(G)$ is triangular. By (2.9), $L^\ell(G)$ is 2-edge-connected, so $L^\ell(G)$ is essentially 2-edge-connected. By Lemma 2.3.2, $L^{\ell+1}(G)$ is both triangular and essentially 3-edge-connected. It follows from Theorem 2.2.3 that $L^{\ell+2}(G)$ is panconnected.

Now assume that $\ell \geq 2$. If for some integer $t \geq 3$, G has a bridge divalent $(3, t)$ -path of length $\ell(G)$, then by Lemma 2.3.3(i), $\text{pc}(G) = \ell + 2$. Therefore we will assume that $\text{pc}(G) = \ell + 2$. Let $Q(G)$ be the collection of all divalent path of G of length ℓ . If every path in $Q(G)$ is not a bridge divalent path, or if every bridge divalent path $Q \in Q(G)$ is an (s, t) path such that either $\min\{s, t\} \geq 4$, or both $\max\{s, t\} \geq 3$ and $\min\{s, t\} = 1$, then by Lemma 2.3.3(ii), $\text{pc}(G) = \ell + 1$, contradicting the assumption that $\text{pc}(G) = \ell + 2$. Hence we must have a bridge divalent $(3, t)$ -path of length $\ell(G)$, for some integer $t \geq 3$. This completes the proof of the theorem. \square

Chapter 3

Supereulerian width of dense graphs

3.1 Main Results

Let \mathcal{F} be a family of graphs with finite number of members whose formal definition will be given in section 3.3. We looked into the problem of finding the supereulerian width $\mu'(G)$ under degree restriction and obtained the following theorem.

Theorem 3.1.1. *For any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for any simple graph G with $n = |V(G)|$, if for any pair of nonadjacent vertices u and v , $\max\{d_G(u), d_G(v)\} \geq an + b$, then $\mu'(G) \geq s + 1$ if and only if G is not contractible to a member in \mathcal{F} .*

Theorem 3.1.2. *For a simple graph G with $|V(G)| = n \geq 141$ and $\kappa'(G) \geq 3$, if for any pair of nonadjacent vertices u and v , $\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$, then $\mu'(G) \geq 3$ if and only if G is not contractible to $K_{3,3}$.*

When $a = \frac{1}{p}$ and $b = -1$, if $\delta(G) \geq \frac{n}{p} - 1$, then for any $u, v \in V(G)$ with $uv \notin E(G)$, $\max\{d_G(u), d_G(v)\} \geq \delta(G) \geq an + b$. Thus the hypothesis of Theorem 1.3.1(ii) implies a special case of the hypothesis of Theorem 3.1.1. Computationally, it takes the same order of computational complexity to examine finitely many graphs. In this sense, Theorem 3.1.1 extends Theorem 1.3.1(ii).

3.2 Reductions and s -Collapsible Graphs

Before we prove our main theorems, we will introduce the \mathcal{C}_s -reduction, which plays an important role in our work. Throughout the following sections, we shall adopt the convention that any graph G is 0-edge-connected, and always assume that $s \geq 1$ is an integer. The maximum number of edge-disjoint spanning trees in a graph G is denoted by $\tau(G)$.

Definition 3.2.1. A graph G is s -**collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, G has a spanning subgraph Γ_R such that

- (i) both $O(\Gamma_R) = R$ and $\kappa'(\Gamma_R) \geq s - 1$, and
- (ii) $G - E(\Gamma_R)$ is connected.

Catlin [12] first introduced collapsible graphs, which are exactly the 1-collapsible graphs defined here. A spanning subgraph Γ_R of G satisfying Definition 3.2.1 (i) and (ii) is an (s, R) -**subgraph** of G . Let \mathcal{C}_s denote the collection of all s -collapsible graphs. Then \mathcal{C}_1 is the collection of all collapsible graphs [12]. By definition, for $s \geq 1$, any $(s + 1, R)$ -subgraph of G is also an (s, R) -subgraph of G . Thus $\mathcal{C}_{s+1} \subseteq \mathcal{C}_s$ for any positive integer s .

For a graph G , and for $X \subseteq E(G)$, the **contraction** G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loops. If H is a subgraph of G , then we write G/H for $G/E(H)$. If H is a connected induced subgraph of G and z is the vertex in G/X onto which H is contracted, then we call H the **(contraction) preimage** of z , and define $PI_G(z) = H$. A vertex $z \in V(G/X)$ with $PI_G(z) \cong K_1$ is often referred as a trivial vertex under the contraction. The following are known.

Proposition 3.2.2. (Li [65], Corollary 2.4 of [66]) Let $s \geq 1$ be an integer. Then \mathcal{C}_s satisfies the following.

- (C1) $K_1 \in \mathcal{C}_s$
- (C2) If $G \in \mathcal{C}_s$ and if $e \in E(G)$, then $G/e \in \mathcal{C}_s$.
- (C3) If H is a subgraph of G and if $H, G/H \in \mathcal{C}_s$, then $G \in \mathcal{C}_s$.

Lemma 3.2.3. (Li [65], Corollary 2.5 of [66]) Let $s \geq 1$ be an integer. If a graph $G \in \mathcal{C}_s$, then $\mu'(G) \geq s + 1$.

A graph is \mathcal{C}_s -**reduced** if it contains no nontrivial subgraph in \mathcal{C}_s . It is shown in [66] that every graph G has a unique collection of maximally s -collapsible subgraphs H_1, H_2, \dots, H_c , and the graph $G'_s = G/(\cup_{i=1}^c E(H_i))$ is \mathcal{C}_s -reduced, which is called the \mathcal{C}_s -**reduction** of G . By the definition of \mathcal{C}_s -reduction and by Proposition 3.2.2, the \mathcal{C}_s -reduction of a graph is always \mathcal{C}_s -reduced.

Lemma 3.2.4. (Li [65], Corollary 2.9 of [66]) Let $s \geq 1$ be an integer, G be a graph and H be a subgraph of G such that $H \in \mathcal{C}_s$. Each of the following holds.

- (i) $G \in \mathcal{C}_s$ if and only if $G/H \in \mathcal{C}_s$.
- (ii) $\mu'(G) \geq s + 1$ if and only if $\mu'(G/H) \geq s + 1$. In particular, if G' is the \mathcal{C}_s -reduction of G , then $\mu'(G) \geq s + 1$ if and only if $\mu'(G') \geq s + 1$.

Let $F(G, s)$ denote the minimum number of additional edges that must be added to G to result in a graph Γ with $\tau(\Gamma) \geq s$. The value of $F(G, s)$ has been determined in [69], whose matroidal versions are proved in [46] and [65].

Theorem 3.2.5. *Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.*

- (i) *(Li [65], Theorem 2.11 of [66]) If $F(G, s+1) \leq 1$, then $G \in \mathcal{C}_s$ if and only if $\kappa'(G) \geq s+1$.*
- (ii) *(Catlin, Han and Lai, Theorem 1.3 of [13]) If $F(G, 2) \leq 2$, then G is 1-collapsible if and only if the \mathcal{C}_1 -reduction of G is not in $\{K_2, K_{2,t} : t \geq 1\}$.*

Lemma 3.2.6. *(Li [65], Corollary 2.13 of [66]) Let G be a connected nontrivial graph, and $s \geq 1$ be an integer.*

- (i) *If $\tau(G) \geq s+1$, then $G \in \mathcal{C}_s$.*
- (ii) *If G is \mathcal{C}_s -reduced, then for any nontrivial subgraph H of G , $\frac{|E(H)|}{|V(H)|-1} < s+1$.*
- (iii) *If $\kappa'(G) \geq s+1$ and G is \mathcal{C}_s -reduced, then*

$$F(G, s+1) = (s+1)(|V(G)| - 1) - |E(G)| \geq 2.$$

Let $\ell > 0$ be an integer and define ℓK_2 to be the graph with two vertices and ℓ edges connecting the two vertices. Catlin [12] showed that $\ell K_2 \in \mathcal{C}_1$ if and only if $\ell \geq 2$ and $K_n \in \mathcal{C}_1$ if and only if $n \geq 3$. Li et al present the following characterization for larger values of s .

Lemma 3.2.7. *(Li et al, Corollary 3.1 and Theorem 3.3 of [66]) Let ℓ, n be integers with $\ell \geq 1$ and $n \geq s \geq 2$. Each of the following holds.*

- (i) *$\ell K_2 \in \mathcal{C}_s$ if and only if $\ell \geq s+1$.*
- (ii) *$K_n \in \mathcal{C}_s$ if and only if $n \geq s+3$;*

Lemma 3.2.8. *(Li et al, Corollaries 2.4 and 2.9 of [66]) Let $s > 0$ be an integer. Each of the following holds.*

- (i) *$\mu'(K_1) \geq s+1$.*
- (ii) *If $e \in E(G)$, then $\mu'(G/e) \geq \mu'(G)$. In particular, if $\mu'(G) \geq s+1$ and $e \in E(G)$, then $\mu'(G/e) \geq s+1$.*

3.3 The General Lower Bound for Supereulerian Width

Following [3], if $V' \subseteq V(G)$ (or $X \subseteq E(G)$, respectively), then $G[V']$ (or $G[X]$) is the subgraph of G induced by V' (by X , respectively). If $v \in V(G)$, let $N_G(v)$ be the vertices of G adjacent to v in G . If H is a graph and Z is a set of edges such that the end vertices of each edge in Z are in $V(H)$, then $H + Z$ denotes the graph with vertex set $V(H)$ and edge set $E(H) \cup Z$. For an integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G , and $d_i(G) = |D_i(G)|$. By Lemma 3.2.6(iii) and with algebraic manipulations, we obtain the following lemma.

Lemma 3.3.1. *If G' is \mathcal{C}_s -reduced and $\kappa'(G') \geq s+1$, then $|E(G')| \leq (s+1)(|V(G')| - 1) - 2$ and*

$$\sum_{i=s+1}^{2s+1} (2s+2-i)d_i \geq \sum_{i \geq 2s+3} (i-2s-2)d_i + 2s+4. \quad (3.1)$$

Proof. By Lemma 3.2.6, for a \mathcal{C}_s -reduced graph H , we have

$$2(s+1) \sum_{i \geq 1} d_i(H) \geq \sum_{i \geq 1} i d_i(H) + 2s + 4. \quad (3.2)$$

Hence

$$\begin{aligned} & 2(s+1) \sum_{i=1}^s d_i(H) + 2(s+1) \sum_{i=s+1}^{2s+1} d_i(H) + 2(s+1) d_{2s+2}(H) + 2(s+1) \sum_{i \geq 2s+3} d_i(H) \\ & \geq \sum_{i=1}^s i d_i(H) + \sum_{i=s+1}^{2s+1} i d_i(H) + (2s+2) d_{2s+2}(H) + \sum_{i \geq 2s+3} d_i(H) + 2s + 4. \end{aligned}$$

Therefore

$$(2s+2-i) \sum_{i=1}^s d_i(H) + (2s+2-i) \sum_{i=s+1}^{2s+1} d_i(H) \geq \sum_{i \geq 2s+3} (i-2s-2) d_i(H) + 2s + 4.$$

As $\kappa'(G') \geq s+1$, we have $\sum_{i=1}^s d_i(H) = 0$, and so we have $(2s+2-i) \sum_{i=s+1}^{2s+1} d_i(H) \geq \sum_{i \geq 2s+3} (i-2s-2) d_i(H) + 2s + 4$. \square

Throughout the rest of this section, we assume that a, b are given real numbers with $0 < a < 1$, s is a fixed positive integer, and G is a simple graph on n vertices. Define

$$W = W_{a,b}(G) = \{v \in V(G) \mid d_G(v) < an + b\}. \quad (3.3)$$

Let G' denote the \mathcal{C}_s -reduction of G and let $n' = |V(G')|$. Then by definition, G' is \mathcal{C}_s -reduced. Define

$$W' = \{z \in V(G') \mid PI_G(z) \cap W \neq \emptyset\}. \quad (3.4)$$

Let $c = 2s + 2$ be a real number. Define

$$X'_c = \{z \in V(G') \mid d_{G'}(z) < c\} \text{ and } X''_c = \{z \in X'_c \mid PI_G(z) \neq K_1\}. \quad (3.5)$$

Let $N_1 = 1 + \max\{s+1, \frac{c-b}{a} + 1, \frac{-(a+2)(b+1)}{a^2}, (2s+3)(\lceil \frac{1}{a} \rceil + s+2) - 2s-4\}$ be an integer. Define $\mathcal{S}(a, b)$ to be the family of simple graphs such that a graph G is in $\mathcal{S}(a, b)$ if and only if G satisfies the following:

$$\text{for any } u, v \in V(G) \text{ with } uv \notin E(G), \max\{d_G(u), d_G(v)\} \geq an + b. \quad (3.6)$$

For a graph $G \in \mathcal{S}(a, b)$, recall that G' is the \mathcal{C}_s -reduction of G . As N_1 is completely determined by the values of a, b and s , N_1 is a finite number when a, b and s are given. Let $\mathcal{F} = \mathcal{F}(a, b, s) = \{G' \mid G \in \mathcal{S}(a, b), \mu'(G') \leq s \text{ and } |V(G')| \leq N_1\}$ to be the family of all \mathcal{C}_s -reductions with order at most N_1 of graphs in $\mathcal{S}(a, b)$ with supereulerian width at most s . Thus a graph $H \in \mathcal{F}$ if and only if H is the \mathcal{C}_s -reduction of a graph G in $\mathcal{S}(a, b)$, such that $\mu'(G') \leq s$ and $|V(G')| \leq N_1$. We have the following observations.

Proposition 3.3.2. *Each of the following holds.*

(i) *If a graph G is contractible to a member in \mathcal{F} , then $\mu'(G) \leq s$.*

(ii) *\mathcal{F} contains only finitely many graphs.*

(iii) *If $\kappa'(G) \leq s$, then G is contractible to a member in \mathcal{F} .*

Proof. As (i) follows from Lemma 3.2.8 (ii), we start our proofs of (ii). By definition, every graph in \mathcal{F} has at most N_1 vertices. By Lemma 3.2.7(i), if $G \in \mathcal{F}$, every edge of G has multiplicity at most s . Thus there are finitely many members in \mathcal{F} and so (ii) holds.

By definition, $\kappa'(H) \geq \mu'(H)$ for any graph H . In particular, for any integer $\ell > 0$, $\mu'(\ell K_2) \leq \ell$ and so as $N_1 \geq s + 2$, we have $\ell K_2 \in \mathcal{F}$, for all $1 \leq \ell \leq s$. By definition, a connected graph G satisfies $\kappa'(G) = \ell > 0$ if and only if G can be contracted to a ℓK_2 . Thus (iii) must hold. \square

Now we are ready to prove Theorem 3.1.1 which we restate below.

Theorem 3.3.3. *For any real numbers a, b with $0 < a < 1$ and any integer $s > 0$, there exists a finite family $\mathcal{F} = \mathcal{F}(a, b, s)$ such that for any simple graph G with $n = |V(G)|$, if for any pair of nonadjacent vertices u and v , $\max\{d_G(u), d_G(v)\} \geq an + b$, then $\mu'(G) \geq s + 1$ if and only if G is not contractible to a member in \mathcal{F} .*

Necessity of Theorem 3.3.3. Let $G \in \mathcal{S}(a, b)$. If G is contractible to a member in \mathcal{F} , then by Proposition 6(i) and by the definition of \mathcal{F} , we have $\mu'(G) \leq s$.

Sufficiency of Theorem 3.3.3. We assume that $G \in \mathcal{S}(a, b)$ and that

$$G \text{ cannot be contracted to a member of } \mathcal{F}, \quad (3.7)$$

to show that $\mu'(G) \geq s + 1$. By (3.7) and by Proposition 6(iii), we in the rest of the proof will assume that

$$\kappa'(G) \geq s + 1 \text{ and } n = |V(G)| \geq N_1. \quad (3.8)$$

Pick any $z \in X_c'' - W'$ and let $H_z = PI_G(z)$. For each $v \in V(H_z)$, by (3.6), we have $|V(H_z)| \geq 1 + d_G(v) - d_{G'}(z) \geq an + b + 1 - c$. We claim that

$$\text{there must be a vertex } v' \in V(H_z) \text{ such that } N_G(v') \cap [V(G) - V(H_z)] = \emptyset \text{ for any } z \in X_c'' - W'. \quad (3.9)$$

If (3.9) does not hold, then every vertex in H_z is adjacent to at least one vertex which is not in H_z . Let $|V(H_z)| = k$. Since $d_{G'}(z) < c$, we have $k \leq d_{G'}(z) \leq c - 1$. Since $n \geq N_1 \geq \frac{c-b}{a} + 1$, we have $an + b \geq c + 1$. This, together with the assumption that $z \in X_c'' - W'$, implies $d_G(v) \geq c + 1$ for any $v \in H_z$.

For any $v \in H_z$, let m_v be the number of edges not in H_z but incident with v . If for any $v \in V(H_z)$, we have $m_v > \frac{c-1}{k}$, then $d_{G'}(z) = \sum_{v \in H_z} m_v > k \times \frac{c-1}{k} = c - 1$ which contradicts our assumption that $d_{G'}(z) \leq c - 1$. Hence there must be a vertex $v_0 \in H_z$ such that $m_{v_0} \leq \frac{c-1}{k}$,

and so we have $k - 1 \geq |N_G(v_0) \cap V(H_z)| = d_G(v_0) - m_{v_0} \geq c + 1 - \frac{c-1}{k}$. Thus we have $k > c + 1$ which contradicts $k \leq c - 1$. Hence, it is impossible that every vertex in H_z is adjacent to a vertex which is not in H_z , implying that (3.9) must hold.

By (3.9), it follows that $|V(H_z)| \geq 1 + d_G(v')$, and so we have

$$\text{for any } z \in X_c'' - W', |V(PI_G(z))| \geq an + b + 1. \quad (3.10)$$

Applying (3.10), we count the number of vertices contained in the preimages of vertices in $X_c'' - W'$ to get $n \geq \sum_{z \in X_c'' - W'} |V(PI_G(z))| \geq |X_c'' - W'| (an + b + 1)$. It follows by $n \geq N_1 > \frac{-(a+2)(b+1)}{a^2}$ that $|X_c'' - W'| \leq \frac{n}{an+b+1} < \frac{1}{a} + \frac{1}{2}$, and so

$$|X_c'' - W'| \leq \lceil \frac{1}{a} \rceil. \quad (3.11)$$

By (3.3) and (3.6), $G[W]$ is a complete subgraph of G . By Lemma 3.2.7 and (3.4), we conclude that

$$|W'| \leq s + 2. \quad (3.12)$$

By (3.5), we have $X_c' - X_c'' \subseteq W'$. Since $c = 2s + 2$, we have $|X_c'| \geq \sum_{i=s+1}^{2s+1} d_i$. It now follows from (3.1) that

$$\begin{aligned} (2s+2)|X_c'| &\geq (2s+2) \sum_{i=s+1}^{2s+1} d_i \geq \sum_{i=s+1}^{2s+1} (2s+2-i)d_i \geq \sum_{i \geq 2s+3} (i-2s-2)d_i + 2s+4 \\ &\geq \sum_{i \geq 2s+3} d_i + 2s+4 = |V(G')| - |X_c'| + 2s+4. \end{aligned}$$

As $X_c' \subseteq W' \cup X_c'' = W' \cup (X_c'' - W')$, this, together with (3.11) and (3.12), implies

$$n' \leq (2s+3)|X_c'| - 2s-4 \leq (2s+3)(|W'| + |X_c'' - W'|) - 2s-4 \leq (2s+3)(\lceil \frac{1}{a} \rceil + s+2) - 2s-4.$$

Hence $|V(G')| \leq N_1$. By (3.7), we must have $\mu'(G') \geq s+1$. It follows by Lemma 3.2.4 that $\mu'(G) \geq s+1$. This completes the proof of the sufficiency of Theorem 3.3.3(i). \square

3.4 The Lower Bound of Supereulerian Width on A Particular Case

To characterize the graphs in \mathcal{F} with more details, we considered the case when $a = \frac{1}{4}$, $b = -\frac{3}{2}$ in Theorem 3.1.1. Throughout this section, we assume $a = \frac{1}{4}$, $b = -\frac{3}{2}$ and $s = 2$ in our discussion. We will use the notation in the previous section, set $c = 2s + 2 = 6$ and follow (3.3), (3.4) and (3.5) to define W , W' , X_c' and W_c'' , respectively. The main goal of this section is to prove Theorem 3.1.2. We will need the following lemmas in this section.

Lemma 3.4.1. (Theorem 4.4 of [66]) Let G be graph on $n \leq 6$ vertices. Then $\mu'(G) \geq 3$ if and only if $G \not\cong K_{3,3}$.

Lemma 3.4.2. (Catlin [12], Corollary 1.) Let G be a connected graph. If for any $e \in E(G)$, G has a collapsible subgraph H_e with $e \in E(H_e)$, then G is collapsible.

Lemma 3.4.3. Let G be a graph with $|V(G)| = n \geq 138$ and $\kappa'(G) \geq 3$ and let G' be the \mathcal{C}_2 -reduced graph of G . If for any $u, v \in V(G)$ with $uv \notin E(G)$,

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2} \quad (3.13)$$

then $|V(G')| \leq 7$.

Proof. By (3.13), the subgraph of G' induced by vertices in W' must be a complete graph. Thus by Lemma 3.2.7 with $s = 2$, we conclude that $|W'| \leq 4$.

Let $k = |V(G') - W'|$. We shall show that $k \leq 4$. Firstly, we claim that

$$\text{for any } z \in V(G') - W', \text{ there exists } v \in V(PI_G(z)) \text{ such that } N_G(v) \subseteq V(PI_G(z)). \quad (3.14)$$

Fix a $z_0 \in V(G') - W'$ which violates (3.14). Then for every $v \in V(PI_G(z_0))$, we have $N_G(v) - V(PI_G(z_0)) \neq \emptyset$. By (3.9), $z_0 \notin X_6'' - W'$. By (3.4), (3.5) and (3.13), for any $z \in X_6'' - W'$, we have $|V(PI_G(z))| \geq \frac{n}{4} - \frac{1}{2}$. It follows that $n \geq (\frac{n}{4} - \frac{1}{2})|X_6'' - W'|$, implying $|X_6'' - W'| \leq 4$.

Since G' is \mathcal{C}_2 -reduced, by Lemma 3.2.6 (iii) with $s = 2$, we have $|E(G')| \leq 3(|V(G')| - 1) - 2 \leq 3(k + 4 - 1) - 2 = 3k + 7$. Denote $|V(PI_G(z_0))| = m$. As $|N_G(v) - V(PI_G(z))| \geq 1$ for each $v \in V(PI_G(z_0))$,

$$\begin{aligned} 0 < m &\leq d_{G'}(z_0) = \sum_{z \in V(G')} d_{G'}(z) - \sum_{z \in V(G') - z_0} d_{G'}(z) \leq 2(3k + 7) - \sum_{z \neq z_0} d_{G'}(z) \\ &\leq 2(3k + 7) - 3|X_6'' - W'| - \sum_{z \neq z_0, z \notin X_6'' - W'} d_{G'}(z) \leq 2(3k + 7) - 3 \times 4 - (k - 5)6 = 33 \end{aligned}$$

It follows that there exists $v \in V(PI_G(z_0))$ such that $|N_G(v) - V(PI_G(z_0))| \leq \frac{33}{m}$. As $n \geq 138$, we have $\frac{n}{4} - \frac{1}{2} \geq 34$. Thus $34 \leq d_G(v) \leq \frac{33}{m} + m - 1$, forcing either $m \leq 0$ or $m \geq 34$, contrary to (3.15). This justifies (3.14).

By (3.14), for each $z \in V(G') - W'$, there exists $v \in V(PI_G(z))$ such that $N_G(v) \subseteq V(PI_G(z))$. It follows by (3.13) that,

$$|V(PI_G(z))| \geq |N_G(v) \cup \{v\}| = d_G(v) + 1 \geq \frac{n}{4} - \frac{1}{2}. \quad (3.15)$$

As $n = |V(G)| \geq \bigcup_{z \in V(G') - W'} |V(PI_G(z))| \geq |V(G') - W'| \left(\frac{n}{4} - \frac{1}{2} \right)$, and our choice of n , we conclude that $|V(G') - W'| \leq 4$.

Thus $|V(G')| = |W'| + |V(G') - W'| \leq 8$, where equality holds if and only if $|W'| = |V(G') - W'| = 4$. If $|V(G')| = 8$, then by the choice of n and by (3.15), we have $8 = |V(G')| \leq 4 + \left[n - 4 \left(\frac{n}{4} - \frac{1}{2} \right) \right] = 6$, a contradiction. Therefore we must have $|V(G')| \leq 7$. \square

Lemma 3.4.4. *Let J be a graph $\kappa'(J) \geq 3$ and $|V(J)| \leq 4$. For any edge subset $X \subseteq E(J)$ with $1 \leq |X| \leq 2$ such that $J[X]$ is a path, each of the following holds.*

- (i) $J - X$ is 1-collapsible if and only if $\kappa'(J - X) \geq 2$.
- (ii) If $J \in \{K_4, K_4 - e\}$ where $e \in E(K_4)$, then J is 1-collapsible.

Proof. Since every edge of K_4 and $K_4 - e$ lies in a triangle, by Lemma 3.4.2, we have (ii) holds. By Lemma 3.2.3, if $J - X$ is 1-collapsible, then $\kappa'(J - X) \geq \mu'(J - X) \geq 2$. It remains to show the sufficiency of (i).

Suppose $\kappa'(J - X) \geq 2$ and assume that $J - X$ is not collapsible. Let $(J - X)'$ be the 1-reduction of $J - X$. Then by Proposition 3.2.2 (C3), we have $2 \leq |V[(J - X)']| \leq 4$. As it is known (Page 38 of [12]) that every 2-edge-connected graph with at most 3 vertices are 1-collapsible, we must have that $|V[(J - X)']| = 4$. Since $\kappa'[(J - X)'] = \kappa'(J - X) \geq 2$, we have $|E(J')| \geq \frac{1}{2} \sum_{v \in V(J')} d_{J'}(v) \geq \frac{4 \times 2}{2} = 4$. It follows by Lemma 3.2.6 (iii) that $F((J - X)', 2) \leq 2$. By Lemma 3.2.5 (ii), we have $(J - X)' \cong K_{2,2}$. This implies that $(J - X)' = J - X$. Since $\kappa'(J) \geq 3$, and since $J - X = K_{2,2}$, X must be a matching of size 2 in J , contrary to the assumption that $J[X]$ is a path in J . This completes the proof. \square

Lemma 3.4.5. *Let H be a graph with $\kappa'(H) \geq 3$ and $|V(H)| = 7$. If H contains a subgraph $L \cong K_4$, then for any distinct $u, v \in H$ there exists a (u, v) -path P in H such that $H - E(P)$ is 1-collapsible.*

Proof. For integers $\ell > 0$ and $t > 0$, let ℓP_t denote the graph obtained from a path $P_t = z_1 z_2 \dots z_t$ by replacing each edge of P_t by a set of ℓ parallel edges. We will use this notation in the proof. Note that if H/L is spanned by a $3P_4$, then by Lemma 3.4.2, Lemma 3.4.5 holds trivially. Hence we assume that

$$H/L \text{ is not spanned by a } 3P_4. \quad (3.16)$$

Fix the vertices $u, v \in V(H)$. If $uv \in E(G)$, then let $P = H[\{uv\}]$, $L_1 = L$ if $uv \notin E(L)$ and $L' = L - uv$ if $uv \in E(L)$. Then $J = H/L' \cong H/L$ is a 3-edge-connected graph with $|V(J)| \leq 4$. It follows by Lemma 3.4.4(i) that $(H - E(P))/L'$ is 1-collapsible. By Lemma 3.4.4(ii) and Proposition 3.2.2 (C3), $H - E(P)$ is 1-collapsible. Hence in the following arguments, we assume that $uv \notin E(H)$, and so $|\{u, v\} \cap V(L)| \leq 1$. Let $J = H/L$ and v_L be the vertex in J onto which L is contracted. We further assume that if $|\{u, v\} \cap V(L)| = 1$, then $v \in V(L)$, in which case we adopt the convention to denote $v = v_L$.

By the assumption of the lemma, $|V(J)| = 4$ with $\kappa'(J) \geq 3$. By (3.16), it is a routine inspection to conclude that J always has a (u, v) -path P' with $|E(P')| \leq 2$ and $\kappa'(J - E(P')) \geq 2$. It follows by Lemma 3.4.4 (i) that $J - E(P')$ is 1-collapsible.

If $v_L \notin V(P')$, then $v \notin V(L)$ and so P' is a path in H , in this case we define $P = P'$. If $v_L \in V(L)$ such that P' is a (u, v_L) -path in J , then $v \in V(L)$. In this case, let $v' \in V(L)$ be the vertex in L such that $H[E(P')]$ is an (u, v') -path; and define $P = P'$ if $v = v'$, and $P = P' + v'v$ if $v \neq v'$. If $v_L \in V(P')$ is an internal vertex of P' , then $v \notin V(L)$ and there exist distinct vertices $v', v'' \in V(L)$ such that $H[E(P')]$ consists of a (u, v') -path T' and a (v'', v) -path T'' . In this case we define $P = T' + v_i v'_i + T''$. In any case, P is a (u, v) -path in H satisfying $|E(P) \cap E(L)| \leq 1$. By Lemma 3.4.4, $L - (E(P) \cap E(L))$ is 1-collapsible. By the definition of contraction,

$$(H - E(P)) / (L - (E(P) \cap E(L))) = (H - E(P')) / L = J - E(P'),$$

is 1-collapsible. We conclude that $H - E(P)$ is 1-collapsible by applying Proposition 3.2.2(C3). \square

With these Lemmas, now we are ready to present the proof of Theorem 3.1.2 which we restate below.

Theorem 3.4.6. *For a simple graph G with $|V(G)| = n \geq 141$ and $\kappa'(G) \geq 3$, if for any pair of nonadjacent vertices u and v , $\max\{d_G(u), d_G(v)\} \geq \frac{n}{4} - \frac{3}{2}$, then $\mu'(G) \geq 3$ if and only if G is not contractible to $K_{3,3}$.*

Proof of Theorem 3.4.6. Necessity: Let G be a graph which is contractible to $K_{3,3}$, then by Lemma 3.4.1 and Lemma 3.2.8 (ii), $\mu'(G) \leq 2$.

Sufficiency: Let G be a graph which is not contractible to $K_{3,3}$. Let G' be the \mathcal{C}_2 -reduction of G . Then by Lemma 3.4.3, $|V(G')| \leq 7$. If $|V(G')| \leq 6$, then since G is not contractible to $K_{3,3}$ and by Lemma 3.4.1 we have $\mu'(G') \geq 3$. If $|V(G')| = 7$, then by Lemma 3.4.5 we have $\mu'(G') \geq 3$. Finally, since $\mu'(G') \geq 3$, by Lemma 3.2.8 we know that $\mu'(G) \geq 3$. \square

3.5 Future Studies

As a natural generalization of supereulerian problem, there are many topics of supereulerian width to be explored. We will investigate lower bounds of supereulerian width of graphs with restrictions of different types of degree conditions and characterize the corresponding exceptional graphical families.

Chapter 4

s -hamiltonicity and s -hamiltonian -connectedness of Line Graphs

4.1 Main Results

We verified the validity of Conjecture 1.4.7 with in the graphs which does not contain an hourglass as an induced subgraph and obtained the following.

Theorem 4.1.1. *Let $L(G)$ be an hourglass-free line graph and s be an integer. Each of the following holds.*

- (i) *If $s \geq 2$, then $L(G)$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.*
- (ii) *If $s \geq 1$, then $L(G)$ is s -hamiltonian-connected if and only if $\kappa(L(G)) \geq s + 3$.*

Throughout this work, we use P_k to denote a path of order k . For integers $s_1, s_2, s_3 \geq 0$, let N_{s_1, s_2, s_3} denote the graph formed by identifying each vertex of a K_3 with an end vertex of three disjoint paths $P_{s_1+1}, P_{s_2+1}, P_{s_3+1}$ of length s_1, s_2 , and s_3 , respectively. We call N_{s_1, s_2, s_3} a net. A graph G is $\{H_1, H_2, \dots, H_s\}$ -free if G contains no induced subgraph isomorphic to any copy of H_i for any i . If $s = 1$, then an $\{H_1\}$ -free graph is simply called an H_1 -free graph. We also verified the validity of Conjecture 1.4.7 with in the graphs which does not contain a net as an induced subgraph and obtained the following result.

Theorem 4.1.2. *Let s be an integer.*

- (i) *Let $\mathcal{N}_1 = \{N_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1 + s_2 + s_3 \leq 6\}$. Then for any $N \in \mathcal{N}_1$, every N -free line graph $L(G)$ with $|V(L(G))| \geq s + 3$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$ for $s \geq 1$.*
- (ii) *Let $\mathcal{N}_2 = \{N_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1 + s_2 + s_3 \leq 4\}$. Then for any $N \in \mathcal{N}_1$, every N -free line graph $L(G)$ with $|V(L(G))| \geq s + 3$ is s -Hamilton-connected if and only if $\kappa(L(G)) \geq s + 3$ for $s \geq 0$.*

Theorem 4.1.2 extends Theorem 1.4.3(i) and furthers the main results in [102].

4.2 Preliminaries

As deleting vertices in $L(G)$ amounts to deleting the corresponding edges in G and then removing the resulting isolated vertices, for simplicity, we use $G - S$ in the discussions instead of $G - S - D_0(G - S)$. Throughout this article, isolated vertices arising from edge deletion will be deleted automatically unless otherwise specified.

A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. The following is well-known.

Theorem 4.2.1. (*Harary and Nash-Williams [36]*) *For a connected graph G with $|E(G)| \geq 3$, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

For a graph G and an integer $k > 0$, a k -edge-cut Y of G is an essential k -edge-cut of G if each component of $G - Y$ has an edge. If a connected graph G does not have an essential k' -edge-cut for any $k' < k$, then G is **essentially k -edge-connected**. The largest integer k such that a connected graph G is essentially k -edge-connected is denoted by $ess'(G)$. It is observed ([92]) that for a graph G , $\kappa(L(G)) \geq k$ if and only if either $L(G)$ is a complete graph of order at least $k + 1$ or $ess'(G) \geq k$.

Definition 4.2.2. *Let $X_1(G) = \{e \in E(G) : e \text{ is incident with a vertex in } D_1(G)\}$. For each vertex $v \in D_2(G)$, let $E_G(v) = \{e_v, e'_v\}$ be the set of edges incident with v . The **core** of G is the graph G_0 defined below.*

$$\begin{aligned} X_2(G) &= \{e_v : v \in D_2(G)\}, X'_2(G) = \{e'_v : v \in D_2(G)\}, \\ G_0 &= G / (X_1(G) \cup X'_2(G)). \end{aligned} \tag{4.1}$$

Following [3], for $u, v \in V(G)$, a uv -**trail** is a trail of G from u to v . For $e, e' \in E(G)$, an (e, e') -trail is a trail of G starting from e and ending at e' . An (e, e') -trail T is **dominating** if each edge of G is incident with at least one internal vertex of T , and T is **spanning** if T is a dominating trail with $V(T) = V(G)$. A graph G is **spanning trailable** if for each pair of edges e_1 and e_2 , G has a spanning (e_1, e_2) -trail.

Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G . If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_e v_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'} v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_e v_1$. As defined in [68], a graph G is **strongly spanning trailable (SST in short)** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$ -trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. Since $e = e'$ is possible, SST graphs are both spanning trailable and supereulerian. The following former tools are useful.

Lemma 4.2.3. (Shao, Lemma 1.4.1 and Proposition 1.4.2 of [92]) Let G be a connected non-trivial graph such that $\kappa(L(G)) \geq 3$ and G_0 be the core of G . Then G_0 is uniquely determined by G with $\delta(G_0) \geq \kappa'(G_0) \geq 3$. Furthermore, each of the following holds.

- (i) $L(G)$ is hamiltonian if and only if G_0 has a dominating eulerian subgraph containing the contraction preimages of the edges in $X_1(G) \cup X'_2(G)$. In particular, if G_0 is supereulerian, then $L(G)$ is hamiltonian.
- (ii) (see also Lemma 2.9 of [56]) If G_0 is strongly spanning trailable, then $L(G)$ is Hamilton-connected.
- (iii) (see also Proposition 2.2 of [56]) $L(G)$ is Hamilton-connected if and only if for any pair of edges $e, e' \in E(G)$, G has a dominating (e, e') -trail.

Let $X \subseteq E(G)$, which is also viewed as a vertex set in the line graph $L(G)$. Imitating the arguments in [36, 92] and in Theorem 2.7 of [55], and by (4.1), we have the following observation.

Proposition 4.2.4. Let $s \geq 0$ be an integer, G be a connected graph with $|E(G)| \geq s + 3$ and $ess'(G) \geq 3$, and G_0 be the core of G .

- (i) (Theorem 2.7 of [55]) The line graph $L(G)$ is s -hamiltonian if and only if for any $X \subseteq E(G)$ with $|X| \leq s$, $G - X$ has a dominating eulerian subgraph.
- (ii) If for any $X \subseteq E(G_0)$ with $|X| \leq s$, $G_0 - X$ is supereulerian, then $L(G)$ is s -hamiltonian.

Let \mathcal{K}_0 denote the connected graphs that does not have any essential edge-cut. Then it is routine to verify that \mathcal{K}_0 consists of connected graphs that are either spanned by a K_3 or contains a vertex incident with all edges. If $G \in \mathcal{K}_0$, then define $ess'(G) = |E(G)| - 1$; otherwise let $ess'(G)$ be the largest integer k such that G is essentially k -edge-connected. We also obtained the following lemma.

Lemma 4.2.5. Suppose $G \notin \mathcal{K}_0$ with $ess'(G) \geq 3$ and let G_0 be the core of G . Let $s \geq 0$ be an integer, $S \subset E(G_0)$ with $|S| \leq s$, and $G_S = (G_0 - S) - D_1(G_0 - S)$.

- (i) If for any $S \subset E(G_0)$ with $|S| \leq s$, G_S is supereulerian, then for any edge subset $Z \subset E(G)$ with $|Z| \leq s$, $G - Z$ has a dominating eulerian trail. Consequently, $L(G)$ is s -hamiltonian.
- (ii) If for any $S \subset E(G_0)$ with $|S| \leq s$, G_S is strongly spanning trailable, then for any edge subset $Z \subset E(G)$ with $|Z| \leq s$, and any $e_1, e_2 \in E(G) - S$, $G - Z$ has a dominating (e_1, e_2) -trail. Consequently, $L(G)$ is s -Hamilton-connected.

Proof. Fix a subset $Z \subseteq E(G)$ with $|Z| \leq s$. Define $X_1(G)$ and $X_2(G)$ as in Definition 4.2.2. Let $S = Z - (X_1(G) \cup X'_2(G))$. We adopt the convention that

$$\text{if } v \in D_2(G) \text{ and } (Z - X_1(G)) \cap E_G(v) = \{e\}, \text{ then we assume that } e \in S \cap X_2(G). \quad (4.2)$$

Thus $S \subset E(G_0)$ with $|S| \leq |Z| \leq s$.

Suppose that G_S is supereulerian, which implies that $G - Z$ has a dominating eulerian trail. By Proposition 4.2.4, $L(G)$ is s -hamiltonian. This proves (i).

Assume that G_S is strongly spanning trailable. Let $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G) - Z$. If both $e_1, e_2 \in E(G_S)$, then by assumption, $G_S(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2}) -trail T' , which, by (4.2), can be lifted and modified to a dominating (e_1, e_2) -trail of $G - Z$. Next, we assume that $e_1 \in E(G_S)$ and $e_2 \notin E(G_S)$. By (4.2), we may assume that $u_2 \in V(G_S)$. By Lemma ?? (ii), there exists an edge $e'_2 = u_2u'_2 \in E_{G_S}$. Hence $G_S(e_1, e'_2)$ has a spanning $(v_{e_1}, v_{e'_2})$ -trail, which can be lifted as a spanning $(v_{e_1}, v_{e'_2})$ -trail T'' of $((G - S) - D_1(G - S))(e_1, e'_2)$. Thus $E(T'' - v_{e'_2}u'_2) \cup \{e'_2, u_2v_{e_2}\}$ (if $u'_2v_{e'_2} \in E(T'')$) or $E(T'' - v_{e'_2}u_2) \cup \{u_2v_{e_2}\}$ (if $u_2v_{e'_2} \in E(T'')$) induces a spanning (v_{e_1}, v_{e_2}) -trail of $((G - S) - D_1(G - S))(e_1, e'_2)$. By (??), $G - Z$ has a dominating (e_1, e_2) -trail. Finally, we assume that $e_1, e_2 \notin E(G_S)$. By Lemma ?? (ii), there exist $e'_1 = u_1u'_1, e'_2 = u_2u'_2 \in E(G_S)$. By assumption, $G_S(e'_1, e'_2)$ has a spanning $(v_{e'_1}, v_{e'_2})$ -trail, which can be lifted as a spanning $(v_{e'_1}, v_{e'_2})$ -trail T''' of $((G - S) - D_1(G - S))(e'_1, e'_2)$. Thus $E(T''' - \{v_{e'_1}u'_1, v_{e'_2}u'_2\}) \cup \{e'_1, e'_2, u_1v_{e_1}, u_2v_{e_2}\}$ (if $v_{e'_1}u'_1, v_{e'_2}u'_2 \in E(T''')$) or $E(T''' - \{v_{e'_1}u'_1, v_{e'_2}u_2\}) \cup \{e'_1, u_1v_{e_1}, u_2v_{e_2}\}$ (if $v_{e'_1}u'_1, v_{e'_2}u_2 \in E(T''')$) or $E(T''' - \{v_{e'_1}u_1, v_{e'_2}u_2\}) \cup \{u_1v_{e_1}, u_2v_{e_2}\}$ (if $v_{e'_1}u_1, v_{e'_2}u_2 \in E(T''')$) induces a spanning (v_{e_1}, v_{e_2}) -trail of $((G - S) - D_1(G - S))(e_1, e'_2)$. By symmetry and by (??), $G - Z$ always has a dominating (e_1, e_2) -trail. This proves (ii). \blacksquare

4.3 Catlin's Reduction Method

In [12], Catlin defined collapsible graphs. A graph is **collapsible** if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod{2}$, G has a spanning connected subgraph Γ such that $O(\Gamma) = R$. For a graph G and an edge subset $X \subseteq E(G)$, G/X denotes the graph obtained from G by contracting each edge in X and then deleting resulting loops. If H is a subgraph of G , then we use G/H for $G/E(H)$. If H is connected, and if v_H is the vertex in G/H onto which H is contracted, then H is the **pre-image** of v_H in G/H . If H_1, H_2, \dots, H_k are the list of all maximal collapsible subgraphs of G , then $G' = G/(\cup_{i=1}^k H_i)$ is the **reduction** of G ; a graph is **reduced** if it is the reduction of some graph. Let C_6^{++} denote the graph obtained from C_6 with $V(C_6) = \{v_1, v_2, \dots, v_6\}$ by adding edges v_2v_5 and v_3v_6 and let $K_{3,3}^- = K_{3,3} - e$ for any edge $e \in E(K_{3,3})$.

The next theorem briefs some of the useful properties related to collapsible graphs.

Theorem 4.3.1. *Let G be a connected graph. Each of the following holds.*

- (i) (Catlin, Corollary of Theorem 3 in [12]) *Let H be a collapsible subgraph of G . Then G is supereulerian (collapsible, respectively) if and only if G/H is supereulerian (collapsible, respectively). In particular, if G' is the reduction of G , then G is supereulerian (collapsible, respectively) if and only if G' is supereulerian (a K_1 , respectively).*
- (ii) (Catlin, Theorem 8 of in [12]) *If a connected graph G is reduced and not in $\{K_1, K_2\}$, then*

$|E(G)| \leq 2|V(G)| - 4$, $\delta(G) \leq 3$ and $g(G) \geq 4$.

(iii) (Catlin, Theorem 5 in [12]) G is reduced if and only if G has no nontrivial collapsible subgraphs. In particular, reduced graphs are simple graphs.

(iv) (Lemma 2.1 of [60]) Let G be a connected simple graph with $n \leq 8$ vertices and with $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$, then the reduction of G is in $\{K_1, K_2, K_{2,3}\}$. Consequently, $K_{3,3}^-$ and C_6^{++} are collapsible.

(v) Let H be a collapsible subgraph of G and let v_H denote the vertex in G/H onto which H is contracted. If G/H has a trail Γ' with $v_H \in V(\Gamma')$, then G has a trail Γ with $E(\Gamma') \subseteq E(\Gamma)$ and $V(H) \subseteq V(\Gamma)$ (The graph Γ is often called a **lift** of Γ').

(vi) (Li et al., Lemma 2.2 of [60]) If G is collapsible, then for any $u, v \in V(G)$, G has a spanning (u, v) -trail.

Proof. It suffices to prove (ii). Let G , H and v_H be given as stated in Lemma 4.3.1 (ii), and assume that Γ' is a trail of G/H with $v_H \in V(\Gamma')$. Let $X = \{w \in V(H) : w \text{ is incident with an odd number of edges in } E(\Gamma')\}$. Since v_H has even degree in Γ' , $|X| \equiv 0 \pmod{2}$. As H is collapsible, there is a spanning connected subgraph R_X of H with $O(R_X) = X$. It follows that $\Gamma = G[E(\Gamma') \cup E(R_X)]$ is a trail of G with $E(\Gamma') \subseteq E(\Gamma)$ and $V(H) \subseteq V(\Gamma)$. This proves (ii). ■

For a graph G , let $\tau(G)$ be the maximum number of edge-disjoint spanning trees packed in G and let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph G_1 with $\tau(G_1) \geq 2$. Thus, $\tau(G) \geq 2$ if and only if $F(G) = 0$.

Theorem 4.3.2. *Let G be a connected graph. Each of the following holds.*

(i) (Catlin et al., Theorem 1.3 in [13]) *If $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.*

(ii) (Theorem 1.1 in [17]) *For any integer $k > 0$, $\kappa'(G) \geq 2k$ if and only if for any edge subset X with $|X| \leq k$, $\tau(G - X) \geq k$.*

(iii) (Catlin et al., Theorem 4 in [16]) *If $F(G) = 0$, then for any $e', e'' \in E(G)$, G has a spanning (e', e'') -trail if and only if $\{e', e''\}$ is not an essential edge-cut of G .*

Lemma 4.3.3. *Let G be a graph, s be an integer, and S be any subset of $E(G)$ with $|S| \leq s$. Each of the following holds.*

(i) *If $s \geq 2$ and $\kappa'(G) \geq s + 2$, then $G - S$ is collapsible.*

(ii) *If $s \geq 1$ and $\kappa'(G) \geq s + 3$, then for any $e', e'' \in E(G - S)$, $(G - S)(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail.*

Proof. Let $S \subseteq E(G)$ be a subset with $|S| \leq s$. If $s \geq 2$ and $\kappa'(G) \geq s + 2$, then pick $S' \subseteq S$ with $|S'| = \min\{s - 2, |S|\}$, and so $\kappa'(G - S') \geq \kappa'(G) - (s - 2) \geq 4$. By Theorem 4.3.2(ii), $\tau(G - S) \geq 2$ and so $F(G - S) = 0$. By Theorem 4.3.2(i), $G - S$ is collapsible. This proves (i).

Now assume that $s \geq 1$ and $\kappa'(G) \geq s + 3$. Pick $S' \subseteq S$ with $|S'| = \min\{s - 1, |S|\}$, and so $\kappa'(G - S') \geq \kappa'(G) - (s - 1) \geq 4$. Let e' and $e'' \in E(G - S)$. As $\kappa'(G - S') \geq 4$, by Theorem 4.3.2(ii), $\tau(G - (S \cup \{e', e''\})) \geq 2$. Since $|S - S'| \leq 1$, we have $\kappa'(G - S) \geq 3$ and $F((G - S)(e', e'')) \leq 1$. It follows by Theorem 4.3.2(ii) that $(G - S)(e', e'')$ is collapsible. By Theorem 4.3.1(vi), $(G - S)(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail. This proves (ii). ■

4.4 s -hamiltonianicity of Hourglass-free line graph

4.4.1 Structural properties of \mathcal{F}_0 -clear graphs

For convenience, we shall use the following notations in the argument. If H is a nonspanning connected subgraph of a graph G , then define

$$A_G(H) = \{v \in V(H) : \text{for some } w \in V(G) - V(H), vw \in E(G)\},$$

called the **vertices of attachments of H in G** . For any edge $e = uv \in E(G)$, denote by $|e|$ the number of parallel edges in G joining u and v .

By the definition of hourglass-free graphs, we observed the following:

Observation 2. *Let G be a nontrivial connected graph. Then $L(G)$ is hourglass-free if and only if for any edge $e = uv \in E(G)$, any distinct $e'_u, e''_u \in E_G(u) - E_G(v)$ and any distinct $e'_v, e''_v \in E_G(v) - E_G(u)$, there must be an $e_u \in \{e'_u, e''_u\}$ and an $e_v \in \{e'_v, e''_v\}$ such that e_u and e_v are adjacent in G , as otherwise, $L(G)[\{e'_u, e''_u, e, e'_v, e''_v\}]$ is an hourglass. Thus, let \mathcal{F}_0 be the family consisting of the three graphs depicted in Figure 1, then*

$L(G)$ is hourglass-free if and only if G does not contain a member in \mathcal{F}_0 as a subgraph. (4.3)

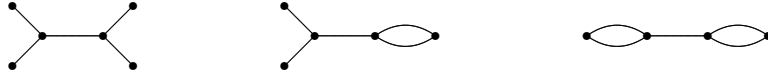


Figure 1: Graphs in \mathcal{F}_0

A graph G is \mathcal{F}_0 -clear if G does not have a (not necessarily induced) subgraph isomorphic to a member in \mathcal{F}_0 . The main goal of this section is to investigate properties of \mathcal{F}_0 -clear graphs for our proofs. In the definition below, we introduce some graphs that would play a role in our arguments.

Definition 4.4.1. *Let $h \geq 1$ and $s, t, \ell \geq 0$ be integers, and define ℓK_2 be the graph with $V(\ell K_2) = \{u_1, u_2\}$ and $E(\ell K_2)$ consisting of ℓ edges joining u_1 and u_2 .*

(i) *Define $\ell K_2(s, t)$ to be the graph obtained from ℓK_2 by attaching s pendant edges to u_1 and attaching t pendant edges to u_2 .*

(ii) *Let $K_{2,h}^\ell(s, t)$ denote the graph obtained from $\ell K_2(s, t)$ by adding h new vertices v_1, v_2, \dots, v_h and $2h$ new edges in $\{u_1 v_1, u_1 v_2, \dots, u_1 v_h, u_2 v_1, u_2 v_2, \dots, u_2 v_h\}$. Define $\mathcal{L}_1 = \{K_{2,0}^{\ell_0}(1, t_0) : \ell_0 \geq$*

$4, t_0 \geq 1\} \cup \{K_{2,1}^{\ell_1}(t_1, 0), K_{2,1}^{\ell_1}(1, 1) : \ell_1 \geq 3, t_1 \geq 1\} \cup \{K_{2,2}^{\ell_2}(1, 0), K_{2,2}^{\ell_2}(0, 0) : \ell_2 \geq 2\} \cup \{K_{2,3}^{\ell_3}(0, 0) : \ell_3 \geq 1\}$. The graphs in \mathcal{L}_1 are depicted in Figures 2.

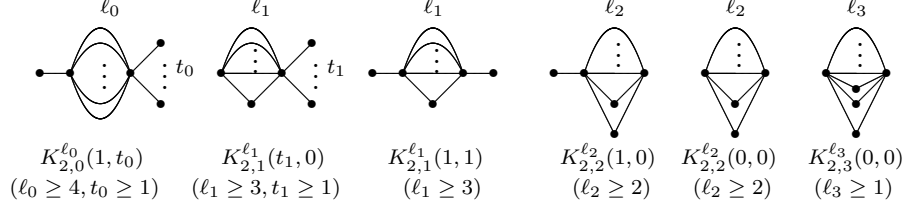


Figure 2: Graphs in \mathcal{L}_1

Lemma 4.4.2. Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$. Let z_1 and z_2 be two adjacent vertices of G with $d_G(z_1) \geq 3$ and $d_G(z_2) \geq 3$. Then each of the following holds:

- (A) Let $d = |N_G(z_1) \cap N_G(z_2)|$. Then $d \leq 3$.
- (B) Let $\ell = |z_1 z_2| \geq 1$, $t_1 = d_G(z_1) - \ell - d$, and $t_2 = d_G(z_2) - \ell - d$. Denote $H = G[\{z_1, z_2\} \cup (N_G(z_1) \cap N_G(z_2))]$, and $\mathcal{L}'_1 = \{K_{2,3}^{\ell_3}(0, 0) : \ell_3 \geq 1\} \cup \{K_{2,2}^{\ell_2}(1, 0), K_{2,2}^{\ell_2}(0, 0) : \ell_2 \geq 2\} \cup \{K_{2,1}^{\ell_1}(1, 1) : \ell_1 \geq 3\}$. Assume that $N_G(z_1) \cap N_G(z_2) \subseteq D_2(G)$. Then the following results hold:
 - (i) Assume that $t_1 \leq t_2$. Either $G \in \mathcal{L}'_1$, or $d = 1$, $t_1 = 0$ and $t_2 \geq 1$, or $d = 0$, $t_1 \leq 1$ and $t_2 \geq 1$.
 - (ii) Assume that $E_G(z_1) \cup E_G(z_2) - E(H) \subseteq X_1(G)$. For latter two cases of (i), $G \in \mathcal{L}_1 \setminus \mathcal{L}'_1$.

Proof. By Observation 2, Lemma 4.4.2(A) holds. Assume that $N_G(z_1) \cap N_G(z_2) \subseteq D_2(G)$. Since $L(G)$ is not a complete graph, if $d \leq 1$, then $t_1 + t_2 \geq 1$. By Observation 2, if $d = 3$, then $t_1 = t_2 = 0$; if $d = 2$, then $t_1 + t_2 \leq 1$; if $d = 1$, then $t_1 = 0$ and $t_2 \geq 1$, or $t_1 = t_2 = 1$; if $d = 0$, then $t_1 \leq 1$ and $t_2 \geq 1$.

Let $H = G[\{z_1, z_2\} \cup (N_G(z_1) \cap N_G(z_2))]$. Then $A_G(H) \subseteq \{z_1, z_2\}$. If $E_G(z_1) \cup E_G(z_2) - E(H) \not\subseteq X_1(G)$, then $E_G(z_1) \cup E_G(z_2) - E(H)$ is an essential edge-cut of G . By $\text{ess}'(G) \geq 4$, $|E_G(z_1) \cup E_G(z_2) - E(H)| \geq 4$. Hence if $d = 1$ and $t_1 = t_2 = 1$, then $E_G(z_1) \cup E_G(z_2) - E(H) \subseteq X_1(G)$. By $\text{ess}'(G) \geq 4$, if $d \in \{2, 3\}$, $G \in \{K_{2,2}^{\ell_2}(t_1, t_2) : \ell_2 \geq 2, t_2 \geq t_1 \geq 0 \text{ and } t_1 + t_2 \leq 1\} \cup \{K_{2,3}^{\ell_3}(0, 0) : \ell_3 \geq 1\}$; if $d = 1$ and $t_1 = t_2 = 1$, $G \in \{K_{2,1}^{\ell_1}(1, 1) : \ell_1 \geq 3\}$. This proves Lemma 4.4.2(B)(i). If $E_G(z_1) \cup E_G(z_2) - E(H) \subseteq X_1(G)$, by $\text{ess}'(G) \geq 4$ and that $L(G)$ is not a complete graph, Lemma 4.4.2(B)(ii) holds. \blacksquare

Lemma 4.4.3. Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$. Assume that J is a component of $G[Y(G)]$ which is spanned by a K_2 with $V(K_2) = \{z_1, z_2\}$. Let $\ell = |z_1 z_2| \geq 1$, and $H = G[\{z_1, z_2\} \cup (N_G(z_1) \cap N_G(z_2))]$. Let

$$t_1 = d_G(z_1) - \ell - |N_G(z_1) \cap N_G(z_2)|, \quad t_2 = d_G(z_2) - \ell - |N_G(z_1) \cap N_G(z_2)|, \quad \text{and } t_1 \leq t_2.$$

Then one of the following holds.

- (i) $G \in \mathcal{L}_1$.

- (ii) For some $\ell \geq 4$, $H \cong \ell K_2$, $t_1 \in \{0, 1\}$, $E_G(z_1) - E(H) \subseteq X_1(G)$ and $|(E_G(z_2) - E(H)) \cap (X_2(G) \cup X'_2(G))| \geq 4$.
- (iii) For some $\ell \geq 3$, $H \cong \ell K_2$, $t_1 = 1$, $E_G(z_1) - E(H) \subseteq X_2(G) \cup X'_2(G)$ and $|(E_G(z_1) \cup E_G(z_2) - E(H)) \cap (X_2(G) \cup X'_2(G))| \geq 4$.
- (iv) For some $\ell \geq 3$, $H \cong K_{2,1}^\ell(0, 0)$, $A_G(H) = \{z_2\}$, and $|(E_G(z_2) - E(H)) \cap (X_2(G) \cup X'_2(G))| \geq 4$.

Proof. Let J be a nontrivial component of $G[Y(G)]$ such that $V(J) = \{z_1, z_2\}$. By the definition of $Y(G)$, both $d_G(z_1) \geq 3$ and $d_G(z_2) \geq 3$, and $N_G(z_1) \cap N_G(z_2) \subseteq D_2(G)$. Then by Lemma 4.4.2(B), either $G \in \mathcal{L}_1$ or $E_G(z_1) \cup E_G(z_2) - E(H) \not\subseteq X_1(G)$. Moreover, if $G \notin \mathcal{L}_1$, $|N_G(z_1) \cap N_G(z_2)| = 0$, $t_1 \leq 1$ and $t_2 \geq 1$, or $|N_G(z_1) \cap N_G(z_2)| = 1$, $t_1 = 0$ and $t_2 \geq 1$. By $\text{ess}'(G) \geq 4$, exactly one of (ii), (iii) and (iv) holds. \blacksquare

Definition 4.4.4. Let K_3 with $V(K_3) = \{z_1, z_2, z_3\}$, K_4 with $V(K_4) = \{z_0, z_1, z_2, z_3\}$ and K_5 with $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$ be complete graphs.

- (i) Define $W_4 = K_5 - \{v_1v_3, v_2v_4\}$ and $F_4 = W_4 - v_1v_4$ (W_4 and F_4 are depicted in Figure 4).
- (ii) For integers $\ell_{12}, \ell_{13}, \ell_{23}, t_1, t_2, t_3 \geq 0$, let $K_3(\ell_{12}, \ell_{13}, \ell_{23}, t_1, t_2, t_3)$ denote the graph obtained from K_3 by replacing each edge $z_i z_j$ by an $\ell_{ij} K_2$, and by attaching t_i pendant edges to z_i , for all $1 \leq i < j \leq 3$. Define $\mathcal{L}_2 = \{K_3(0, \ell_{13}, \ell_{23}, t_1, t_2, t_3) : t_1, t_2 \in \{0, 1\}, t_3 \geq 0, \ell_{13} - t_1 \geq 3, \text{ and } \ell_{23} - t_2 \geq 3\} \cup \{K_3(\ell_{12}, \ell_{13}, \ell_{23}, t_1, t_2, t_3) : t_1, t_2, t_3 \in \{0, 1\}, t_1 + t_2 + t_3 \geq 1, \text{ and } \ell_{12} \geq 1, \ell_{13} \geq 1, \ell_{23} \geq 1 \text{ with } \sum_{j \neq i, t_i=1} \ell_{ij} \geq 4\}$.
- (iii) For integers $\ell_1, \ell_2, \ell_3, \ell_{23} > 0$ and $t_0, t_3 \geq 0$, let $L = K_4 - z_1 z_3$ and let $L(\ell_1, \ell_2, \ell_3, \ell_{23}, t_0, t_3)$ denote the graph obtained from L by replacing each edge $z_0 z_1$, $z_0 z_2$, $z_0 z_3$ and $z_2 z_3$ by $\ell_1 K_2$, $\ell_2 K_2$, $\ell_3 K_2$, and $\ell_{23} K_2$, respectively, and by attaching t_0 and t_3 pendant edges to z_0 and z_3 , respectively. Define $\mathcal{L}_3 = \{L(\ell_1, \ell_2, \ell_3, \ell_{23}, t_0, t_3) : \ell_3 \geq \ell_2 \geq \ell_1 > 0 \text{ and } t_0, t_3 \in \{0, 1\}\}$.
- (iv) For integers $\ell_2, \ell_3 > 0$, let $F_4(\ell_2, \ell_3)$ denote the graph obtained from F_4 by replacing each edge $v_0 v_i$ by an $\ell_i K_2$, for $i \in \{2, 3\}$. Define $\mathcal{L}_4 = \{F_4(\ell_2, \ell_3) : \ell_3 \geq \ell_2 \geq 2\}$.
- (v) For an integer $\ell_3 > 0$, let $F_4^+(\ell_3)$ denote the graph obtained from $F_4 + v_1 v_3$ by replacing $v_0 v_3$ by an $\ell_3 K_2$. Define $\mathcal{L}'_4 = \{F_4^+(\ell_3) : \ell_3 \geq 1\}$.

The graphs defined in Definition 4.4.4 (ii) - (iv) are depicted in Figure 4, and those defined in Definition 4.4.4 (v) are depicted in Figure 5.

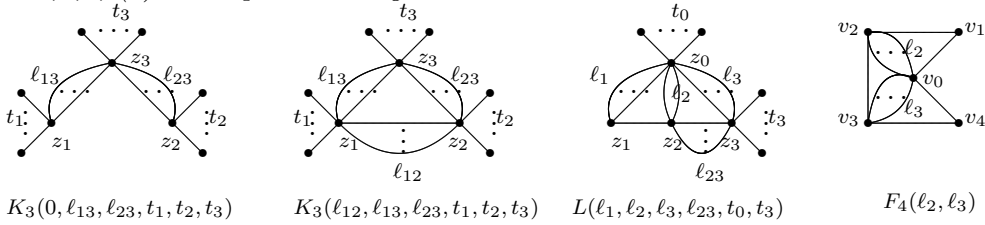


Figure 3: Some graphs in Definition 4.4.4.

Lemma 4.4.5. Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$. If J is a component of $G[Y(G)]$ with $V(J) = \{z_1, z_2, z_3\}$, then there exist integers $\ell_{12} \geq 0$ and

$\ell_{13}, \ell_{23} > 0$ such that one of the following holds.

- (i) $G \in \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$.
- (ii) For $\ell_{23} \geq \ell_{13} \geq 3$, $J = K_3(0, \ell_{13}, \ell_{23}, 0, 0, 0)$, $A_G(J) \subseteq \{z_1, z_2, z_3\}$, and $E_G(z_1) \cup E_G(z_2) \cup E_G(z_3) - E(J) \not\subseteq X_1(G)$.
- (iii) For $\ell_{23} \geq \ell_{13} \geq 2$ and $\ell_{12} > 0$, $J = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$, $A_G(J) = \{z_3\}$ and $E_G(z_3) \not\subseteq X_1(G)$.

Proof. Let J be a component of $G[Y(G)]$ with $V(J) = \{z_1, z_2, z_3\}$. By the definition of $Y(G)$, for each $1 \leq i \leq 3$, $d_G(z_i) \geq 3$. For $1 \leq i < j \leq 3$, let $\ell_{ij} = |z_i z_j|$. Assume that $d_J(z_1) \leq d_J(z_2) \leq d_J(z_3)$. Then we have $\ell_{23} + \ell_{13} = d_J(z_3) \geq d_J(z_2) = \ell_{12} + \ell_{23}$, and so $\ell_{13} \geq \ell_{12}$. Similarly, $\ell_{23} \geq \ell_{13}$. Thus $J = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$ with $\ell_{23} \geq \ell_{13} \geq \ell_{12}$. For $1 \leq i \leq 3$, let $s_i = d_G(z_i) - d_J(z_i)$.

Claim 1. *If $\ell_{12} = 0$, then both $s_1 \leq 1$ and $s_2 \leq 1$, and one of the following holds.*

- (i) $G \in \{K_3(0, \ell_{13}, \ell_{23}, s_1, s_2, s_3) : s_1, s_2 \in \{0, 1\}, s_3 \geq 0, \ell_{13} - s_1 \geq 3 \text{ and } \ell_{23} - s_2 \geq 3\}$.
- (ii) For $\ell_{23} \geq \ell_{13} \geq 3$, $J = K_3(0, \ell_{13}, \ell_{23}, 0, 0, 0)$, $A_G(J) \subseteq \{z_1, z_2, z_3\}$ and $E_G(z_1) \cup E_G(z_2) \cup E_G(z_3) - E(J) \not\subseteq X_1(G)$.

Since $\ell_{12} = 0$ and J is connected, $\ell_{13}, \ell_{23} \geq 1$. Note that $d_G(z_2) \geq 3$, $d_G(z_3) \geq 3$, $N_G(z_2) \cap N_G(z_3) \subseteq D_2(G)$ and $G \notin \mathcal{L}_1$. If $s_2 \geq 2$, then by Lemma 4.4.2(B)(ii), $s_3 + \ell_{13} \leq 1$. Since $\ell_{13} \geq 1$, we must have $\ell_{13} = 1$ and $s_3 = 0$. Since $d_G(z_1) \geq 3$ and $d_J(z_3) = d_G(z_3) \geq 3$, we have $s_1 \geq 2$ and $\ell_{23} \geq 2$. As $\ell_{12} = 0$, $G[E_G(z_1) \cup E_G(z_3)]$ contains a graph in \mathcal{F}_0 , a contradiction. Hence we must have $s_2 \leq 1$. Similarly, $s_1 \leq 1$.

Fix an $i \in \{1, 2\}$. As $s_i + \ell_{i3} = d_G(z_i) \geq 3$, $\ell_{i3} \geq 3 - s_i$. If $s_i = 0$, then $\ell_{i3} \geq 3$. Assume that $s_i = 1$ and let $e_i \in E_G(z_i) - E_J(z_i)$. By the definition of $Y(G)$, $e_i \in X_1(G) \cup X_2(G) \cup X'_2(G)$. If $e_i \in X_1(G)$, then by $\text{ess}'(G) \geq 4$, $\ell_{i3} \geq 4$. If $e_i \in X_2(G) \cup X'_2(G)$, then for some vertex $z \in D_2(G)$, $E_G(z) = \{e_i, e'_i\}$ and $e_i = z_i z$. Thus $E_J(z_i) \cup \{e'_i\}$ is an essential edge-cut of G , and so $\ell_{i3} + 1 \geq 4$. It follows that in any case, we always have $\ell_{i3} \geq 3$. Assume that $\ell_{23} \geq \ell_{13}$. If $E_G(z_1) \cup E_G(z_2) \cup E_G(z_3) - E(J) \subseteq X_1(G)$, then Claim 1(i) holds. If $E_G(z_1) \cup E_G(z_2) \cup E_G(z_3) - E(J) \not\subseteq X_1(G)$, Claim 1(ii) must hold. This justifies Claim 1.

Claim 2. *If $\ell_{12} > 0$, then each of the following holds.*

- (i) $\ell_{23} \geq \ell_{13} \geq 2$, and $\max\{s_1, s_2\} \leq 1$.
- (ii) If $s_1 = 1$ or if $s_2 = 1$, then Lemma 4.4.5(i) holds; if $(s_1, s_2) = (0, 0)$, then either $G \in \mathcal{L}_2$ or $J = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$, $A_G(J) = \{z_3\}$ and $E_G(z_3) \not\subseteq X_1(G)$.

We first prove Claim 2(i). As $\ell_{23} \geq \ell_{13} \geq \ell_{12} \geq 1$, if $\ell_{13} = 1$, then $\ell_{12} = 1$, whence $d_J(z_1) = \ell_{12} + \ell_{13} = 2$. By the definition of $Y(G)$, we have $d_G(z_1) \geq 3$ and $d_G(z_2) \geq 3$. If $E_G(z_1) - E(J) \subseteq X_1(G)$, then $E_J(z_1)$ is an essential 2-edge-cut of G , contrary to $\text{ess}'(G) \geq 4$. Again by $\text{ess}'(G) \geq 4$, we must have $|(E_G(z_1) - E(J)) \cap (X_2(G) \cup X'_2(G))| \geq 2$. Since G is

\mathcal{F}_0 -clear and since $\ell_{23} \geq 1$, we deduce that $\ell_{23} = 1$, $s_1 = s_2 = 2$, every edge in $E_G(z_1) - E(J)$ must be adjacent to an edge in $E_G(z_2) - E(J)$ and that every edge in $E_G(z_2) - E(J)$ must be adjacent to an edge in $E_G(z_1) - E(J)$. This further forces that $s_3 = 0$ and so $d_G(z_3) = 2$, contrary to $d_G(z_3) \geq 3$. This shows that $\ell_{23} \geq \ell_{13} \geq 2$. As G is \mathcal{F}_0 -clear, $\max\{s_1, s_2\} \leq 1$. This proves Claim 2(i).

Fix an $i \in \{1, 2\}$ and suppose that $s_i = 1$. By $\text{ess}'(G) \geq 4$, either $E_G(z_i) \cap X_1(G) \neq \emptyset$ and $\ell_{i3-i} + \ell_{i3} \geq 4$, or $E_G(z_i) \cap (X_2(G) \cup X'_2(G)) \neq \emptyset$ and $\ell_{i3-i} + \ell_{i3} \geq 3$. Since $G[(E_G(z_3) \cup E_G(z_i)) - E(J)] \cup \{z_i z_3\}$ cannot contain a member in \mathcal{F}_0 , we have $s_3 \leq 2$, where $s_3 = 2$ only if there exists a vertex $z^i \in D_2(G)$ with $z^i z_j \in E_G(z_j) - E_J(z_j)$ for $j \in \{i, 3\}$ and $\ell_{i3} \geq 2$.

If $s_3 = 2$ and there exist $z^1, z^2 \in D_2(G)$ with $z^j \in N_G(z_j) \cap N_G(z_3)$, then $G[z^1, z^2, z_1, z_2, z_3] = F_4(\ell_{13}, \ell_{23})$. Since G is \mathcal{F}_0 -clear, we must have $G = F_4(\ell_{13}, \ell_{23}) \in \mathcal{L}_4$. Next, suppose that $s_3 = 2$ and for an $i \in \{1, 2\}$, there exists $z^i \in D_2(G)$ with $z^i \in N_G(z_i) \cap N_G(z_3)$ but $N_G(z_{3-i}) \cap N_G(z_3) = \emptyset$. If $s_{3-i} = 1$, then $G[E_G(z_3) \cup E_G(z_{3-i})]$ contains a graph in \mathcal{F}_0 . Hence we must have $s_{3-i} = 0$ and so $G[E_G(z_1) \cup E_G(z_2) \cup E_G(z_3)] = L(1, \ell_{13}, \ell_{23}, 1, 1, 0)$. By $\text{ess}'(G) \geq 4$, $G = L(1, \ell_{13}, \ell_{23}, 1, 1, 0) \in \mathcal{L}_3$. Since we assume that G is \mathcal{F}_0 -clear and Lemma 4.4.5(i) fails, we must have $s_3 \leq 1$, and so $G[E_G(z_1) \cup E_G(z_2) \cup E_G(z_3)] \in \{L(1, \ell_{13}, \ell_{23}, 1, 0, 0), L(1, \ell_{13}, \ell_{23}, 1, 0, 1), K_3(\ell_{12}, \ell_{13}, \ell_{23}, s_1, s_2, s_3)\}$, with $s_1, s_2, s_3 \in \{0, 1\}$. By $\text{ess}(G') \geq 4$, we must have $G = G[E_G(z_1) \cup E_G(z_2) \cup E_G(z_3)]$, and so Lemma 4.4.5(i) holds.

Now assume that $s_1 = s_2 = 0$. Then $J = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$ and $A_G(J) \subseteq \{z_3\}$. Thus Claim 2(ii). This completes the proof of Lemma 4.4.5. \blacksquare

Definition 4.4.6. Let K_4 with $V(K_4) = \{z_0, z_1, z_2, z_3\}$ and K_5 with $V(K_5) = \{v_0, v_1, v_2, v_3, v_4\}$ be complete graphs.

- (i) Let $H_4 = K_5 - \{v_1 v_3, v_2 v_4, v_0 v_4\}$ be the graph depicted in Figure 4.
- (ii) If H is a subgraph of G , then define $[H, G] = \{\Gamma : H \subseteq \Gamma \subseteq G\}$.
- (iii) Let $\ell_1, \ell_2, \ell_3, \ell_{23}$ and t be positive integers satisfying $\ell_3 \geq \ell_2 \geq \ell_1 \geq 1$, $\ell_{23} \geq 1$ and $t \geq 0$. Define $K_4(t, \ell_1, \ell_2, \ell_3, \ell_{23})$ to be the graph obtained from K_4 by replacing each edge $z_0 z_i$ with an $\ell_i K_2$, replacing $z_2 z_3$ with an $\ell_{23} K_2$ and attaching t additional edges incident with z_0 . In particular, $K_4(0, 1, 1, 1, 1) = K_4$. Denote $\mathcal{L}_5 = \{K_4(t, \ell_1, \ell_2, \ell_3, 1) : \ell_3 \geq \ell_2 \geq \ell_1 \geq 1, t \in \{0, 1\}, \text{ and if } t = 1, \ell_3 \geq 2\} \cup \{K_4(0, 1, \ell_2, \ell_3, \ell_{23}) : \ell_2, \ell_3, \ell_{23} \geq 2\}$.
- (iv) For integers t, t' with $t \geq 2t' > 0$, let S be a star on $t + 1$ vertices with center vertex u_0 and pendant vertices $\{u_1, u_2, \dots, u_t\}$. Let $S_{t,t'}^+$ be the graph obtained from S by adding t' edges in $\{u_1 u_2, u_3 u_4, \dots, u_{2t'-1} u_{2t'}\}$. Define $S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t)$ be the graph obtained from $S_{t,t'}^+$ by replacing $u_0 u_i$ with a $\ell_i K_2$ for each $1 \leq i \leq t$, where $\ell_i \geq 2$. Denote $\mathcal{L}_6 = \{S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t) : \text{for } 1 \leq i \leq 2t', \ell_i \geq 2, \text{ and for } 2t' + 1 \leq i \leq t, \ell_i \geq 3\}$.

Graphs defined in Definition 4.4.6(iii) and (iv) are depicted in Figure 5.

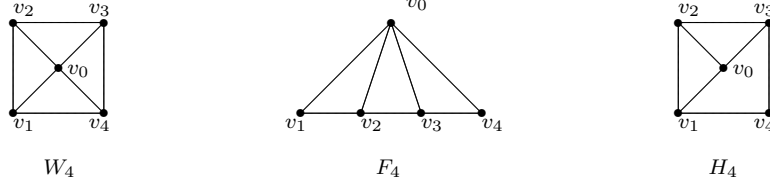


Figure 4: Graphs W_4 , F_4 and H_4

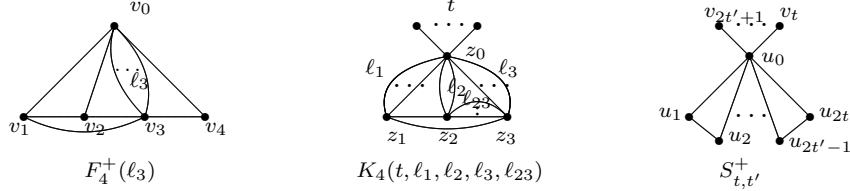


Figure 5: Graphs $F_4^+(\ell_3)$, $K_4(t, \ell_1, \ell_2, \ell_3, \ell_{23})$ and $S_{t,t'}^+$

Lemma 4.4.7. *Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$. Let J be a component of $G[Y(G)]$. Each of the following holds.*

- (i) *If $W_4 \subseteq J$, then $G \in [W_4, K_5]$.*
- (ii) *Assume that G has a subgraph $H \cong F_4$. If $|V(H) \cap D_2(G)| = 0$, then $G \in [W_4, K_5]$; if $|V(H) \cap D_2(G)| = 1$, $G \in \mathcal{L}'_4$; if $|V(H) \cap D_2(G)| = 2$, $G \in \mathcal{L}_4$.*
- (iii) *There is no subgraph of G isomorphic to H_4 .*

Proof. (i) Suppose that $W_4 \subseteq G$. Since G is \mathcal{F}_0 -clear, we must have $V(G) = V(W_4)$. As G is spanned by W_4 , again by the assumption that G is \mathcal{F}_0 -clear, G must be a simple graph and so $G \in [W_4, K_5]$.

(ii) Using the notation in Definition 4.4.6, let $V(H) = \{v_0, v_1, v_2, v_3, v_4\}$ and $H = K_5 - \{v_1v_4, v_1v_3, v_2v_4\}$. Since G is \mathcal{F}_0 -clear, $V(G) = V(H)$ and $|v_2v_3| = 1$. If $d_G(v_1) \geq 3$, then there exist an edge $e'_1 \in E_G(v_1) - E(H)$ such that $e'_1 \in E_G(v_i)$ for some $0 \leq i \leq 4$. If $e'_1 \in E_G(v_j)$, where $j \in \{0, 2\}$, then $H[e'_1, v_1v_j, v_2v_3, v_0v_3, v_3v_4] \in \mathcal{F}_0$, contrary to the fact that G is \mathcal{F}_0 -clear. Hence $e'_1 \in E_G(v_4) \cup E_G(v_3)$.

If $|V(H) \cap D_2(G)| = 0$, then $d_G(v_1) \geq 3$ and $d_G(v_4) \geq 3$, and so there exist an edge $e'_1 \in E_G(v_1) - E(H)$ and an edge $e'_4 \in E_G(v_4) - E(H)$. Arguing as above, we have $e'_1 \in E_G(v_4) \cup E_G(v_3)$ and $e'_4 \in E_G(v_1) \cup E_G(v_2)$. If $e'_1 \in E_G(v_4)$, then $H + e'_1 = W_4$. If $e'_4 \in E_G(v_1)$, then $H + e'_4 = W_4$. If $e'_1 \notin E_G(v_4)$ and $e'_4 \notin E_G(v_1)$, then $e'_1 \in E_G(v_3)$, $e'_4 \in E_G(v_2)$, and $H - v_2v_3 + \{e'_1, e'_4\} = W_4$. Thus in any case, we always have $W_4 \subseteq J$. By (i), $G \in [W_4, K_5]$.

If $|V(H) \cap D_2(G)| = 1$, without loss of generality, we assume that $d_G(v_1) = 3$ and $d_G(v_4) = 2$. Then $v_1v_3 \in E(G)$. Since G is \mathcal{F}_0 -clear, $|v_1v_3| = |v_1v_2| = |v_0v_1| = |v_0v_2| = 1$. Hence by $\text{ess}'(G) \geq 4$, $G \in \mathcal{L}'_4$. If $|V(H) \cap D_2(G)| = 2$, then $d_G(v_1) = d_G(v_4) = 2$. By $\text{ess}'(G) \geq 4$, $G \in \mathcal{L}_4$.

- (iii) Assume that G has a subgraph H isomorphic to H_4 . Using the notation in Definition

4.4.6, let $V(H) = \{v_0, v_1, v_2, v_3, v_4\}$ and $H = K_5 - \{v_0v_4, v_1v_3, v_2v_4\}$. Since G is \mathcal{F}_0 -clear, $V(G) = V(H)$ and $|e| = 1$ for any $e \in E(H) - v_0v_2$. But then $\{v_1v_2, v_0v_1, v_3v_4\}$ is an essential edge cut, contrary to $\text{ess}'(G) \geq 4$. Hence there is no subgraph of G isomorphic to H_4 . ■

Following [3], for a graph J , $\omega(J)$ denotes the clique number of J , which is the largest integer k such that J has a subgraph isomorphic to K_k .

Lemma 4.4.8. *Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$, and let J be a component of $G[Y(G)]$ with clique number $c = \omega(J)$. Then the following hold.*

- (i) $c \leq 5$.
- (ii) $c = 5$ if and only if $G = K_5$.
- (iii) $c = 4$ if and only if $G \in \mathcal{L}'_4 \cup \mathcal{L}_5 \cup [W_4, K_5]$.

Proof. Since a K_6 contains a graph in \mathcal{F}_0 , we must have $c \leq 5$. Suppose that $c = 5$ and so J contains a subgraph $H \cong K_5$ with $V(H) = \{v_1, v_2, v_3, v_4, v_5\}$. If for some $\ell \geq 2$, $H[\{v_1, v_2\}] \cong \ell K_2$, then $H[E(H[\{v_1, v_2\}]) \cup \{v_1v_3, v_3v_4, v_3v_5\}]$ is in \mathcal{F}_0 . Hence H must be a simple graph. If $V(G) \neq V(H)$, then since G is connected, by symmetry, we may assume that there exists an edge $e \in N_G(v_1) - E(H)$, and so $G[e, v_1v_2, v_1v_3, v_3v_4, v_3v_5] \in \mathcal{F}_0$, contrary to the fact that G is \mathcal{F}_0 -clear. Hence $G = K_5$ in this case. This proves (ii).

Assume that $c = 4$ and J contains a subgraph $H' \cong K_4$ with $V(H') = \{z_0, z_1, z_2, z_3\}$. Since G is \mathcal{F}_0 -clear, H' does not have a matching $\{e_i, e_j\}$ with both $|e_i| \geq 2$ and $|e_j| \geq 2$. By symmetry, we may assume that $G[V(H')] = K_4(0, \ell_1, \ell_2, \ell_3, 1)$ with $\ell_3 \geq \ell_2 \geq \ell_1 \geq 1$, or $G[V(H')] = K_4(0, 1, \ell_2, \ell_3, \ell_{23})$ with $\ell_3 \geq \ell_2 \geq 2$ and $\ell_{23} \geq 2$.

If $G[V(H')] = K_4(0, 1, \ell_2, \ell_3, \ell_{23})$ with $\ell_3 \geq \ell_2 \geq 2$ and $\ell_{23} \geq 2$, then as G is \mathcal{F}_0 -clear, $G = K_4(0, 1, \ell_2, \ell_3, \ell_{23}) \in \mathcal{L}_5$. Now we assume that $G[V(H')] = K_4(0, \ell_1, \ell_2, \ell_3, 1)$ with $\ell_3 \geq \ell_2 \geq \ell_1 \geq 1$. If $V(G) \neq V(H')$, then as G is \mathcal{F}_0 -clear, every vertex $z_i \in V(H')$ must be incident with at most one edge in $E(G) - E(G[V(H')])$; and by the same reason, the edges in $E(G) - E(G[V(H')])$ are adjacent to the same vertex not in $V(H')$. Moreover, when $|E(G) - E(G[V(H')])| = 1$ and when $\ell_3 > 1$, the only edge in $E(G) - E(G[V(H')])$ must be incident with z_0 , i.e., $G \in \mathcal{L}_5$; when $|E(G) - E(G[V(H')])| \geq 2$, G has a subgraph H'' isomorphic to F_4 with $|V(H'') \cap D_2(G)| \leq 1$. By Lemma 4.4.7, $G \in [W_4, K_5] \cup \mathcal{L}'_4$. This completes the proof of (iii). ■

A path $P = v_1v_2 \dots v_t$ of G is an **induced path** if $t \geq 2$ and for any $1 \leq i, j \leq t$, we have $v_1 \neq v_t$ and $v_iv_j \notin E(G)$. Thus a graph G has no induced paths if and only if for any $e \in E(G)$, $|e| \geq 2$.

Lemma 4.4.9. *Let $G \notin \mathcal{K}_0$ be a connected \mathcal{F}_0 -clear graph with $|E(G)| \geq 4$ and $\text{ess}'(G) \geq 4$ and let J be a component of $G[Y(G)]$ with $|V(J)| \geq 4$ and clique number $\omega(J) \in \{2, 3\}$. Then one of the following holds:*

- (i) $G \in \{L(\ell_1, \ell_2, \ell_3, 1, t, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1 \text{ and } t \in \{0, 1\}\}$.

(ii) $G \in \mathcal{L}_6$.

(iii) For some integers $t' \geq 1$ and $\ell_1, \ell_2, \dots, \ell_t \geq 2$, $J = S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t)$ (as defined in Definition 4.4.6) with $A_G(J) \subseteq \{u_0, u_{2t'+1}, u_{2t'+2}, \dots, u_t\}$.

Proof. Assume first that J has no induced paths. Let t denote the number of vertices in a longest path of J . Since G is \mathcal{F}_0 -clear, $2 \leq t \leq 3$. As $|V(J)| \geq 4$, we have $t = 3$. Let $u_1 u_0 u_2$ be a longest path of J , and let u be any vertex in $V(J) - \{u_0, u_1, u_2\}$. Since $t = 3$, we have $u \in N_G(u_0)$ and $|uu_0| \geq 2$. As G is \mathcal{F}_0 -clear, $N_G(u) = \{u_0\}$. Then $J = S_{t,0}^+(\ell_1, \ell_2, \dots, \ell_t)$ for some integers $\ell_1, \ell_2, \dots, \ell_t \geq 2$. If $V(G) = V(J)$, by $\text{ess}'(G) \geq 4$, $G \in \mathcal{L}_6$.

Now we assume that J has a longest induced path $P = v_1 v_2 \dots v_t$ with $v_1 \neq v_t$. Denote $e_i = v_i v_{i+1}$ for $1 \leq i \leq t-1$. Since $V(P) \subseteq V(J)$, by the definition of $Y(G)$, $d_G(v_i) \geq 3$ for $1 \leq i \leq t$. Since P is an induced path, there exist distinct edges $e'_1, e''_1 \in E_G(v_1) - E(P)$, $e'_i \in E_G(v_i) - E(P)$ with $2 \leq i \leq t-1$, and $e'_t, e''_t \in E_G(v_t) - E(P)$. Since $G[e_1, e'_1, e''_1, e_2, e'_2] \notin \mathcal{F}_0$ and P is an induced path, we may assume that there exists a vertex $v_0 \in V(G) - V(P)$ such that $e'_1, e'_2 \in E_G(v_0)$. For $i \in \{2, \dots, t-1\}$, if $e'_i \in E_G(v_0)$, then as $G[e_{i-1}, e_i, e'_i, e_{i+1}, e'_{i+1}] \notin \mathcal{F}_0$, we may assume that $e'_{i+1} \in E_G(v_0)$. It follows that $e'_1, e'_2, \dots, e'_t \in E_G(v_0)$.

Claim 3. $t \in \{2, 3\}$.

If $t \geq 5$, then $J[e'_1, e'_t, e'_3, e_2, e_3] \in \mathcal{F}_0$. Hence $2 \leq t \leq 4$. Suppose that $t = 4$. If $e''_1 \notin E_G(v_0)$, then as P is an induced path, $e''_1 \notin \cup_{i=1}^4 E_G(v_i)$, and so $J[e_1, e'_1, e''_1, e'_3, e'_4] \in \mathcal{F}_0$. Hence we must have $e''_1 \in E_G(v_0)$. In this case, $J[e'_1, e''_1, e'_3, e_2, e_3] \in \mathcal{F}_0$. Thus $t = 4$ is impossible.

Claim 4. If $t = 3$, then $G \in \{L(\ell_1, \ell_2, \ell_3, 1, t, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1 \text{ and } t \in \{0, 1\}\}$.

Assume that $t = 3$ and $e''_1 \in E_G(v_0)$. As G is \mathcal{F}_0 -clear, $e''_3 \in E_G(v_0)$. By Definition 4.4.4(iii) (viewing each v_i here as a z_i in Definition 4.4.4), $L(2, 1, 2, 1, 0, 0)$ is a subgraph of G with vertex set $\{v_0, v_1, v_2, v_3\}$. Let $H = G[\{v_0, v_1, v_2, v_3\}]$. Then $H = L(\ell_1, \ell_2, \ell_3, 1, 0, 0)$ for some $\ell_3 \geq \ell_1 \geq 2$ and $\ell_2 \geq 1$. If $G = H$, then Claim 5 holds. Hence we assume that $V(G) \neq V(H)$. Thus for some $v \in V(H)$, there exists an $e \in (E(G) - E(H)) \cap E_G(v)$. If $v = v_i$ with $i \in \{1, 3\}$, then $G[\{e, v_i v_2, e'_i, e'_{4-i}, e''_{4-i}\}] \in \mathcal{F}_0$; if $v = v_2$, then $G[\{e, e'_1, e''_1, v_1 v_2, v_2 v_3\}] \in \mathcal{F}_0$. Hence we must have $v = v_0$. Again as G is \mathcal{F}_0 -clear, $|(E(G) - E(H)) \cap E_G(v_0)| \leq 1$. It follows by $\text{ess}'(G) \geq 4$ that $G \in \{L(\ell_1, \ell_2, \ell_3, 1, t, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1 \text{ and } t \in \{0, 1\}\}$.

Assume that $t = 3$ and $\{e''_1, e''_3\} \cap E_G(v_0) = \emptyset$. If e''_1 and e''_3 are adjacent, then G has a subgraph isomorphic to H_4 , contrary to Lemma 4.4.7(iii). Hence e''_1 and e''_3 are not adjacent. Let $H = G[\{v_0, v_1, v_2, v_3\}]$. As G is \mathcal{F}_0 -clear, $A_G(H) = \{v_1, v_3\}$, $E_G(v_0) \cup E_G(v_2) \subset E(H)$ and for $i \in \{1, 3\}$, we have $E_G(v_i) - E(H) = \{e''_i\}$. Thus $\{v_0 v_3, v_2 v_3, e''_1\}$ is an essential 3-edge-cut, contrary to $\text{ess}'(G) \geq 4$.

Claim 5. Assume that $t = 2$, and $P = v_1 v_2$ is a longest induced path of J . Denote $|N_G(v_1) \cap N_G(v_2)| = d$ and $N_G(v_1) \cap N_G(v_2) = \{u_1, u_2, \dots, u_d\}$. Then the following results hold:

- (i) There are no vertices in $V(J) - N_G(v_1) \cap N_G(v_2)$ adjacent to v_1 or v_2 .
- (ii) $d = 1$.
- (iii) Let $N_G(v_1) \cap N_G(v_2) = \{w_0\}$. Then $|v_1v_2| = 1$, $|v_1w_0| \geq 2$, $|v_2w_0| \geq 2$, and $A_G(G[\{w_0, v_1, v_2\}]) \subseteq \{w_0\}$.
- (iv) For any edge $e = w_1w_2 \in E(J)$, then either w_1w_2 is a longest induced path with $G[\{w_0, w_1, w_2\}]$ being spanned by a K_3 , or $w_0 \in \{w_1, w_2\}$.
- (v) Either $G \in \mathcal{L}_6$ or $J = S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t)$ for some integers $t' \geq 1$ and $\ell_1, \ell_2, \dots, \ell_t \geq 2$ (as defined in Definition 4.4.6) with $A_G(J) \subseteq \{u_0, u_{2t'+1}, u_{2t'+2}, \dots, u_t\}$.

Since $d_G(v_1) \geq 3$, $d_G(v_2) \geq 3$ and G is \mathcal{F}_0 -clear, $N_G(v_1) \cap N_G(v_2) \neq \emptyset$. Hence $|N_G(v_1) \cap N_G(v_2)| = d \geq 1$. Assume that there is a vertex $w \in V(J) - N_G(v_1) \cap N_G(v_2)$ adjacent to v_1 . As $P = v_1v_2$ is the longest induced path of J , $|wv_1| \geq 2$. As G is \mathcal{F}_0 -clear, $d = 1$, $|v_2u_1| = 1$ and $A_G(G[\{v_1, v_2, u_1, w\}]) \subseteq \{v_1, w\}$. Recall that $|v_1v_2| = 1$. We have $d_G(v_2) = 2$, contrary to $v_2 \in V(J)$. Hence we conclude that there are no vertices in $V(J) - N_G(v_1) \cap N_G(v_2)$ adjacent to v_1 or v_2 . This proves Claim 5(i).

By $\omega(J) \in \{2, 3\}$, if $d \geq 2$, then for any $1 \leq i, j \leq d$, u_i and u_j are not adjacent. As G is \mathcal{F}_0 -clear, $d \leq 3$. Moreover, if $d = 3$, then again by G being \mathcal{F}_0 -clear, $V(G) = \{v_1, v_2\} \cup N_G(v_1) \cap N_G(v_2)$, and for each $1 \leq i \leq d$, $|u_i v_1| = |u_i v_2| = 1$. It follows that $|V(J)| = 2$, contrary to the assumption $|V(J)| \geq 4$. Hence $1 \leq d \leq 2$.

Assume that $d = 2$ and $u_1, u_2 \in V(J)$. As $\omega(J) \leq 3$, we conclude that $u_1u_2 \notin E(G)$. Since $u_1v_1u_2$ is not an induced path of J , we may assume that $|v_1u_2| \geq 2$. Similarly, as $u_1v_2u_2$ is not an induced path of J , either $|v_2u_1| \geq 2$ or $|v_2u_2| \geq 2$. If $|v_2u_1| \geq 2$, then $G[\{u_1, u_2, v_1, v_2\}]$ contains a subgraph in \mathcal{F}_0 , a contradiction. Hence $|v_2u_2| \geq 2$, and so as G is \mathcal{F}_0 -clear, we have $|u_1v_1| = |u_1v_2| = 1$. Since $u_1 \in V(J)$, there must be a vertex $w \in V(G) - \{u_1, u_2, v_1, v_2\}$ such that $u_1w \in E(G)$. It follows that $G[\{u_1, u_2, v_1, v_2, w\}]$ contains a subgraph in \mathcal{F}_0 , contrary to that G is \mathcal{F}_0 -clear. Hence it is impossible that $d = 2$ and $u_1, u_2 \in V(J)$.

Let $w_0 \in (N_G(v_1) \cap N_G(v_2)) \cap V(J)$. Since $|V(J)| \geq 4$ and by Claim 5 (i), there must be a vertex $u \in (V(J) \cap N_G(w_0)) - (N_G(v_1) \cup N_G(v_2))$. Assume that $A_G(G[\{w_0, v_1, v_2\}]) \not\subseteq \{w_0\}$. Then we may assume that there exists an edge $e_1 \in E_G(v_1) - (E_G(v_2) \cup E_G(w_0))$. As the subgraph induced by $G[\{e_1\} \cup \{v_1, v_2, w_0, u\}]$ cannot contain a subgraph in \mathcal{F}_0 , we have $|w_0u| = 1$. Since v_1w_0u and v_2w_0u are not induced paths of J , we have $|v_1w_0| \geq 2$ and $|v_2w_0| \geq 2$. By $|w_0u| = 1$, $d_G(u) \geq 3$ and by $u \notin N_G(v_1) \cup N_G(v_2)$, there must be two edges $e, e' \in E_G(u) - (E_G(w_0) \cup E_G(v_1) \cup E_G(v_2))$. Hence $G[\{v_1w_0, v_2w_0, w_0u, e, e'\}] \in \mathcal{F}_0$, contrary to that G is \mathcal{F}_0 -clear. Hence $A_G(G[\{w_0, v_1, v_2\}]) \subseteq \{w_0\}$. This justifies Claim 5 (ii). Then by $|v_1v_2| = 1$ and $v_1, v_2 \in V(J)$, (iii) must hold.

Let $e = w_1w_2 \in E(J) - E(G[\{w_0, v_1, v_2\}])$. We assume that $|e| \geq 2$. As J is connected, by (iii), we may assume that exists a shortest path $Q = z_1z_2z_3\dots z_s$ with $z_1 = w_1$, $z_s = w_0$,

and $w_2 \notin V(Q)$. By (iii), for every $i \geq 1$, we have $|z_i z_{i+1}| = 1$. Thus if $s \geq 3$, then $G[\{z_{s-2}, z_{s-1}, w_0, v_1\}]$ contains a member in \mathcal{F}_0 , contrary to the assumption that G is \mathcal{F}_0 -clear; if $s = 2$, then $G[\{w_2, z_{s-1}, w_0, v_1\}]$ contains a member in \mathcal{F}_0 , contrary to the assumption that G is \mathcal{F}_0 -clear. Hence $s = 1$, and so $w_0 \in \{w_1, w_2\}$. This proves (iv). If $|e| = 1$, then (iv) follows from (iii).

By (iv), every block of J is either an ℓK_2 for some integer $\ell \geq 1$ with $w_0 \in A_G(\ell K_2)$ or is spanned by a K_3 with vertex set $\{w_0, w_1, w_2\}$ and $A_G(G[\{w_0, w_1, w_2\}]) \subseteq \{w_0\}$. It follows by $\text{ess}'(G) \geq 4$ that (v) must hold. This completes the proof of Lemma 4.4.9. \blacksquare

Define

$$\mathcal{L} = \{G : \text{ess}'(G) \geq 4 \text{ and } G \in [W_4, K_5] \cup (\cup_{i=1}^6 \mathcal{L}_i) \cup \mathcal{L}'_4\}. \quad (4.4)$$

Theorem 4.4.10. *Let G be an \mathcal{F}_0 -clear graph such that $\text{ess}'(G) \geq 4$, $G \notin \mathcal{K}_0 \cup \mathcal{L}$, and let G_0 be the core of G and $(X_1(G), X_2(G), X'_2(G), Y(G))$ be the partition of $E(G)$ as defined in Definition 4.2.2. If B is a block of G_0 with $\kappa'(B) \leq 3$, then one of the following holds.*

- (i) $B = 3K_2$ with $|A_G(B)| = 1$.
- (ii) $B = G_0$ and $G \in \{L(\ell_1, \ell_2, \ell_3, 1, t, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1 \text{ and } t \in \{0, 1\}\}$.
- (iii) For some integers $\ell_{12}, \ell_{13}, \ell_{23}$ satisfying $0 < \ell_{12}$ and $2 \leq \ell_{13} \leq \ell_{23}$, $B = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$ with $|A_G(B)| = 1$.

Proof. Let B be a block of G_0 with $\kappa'(B) \leq 3$, and X be an edge cut of B with $|X| \leq 3$. Since B is a block of G_0 , X is also an edge cut of G_0 as well as an edge cut of G . By Lemma ??, $|X| \geq \kappa'(G_0) \geq 3$ and so $|X| = 3$. Let $X = \{e_1, e_2, e_3\}$.

As $\text{ess}'(G) \geq 4$, X cannot be an essential edge-cut of G , and so there must be a vertex $x \in V(G)$ such that $X = E_G(x)$. By Definition 4.2.2, we may assume $x \in V(G_0)$ with $X = E_{G_0}(x)$. Hence $x \in V(B)$, and $N_B(x) = N_{G_0}(x)$. If $|N_B(x)| = 1$, then $B[X] \cong 3K_2$. As $G \notin \mathcal{K}_0$ and $\text{ess}'(G) \geq 4$, we must have $|A_G(B)| = 1$. Hence Theorem 4.4.10 (i) follows.

Hence we assume that $B \neq 3K_2$ and $|N_B(x)| \in \{2, 3\}$. Fix an $i \in \{1, 2, 3\}$. If $e_i \in X \cap (X_2(G) \cup X'_2(G))$, then by Definition 4.2.2, there exists a vertex $v \in D_2(G)$ such that $E_G(v) = \{e_i, e'_i\}$ with $e_i \in E_G(x)$. It follows that $\{e'_i\} \cup (X - \{e_i\})$ is an essential edge-cut of G , contrary to the assumption of $\text{ess}'(G) \geq 4$. Hence $e_i \in Y(G)$, and $X = E_G(x) \subseteq Y(G)$. Let J be the component of $G[Y]$ that contains X . Then as $|N_B(x)| \geq 2$, we have $|V(J)| \geq 3$.

Suppose first that $|V(J)| \geq 4$. If $\omega(J) \geq 4$, then by Lemma 4.4.8, $G \in \mathcal{L}'_4 \cup \mathcal{L}_5 \cup [W_5, K_5] \subseteq \mathcal{L}$, contrary to the assumption that $G \notin \mathcal{K}_0 \cup \mathcal{L}$. Hence we must have $\omega(J) \in \{2, 3\}$. Then as $G \notin \mathcal{K}_0 \cup \mathcal{L}$, it follows by Lemma 4.4.9 that either Lemma 4.4.9 (i) holds, whence Theorem 4.4.10 (ii) follows; or Lemma 4.4.9 (iii) holds, whence $J = S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t)$ and B is a block of J . Since $B \neq 3K_2$, we must have Theorem 4.4.10 (iii).

Assume that $|V(J)| = 3$. As $B \neq 3K_2$, Theorem 4.4.10(iii) follows from Lemma 4.4.5(iii). This completes the proof of the theorem. \blacksquare

4.4.2 s -Hamitonicity and s -Hamitonian-connectedness of Hourglass Free Graphs

We adopt the convention that $\kappa'(K_1) = \infty$. Define $\mathcal{G}(4)$ to be a graph family such that a graph $G \in \mathcal{G}(4)$ if and only if $\kappa'(G) \geq 3$, $ess'(G) \geq 4$ and every block B of G satisfies one of the following.

(G1) $B = 3K_2$ or $\kappa'(B) \geq 4$.

(G2) For some integers $\ell_{12}, \ell_{13}, \ell_{23}$ satisfying $0 < \ell_{12}$ and $2 \leq \ell_{13} \leq \ell_{23}$, $B = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$.

Observation 3. *By the definition of $\mathcal{G}(4)$ and by Theorem 4.3.1, we have these observations.*

(i) *If $G \in \mathcal{G}(4)$, then for any block B of G , $G/E(B) \in \mathcal{G}(4)$.*

(ii) *If $G \in \mathcal{G}(4)$, then every block of G is collapsible.*

Lemma 4.4.11. *Let G be a graph $\mathcal{G}(4)$ with $|E(G)| \geq 5$. Let $S \subseteq E(G)$ and $G_S = (G - S) - D_1(G - S)$. Then each of the following holds.*

(i) *If $|S| \leq 2$, then G_S is supereulerian.*

(ii) *If $|S| \leq 1$, then for any $e', e'' \in E(G_S)$, either $G_S(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail, or G has a block $B = 3K_2$ with $|A_G(B)| = 1$ such that $E(B) = S \cup \{e', e''\}$, and $G_S(e', e'') - (V(B) - A_G(B))$ has a spanning $(v_{e'}, v_{e''})$ -trail.*

Proof. Since $G \in \mathcal{G}(4)$, we have $ess'(G) \geq 4$. Let B_1, B_2, \dots, B_b denote the blocks of G , where $b = b(G)$ denotes the number of blocks of G . Let $S \subseteq E(G)$.

We argue by induction on b . Suppose $b = 1$. Then $G = B_1$. Since $G \in \mathcal{G}(4)$ with $|E(G)| \geq 5$, if (G1) holds, then $\kappa'(G) \geq 4$, and so by Lemma 4.3.3, if $|S| \leq 2$, then $G - S$ is supereulerian; and if $|S| \leq 1$, then for any $e', e'' \in E(G_S)$, $G_S(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail.

If (G2) holds, then $G = K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0)$ with $0 < \ell_{12}$ and $2 \leq \ell_{13} \leq \ell_{23}$, and so it is routine to verify that for any subset $S \subset E(G)$, if $|S| \leq 2$, then G_S is supereulerian; and if $|S| \leq 1$, then for any $e', e'' \in E(G_S)$, $G_S(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail. Thus Lemma 4.4.11 holds if $b = 1$. Hence we assume that $b \geq 2$ and Lemma 4.4.11 holds for smaller values of b .

Claim 6. *If there exists a block B_b (say) with $E(B_b) \cap S = \emptyset$, then Lemma 4.4.11 holds.*

Let $G' = G/B_b$ and v_b be the vertex in G' onto which B_b is contracted. By Observation 3, $G' \in \mathcal{G}(4)$.

Assume first $|S| \leq 2$. If $|E(G')| \geq 5$, then by induction, $(G' - S) - D_1(G' - S)$ has a spanning eulerian subgraph Γ' . Since $ess'(G') \geq 4$, $v_b \in V(\Gamma')$. If $|E(G')| \leq 4$, by definition of $\mathcal{G}(4)$, G' is spanned by a $3K_2$, and so $(G' - S) - D_1(G' - S)$ also has a spanning eulerian subgraph Γ' such that $v_b \in V(\Gamma')$. By Observation 3, B_b is collapsible. Thus by Theorem 4.3.1(v), Γ' can be lifted to an eulerian subgraph Γ of G with $E(\Gamma') \subseteq E(\Gamma)$ and $V(B_b) \subseteq V(\Gamma)$. Thus Γ is a spanning eulerian subgraph of G_S , and so Lemma 4.4.11(i) holds.

Next, we assume that $|S| \leq 1$ and $e', e'' \in E(G - S)$. If $|E(G')| = 3$, then as $G' \in \mathcal{G}(4)$, $G' = 3K_2$ and so G' is a block of G . As $G \in \mathcal{G}(4)$, we have $\text{ess}'(G) \geq 4$ and so $|A_G(G')| = 1$. If $e', e'' \in E(G')$, then Lemma 4.4.11(i) holds. If $e' \in E(G')$ and $e'' \in E(B_b)$, then by (G1) or (G2), both $(G' - S)(e')$ and $B_b(e'')$ are collapsible. If $e', e'' \in E(B_b)$ and $B_b = 3K_2$, then $F(B_b(e', e'')) = 1$, and so by Theorem 4.3.2(i), $B_b(e', e'')$ is collapsible. If $e', e'' \in E(B_b)$ and $\kappa'(B_b) \geq 4$, then by Theorem 4.3.2(ii) and (i), $B_b(e', e'')$ is collapsible. It follows that in any case, $G_S(e', e'')$ is collapsible. By Theorem 4.3.1 (vi), $G_S(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail.

If $|E(G')| = 4$, then as $G' \in \mathcal{G}(4)$, $G' = 4K_2$, and so Lemma 4.4.11(ii) follows straightforwardly. Assume that $|E(G')| \geq 5$. If $e', e'' \in E(G')$, then by induction, $(G' - S)(e', e'')$ has a spanning $(v_{e'}, v_{e''})$ -trail T' . As $v_b \in V(T') - \{v_{e'}, v_{e''}\}$, and as B_b is collapsible, by Theorem 4.3.1(v), T' can be lifted to a spanning $(v_{e'}, v_{e''})$ -trail of $G_S(e', e'')$. With a similar argument, when $|\{e', e''\} \cap E(G')| \leq 1$, $G_S(e', e'')$ always has a spanning $(v_{e'}, v_{e''})$ -trail. Thus Claim 6 must hold.

As the Lemma holds when $b = 1$ and by Claim 6, Lemma 4.4.11(ii) is justified. It remains to prove Lemma 4.4.11(i) for the case when $b = |S| = 2$. Let $S = \{e^1, e^2\}$. By Claim 6, we may assume that $e^i \in E(B_i)$, for $i \in \{1, 2\}$. By (G1) or (G2) and by Theorem 4.3.2(i) and (ii), for $i \in \{1, 2\}$, $B_i - e^i$ is collapsible. It follows that $G - S$ is collapsible. By Theorem 4.3.1 (i), $G - S$ is supereulerian. This completes the proof of the lemma. \blacksquare

Lemma 4.4.12. *If $G \in \mathcal{L}$, then $L(G)$ is 2-hamiltonian and 1-hamiltonian-connected.*

Proof. Suppose that $G \in \mathcal{L}$. By Theorem 4.2.1 and Proposition 4.2.4, it suffices to prove each of the following statements:

- (L1) For any $S \subseteq E(G)$ with $|S| \leq 2$, $G - S$ has a dominating eulerian subgraph.
- (L2) For any $S \subseteq E(G)$ with $|S| \leq 1$ and any $e', e'' \in E(G) - S$, $G - S$ has a dominating (e', e'') -trail.

By (4.4), we will justify Lemma 4.4.12 according to which family G belongs to. Let $S \subseteq E(G)$. Suppose that $G = W_4$. If $|S| \leq 2$, then either $(W_4 - S) - D_1(W_4 - S)$ is collapsible or $G - S$ is isomorphic to $K_{2,3}$. Hence (L1) holds for any graph $G \in [W_4, K_5]$. If $|S| \leq 1$, then $F(G - S) = 0$ and $\text{ess}'(G - S) \geq 3$. By Theorem 4.3.2 (iii), for any $e', e'' \in E(G) - S$, $G - S$ has a dominating (e', e'') -trail. Hence (L2) holds for any graph $G \in [W_4, K_5]$.

Let $G \in \mathcal{L} - [W_4, K_5]$. As $\text{ess}'(G) \geq 4$, we also have $\text{ess}'(G_0) \geq 4$. By Definitions 7.4.4.4 and 4.4.6, and by $\text{ess}'(G) \geq 4$, we have $G_0 \in \{K_{2,0}^\ell(0, 0) : \ell \geq 4\} \subseteq \mathcal{G}(4)$ (if $G \in \mathcal{L}_1$); or $G_0 \in \{K_3(0, \ell_{13}, \ell_{23}, 0, 0, 0) : \ell_{23} \geq \ell_{13} \geq 3\} \cup \{K_3(\ell_{12}, \ell_{13}, \ell_{23}, 0, 0, 0) : \ell_{12} > 0, \ell_{23} \geq \ell_{13} \geq 2\} \subseteq \mathcal{G}(4)$ or $G_0 \in \{L(\ell_1, \ell_2, \ell_3, \ell_{23}, 0, 0) : \ell_{23} > 0, \ell_3 \geq \ell_2 \geq \ell_1 \geq 2\}$ (if $G \in \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$); or $G_0 \in \{K_4(0, \ell_1, \ell_2, \ell_3, \ell_{23}) : \ell_3 \geq \ell_2 \geq \ell_1 \geq 1\}$ (if $G \in \mathcal{L}'_4 \cup \mathcal{L}_5$) or $G_0 \in \{S_{t,t'}^+(\ell_1, \ell_2, \dots, \ell_t) : \text{for } 1 \leq i \leq 2t', \ell_i \geq 2, \text{ and for } 2t' + 1 \leq i \leq t, \ell_i \geq 3\} \subseteq \mathcal{G}(4)$ (if $G \in \mathcal{L}_6$). If $G_0 \in \mathcal{G}(4)$ and

$|E(G)| \geq 5$, by Lemmas 4.4.11 and 4.2.5, (L1) and (L2) hold. Otherwise, by Lemma 4.2.5 and by checking each of these graphs, (L1) and (L2) hold. \blacksquare

In the rest of this section, we prove Theorem 4.5.6 below, which, by (4.3), would imply the validity of Theorem 4.1.1.

Theorem 4.4.13. *Let G be an \mathcal{F}_0 -clear graph and $s \geq 1$ be an integer. Each of the following holds.*

- (i) *If $s \geq 2$, then $L(G)$ is s -hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.*
- (ii) *If $s \geq 1$, then $L(G)$ is s -hamiltonian-connected if and only if $\kappa(L(G)) \geq s + 3$.*

Proof. It suffices to prove the sufficiency in either statement of the theorem. If $L(G)$ is a complete graph, then $L(G)$ is s -hamiltonian (in (i)) and $L(G)$ is s -hamiltonian-connected (in (ii)). Thus we assume that $G \notin \mathcal{K}_0$.

We argue by induction on s . If $G \in \mathcal{L}$, then by Lemma 4.4.12, both (i) and (ii) of Theorem 4.5.6 hold. Hence we assume that $G \notin \mathcal{L}$. Hence by Theorem 4.4.10, $G_0 \in \mathcal{G}(4)$ or $G_0 \in \{L(\ell_1, \ell_2, \ell_3, 1, 0, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1\}$. If $G_0 \in \mathcal{G}(4)$, by Lemma 4.4.11, Theorem 4.5.6 (i) holds for $s = 2$ and Theorem 4.5.6 (ii) holds for $s = 1$. If $G_0 \in \{L(\ell_1, \ell_2, \ell_3, 1, 0, 0) : \ell_1 \geq 2, \ell_3 \geq 2, \ell_2 \geq 1\}$, it is routine to verify that Theorem 4.5.6 (i) holds for $s = 2$ and Theorem 4.5.6 (ii) holds for $s = 1$. Assume that $s \geq 3$ for (i) and $s \geq 2$ for (ii), and that Theorem 4.5.6 holds for smaller values of s . Since $L(G)$ is hourglass-free, G is \mathcal{F}_0 -clear. Let $X \subseteq E(G)$ be an edge set with $1 \leq |X| \leq s$. Choose a subset $X' \subseteq X$ such that $|X'| \leq s - 1$. Let $X'' = X - X'$, and $G' = G - X''$. Since G is \mathcal{F}_0 -clear, G' is also \mathcal{F}_0 -clear. By the definition of a line graph, $L(G') = L(G) - X''$ satisfies $\kappa(L(G')) \geq \kappa(L(G)) - 1$.

For (i), assume that $\kappa(L(G)) \geq s + 2$. Then $\kappa(L(G')) \geq (s - 1) + 2$, and so by induction on s and as $|X'| \leq s - 1$, we deduce that $L(G) - X = L(G') - X'$ is hamiltonian. For (ii), assume that $\kappa(L(G)) \geq s + 3$. Then $\kappa(L(G')) \geq (s - 1) + 3$, and so by induction on s and as $|X'| \leq s - 1$, we deduce that $L(G) - X = L(G') - X'$ is hamiltonian-connected. This proves the theorem. \blacksquare

4.5 On s -hamiltonicity of net-free line graphs

4.5.1 Supereulerian graphs with small circumference

Definition 4.5.1. *Let $P(10)$ denote the Petersen graph and $P(10)^- = P(10) - e$ for an edge $e \in E(P(10))$, and let $K \cong K_{1,3}$ with $D_3(K) = \{a\}$ (the center of K) and $D_1(K) = \{a_1, a_2, a_3\}$. For integers $s_1, s_2, s_3, \ell, m, t$ with $\ell \geq 1$ and $m, t \geq 2$, we make the following definitions.*

- (i) *Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from K by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod{3}$.*
- (ii) *Define $C^6(s_1, s_2, s_3) = K_{1,3}(s_1, s_2, s_3) - a$, where $s_2 \geq s_1 \geq 1$ and $s_3 \geq 2$. Furthermore,*

denote

$$\begin{aligned}
N_{C^6(s_1, s_2, s_3)}(a_1) \cap N_{C^6(s_1, s_2, s_3)}(a_2) &= \{v_1, v_2, \dots, v_{s_1}\}, \\
N_{C^6(s_1, s_2, s_3)}(a_2) \cap N_{C^6(s_1, s_2, s_3)}(a_3) &= \{w_1, w_2, \dots, w_{s_2}\}, \\
N_{C^6(s_1, s_2, s_3)}(a_1) \cap N_{C^6(s_1, s_2, s_3)}(a_3) &= \{u_1, u_2, \dots, u_{s_3}\}.
\end{aligned} \tag{4.5}$$

(iii) Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t . Let $S_{m,\ell}$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K_{2,\ell}(w, w')$ by identifying u with w , and joining u' and w' by a new edge $u'w'$.

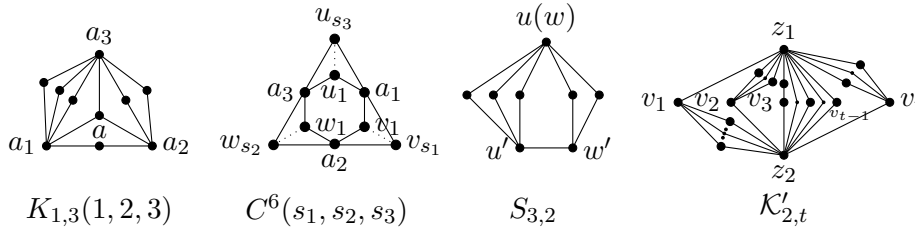


Figure 1. Graphs in Definitions 2.2 and 2.3

Definition 4.5.2. Let $t \geq 2$, $r_1 \geq r_2 \geq \dots \geq r_t \geq 0$ be integers such that $r_2 > 0$, K be a graph isomorphic to $K_{2,t}$ with $\{z_1, z_2\}$ and $\{v_1, v_2, \dots, v_t\}$ being the bipartition of K . For each i with $1 \leq i \leq t$,

(i) denote $E_{K_{2,t}}(v_i) = \{e_i, e'_i\}$;

(ii) if $r_i > 0$, define $K_{2,r_i}(x_i, y_i)$ to be the bipartite graphs with x_i and y_i being the two nonadjacent vertices of degree r_i ;

(iii) if $r_i = 0$, define $K_{2,0}(x_i, y_i) = K_2(x_i, y_i)$, which consists of an edge with end vertices x_i and y_i .

(K1) Define $K'_{2,t}(r_1, r_2, \dots, r_t)$ to be a graph formed by, for each $i \in \{1, 2, \dots, t\}$, replacing exactly one of e_i, e'_i by a $K_{2,r_i}(x_i, y_i)$ by identifying x_i and v_i and by identifying y_i with exactly one of z_1 or z_2 . (See the fourth graph in Figure 1 for an example). Let $\mathcal{K}'_{2,t}$ denote the family of all such defined $K'_{2,t}(r_1, r_2, \dots, r_t)$'s. For notational convenience, when there is no confusion arises, we often use $K'_{2,t}$ to denote an arbitrary member in $\mathcal{K}'_{2,t}$.

(K2) Let $\mathcal{B}_t = \{K'_{2,t}(r_1, r_2, \dots, r_t) + z_1 z_2 : K'_{2,t}(r_1, r_2, \dots, r_t) \in \mathcal{K}'_{2,t}\}$.

By definition, the 6-cycle $C_6 = K'_{2,t}(1, 1)$ is a member in $\mathcal{K}_{2,2}$. Following [3], for a given graph $K'_{2,t}$, a (z_1, z_2) -**component** of this $K'_{2,t}$ is a subgraph of the form $K'_{2,t}[\{z_1, z_2, v_i\} \cup N_{K'_{2,t}}(v_i)]$ for some i with $1 \leq i \leq t$. Throughout the rest of the paper, we define

$$\begin{aligned}
\mathcal{G} &= \{K_{2,t} : t \geq 1\} \cup \{S_{m,\ell} : \ell \geq m \geq 1\} \cup \{K_{1,3}(s_1, s_2, s_3) : s_1 \geq s_2 > 0 \text{ and } s_3 \geq 0\} \\
&\cup \{K_1\} \cup \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t \geq 2} (\mathcal{B}_t \cup \mathcal{K}'_{2,t})).
\end{aligned} \tag{4.6}$$

In this section, we obtained the following two main results characterizing the structure of supereulerian graphs and reduced graphs with small circumference. These results are important tools in proving the main theorem of this work.

Lemma 4.5.3. *Let G be a noncollapsible reduced graph with $\kappa(G) \geq 2$. Then each of the following holds.*

- (i) $c(G) \leq 6$ if and only if $G \in \mathcal{G}$.
- (ii) If $c(G) \leq 6$, then $|D_2(G)| \geq 3$. Furthermore $|D_2(G)| = 3$ if and only if $G \in \{K_{2,3}, K_{1,3}(1, 1, 1)\}$.

Theorem 4.5.4. *Let G be a reduced graph. If $\kappa'(G) \geq 2$, $c(G) \leq 8$, $|D_2(G)| \leq 2$ and $\text{ess}'(G) \geq 3$, then G is collapsible.*

We prove a lemma so that can be used it in the prove of next theorem.

Lemma 4.5.5. *Let G be a connected graph with a 2-edge-cut X and let G_1 and G_2 be the two components of $G - X$. If both G/G_1 and G/G_2 are supereulerian, then G is also supereulerian.*

Proof. For $i \in \{1, 2\}$, let v_i denote the vertex in G/G_i onto which G_i is contracted. Then in G/G_i , the set of edges incident with v_i is X . Let L_i be a spanning eulerian subgraph of G/G_i . As $|X| = 2$ and as $v_i \in V(L_i)$, it follows that $X \subseteq E(L_i)$, and so $G[E(L_1) \cup E(L_2)]$ is a spanning eulerian subgraph of G . ■

By assuming the validity of Theorem 4.5.4, we obtain the following extension of Theorem 4 in [57].

Theorem 4.5.6. *Let G be a 2-edge-connected graph. Each of the following holds.*

- (i) Let Γ be a graph with $\kappa'(\Gamma) \geq 3$ and $e \in E(\Gamma)$. If $G = \Gamma - e$ and $c(G) \leq 8$, then G is supereulerian.
- (ii) If $c(G) \leq 8$ and G has at most two edge-cuts of size 2, then G is supereulerian.

Proof of Theorem 4.5.6. We argue by contradiction to prove Theorem 4.5.6(i), and assume that there exists a 3-edge-connected graph Γ , an edge $e = u_1u_2 \in E(\Gamma)$ such that $c(\Gamma - e) \leq 8$ and $G := \Gamma - e$ is not supereulerian with $|V(\Gamma)|$ minimized.

As $\kappa'(\Gamma) \geq 3$, we have $|D_2(G)| \leq 2$ and $\kappa'(G) \geq 2$. If $|V(G)| \leq 8$, then by Theorem 4.3.1(iv), G is supereulerian. Hence we assume that $|V(G)| \geq 9$. Suppose that G has an essential edge cut X with $|X| = 2$. Let G_1, G_2 be the two components of $G - X = \Gamma - (X \cup e)$ with $\min\{|E(G_1)|, |E(G_2)|\} \geq 1$ and $u_i \in V(G_i)$. Let $\Gamma_1 = \Gamma/G_i$. By the minimality of $|V(\Gamma)|$, G/G_i is supereulerian. By Lemma 4.5.5, G is supereulerian, contrary to the choice of G . Hence $\text{ess}'(G) \geq 3$.

If G is reduced, then by Theorem 4.5.4, G must be collapsible, and so supereulerian. Hence we assume that G contains a nontrivial collapsible subgraph H . Since $\text{ess}'(G) \geq 3$, we conclude that $|D_2(G/H)| \leq 2$. As $c(G/H) \leq c(G) \leq 8$ and $G/H = (\Gamma - e)/H = \Gamma/H - e$, it follows

by the minimality of $|V(\Gamma)|$ that G/H has a spanning eulerian subgraph, and so by Theorem 4.3.1(i), G is supereulerian, contrary to the assumption that G is a counterexample. This proves Theorem 4.5.6(i).

We again argue by contradiction to prove Theorem 4.5.6(ii) and assume that G is a counterexample to Theorem 4.5.6(ii) with $|V(G)|$ minimized. By the minimality of G and by Theorem 4.3.1(i), we may assume that G is reduced. By the minimality of G and by Lemma 4.5.5, we may assume that G does not have any essential edge cut of size 2. It follows that there exist vertices u_1 and u_2 in $V(G)$ such that every 2-edge-cut of G must be the set of edges incident with u_1 or u_2 . This implies that we can choose an edge $e = u_1u_2$ not in G such that adding e to G joining u_1 and u_2 will result in a graph Γ with $\kappa'(\Gamma) \geq 3$. As G is reduced, it follows by Theorem 4.5.6(i) that G is collapsible, and so supereulerian, contrary to the assumption that G is a counterexample. This completes the proof of the theorem. ■

4.5.2 Proof of Lemma 4.5.3

For sets X and Y , the **symmetric difference** of X and Y is $X\Delta Y = (X \cup Y) - (X \cap Y)$. If an edge $e = uv \notin E(G)$ but $u, v \in V(G)$, then let $G + e$ be the graph containing G as a spanning subgraph with edge set $E(G) \cup \{e\}$. For $v \in V(G)$ and $e \in E(G)$, define We first study reduced graphs with circumferences at most 6.

We have the following observations and facts. The first two are from the definition of \mathcal{G} in (4.6).

Observation 4. *Let G be a nontrivial connected graph.*

- (i) *If $|D_2(G)| \leq 2$ or $|D_2(G)| = 3$ and G contains two adjacent degree 2 vertices, we have $G \notin \mathcal{G}$.*
- (ii) *If $|D_2(G)| = 3$, then $G \in \mathcal{G}$ if and only if $G \in \{K_{2,3}, K_{1,3}(1, 1, 1)\}$.*
- (iii) *(Theorem 3 of [54]) If G is reduced with diameter 2, then $G \in \{K_{1,t}, K_{2,t}, S_{m,t}, P(10)\}$ where $t \geq 2$.*

Lemma 4.5.7. *Suppose $G \in \mathcal{K}'_{2,t}$. Let $x, y \in V(G)$ such that $d_G(x, y) \geq 2$ and $\{x, y\} \cap \{z_1, z_2\} = \emptyset$. Then there exists a cycle C of G such that $|E(C)| \geq 5$ and $|V(C) \cap \{x, y\}| = 1$ unless, up to isomorphism, $G \in K'_{2,2}$ and $\{x, y\} = \{v_1, v_2\}$.*

Proof. Suppose first that x, y are in the same (z_1, z_2) -component of G , then by Definition 4.5.2, $G - x$ is also in $\mathcal{K}'_{2,t}$, and so a cycle C of length at least 5 containing y exists in $G - x$. If $t \geq 3$ and x and y are in different (z_1, z_2) -component of G , then by Definition 4.5.2, the graph G' formed by deleting the component of $G - \{z_1, z_2\}$ containing x is also in $\mathcal{K}'_{2,t}$, and so a cycle C of length at least 5 containing y exists in G' . Therefore, we may assume that $t = 2$, $G \neq C_6$, and x and y are in different (z_1, z_2) -component of G . By symmetry, we may further assume that $d_G(v_1) \geq 3$, x and v_1 are in the same (z_1, z_2) -component and y and v_2 are in the same

(z_1, z_2) -component. If $x \neq v_1$, then $G - x$ is also in $\mathcal{K}'_{2,t}$ and so a cycle of length at least 5 containing y but not x exists. Hence $x = v_1$. Similarly, $y = v_2$. ■

Throughout the rest of this work, suppose that $P = v_1v_2\dots v_n$ denotes a v_1v_n -path and $1 \leq i < j \leq n$. We define $P[v_i, v_j] = v_i v_{i+1} \dots v_j$ and $P^{-1}[v_i, v_j] = v_j v_{j-1} \dots v_i$. Thus $P = P[v_1, v_n]$. Similarly, suppose that $C = v_1v_2\dots v_nv_1$ denotes a cycle and $1 \leq i < j \leq n$. Define $C[v_i, v_j] = v_i v_{i+1} \dots v_j$ and $C^{-1}[v_i, v_j] = v_j v_{j+1} \dots v_n v_1 \dots v_i$ to be the subpaths of C . We have the following observation.

Observation 5. *Let $C = v_1v_2\dots v_nv_1$ be a cycle of G , P_1 be a $v_i v_k$ -path of G satisfying $V(P_1) \cap V(C) = \{v_i, v_k\}$, and P_2 be a $v_j v_\ell$ -path of G satisfying $V(P_2) \cap V(C) = \{v_j, v_\ell\}$. Suppose that $1 \leq i < j < k < \ell < n$. If $|E(P_1)| + |E(P_2)| > |E(C[v_k, v_\ell])| + |E(C[v_i, v_j])|$, then C is not a longest cycle of G .*

Proof of Lemma 4.5.3. As (ii) follows immediately from (i) and Observation 4, it suffices to justify (i). It is routine to verify that graphs in \mathcal{G} are reduced and if $G \in \mathcal{G}$, then $c(G) \leq 6$. If $c(G) \leq 5$, then the diameter of G is at most 2 and so by Observation 4(iii), $G \in \mathcal{G}$. Hence we assume that $c(G) = 6$.

Claim 7. *The graph G is spanned by H where $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t \geq 2} \mathcal{K}'_{2,t})$.*

Since a cycle of order 6 is in $\mathcal{K}'_{2,2}$ and $c(G) = 6$, we conclude that G contains a member in $\mathcal{K}'_{2,t}$ as a subgraph. Choose an $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t \geq 2} \mathcal{K}'_{2,t})$ such that

$$H \text{ is a subgraph of } G \text{ with } |V(H)| + |E(H)| \text{ maximized.} \quad (4.7)$$

If $V(G) = V(H)$, then done. Therefore there must be a vertex $u \in V(G) - V(H)$. As $\kappa(G) \geq 2$, G has a uv -path P_1 and a uw -path P_2 with $V(P_1) \cap V(P_2) = \{u\}$, $V(P_1) \cap V(H) = \{v\}$, $V(P_2) \cap V(H) = \{w\}$ for distinct vertices v and w . If $vw \in E(H)$, then since each edge of H lies in a cycle with length at least 5, $H \cup P_1 \cup P_2$ contains a cycle with length greater than 6, contrary to $c(G) = 6$. Hence $d_H(v, w) \geq 2$. In the arguments below, we will use the notations in Figure 1.

Assume first that $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\}$. By (4.7) and Observation 5, we have $\{v, w\} \in \{\{u_p, u_q\}, \{v_p, v_q\}, \{w_p, w_q\} : p \neq q\}$. If $\{v, w\} = \{u_p, u_q\}$, then $G[\{u_p a_1, a_1 v_1, v_1 a_2, a_2 w_1, w_1 a_3, a_3 u_q\} \cup E(P_1) \cup E(P_2)]$ contains a cycle of length longer than 6, contrary to $c(G) = 6$. Hence $\{v, w\} \neq \{u_p, u_q\}$. By symmetry, we also conclude that $\{v, w\} \neq \{w_p, w_q\}$ and $\{v, w\} \neq \{v_p, v_q\}$.

Therefore, we may assume that $H \in \mathcal{K}'_{2,t}$. If $\{v, w\} = \{z_1, z_2\}$, then $G[H + u] \in \mathcal{K}'_{2,t+1}$, violating (4.7). Hence we must have $\{v, w\} \neq \{z_1, z_2\}$.

Suppose that $\{v, w\} \cap \{z_1, z_2\} = \emptyset$. By Lemma 4.5.7, either $t = 2$ and $\{v, w\} = \{v_1, v_2\}$, whence $G[H + u] \in K'_{2,3}$ or $G[H + u] \cong C^6(s_1, s_2, s_3)$, contrary to (4.7); or there exists a cycle C with $|V(C)| \geq 5$ such that (by symmetry) $V(C) \cap \{v, w\} = \{w\}$. As such a cycle C must contain both z_1 and z_2 , we may assume that $wz_1 \in E(H)$. Let P be the shortest vz_1 -path in H . Then $C' = P[v, z_1]z_1wP_2^{-1}[w, u]$ is a cycle of length at least 4 and $|E(C \cap C')| = 1$. Thus $C \triangle C'$ is a cycle of length greater than 6, contrary to $c(G) = 6$. These contradictions indicate that we must have $|\{v, w\} \cap \{z_1, z_2\}| = 1$.

By symmetry, we assume $w = z_1$. Since G is reduced and $c(G) = 6$, both $uw, uv \in E(G)$. If $v = v_i$ for some $i \in \{1, 2, \dots, t\}$, then $G[H + u] \in K'_{2,t}$, violating (4.7). Hence we have $v \in V(H) - \{v_1, \dots, v_t, z_1, z_2\}$, whence $N_H(v) = \{z_2, v_i\}$ for some $1 \leq i \leq t$. If $d_H(v_i) = 2$, then $G[H + u] \in K'_{2,t}$, again violating (4.7). Therefore we have $d_H(v_i) \geq 3$, and so by Observation 5, $G[H + u]$ contains cycle with length greater than 6. This justifies Claim 1.

By Claim 7, G is spanned by an H where $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup K'_{2,t}$. Suppose $xy \in E(G) - E(H)$. Then since G is reduced, we have $d_H(x, y) \geq 3$.

Assume first that $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\}$. Since $d_H(x, y) \geq 3$, we have $\{x, y\} \in \{\{a_2, u_p\}, \{a_1, w_p\}, \{a_3, v_p\} : p \geq 1\}$. But any such case implies that $G[H + xy] \cong K_{1,3}(s'_1, s'_2, s'_3)$ where $s_1 + s_2 + s_3 = s'_1 + s'_2 + s'_3 + 2$. If $G = G[H + xy]$, then $G \in \mathcal{G}$ and we are done. Assume that there exists an edge $e' \in E(G) - E(G[H + xy])$. By the definition of $K_{1,3}(s'_1, s'_2, s'_3)$, the graph $K_{1,3}(s'_1, s'_2, s'_3) + e'$ must create a cycle of length at most 3 or an C_6^{++} . By Theorem 4.3.1(iv), G is not reduced, contrary to the assumption that G is reduced.

Thus we must have $H \in K'_{2,t}$. Recall that $xy \in E(G) - E(H)$ is an edge not in H . By Definition 4.5.2, any $z' \in V(G) - \{z_1, z_2\}$ has distance at most two to z_1 and z_2 . Thus if $x \in \{z_1, z_2\}$ and $y \notin \{z_1, z_2\}$, then $H + xy$ contains a cycle of length at most 3, contrary to the assumption that G is reduced. Thus either $\{x, y\} = \{z_1, z_2\}$ or $\{x, y\} \cap \{z_1, z_2\} = \emptyset$.

If $\{x, y\} = \{z_1, z_2\}$, then since G is reduced, by (4.7) and by Theorem 4.3.1(iv), we have $G[E(H) \cup xy] \in \mathcal{B}$. Assume that $\{x, y\} \cap \{z_1, z_2\} = \emptyset$. If $H \in \mathcal{K}'_{2,2}$ with $\{x, y\} = \{v_1, v_2\}$, then $G[E(H) \cup xy] \in \mathcal{B}$. As any additional edge added to a graph in \mathcal{B} will result in a cycle of length at most 3, contrary to the assumption that G is reduced.

Hence we must have that $\{x, y\} \cap \{z_1, z_2\} = \emptyset$ and if $H \in \mathcal{K}'_{2,2}$, then $\{x, y\} \neq \{v_1, v_2\}$. By Lemma 4.5.7, there exists a cycle C of H such that $|V(C)| \geq 5$ and $|V(C) \cap \{x, y\}| = 1$. Assume first that $t \geq 3$. Then we may assume that for some i , $v_i \notin V(C)$ and $V(C) \cap \{x, y\} = \{y\}$. By the definition of $K'_{2,t}$, as $|V(C)| \geq 5$ and as any cycle of a $K'_{2,t}$ with length at least 5 must contain both z_1 and z_2 , it follows that $\{z_1, z_2\} \subseteq V(C)$. Since $d_H(y, z_1) + d_H(y, z_2) \leq 3$, we may assume $d_H(y, z_1) = 1$, and so $yz_1 \in E(H)$. Let Q be the shortest xz_1 -path in H . As G is reduced, $|V(Q)| \geq 3$ and so $C'' = Q[x, z_1]z_1yx$ is a cycle with length at least 4 with $|E(C \cap C'')| = 1$. It follows that $C \triangle C''$ is a cycle with length at least 7, contrary to assumption of $c(G) = 6$.

Hence we must have $t = 2$ but $y \notin \{v_1, v_2\}$. Again by Definition 4.5.2 and by $|V(C)| \geq$

5, we have $\{z_1, z_2\} \subseteq V(C)$ and we by symmetry may assume that $v_2y, z_2y \in E(H)$ with $N_G(z_2) \cap N_G(v_2) - \{y\} \neq \emptyset$. Since G is reduced, we may assume that either $x = v_1$ or $N_G(z_1) \cap N_G(v_1) = \{x\}$, whence G contains a $K_{1,3}(1, s_2, s_3)$, violating (4.7); or there exist distinct $x, x' \in N_G(z_1) \cap N_G(v_1)$, whence for any $y' \in N_G(z_2) \cap N_G(v_2) - \{y\}$, the cycle $y'v_2yxz_1x'v_1v_2y'$ has length at least 7, contrary to the assumption. This completes the proof of the lemma. ■

Definition 4.5.8. Let $C = x_1x_2y_1y_2x_1$ be a 4-cycle in G with a partition $\pi(C) = \langle \{x_1, y_1\}, \{x_2, y_2\} \rangle$.

(i) (Catlin [15]) Let $G/\pi(C)$, the $\pi(C)$ -**reduction** of G , be the graph obtained from $G - E(C)$ by identifying x_1 and y_1 to form a vertex v_1 , by identifying x_2 and y_2 to form a vertex v_2 , and by adding an edge $e_{\pi(C)} = v_1v_2$.

(ii) The 4-cycle C is a **reducible 4-cycle** of G if $G/\pi(C)$ has a cycle containing the edge $e_{\pi(C)} = v_1v_2$. (In other words, $e_{\pi(C)}$ is not a cut edge of $G/\pi(C)$.)

Theorem 4.5.9. Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as above. Each of the following holds.

(i) (Catlin, Corollary 1 of [15]) If $G/\pi(C)$ is collapsible, then G is collapsible.

(ii) (Catlin, Corollary 2 of [15]) If $G/\pi(C)$ is supereulerian, then G is supereulerian.

(iii) $c(G/\pi(C)) \leq c(G)$.

Proof. We adopt the notation in Definition 4.5.8 to justify (iii). Let C' be a longest cycle of $G/\pi(C)$. If $e_{\pi(C)} = v_1v_2$ is not an edge of C' , then C' is a cycle of G and so $c(G/\pi(C)) \leq c(G)$. Assume that $e_{\pi(C)}$ is an edge of C' . Then by the definition of $e_{\pi(C)} = v_1v_2$, C' can be modified into a cycle of G of length at least $|E(C')|$ by adding a path joining a vertex in $\{x_1, y_1\}$ to a vertex in $\{x_2, y_2\}$ to $C' - v_1v_2$. Again we have $c(G/\pi(C)) \leq c(G)$, and so (iii) must hold. ■

4.5.3 Proof of Theorem 4.5.4

By contradiction, we assume that

$$G \text{ be a counterexample to Theorem 4.5.4 with } |V(G)| \text{ minimized.} \quad (4.8)$$

We shall make a number of claims in our proofs.

Claim 8. Each of the following holds.

(i) G is simple, $\kappa(G) \geq 2$, $c(G) \leq 8$, $|D_2(G)| \leq 2$, $g(G) \geq 4$, and G does not have essential 2-edge-cuts.

(ii) $|V(G)| \geq c(G) \geq 7$.

(iii) G does not contain a reducible 4-cycle.

As Claim 8 (i) and (ii) follow from assumption of Theorem 4.5.4, Theorem 4.3.1 and Lemma 4.5.3, it remains to prove Claim 8(iii). By contradiction, assume that G has a reducible 4-cycle $C' = x_1x_2y_1y_2x_1$. In the arguments below, let $G_\pi = G/\pi(C')$, G'_π be the reduction

of G_π and we adopt the notation in Definition 4.5.8 with $e_{\pi(C')} = v_1v_2$, and view $E(G_\pi) = (E(G) - E(C')) \cup \{v_1v_2\}$ and $V(G_\pi) = (V(G) - V(C')) \cup \{v_1, v_2\}$. Then for each $i \in \{1, 2\}$, $d_{G_\pi}(v_i) = d_G(x_i) + d_G(y_i) - 3$. As C' is a reducible 4-cycle, $d_{G_\pi}(v_i) \geq 2$, where equality holds if and only if exactly one of $d_G(x_i)$ and $d_G(y_i)$ equals 2 and the other equals 3. We have the following subclaims.

(2A) $|D_2(G_\pi)| \leq |D_2(G)|$.

If $d_{G_\pi}(v_i) > 2$, then $D_2(G_\pi) \subseteq D_2(G)$, and so (2A) holds. Assume that $d_{G_\pi}(v_i) = 2$. By symmetry, we may assume that $i = 2$ and $d_G(x_2) = 2$ and $d_G(y_2) = 3$. Then $D_2(G_\pi) = (D_2(G) - \{x_2\}) \cup \{v_2\}$, and so (2A) follows.

By Theorems 4.3.1, 4.5.9 and Claim 8 (i), we have the following observation (2B).

(2B) Each of the following holds.

- (i) The edge e_π cannot be contained in any collapsible subgraph of G_π and G'_π is nontrivial.
- (ii) Any essential 2-edge-cut of G'_π must contain e_π .

Thus $e_\pi = v'_1v'_2 \in E(G'_\pi)$, where v'_i denotes the vertex of the contraction image in G_π that contains v_i . If $\text{ess}'(G'_\pi) \geq 3$, then G'_π satisfies the hypotheses of Theorem 4.5.4, and so by (4.8), G'_π is collapsible. By Theorems 4.3.1(i) and 4.5.9(i), G is collapsible, contrary to (4.8). Hence

$$G'_\pi \text{ has an essential edge cut } X \text{ with } |X| = 2. \quad (4.9)$$

By Claim 8 (i), we may assume that $X = \{v'_1v'_2, w_1w_2\}$ for some vertices $w_1, w_2 \in V(G_\pi)$. Let L_1, L_2 be the two components of $G - (E(C') \cup \{w_1w_2\})$ and we assume that for $i \in \{1, 2\}$, $w_i, x_i, y_i \in V(L_i)$. Thus $|V(L_i)| \geq 2$ where equality holds if and only if $w_i \in \{x_i, y_i\}$. By symmetry, assume that $|E(L_1 - \{w_1, x_1, y_1\})| \geq |E(L_2 - \{w_2, x_2, y_2\})|$. Throughout the rest of the proof, B' denotes the block of G'_π with $e_\pi \in E(B')$.

(2C) Each of the following holds.

- (i) Any vertex $v \in D_2(B') - D_2(G)$ must be adjacent to e_π , and $|D_2(B')| \leq 4$.
- (ii) $|\{v'_1, v'_2\} \cap D_2(B')| \leq 1$.

As (2C)(i) follows from $|D_2(G)| \leq 2$, it suffices to show (2C)(ii). Suppose $d_{B'}(v'_1) = d_{B'}(v'_2) = 2$. Let $E_{B'}(v'_1) = \{e_{v'_1}, e_\pi\}$, $E_{B'}(v'_2) = \{e_{v'_2}, e_\pi\}$. By (2C)(i), $\{e_{v'_1}, e_{v'_2}\}$ cannot be an essential 2-edge-cut of B' , and so $G[\{v'_1v'_2, e_{v'_1}, e_{v'_2}\}]$ is a 3-cycle, contrary to (2B)(i).

(2D) Each of the following holds.

- (i) $|E(L_1 - \{w_1, x_1, y_1\})| \neq 0$.
- (ii) $|E(L_2 - \{w_2, x_2, y_2\})| = 0$.
- (iii) x_i, y_i, w_i are mutually distinct.

Suppose $|E(L_2 - \{w_2, x_2, y_2\})| = |E(L_1 - \{w_1, x_1, y_1\})| = 0$. Since G is reduced, it cannot contain $K_{3,3}^-$ as a subgraph, and so we must have $V(L_2) - \{w_2, x_2, y_2\} \subseteq D_2(G)$. Furthermore, $|V(L_2) - \{w_2, x_2, y_2\}| = |D_2(G)| = 2$ by $|D_2(G)| \leq 2$. Then $V(L_1) - \{w_1, x_1, y_1\} \subseteq D_3(G)$ and

$|V(L_1) - \{w_1, x_1, y_1\}| \geq 2$. Let $\{u, v\} \subseteq V(L_1) - \{w_1, x_1, y_1\}$. Then $G[\{x_1x_2, x_2y_1, ux_1, uy_1, vx_1, vy_1, uw_1, vw_1\}]$ is a $K_{3,3}^-$, a contradiction. This proves (2D)(i).

Hence $L_1 - \{w_1, x_1, y_1\}$ must contain an edge $e_1 = z'_1z'_2$ (say). To prove (2D)(ii), assume that $L_2 - \{w_2, x_2, y_2\}$ has an edge $e_2 = z''_1z''_2$. As $\kappa(G) \geq 2$, G has a cycle C_1 containing e_1 and e_2 . By the choice of e_1 and e_2 , $|E(C_1)| \geq 8$. Moreover, if $|E(C_1) \cap E(C')| = 1$, then $C_1 \triangle C'$ is a cycle of length $|E(C_1)| + 2 > 8$. Since $c(G) \leq 8$, we may assume that $C_1 = z'_1z'_2y_1y_2z''_2z''_1x_2x_1z'_1$ is a cycle of length 8. Again by $\kappa(G) \geq 2$, G has a cycle C'_1 containing $z'_1z'_2$ and w_1w_2 . We may assume by symmetry that C'_1 has a $w_1z'_2$ -path Q_1 not containing z'_1 and a $w_2z''_2$ -path Q_2 not containing z''_1 . But then G contains a cycle containing e_1 and e_2 , and intersecting C' at only one edge, implying the existence of a cycle of length at least 9 in G , contrary to $c(G) \leq 8$. This proves (2D)(ii).

As $x_2 \neq y_2$, we first assume that $w_2 \in \{x_2, y_2\}$ (say $w_2 = y_2$). Since G is reduced and since $E(L_2) \neq \emptyset$, there must be a vertex $w \in V(L_2) - \{x_2, y_2\}$ satisfying $wx_2, wy_2 \in E(G)$. Then $w \in D_2(G)$, and $G_\pi[\{v'_2, w\}]$ contains a 2-cycle, and so $d_{G'_\pi}(v'_2) = 2$. By (2B)(ii), $\text{ess}'(G'_\pi) \geq 3$, contrary to (4.9). This proves that $|\{w_2, x_2, y_2\}| = 3$. Next, as $x_1 \neq y_1$, we assume that $w_1 \in \{x_1, y_1\}$ (say $w_1 = y_1$). If there exists $u \in V(L_2) - \{x_2, y_2, w_2\}$ such that $d_G(u) = 3$, then $G_\pi[u, v'_1, v'_2]$ is a 3-cycle, a collapsible subgraph containing e_π , contrary to (B)(i). Hence $V(L_2) - \{x_2, y_2, w_2\} \subseteq D_2(G)$, and so $|V(L_2) - \{x_2, y_2, w_2\}| \leq |D_2(G)| \leq 2$. This implies that $D_2 \subseteq V(L_2)$. As every vertex in $V(L_2) - \{x_2, y_2, w_2\}$ must be adjacent to two vertices in $\{x_2, y_2, w_2\}$, that $|V(L_2) - \{x_2, y_2, w_2\}| = 1$ would imply that $|D_2(G)| > 2$. Hence $|V(L_2) - \{x_2, y_2, w_2\}| = |D_2(G)| = 2$. Let $\{u, v\} = V(L_2) - \{x_2, y_2, w_2\} = D_2(G)$. We may assume $\{ux_2, uw_2, vy_2, vw_2\} \in E(G)$. Let $G_1 = G[V(G) - \{u, v, w_2\}]$. Then G_1 satisfies the hypotheses of Theorem 4.5.4, and so by (4.8), G_1 is collapsible. By Theorem 4.3.1, G is also collapsible, contrary to (4.8). This completes the proof of (2D).

By (2D)(i), in the rest of the arguments, we assume that $z'_1, z'_2 \in V(L_1 - \{w_1, x_1, y_1\})$ such that $z'_1z'_2 \in E(L_1)$.

(2E) $c(B') \leq 6$.

Let H be the block of G_π with $e_\pi \in E(H)$. Choose a longest cycle C in H such that $|\{e_\pi\} \cap E(C)|$ is maximized. By contradiction, assume that $|E(C)| \geq 7$. If $e_\pi \in E(C)$, then we may assume that $G[E(C - e_\pi)]$ is an x_1x_2 -path in G . It follows that $G[E(C - e_\pi) \cup \{x_1y_2, y_2y_1, y_1x_2\}]$ is a cycle of G with length at least 9, contrary to $c(G) \leq 8$. Hence e_π is not on any longest cycle of H , and so $|\{v_1, v_2\} \cap V(C)| \leq 1$.

Suppose $|\{v_1, v_2\} \cap V(C)| = 0$. As $\kappa(H) \geq 2$, for $j \in \{1, 2\}$, H contains disjoint $v_iu_{i_j}$ -path P'_j such that u_{i_1}, u_{i_2} are distinct vertices of C and $V(P'_j) \cap V(C) = u_{i_j}$. By symmetry, we may assume $G[E(P'_j)]$ is an $x_ju_{i_j}$ -path P_j . Since $|E(C)| \geq 7$, C contains an $u_{i_1}u_{i_2}$ -path P_3 such that $|E(P_3)| \geq 4$. Therefore $u_{i_1}P_3[u_{i_1}, u_{i_2}]u_{i_2}P_2^{-1}[x_2, u_{i_2}]x_2y_1y_2x_1P_1[x_1, u_{i_1}]u_{i_1}$ is a cycle of G with length at least 9, contrary to $c(G) \leq 8$. Hence $|\{v_1, v_2\} \cap V(C)| = 1$. By (2D)(ii),

we have $\{v_1, v_2\} \cap V(C) = \{v_1\}$. By $\kappa(H) \geq 2$, $H - v_1$ contains a $v_2 u_k$ -path P_4 such that $V(P_4) \cap V(C - v_1) = \{u_k\}$. By definition of L_1 , $u_k \in V(L_1)$. By (2D)(iii), $|V(P_4)| \geq 3$. As e_π is not on any longest cycle of H , replacing edges in a $v_1 u_k$ -path on C by $E(P_4) \cup \{e_\pi\}$ will not result in a longest cycle of H , and so $|E(C)| = 8$. If $x_1, y_1 \in V(G[E(C)])$, then $G[E(C) \cup \{x_1 x_2, x_2 y_1\}]$ is a cycle of length at least 9, a contradiction. Hence we may assume that $V(C') \cap V(G[E(C)]) = \{x_1\}$. By symmetry, we assume that P_4 is a $y_2 u_k$ -path in G . Let P_5 be a longest $x_1 u_k$ -path on C with $|E(P_5)| \geq 4$. It follows that $x_1 x_2 y_1 y_2 P_4[y_2, u_k] u_k P_5^{-1}[x_1, u_k] x_1$ is a cycle of G with length at least 9. This proves (2E).

(2F) $|V(L_2) - \{x_2, y_2, w_2\}| \leq 1$.

Suppose $V(L_2) - \{x_2, y_2, w_2\}$ contains two vertices a_1, a_2 with $d_G(a_1) \geq d_G(v)$ for any $v \in V(L_2) - \{x_2, y_2, w_2\}$. If $V(L_2) - \{x_2, y_2, w_2\} \subseteq D_2(G)$, then $V(L_2) - \{x_2, y_2, w_2\} = \{a_1, a_2\}$ and $\{a_1, a_2\} \subseteq N_G(w_2)$. Assume $a_1 x_2 \in E(G)$ by symmetry. Suppose first that $\{x_1, y_1\}$ is a vertex-cut of G and let S' be the (x_1, y_1) -component contained in L_1 . Then $G[E(S') \cup E(C')]$ satisfies each hypotheses of Theorem 4.5.4, whence by (4.8), $G[E(S') \cup E(C')]$ is collapsible, contrary to the assumption that G is reduced. Hence $\{x_1, y_1\}$ is not a vertex-cut of G , and so $G - y_1$ has two internally disjoint $z'_1 a_1$ -paths. Thus $L_1 - \{y_1\}$ contains internally disjoint $z'_1 x_1$ -path Q and $z'_2 w_1$ -path Q' . It follows that $z'_1 z'_2 Q'[z'_2, w_1] w_1 w_2 a_1 x_2 y_1 y_2 x_1 Q^{-1}[z'_1, x_1] z'_1$ is a cycle of length at least 9, contrary to $c(G) \leq 8$.

Hence $d_G(a_1) \geq 3$ and $\{a_1 x_2, a_1 y_2, a_1 w_2\} \subseteq E(G)$. Since $\kappa(G) \geq 2$, G contains a cycle C'' with $z'_1 z'_2, a_1 w_2 \in E(C'')$. As $\{z'_1, z'_2\} \cap \{x_1, y_1, w_1\} = \emptyset$, C'' must use at least 3 edges in $E(L_1)$ and two edges incident with a_1 , and so $|E(C'')| \geq 7$. By $c(G) \leq 8$, it follows that $|V(C'') \cap V(C')| = 2$ and $|E(C'') \cap E(C')| = 1$. Hence $C'' \triangle C'$ is a cycle of length at least 9, a contradiction to the assumption $c(G) \leq 8$. This proves (2F).

By (2F), we use \bar{a} to denote the possible vertex in $V(L_2) - \{x_2, y_2, w_2\}$. By (2D), (2F) and by $|D_2(G)| \leq 2$, we conclude that L_2 must contain one of the following graphs H_i , ($1 \leq i \leq 5$), depicted in Figure 2, as a subgraph.

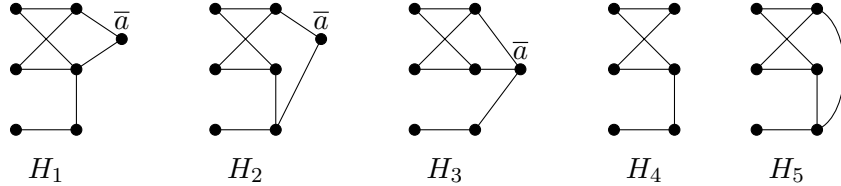


Figure 2: The possible subgraphs in L_2 .

(2G) None of H_1, H_2, H_4 can be a subgraph of L_2 .

By contradiction, suppose that L_2 contains $H' \in \{H_1, H_2, H_4\}$ as a subgraph. Then we have $|D_2(B')| \leq 3$ and $d_{B'}(v'_2) = 2$. By (2E), $c(B') \leq 6$. If $|D_2(B')| \leq 2$, then by Lemma 4.5.3 and

Observation 4 (i), B' is collapsible, contrary to (B)(i). Hence we may assume $|D_2(G'_\pi)| = 3$ and $H' \neq H_2$, and so by (2C)(i) $d_{B'}(v'_1) = d_{B'}(v'_2) = 2$, contrary to (2C) (ii).

By (2G), either H_3 or H_5 is a subgraph of L_2 . If $|D_2(B')| = 4$, then by (2C)(i), $d_{B'}(v'_1) = d_{B'}(v'_2) = 2$, violating (2C)(ii). This implies that $|D_2(B')| = 3$, $d_{B'}(v'_2) = 2$ and $d_{B'}(v'_1) \geq 3$. Let G_1 be a graph contains H_3 as a subgraph, and $G_2 = G_1 - \bar{a} + x_2w_2 + y_2w_2$. Then G_2 is obtained from G_1 by replacing H_3 by H_5 and so $c(G_1) \geq c(G_2)$. It is suffice to show $c(G_2) \geq 9$ to complete the proof of claim 3. Hence we may assume that G contains H_5 as a subgraph.

(2H) $B' = G'_\pi$.

Assume that $G'_\pi - v'_2$ has a block $B'' \neq B'$. Then by (2D), $V(B') \cap V(B'') = \{v'_1\}$. It follows that $H'' = G[E(B'') \cup E(C')]$ is also a 2-edge-connected subgraph of G . As $|D_2(B')| = 3$, $D_2(G) \cap V(H'') = \emptyset$, and so $D_2(H'') = \{x_2, y_2\}$. Furthermore, any edge-cut of B'' not intersecting $E(C')$ is also an edge-cut of G , and any edge-cut of B'' intersecting C must be either the two edges incident with x_2 or y_2 , or of size at least 3. Hence $ess'(B'') \geq 3$. Since $c(H'') \leq c(G) \leq 8$, it follows by (4.8) that H'' is collapsible, contrary to the assumption that G is reduced. This proves (2H).

By (2E) and (2H), $c(G'_\pi) = c(B') \leq 6$. It follows by Lemma 4.5.3, Observation 4 and $|D_2(B')| \leq 3$ that $G'_\pi \in \{K_{2,3}, K_{1,3}(1,1,1)\}$.

By $|D_2(G)| \leq 2$, (2C) and the structures of $K_{2,3}$ and $K_{1,3}(1,1,1)$, G_π contains only one maximal nontrivial collapsible subgraph S_1 with $v_1 \in V(S_1)$ in each of these two cases. Hence G must have one of the following structures.

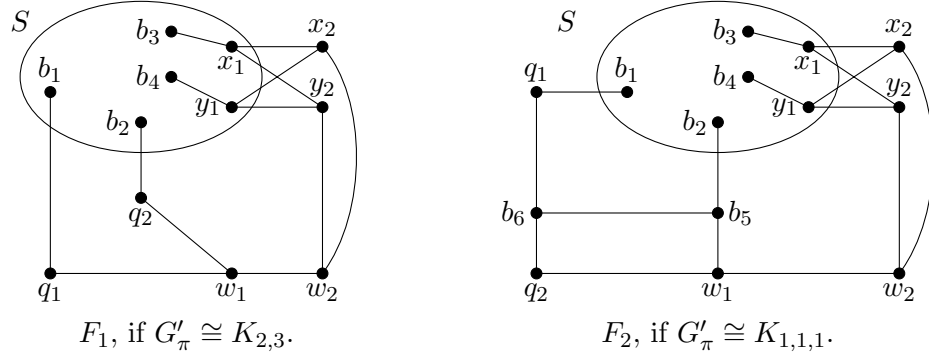


Figure 3: The two possible structures of G .

In the following, we adopt the notation in Figure 3, and so S denotes the preimage of S_1 , $D_2(G) = \{q_1, q_2\}$ and $b_3 \in N_{L_1}(x_1)$, $b_4 \in N_{L_1}(y_1)$ in both of F_1, F_2 , $N_G(q_1) = \{b_1, w_1\}$ in F_1 and $N_{L_1}(q_1) = \{b_1, b_6\}$ in F_2 , $N_{L_1}(w_1) = \{q_1, q_2\}$ in F_1 and $N_{L_1}(w_1) = \{q_2, b_5\}$ in F_2 .

Suppose that G has structure F_2 . By symmetry we may assume that $d_G(b_1, x_1) \leq d_G(b_1, y_1)$. Let P_8 be a shortest b_1x_1 -path in S . Then $x_1x_2y_1y_2w_2w_1b_5b_6q_1b_1P_8[b_1, x_1]x_1$ is a cycle of G with length at least 10, contrary to $c(G) \leq 8$. Hence G must have structure F_1 .

(2I) $b_3 \neq b_4$ and $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$.

If $b_3 = b_4$, then $G[\{b_3x_1, b_3y_1, w_2x_2, w_2y_2\} \cup E(C')] \cong K_{3,3}^-$ is collapsible, contrary to the assumption that G is reduced. Thus $b_3 \neq b_4$.

Assume that $\{b_1, b_2\} \cap \{x_1, y_1\} \neq \emptyset$. Then by symmetry we assume that $b_1 = x_1$. Let $C_1 = y_1x_2w_2y_2y_1$. Using the notation in Definition 4.5.8, we let e'_π be the new edge in $G_{\pi(C_1)}$, the $\pi(C_1)$ -reduction of G , and B'' be the block of $G'_{\pi(C_1)}$ containing e'_π . As $|D_2(G)| \leq 2$ and by applying (2C) and (2E) to $G_{\pi(C_1)}$ with B'' replacing B' , we observe that $|D_2(B'')| \leq 3$ and $c(B'') \leq 6$.

Let $G'_{\pi(C_1)}$ be the reduction of $G_{\pi(C_1)}$, v_0 be the vertex onto which the collapsible subgraph of $G_{\pi(C_1)}$ containing x_1 is contracted. If $d_{G'_{\pi(C_1)}}(v) = 2$, then $q_1v \in E(G'_{\pi(C_1)})$ with $d_{G'_{\pi(C_1)}}(q_1) = d_{G'_{\pi(C_1)}}(v) = 2$. As $c(B'') \leq 6$, by Lemma 4.5.3 and Observation 4, B'' is collapsible, and so is $G'_{\pi(C_1)}$. Hence by Theorems 4.3.1(i) and 4.5.9(i), G is collapsible, contrary to (4.8). Hence $d_{G'_{\pi(C_1)}}(v) \geq 3$, and so by (2C), $|D_2(B'')| \leq 2$. As $c(B'') \leq 6$, by Lemma 4.5.3 and Observation 4(i), B'' is collapsible, which implies that G is collapsible, contrary to (4.8). This proves that $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$ and so (2I) is justified.

Let P_9 be a longest b_1v_1 -path contained in S_1 . By (2I), $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$, and so $|E(P_9)| \geq 1$. Suppose $|E(P_9)| = 1$. Let B_1 be the block of S_1 which contains P_9 and e_a, e_b be two edges incident with b_1 in B_1 . Since any longest b_1v_1 -path in S_1 has length 1, by $g(G) \geq 4$, we may assume that $e_a = b_1x_1$ and $e_b = b_1y_1$ in G . Then $G[\{b_1, x_1, y_1, x_2y_2, w_2\}] \cong K_{3,3}^-$ is collapsible, contrary to the assumption that G is reduced. Hence $|E(P_9)| \geq 2$. By symmetry, we may assume $G[E(P_9)]$ is a b_1, x_1 -path P'_9 . Thus $x_1x_2y_1y_2w_2w_1q_1b_1P'_9[b_1, x_1]x_1$ is a cycle of G of length at least 9, contrary to $c(G) \leq 8$. This completes the proof of Claim 8.

Let $c = c(G)$ and $C = z_1z_2\dots z_cz_1$ be a longest cycle of G . As C is longest, for $z_i, z_j \in V(C)$ with $1 \leq i < j \leq c$, we have:

$$\text{any } (z_i, z_j)\text{-path in } G \text{ internally disjoint from } V(C) \text{ has length at most } d_C(z_i, z_j). \quad (4.10)$$

Claim 9. $|E(G[V(C)]) - E(C)| \leq 2$ and $V(G) - V(C) \neq \emptyset$.

Suppose there exist three edges $e_1, e_2, e_3 \in E(G[V(C)]) - E(C)$. If $c(G) = 7$, then as $g(G) \geq 4$, $G[E(C) \cup \{e_1\}]$ contains a reducible 4-cycle, contrary to Claim 8(iii). Hence $c(G) = 8$. By Claim 8(iii), we must have $\{e_1, e_2, e_3\} \subset \{z_1z_5, z_2z_6, z_3z_7, z_4z_8\}$. This forces that $E(C) \cup \{e_1, e_2, e_3\}$ contains a reducible 4-cycle, which is also contrary to Claim 8(iii). Hence $|E(G[V(C)]) - E(C)| \leq 2$. Since $|D_2(G)| \leq 2$, we must have $V(G) - V(C) \neq \emptyset$. This proves Claim 9.

Claim 10. *There exists $v \in V(G) - V(C)$ such that $d_G(v) \geq 3$.*

Suppose $d_G(v) = 2$ for any $v \in V(G) - V(C)$. Then $|V(G) - V(C)| \leq |D_2(G)| \leq 2$, and so there exists $z_iz_j \in E(G[V(C)]) - E(C)$ where $1 \leq i < j \leq c$. As $|V(C)| = 7$ would imply that

$G[E(C) \cup \{z_i z_j\}]$ contains a reducible 4-cycle, violating Claim 8(iii), we must have $c = 8$, and so by $|D_2(G)| \leq 2$ and Claim 9, $|E(G[V(C)]) - E(C)| = 2$. It follows by $V(G) - V(C) \subseteq D_2(G)$ that $|V(G) - V(C)| = 2$. Suppose that $E[G(V(C))] - E(C) = \{z_{i_1} z_{i_2}, z_{i_3} z_{i_4}\}$, $V(G) - V(C) = D_2(G) = \{u, v\}$ and $uz_{i_5}, uz_{i_6}, vz_{i_7}, vz_{i_8} \in E(G)$. Since G is reduced and by Claim 8 (iii), each pair of vertices in $\{z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}\}$ must have distance 2 on C , and so we may assume $i_1 = 1, i_3 = 3, i_2 = 5, i_4 = 7$. Then by Claim 8 (iii), we must have $\{i_5, i_6\}, \{i_7, i_8\} = \{2, 6\}, \{4, 8\}$. It follows that $G = P(10)^-$. This implies that $c(G) = 9$, contrary to $c(G) = 8$, and so Claim 10 follows.

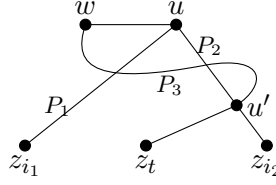


Figure 4: Proof of Claim 11.

Claim 11. *There exists a vertex $v \in V(G) - V(C)$ such that there are three internally disjoint vz_{i_j} -path P_j where $z_{i_j} \in V(C)$ and $V(P_j) \cap V(C) = \{z_{i_j}\}$ for $j \in \{1, 2, 3\}$.*

By Claim 10, there exists a vertex $u \in V(G) - V(C)$ with $d_G(u) \geq 3$. As $\kappa(G) \geq 2$, G contains a uz_{i_1} -path Q_1 and a uz_{i_2} -path Q_2 with $z_{i_1} \neq z_{i_2}$, $V(Q_1) \cap V(Q_2) = \{u\}$ and for $j \in \{1, 2\}$, $V(Q_j) \cap V(C) = \{z_{i_j}\}$. Let f_j be the edge in Q_j incident with z_{i_j} . Since $d_G(u) \geq 3$, there exists an edge $uw \in E(G) - E(Q_1) \cup E(Q_2)$. If $w \in V(C)$, then done. Assume that $w \notin V(C)$. As $\text{ess}'(G) \geq 3$, $G - \{f_1, f_2\}$ has a uz_t -path Q_3 with $V(Q_3) \cap V(C) = \{z_t\}$. We may assume that $V(Q_3) \cap V(Q_1) \cup V(Q_2) - (V(C) \cup \{u\}) \neq \emptyset$, as otherwise the claim holds. Let $v \in V(Q_3) \cap V(Q_1) \cup V(Q_2) - (V(C) \cup \{u\})$ such that for $j \in \{1, 2\}$, if $v \in V(Q_j)$, then $V(Q_j[v, z_{i_j}]) \cap V(Q_3) \subseteq V(C) \cup \{v\}$. Assume that $v \in V(Q_1)$. Let $P_1 = Q_1[v, z_{i_1}]$, $P_2 = Q_1^{-1}[v, u]Q_2[u, z_{i_2}]$ and $P_3 = Q_3[v, z_t]$. Then P_1, P_2, P_3 are the paths satisfying the claim. This justifies Claim 11.

Let $v \in V(G) - V(C)$ and P_j be vz_{i_j} -path where $z_{i_j} \in V(G)$ and $V(P_j) \cap V(C) = \{z_{i_j}\}$ for $j \in \{1, 2, 3\}$. By Claim 11, there exists a vertex $v \in V(G) - V(C)$ and three internally disjoint vz_{i_j} -paths P_j , $j \in \{1, 2, 3\}$, with $V(P_j) \cap V(C) = \{z_{i_j}\}$. We label the P_i 's so that $|E(P_1)| \leq |E(P_2)| \leq |E(P_3)|$.

Claim 12. *Each of the following holds.*

- (i) $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| < 3$, $|E(P_2)| \geq 2$ and $|E(P_3)| \leq 3$.
- (ii) If $|E(P_1)| \geq 2$, then $|E(P_1)| = |E(P_2)| = |E(P_3)| = 2$.
- (iii) If $|E(P_1)| = 1$, then $z_{i_2} = z_{i_3}$.

Assume by contradiction that $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| = 3$. If $|E(P_2)| \geq 2$, then by (4.10), we have $8 \geq c(G) \geq d_C(z_{i_3}, z_{i_1}) + d_C(z_{i_3}, z_{i_2}) + d_C(z_{i_1}, z_{i_2}) = 3 + 4 + 3 = 10$, a contradiction. Thus we may assume that $|E(P_2)| = 1$. Then no matter whether $|E(P_3)| \geq 2$ or $|E(P_3)| = 1$, as G is reduced and $c(G) \leq 8$ and by (4.10), either P_1 or P_2 is always in a reducible 4-cycle of G , contrary to Claim 8 (iii). Hence $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| < 3$. Next we assume that $|E(P_2)| = |E(P_1)| = 1$. As $g(G) \geq 4$, we cannot have $z_{i_1} = z_{i_2}$, and so by the symmetry between P_1 and P_2 , we may assume that $z_{i_1} = z_{i_3}$. By $g(G) \geq 4$, we have $|E(P_3)| \geq 3$. If $|E(P_3)| = 3$, then $E(P_1) \cup E(P_3)$ induces a reducible 4-cycle of G , contrary to Claim 8 (iii). Hence $|E(P_3)| \geq 4$. But then $E(P_3) \cup E(P_2)$ induces a path of length at least 5 with both ends on $V(C)$, contrary to (4.10). This proves that $|E(P_2)| \geq 2$. If $|E(P_3)| \geq 4$, then for some $j \in \{1, 2\}$, $E(P_3) \cup E(P_j)$ induces a path of length at least 5 with end vertices on $V(C)$, contrary to (4.10), and so (i) is justified.

Now assume that $|E(P_1)| \geq 2$. Then for $j \in \{2, 3\}$, $E(P_1) \cup E(P_j)$ is either a cycle or a path. If $E(P_1) \cup E(P_j)$ is a path, then both ends of this path are on $V(C)$. As $c(G) \leq 8$, $4 \leq 2|E(P_1)| \leq |E(P_1) \cup E(P_j)| \leq 4$, implying $|E(P_1)| = |E(P_j)| = 2$, and so we may assume that $j = 2$ and $E(P_1) \cup E(P_j)$ is a cycle. But then, $E(P_2) \cup E(P_3)$ is a path with both ends of it on $V(C)$. As $c(G) \leq 8$, we also have $|E(P_2)| = |E(P_3)| = 2$, and so (ii) follows.

Assume that $|E(P_1)| = 1$. If $z_{i_1} = z_{i_j}$ for some $j \in \{2, 3\}$. Then as $g(G) \geq 4$, we have $|E(P_3)| \geq |E(P_j)| \geq 3$. It follows by Claim 12(i) that $E(P_3) \cup E(P_2)$ induces a path of length at least 5 with both ends on $V(C)$, contrary to (4.10). Hence we must have $z_{i_2} = z_{i_3}$. This completes the proof of the claim.

By Claim 12, we may assume $z_{i_2} = z_{i_3}$, and there exist vertices $u^2 \in V(P_3) - \{v, z_{i_2}\}$ and $u^3 \in V(P_2) - \{v, z_{i_3}\}$ such that $\{vu^2, vu^3\} \subseteq E(G)$. By (4.10), $|V(P_1)| \leq 3$. Let p be the possible vertex of P_1 such that $p \notin \{v, z_{i_1}\}$.

Claim 13. *Each of the following holds.*

- (i) For $j \in \{2, 3\}$, if $|E(P_j)| = 2$, then $u^j \in D_2(G)$.
- (ii) $D_2(G) = \{u^2, u^3\}$.

Let $j \in \{j, j'\} = \{2, 3\}$ and $d_G(u^j) \geq 3$. By $\kappa(G) \geq 2$, u^j is adjacent to a vertex $q \in V(G) - V(P_j)$ such that $u^j q$ is on a $u^j z$ -path P_4 with $V(P_4 - u^j) \cap (V(C) \cup V(P_1) \cup V(P_2) \cup V(P_3)) = \{z\}$. By Claim 12 with v being replaced by u^j , we conclude that $z \in V(P_1) \cup V(P_2) \cup V(P_3)$.

Assume that $|E(P_j)| = 2$. As $g(G) \geq 4$ and $c(G) \leq 8$, either $z = z_{i_1}$, $|E(P_4)| = 1$ and $|E(P_1)| = 2$, whence $vpzu^jv$ is a reducible 4-cycle of G , contrary to Claim 8 (iii); or $z = z_{i_1}$, $|E(P_4)| \geq 2$, whence $P_{j'}[z_{i_2}, v]vu^jP_4[u_j, z]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (4.10); or $z = z_{i_2}$ and $|E(P_4)| \geq 3$ whence $P_4[z_{i_2}, u^j]u^jvP_1[v, z_{i_1}]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (4.10). This proves (i).

If $|E(P_3)| = 2$, then by Claim 12, $|E(P_2)| = |E(P_3)| = 2$, and so Claim 13(i) implies (ii). Thus we assume that $|E(P_3)| = 3$ to show that $u^3 \in D_2(G)$. By (4.10), $|E(P_1)| = 1$. If $z = z_{i_1}$,

then by $g(G) \geq 4$ and by Claim 8 (iii), $|E(P_4)| \geq 3$. It follows that $P_3[z_{i_2}, u^3]P_4[u^3, z_{i_1}]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (4.10). This proves (ii), as well as Claim 13.

We now complete the proof of the Theorem 4.5.4 by finding a contradiction. If there exists a vertex $v' \in V(G) - V(C) \cup \{v, u, w\}$, then by Claim 13, $d_G(v') \geq 3$. Applying Claims 10, 11, 12 and 13 to the case when v is replaced by v' , we are led to the conclusion that v' must be adjacent to both vertices in $D_2(G)$. It follows that for $j \in \{2, 3\}$, the vertex u^j must be adjacent to distinct vertices v, v' and a vertex in $V(P_j) - \{v\}$, contrary to the fact that $u^j \in D_2(G)$. Hence we must have $V(G) = V(C) \cup \{v, u^2, u^3\}$. As $D_2(G) = \{u^2, u^3\}$, we must have $|E(G[V(C)]) - E(C)| \geq 3$, contrary to Claim 9. This completes the proof of the theorem. \blacksquare

4.6 Proof of Theorem 4.1.2

For an integer $m > 0$, we use \mathbb{Z}_m to denote the cyclic group of order m . For integers $s_1 \geq s_2 \geq s_3 \geq 1$, let Y_{s_1, s_2, s_3} be the graph obtained from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths. (See Figure 1 in [102] for an example.) By definition, $N_{s_1, s_2, s_3} = L(Y_{s_1, s_2, s_3})$. Define

$$\begin{aligned} \mathcal{Y}_1 &= \{Y_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0, s_1 + s_2 + s_3 \leq 6\}. \\ \mathcal{Y}_2 &= \{Y_{s_1, s_2, s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0, s_1 + s_2 + s_3 \leq 4\}. \end{aligned} \quad (4.11)$$

By definition of line graphs, a line graph $L(G)$ is N_{s_1, s_2, s_3} -free if and only if G does not have a Y_{s_1, s_2, s_3} as a subgraph. To complete the proof for Theorem 5.2.18, the following additional lemmas for a generic graph G will be needed.

Let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning trees. Catlin (Theorem 7 of [15], see also Corollary 2.13 of [66]) indicated that if G is connected, reduced and $G \notin \{K_1, K_2\}$, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \quad (4.12)$$

Lemma 4.6.1. (Theorem 2.4 of [21]) *If G is a reduced graph with $\kappa'(G) \geq 2$, $|V(G)| \leq 11$, $F(G) \leq 3$ and $|D_2(G)| \leq 2$, then G is collapsible.*

4.6.1 Proof of Theorem 4.1.2(i)

Lemma 4.6.2. *For any $Y \in \mathcal{Y}_1$. Let G be a connected graph with $\kappa'(G) \geq 3$ and $|E(G)| \geq 4$. If G does not contain Y as a subgraph, then for any $e \in E(G)$, $G - e$ is supereulerian.*

Proof. The lemma holds trivially if $n = |V(G)| \leq 3$ and $|E(G)| \geq 4$. We argue by contradiction and assume that

$$G \text{ is a counterexample graph with } |V(G)| \text{ minimized.} \quad (4.13)$$

Claim 14. *There exists an edge $e_0 \in E(G)$ such that*

- (i) $G - e_0$ is not supereulerian.
- (ii) $G - e_0$ is reduced, $g(G - e_0) \geq 4$ and $c(G - e_0) \geq 9$.

Claim 14 (i) follows from (4.13). If $G - e_0$ has a nontrivial collapsible subgraph H , then $|V(G/H)| < |V(G)|$ and so by (4.13), $(G - e_0)/H = G/H - e_0$ is supereulerian. By Theorem 4.3.1 (i), $G - e_0$ is supereulerian, contrary to (4.13). Hence $G - e_0$ must be reduced. By Theorem 4.3.1 (ii), $g(G - e_0) \geq 4$. By Theorem 4.5.6 (i), $c(G - e_0) \geq 9$. This proves the claim.

Let $C = v_1 v_2 \dots v_c v_1$ with $c = |E(C)| \geq c(G - e_0) \geq 9$ be a longest cycle of $G - e_0$. Since C is not spanning G , we assume that there exists a vertex $u_1 \in V(G) - V(C)$ such that $u_1 v_1 \in E(G - e_0)$. By definition, we observe that, as $c \geq 9$, the subgraph $G[E(C) \cup \{u_1 v_1\}]$ contains every member in $\{Y_{s_1, s_2, 0} : s_1 \geq s_2 \geq 0, 1 \leq s_1 + s_2 \leq 6\}$ with v_1 being the unique vertex of degree 3 in these subgraphs. In the following, we shall show that either G has a longer cycle than C , or G contains every member in \mathcal{Y}_1 as defined in (4.11). These contradictions will then justify the lemma.

If there exists a $u_2 \in N_G(u_1) - V(C)$, then as $c \geq 9$, the subgraph $G[E(C) \cup \{u_1 v_1, u_1 u_2\}]$ contains every member in \mathcal{Y}_1 with $1 \in \{s_1, s_2, s_3\}$ and with v_1 being the unique vertex of degree 3 in these subgraphs. It remains to show that G also contains a $Y_{2,2,2}$. If $N_G(u_2) - V(C)$ has at least two vertices, then there exists a $u_3 \in N_G(u_2) - (V(C) \cup \{u_1\})$, and so $G[E(C) \cup \{u_1 v_1, u_1 u_2, u_2 u_3\}]$ contains a $Y_{2,2,2}$ as a subgraph. Hence $N_G(u_2) - \{u_1\} \subseteq V(C)$. By $\kappa'(G) \geq 3$, $g(G - e_0) \geq 4$ and the choice of C , we assume that $v_{j_1}, v_{j_2} \in N_G(u_2) \cap V(C)$ with $5 \leq j_1 + 1 < j_2 \leq c - 2$. Then $C^1 = v_1 u_1 u_2 v_{j_1} v_{j_1+1} \dots v_c v_1$ is a cycle of length $c - (j_1 - 1) + 3 = (c - j_2) + (j_2 - j_1) + 4 \geq 2 + 2 + 4 = 8$. If $j_1 \geq 5$, $G[E(C^1) \cup \{v_1 v_2, v_2 v_3, v_3 v_4\}]$ contains every member in \mathcal{Y}_1 with v_1 being the unique vertex of degree 3 in these subgraphs. By symmetry, we assume that $j_1 = 4$ and $j_2 = c - 2$. As $c \geq 9$, $j_2 \geq j_1 + 3$. Thus $G[E(C^1) \cup \{v_1 v_2, v_2 v_3, v_3 v_4\}]$ contains $Y_{4,1,1}$ and $Y_{3,2,1}$, and $G[(E(C) - \{v_1 v_2, v_{j_1} v_{j_1+1}, v_{j_2} v_{j_2+1}, v_{j_2+1} v_{j_2+2}\}) \cup \{v_1 u_1, u_2 u_2, u_2 v_{j_1}, u_2 v_{j_2}\}] = Y_{2,2,2}$. Thus in any case, G contains every member in \mathcal{Y}_1 . This contradiction shows that

$$\text{for any } u \in V(G) - V(C), N_G(u) \subseteq V(C). \quad (4.14)$$

Let $v_1, v_i, v_j \in N_{G_0}(u_1)$ with $1 < i < j$. By $g(G) \geq 4$, we have $3 < i + 1 < j \leq c - 1$. Let $k = \max\{i - 1, j - i, c - j + 1\}$.

Assume that $k \geq 4$. Without loss of generality, we assume that $i - 1 \geq 4$. Then $C^4 = G[E(C - \{v_2, \dots, v_{i-1}\}) \cup \{v_1 u_1, u_1 v_i\}]$ is a cycle of length at least 6. If the length of C^4 is at least 8, then $G[(E(C) - \{v_{i-1} v_i\}) \cup \{v_1 u_1, u_1 v_i\}]$ contains every member in \mathcal{Y}_1 with v_1 being the unique vertex of degree 3 in these subgraphs. If the length of C^4 is 7, without loss of generality, we assume that $j = i + 2$ and $c - j = 2$. Then $G[(E(C) - \{v_i v_{i-1}, v_j v_{j-1}\}) \cup \{v_1 u_1, u_1 v_i\}]$ contains the subgraph $Y_{2,2,2}$, $G[(E(C) - \{v_i v_{i-1}, v_i v_{i+1}\}) \cup \{v_1 u_1, u_1 v_i\}]$ contains the subgraph $Y_{3,2,1}$, and

$G[(E(C) - \{v_i v_{i-1}, v_1 v_c\}) \cup \{v_1 u_1, u_1 v_j\}]$ contains the subgraph $Y_{4,1,1}$. If the length of C^4 is 6, then $c = j + 1$, $j = i + 2$ and $i \geq 6$. Thus $G[(E(C) - \{v_j v_{j-1}, v_1 v_c, v_i v_{i-1}\}) \cup \{u_1 v_1, u_1 v_i, u_1 v_j\}]$ contains the subgraph $Y_{4,1,1}$, $G[(E(C) - \{v_i v_{i+1}, v_{i-1} v_{i-2}\}) \cup \{v_1 u_1, u_1 v_i\}]$ contains the subgraph $Y_{2,2,2}$, and $G[(E(C) - \{v_i v_{i+1}, v_i v_{i-1}\}) \cup \{v_1 u_1, u_1 v_i\}]$ contains the subgraph $Y_{3,2,1}$. Therefore, $k \leq 3$. As $c \geq 9$, we have $i = 4, j = 7$ and $c = 9$. Then $G[(E(C) - \{v_3 v_4\}) \cup \{u_1 v_1, v_1 v_i\}]$ contains every member in \mathcal{Y}_1 with $1 \in \{s_1, s_2, s_3\}$ and with v_1 being the unique vertex of degree 3 in these subgraphs, and $G[(E(C) - \{v_1 v_2, v_i v_{i+1}, v_j v_{j+1}\}) \cup \{u_1 v_1, u_1 v_i, u_1 v_j\}] \cong Y_{2,2,2}$. All these contradictions indicate the truth of the lemma. ■

Proof of Theorem 4.1.2(i). By the definition of line graph, a graph Γ has a subgraph in \mathcal{Y}_1 if and only if $L(G)$ has a member as an induced subgraph in $L(G)$. Therefore, to prove Theorem 4.1.2(i), it suffices to show that, for any fixed $Y \in \mathcal{Y}_1$ and for an integer $s \geq 1$, if G does not have Y as a subgraph, then

$$\kappa(L(G)) \geq s + 2 \text{ implies that } L(G) \text{ is } s\text{-hamiltonian.} \quad (4.15)$$

We argue by induction on s to prove (4.15), and assume that $s = 1$. Let G be a graph with $\kappa(L(G)) \geq 3$, and let G_0 be the core of G . Since G does not have Y as a subgraph, G_0 also contains no subgraph isomorphic to Y . By Lemma 4.2.3, $\kappa'(G_0) \geq 3$ and so by Lemma 4.6.2, for any $e_0 \in E(G_0)$, $G - e_0$ is supereulerian. By Lemma 4.2.4(ii), (4.15) holds for $s = 1$.

Assume that $s \geq 2$ and (4.15) holds for smaller values of s . For any edge subset $X \subseteq E(G)$ with $|X| = s$. Pick $e_0 \in X$. Define $G_1 = G - e_0$ and $X_1 = X - \{e_0\}$. As G does not have Y as a subgraph, G_1 also contains no subgraph isomorphic to Y , with $\kappa(L(G_1)) = \kappa(G - e_0) \geq (s + 2) - 1 = (s - 1) + 2$. By induction, G_1 is $(s - 1)$ -hamiltonian, and so $L(G) - X = L(G_1) - X_1$ is hamiltonian. Thus (4.15) holds for all integer $s \geq 1$, and so Theorem 4.1.2 (i) is justified. ■

4.6.2 Proof of Theorem 4.1.2(ii)

Define H_8 to be the graph with $V(H_8) = \{v_i : i \in \mathbb{Z}_8\}$ and $E(H_8) = \{v_i v_{i+1}, v_i v_{i+4} : i \in \mathbb{Z}_8\}$. The graph H_8 is known as the Wagner graph ([84]) in the literature. It is routine to verify that

$$\text{for any } Y \in \mathcal{Y}_2, H_8 \text{ contains } Y \text{ as a subgraph.} \quad (4.16)$$

Lemma 4.6.3. *Let $Y \in \mathcal{Y}_2$ and G be a graph with $\kappa'(G) \geq 3$ such that*

$$G \text{ does not contain } Y \text{ as a subgraph.} \quad (4.17)$$

Then each of the following holds.

- (i) *For any $e', e'' \in E(G)$, $G(e', e'')$ is collapsible.*
- (ii) *G is strongly spanning trailable.*

Proof. By Theorem 4.3.1, (i) implies (ii) and so it suffices to prove (i). We argue by contradiction and assume that

$$G \text{ is a counterexample to Lemma 4.6.3(i) with } |V(G)| \text{ minimized.} \quad (4.18)$$

Then there must be edges $e^1, e^2 \in E(G)$ such that $G(e^1, e^2)$ is not collapsible. Let $J = G(e^1, e^2)$. By (4.18), we may assume J is reduced. Since $G(e^1, e^2)$ is not collapsible, $J \neq K_1$.

Suppose that $c(J) \leq 8$. By Theorem 4.5.4, J must have an essential edge-cut X with $X = \{f_1, f_2\}$. For each $i \in \{1, 2\}$, if f_i is incident with v_{e^j} , for some $j \in \{1, 2\}$, then define $f'_i = e^j$, otherwise set $f'_i = f_i$. By definition, $f'_1, f'_2 \in E(G)$ and so $\{f'_1, f'_2\}$ would be an essential 2-edge-cut of G , contrary to $\kappa'(G) \geq 3$. Hence we must have $|c(J)| \geq 9$. Let C' be a longest cycle of J . We lift C' to a cycle C'' in $G(e^1, e^2)$ and convert C'' to a cycle C of G by undoing the subdivisions on e^1 and e^2 if $\{v_{e^1}, v_{e^2}\} \cap V(C') \neq \emptyset$. As v_{e^1}, v_{e^2} might be in $V(C')$, we have $|E(C)| \geq 7$.

Assume first that $V(G) - V(C) \neq \emptyset$. Since G is connected, there must be a vertex $v \in V(G) - V(C)$ with $uv \in E(G)$ for some $u \in V(C)$. Since $|E(C)| \geq 7$, $G[E(C) \cup \{uv\}]$ contains $Y_{4,0,0}$, $Y_{3,1,0}$ and $Y_{2,2,0}$ as subgraphs, in each of which u is the only degree 3 vertex. We are to show that G also contains $Y_{2,1,1}$ as a subgraph to find a contradiction to (4.17). Suppose there exists $w \in N_G(v) - V(C)$. Then $G[E(C) \cup \{uv, vw\}]$ contains $Y_{2,1,1}$ as a subgraph that takes u as the only degree 3 vertex. Hence we have $N_G(v) \subseteq V(C)$. Since $\kappa'(G) \geq 3$, we may assume $\{v_1, v_2, v_3\} \subseteq N_G(v)$. As C is the longest cycle of G , we have $2 \leq d_C(v_i, v_j) \leq 3$ for $1 \leq i < j \leq 3$. Then $G[E(C) \cup \{vv_1, vv_2, vv_3\}]$ contains $Y_{2,1,1}$ as a subgraph that takes v as the only degree 3 vertex, a contradiction. Hence we must have $V(G) = V(C)$. Let $n = |V(G)|$. Denote $V(C) = \{v_i : i \in \mathbb{Z}_n\}$ with $E(C) = \{v_i v_{i+1} : i \in \mathbb{Z}_n\}$. If $n = 7$, then by (4.12), J satisfies the hypotheses of Lemma 4.6.1, and so J is collapsible, a contradiction. Therefore we must have $n \geq 8$.

Claim 15. *If $n = 8$, then (4.17) is violated.*

We assume that $n = 8$ to justify the claim. By (4.12), if $\Delta(G) \geq 4$, then $F(J) \leq 3$, and so by Lemma 4.6.1, J must be collapsible, a contradiction. Thus G must be a 3-regular graph with C being a Hamilton cycle of G . For any $t \in \mathbb{Z}_8$, there exists an $i(t) \in \mathbb{Z}_8 - \{t\}$ such that $v_t v_{i(t)} \in E(G) - E(C)$. Since $\kappa'(G) \geq 3$ and G is 3-regular, G cannot have parallel edges, and so $i(t) \notin \{t-1, t+1\}$ in \mathbb{Z}_8 .

If there is a $t \in \mathbb{Z}_8$ with $i(t) = t+2$ in \mathbb{Z}_8 , then by symmetry, we may assume that $i(1) = 3$. If, in addition, $i(2) = 4$, then as G is 3-regular, $\{v_8 v_1, v_4 v_5\}$ is a 2-edge-cut of G , contrary to $\kappa'(G) \geq 3$. Thus in \mathbb{Z}_8 , by symmetry $i(2) \notin \{4, 8\}$, and so $i(2) \in \{5, 6, 7\}$. Suppose $i(2) = 5$. Then as $\mathcal{Y}_2 = \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$, for each $Y \in \mathcal{Y}_2$, $G[E(C) \cup \{v_1 v_3, v_2 v_5\}]$ contains a Y as a subgraph with v_1 being the only vertex of degree 3, contrary to (4.17). Hence by

symmetry, $i(2) \notin \{5, 7\}$, forcing $i(2) = 6$. It follows that for any $Y \in \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}\}$, $G[E(C) \cup \{v_1v_3, v_2v_6\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3. Furthermore, $G[E(C) \cup \{v_1v_3, v_2v_6\}]$ contains $Y_{2,1,1}$ as a subgraph with v_6 being the only degree 3 vertex, and so (4.17) is violated. We conclude that by symmetry, for any $t \in \mathbb{Z}_8$, $i(t) \notin \{t-2, t-1, t, t+1, t+2\}$, or equivalently,

$$\text{for any } t \in \mathbb{Z}_8, i(t) \in \{t+3, t+4, t+5\} \text{ in } \mathbb{Z}_8. \quad (4.19)$$

If there is a $t \in \mathbb{Z}_8$ with $i(t) = t+3$ in \mathbb{Z}_8 , then by symmetry, we may assume that $i(1) = 4$. If $i(2) = 5$, then by (4.19) and as G is 3-regular, we must have $i(3) = 7$, forcing $i(6) = 8$ violating (4.19). Thus by symmetry, in \mathbb{Z}_8 , we must have $i(2) \notin \{5, 7\}$, and so $i(2) = 6$. It follows that for any $Y \in \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}\}$, $G[E(C) \cup \{v_1v_4, v_2v_6\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3. As $G[E(C) \cup \{v_1v_4, v_2v_6\}]$ also contains $Y_{2,1,1}$ as a subgraph with v_6 being the only degree 3 vertex, (4.17) is violated. We now conclude that by symmetry, we must have $i(t) = t+4$ for any $t \in \mathbb{Z}_8$, and so $G \cong H_8$. By (4.16), (4.17) is violated. This completes the proof for the claim.

By Claim 15, we must have $n \geq 9$. We first prove that

$$G \text{ always contains } Y_{4,0,0} \text{ as a subgraph.} \quad (4.20)$$

Since $\kappa'(G) \geq 3$, we may assume that $v_1v_j \in E(G)$ for some j with $1 < j \leq n/2 + 1$. If $n \geq 11$, then $G[E(C[v_{j-1}, v_{j+5}]) \cup \{v_1v_j\}] \cong Y_{4,0,0}$. Assume that $n = 10$. If there exists an $t \in \mathbb{Z}_{10}$, and a $t' \in \mathbb{Z}_{10} - \{t-1, t, t+1\}$ with $v_tv_{t'} \in E(G) - E(C)$, such that v_t and $v_{t'}$ are of distance at most 4 on C , then as $G[E(C) \cup \{v_tv_{t'}\}]$ contains a cycle of length at least 7 other than C , $Y_{4,0,0}$ is a subgraph of $G[E(C) \cup \{v_tv_{t'}\}]$. It follows that we must have $t' = t+5$ in \mathbb{Z}_{10} , whence $G[(E(C - \{v_8, v_9, v_{10}\}) - \{v_3v_4\}) \cup \{v_2v_7, v_4v_9\}] \cong Y_{4,0,0}$. Now assume that $n = 9$. We observe that to avoid a $Y_{4,0,0}$, any chord of C must have the form v_iv_{i+4} , and so $G[(E(C - \{v_9, v_{10}\}) - \{v_3v_4, v_6v_7\}) \cup \{v_1v_5, v_3v_7\}] \cong Y_{4,0,0}$. Hence (4.20) must hold.

By (4.20), it suffices to show that any $Y \in \{Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$ is a subgraph of G . Let $e \in E(G) - E(C)$ be an edge. Since C is a Hamilton cycle of G , e is a chord of C . Let $g(C+e)$ be the length of a shortest cycle of $G[E(C) \cup \{e\}]$. Since $J = G(e^1, e^2)$ is reduced, and since cycles of length at most 3 is collapsible, it follows that every cycle of length at most 3 contains either e^1 or e^2 , and a cycle of length 2 in G must be induced by $\{e^1, e^2\}$. Since $n \geq 9$ and $\kappa'(G) \geq 3$, C has at least $\lceil \frac{n}{2} \rceil = 5$ chords. It follows that there must be a chord $e \in E(G) - E(C)$ such that $g(C+e) \geq 4$. By symmetry, assume that $e = v_1v_j$ with $4 \leq j \leq 7$. Then for any $Y \in \{Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$, $G[E(C) \cup \{v_1v_j\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3 in Y . This, together with (4.20), implies that (4.17) is violated. This completes the proof of the Lemma. \blacksquare

Proof of Theorem 4.1.2(ii). It suffices to prove that for $s \geq 0$,

$$\text{if (4.17) and } \kappa(L(G)) \geq s + 3, \text{ then } L(G) \text{ is } s\text{-Hamilton-connected.} \quad (4.21)$$

We argue by induction on s to prove (4.21), and assume that $s = 0$. Let G be a graph with $\kappa(L(G)) \geq 3$, and let G_0 be the core of G . By Lemma 4.2.3(ii), it suffices to show that G_0 is strongly spanning trailable. By Lemma 4.2.3(i), $\kappa'(G_0) \geq 3$. By (4.17), G_0 also does not contain any $Y \in \mathcal{Y}_2$ as a subgraph. It follows by Lemma 4.6.3 that G_0 is strongly spanning trailable. Hence (4.21) holds for $s = 0$.

Assume that $s > 0$ and (4.21) holds for smaller values of s . For any edge subset $X \subseteq E(G)$ with $0 < |X| \leq s$. Pick $e_0 \in X$. Define $G_1 = G - e_0$ and $X_1 = X - \{e_0\}$. By (4.17), G_1 does not have $Y_{4,0,0}$ as a subgraph, with $\kappa(L(G_1)) = \kappa(G - e_0) \geq (s + 3) - 1 = (s - 1) + 3$. By induction, G_1 is $(s - 1)$ -Hamilton-connected, and so $L(G) - X = L(G_1) - X_1$ is Hamilton-connected. This completes the proof of Theorem 4.1.2(ii).

Chapter 5

Characterizations of matroids with an element lying in a restricted number of circuits

5.1 Main Results

A matroid M with a distinguished element $e_0 \in E(M)$ is a **rooted matroid** with e_0 being the root. We often use $M(e_0)$ to emphasize the root e_0 . Two rooted matroids $M(e_0)$ and $N(f_0)$ are isomorphic if e_0 corresponds to f_0 under the matroid isomorphism. When f_0 is not emphasized, we often just say that M or $M(e_0)$ is isomorphic to N . Given a matroid $M(e_0)$, define $\mathcal{C}_{M,e_0} = \{C \in \mathcal{C}(M) : e_0 \in C\}$. We obtained the following.

Theorem 5.1.1. *Let M be a binary matroid. Each of the following holds.*

- (i) *There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_1 - \{M(K_3P_3)\}$ with $e_0 \in E(M)$; or to $M(K_3P_3)$ with e_0 being any edge of K_3P_3 lying in a 2-circuit.*
- (ii) *There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_2 - \{M(C_4P_4)\}$ with $e_0 \in E(M)$; or to $M(C_4P_4)$ with e_0 being the only edge not lying in a 2-circuit.*

5.2 Binary matroids with an element in restricted number of circuits

The main purpose of this section is to characterize all connected binary rooted matroids whose root is lying in at most three circuits, and all connected binary rooted matroids whose root is

lying in all but at most three circuits. Let

$$\mathcal{F}_1 = \{M = M(e_0) : |\mathcal{C}_{M,e_0}| \leq 3\}, \text{ and } \mathcal{F}_2 = \{M = M(e_0) : |\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3\}. \quad (5.1)$$

Throughout this section, for fixed $i \in \{1, 2\}$, if M is such a matroid that for any $e_0 \in E(M)$, $M(e_0)$ is in \mathcal{F}_1 , then we simply say that $M \in \mathcal{F}_i$ without indicating the root.

Excluded minor characterizations will be developed in this section. Let \mathcal{F} be a collection of matroids. Define $EX(\mathcal{F})$ to be the family of matroids such that $M \in EX(\mathcal{F})$ if and only if M does not have a minor isomorphic to a member in \mathcal{F} . When $\mathcal{F} = \{N_1, N_2, \dots, N_k\}$ is a finite collection, we also use $EX(N_1, N_2, \dots, N_k)$ for $EX(\{N_1, N_2, \dots, N_k\})$. Following [84], F_7 and F_7^* are the two binary vector matroids $F_7 = M_2[I_3|D]$ and $F_7^* = M_2[D^T|I_4]$, where

$$[I_3|D] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } [D^T|I_4] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.2)$$

Let M and N be matroids. If for some element $f \in E(M)$, f lies in a 2-circuit of M and $M - f = N$, then M is a **single element parallel extension** of N and N is a **single parallel deletion** of M . If M is obtained from N by taking a finite number of single element parallel extensions, then M is a **parallel extension** of N . If for some element $f \in E(M)$, f lies in a 2-cocircuit of M and $M/f = N$, then M is a **single element serial extension** of N and N is a **serial contraction** of M . If M is obtained from N by taking a finite number of single element serial extensions, then M is a **serial extension** of N . A subset $X \subseteq E(M)$ is a **serial class** if every pair of elements in X form a cocircuit of M such that X is a maximal subset of $E(M)$ with this property.

Proposition 5.2.1. (*Li and Liu, Lemma 6 of [63]*) Suppose that $e, e' \in E(M)$ and $\{e, e'\} \in \mathcal{C}(M^*)$.

(i) For any element $e_0 \neq e'$, $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if $|\mathcal{C}_{M/e',e_0}| \leq 3$; and $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if $|\mathcal{C}(M/e')| - |\mathcal{C}_{M/e',e_0}| \leq 3$.

(ii) Consequently, if M is a serial extension of a matroid N , and if $e_0 \in E(N)$, then $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if $|\mathcal{C}_{N,e_0}| \leq 3$; and $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if $|\mathcal{C}(N)| - |\mathcal{C}_{N,e_0}| \leq 3$.

5.2.1 Rooted matroid minors

Let $M(e_0)$ be a rooted matroid. A **rooted minor** of $M(e_0)$ is a rooted matroid $N = N(e_0)$ such that for some disjoint subsets $S, T \subseteq E(M - e_0)$, $N = M/S - T$. Proposition 5.2.1 can be slightly extended to Lemma 5.2.2 below, showing that the properties of satisfying $|\mathcal{C}_{M,e_0}| \leq 3$ and of satisfying $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ are in fact closed under taking rooted minors.

Lemma 5.2.2. *Let $M = M(e_0)$ be a matroid rooted at e_0 .*

- (i) *If $|\mathcal{C}_{M,e_0}| \leq 3$, then for any $x \in E(M) - e_0$, $|\mathcal{C}_{M-x,e_0}| \leq 3$.*
- (ii) *If $|\mathcal{C}_{M,e_0}| \leq 3$, then for any $x \in E(M) - e_0$, $|\mathcal{C}_{M/x,e_0}| \leq 3$.*
- (iii) *If $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$, then for any $x \in E(M) - e_0$, $|\mathcal{C}(M-x)| - |\mathcal{C}_{M-x,e_0}| \leq 3$.*
- (iv) *If $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$, then for any $x \in E(M) - e_0$, $|\mathcal{C}(M/x)| - |\mathcal{C}_{M/x,e_0}| \leq 3$.*

Proof. Let $M = M(e_0) \in \mathcal{F}_1$, and let $x \in E(M) - e_0$. By definition, $|\mathcal{C}_{M,e_0}| \leq 3$. As $\mathcal{C}(M-x) \subseteq \mathcal{C}(M)$, we have $\mathcal{C}_{M-x,e_0} \subseteq \mathcal{C}_{M,e_0}$. Moreover, for any $C \in \mathcal{C}(M-x) - \mathcal{C}_{M-x,e_0}$, as $\mathcal{C}(M-x) \subseteq \mathcal{C}(M)$ and $e_0 \notin C$, we have $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_0}$, implying that $\mathcal{C}(M-x) - \mathcal{C}_{M-x,e_0} \subseteq \mathcal{C}(M) - \mathcal{C}_{M,e_0}$. Therefore, we have both $|\mathcal{C}_{M-x,e_0}| \leq |\mathcal{C}_{M,e_0}| \leq 3$ and $|\mathcal{C}(M-x) - \mathcal{C}_{M-x,e_0}| \leq |\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \leq 3$, and so (i) and (iii) must hold.

We now prove (ii). As $\mathcal{C}(M/x)$ consists of the minimal members of $\{C - x : C \in \mathcal{C}(M)\}$, for each $C' \in \mathcal{C}_{M/x,e_0}$, there exists a circuit $C \in \mathcal{C}_{M,e_0}$ with $C' = C - x$. Thus the mapping $f(C') = C$ is injective. This implies that $|\mathcal{C}_{M/x,e_0}| \leq |\mathcal{C}_{M,e_0}| \leq 3$, and so (ii) holds. Similarly, for each $C' \in \mathcal{C}(M/x) - \mathcal{C}_{M/x,e_0}$, there exists a $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_0}$ with $C' = C - x$. As the mapping from C' to C is injective, it follows that $|\mathcal{C}(M/x) - \mathcal{C}_{M/x,e_0}| \leq |\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \leq 3$, implying (iv). \blacksquare

The following theorem of Brylawski and Seymour will be needed in our arguments.

Theorem 5.2.3. (Brylawski [11] and Seymour [93]) *Let N be a connected minor of a connected matroid M . For any $f \in E(M) - E(N)$, one of $M - f$ and M/f is connected and contains N as a minor.*

Lemma 5.2.4. *Let M, N be a connected matroids such that N is a minor of M , and let $e_0 \in E(M) - E(N)$. Each of the following holds.*

- (i) *Either $|E(M)| = |E(N)| + 1$, or M has a connected proper minor L with $e_0 \in E(L)$ such that L contains N as a minor.*
- (ii) *$M(e_0)$ contains a connected rooted minor $L(e_0)$ such that $L(e_0) - e_0 = N$.*

Proof. As (ii) follows from (i), we argue by induction on $|E(M)|$ to prove (i). By assumption, $|E(M)| \geq |E(N) \cup e_0| = |E(N)| + 1$. If $|E(M)| = |E(N)| + 1$, then $L = M$. Assume that $|E(M)| > |E(N)| + 1$ and the lemma holds for smaller values of $|E(M)|$. Pick $f \in E(M) - (E(N) \cup e_0)$. By Theorem 5.2.3, either $M - f$ or M/f is connected, contains e_0 as an element and N as a minor. Thus by induction, either $M - f$ or M/f has a connected minor L with $e_0 \in E(L)$ such that L contains N as a minor. \blacksquare

We need a few more notational conventions.

Notation 5.2.5. *For an integer $r > 0$, let $V(r, 2)$ denote the r -dimensional vector space over the 2-element field $GF(2)$. Suppose that $M = M_2[I_r|D]$ is a binary matroid with $E(M) = \{e_1, e_2, \dots, e_m\}$ such that, for $1 \leq i \leq m$, e_i is the label of the i th column vector v_i of $[I_r|D]$.*

Then $B = \{e_1, e_2, \dots, e_r\}$ is a basis of M and $\{v_1, v_2, \dots, v_r\}$ is the standard basis of $V(r, 2)$. For any nonzero vector $v = (x_1, x_2, \dots, x_r) \in V(r, 2) - \{0\}$,

$$S(v) = \{i : x_i \neq 0\} \text{ and } B(v) = \{e_i : 1 \leq i \leq r \text{ and } x_i \neq 0\}. \quad (5.3)$$

Thus $B(v)$ is the unique minimum subset of B such that the vectors $\{v\} \cup \{v_i : e_i \in B(v)\}$ is a linearly dependent set in $\{v_1, v_2, \dots, v_r, v\}$ that contains v .

Using the notation in Definition 5.2.5, we have the following observations. Observation 6 follows immediately from the definition of a vector matroid and from (5.3).

Observation 6. Let $M = M_2[I_r|D]$ denote a binary matroid.

(i) M is simple if and only if $[I_r|D]$ does not have an all zero column and does not have two identical columns. Consequently, if M is simple, then for any $j \geq r+1$, $|S(v_j)| \geq 2$.

(ii) For vectors $w_1, w_2 \in V(r, 2)$, $B(w_1) = B(w_2)$ if and only if $w_1 = w_2$.

Observation 7. Let $M = M_2[I_r|D]$ be a simple binary matroid, let $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ be distinct column vectors of D , and suppose that $\{e_{i_1}, e_{i_2}, \dots, e_{i_t}\} \in \mathcal{I}(M)$. Let $v = v_{i_1} + v_{i_2} + \dots + v_{i_t}$. Then the following are equivalent.

(i) $B(v) \cup \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\}$ is a circuit of M .

(ii) For any partition of the set $\{i_1, i_2, \dots, i_t\}$ into two disjoint nonempty sets J_1 and J_2 , we have $S(\sum_{i \in J_1} v_i) \cap S(\sum_{j \in J_2} v_j) \neq \emptyset$.

Proof. Let $X = B(v) \cup \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\}$ and $J = \{i_1, i_2, \dots, i_t\}$. Since M is binary and since $v \neq 0$, it follows by (5.3) that X is a disjoint union of circuits, and so there exist disjoint circuits C_1, C_2, \dots, C_s such that $X = \cup_{i=1}^s C_i$.

Assume (i) holds. Then $s = 1$. To show (ii), we argue by contradiction and assume that J can be partitioned into two disjoint nonempty sets J_1 and J_2 satisfying $S(\sum_{i \in J_1} v_i) \cap S(\sum_{j \in J_2} v_j) = \emptyset$. Let $w_1 = \sum_{i \in J_1} v_i$ and $w_2 = \sum_{j \in J_2} v_j$. Since $\{e_{i_1}, e_{i_2}, \dots, e_{i_t}\} \in \mathcal{I}(M)$, we have $w_1 \neq 0$ and $w_2 \neq 0$. By (5.3), each of $B(w_1) \cup \{e_i : i \in J_1\}$ and $B(w_2) \cup \{e_i : i \in J_2\}$ is a disjoint union of circuits of M contained in X , contrary to the assumption that $s = 1$. Hence (i) implies (ii).

We shall show that (ii) implies $s = 1$. By contradiction, we assume that $s \geq 2$. Define $J'_1 = \{i : e_i \in C_1\}$ and $J'_2 = \{i : e_i \notin C_1\}$. Since B is a basis, we must have $J_1 = J'_1 - \{1, 2, \dots, r\} \neq \emptyset$. With a similar argument, we also have $J_2 = J'_2 - \{1, 2, \dots, r\} \neq \emptyset$. Since $C_1 \cap (\cup_{i=2}^s C_i) = \emptyset$, we have $J_2 = J - J_1$. Define $w_1 = \sum_{i \in J_1} v_i$ and $w_2 = \sum_{i \in J_2} v_i$. By (5.3), $B(w_1) = \{e_i : i \in J'_1 \cap \{1, 2, \dots, r\}\}$ and $B(w_2) = \{e_i : i \in J'_2 \cap \{1, 2, \dots, r\}\}$. Thus for any $1 \leq j \leq r$, if $j \in S(w_1)$, then $e_j \in B(w_1) \subset C_1$; and if $j \in S(w_2)$, then $e_j \in B(w_2) \subset X - C_1$. It follows that $S(w_1) \cap S(w_2) = \emptyset$, contrary to (ii). This shows that (ii) implies (i). ■

Corollary 5.2.6. *Suppose that $M = M_2[I_r|D]$ is connected and simple such that D is an r by $m - r$ matrix with $m - r \geq 3$. If there exist distinct $h, k, \ell \in \{r + 1, r + 2, \dots, m\}$ satisfying*

$$S(v_\ell) \cap S(v_h) \neq \emptyset, S(v_\ell) \cap S(v_k) \neq \emptyset, \text{ and } S(v_h) \cap S(v_k) = \emptyset, \quad (5.4)$$

then either $B(v_\ell + v_h + v_k) \cup \{e_\ell, e_h, e_k\} \in \mathcal{C}(M)$ (if $v_\ell + v_h + v_k \neq 0$), or $\{e_h, e_k, e_\ell\} \in \mathcal{C}(M)$ (if $v_\ell + v_h + v_k = 0$).

Proof. Since M is simple, e_h, e_k, e_ℓ are mutually distinct non-zero vectors, and so if $v_\ell + v_h + v_k = 0$, then $\{e_h, e_k, e_\ell\} \in \mathcal{C}(M)$. Hence we assume that $\{e_h, e_k, e_\ell\} \notin \mathcal{C}(M)$. Again as M is simple, M contains no circuit of length at most 2, and so $\{e_h, e_k, e_\ell\} \in \mathcal{I}(M)$. For any partition of $\{e_h, e_k, e_\ell\}$ into two nonempty parts J_1 and J_2 , (5.4) implies that $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_\ell) \cap S(v_h)$ or $S(v_\ell) \cap S(v_k)$. Hence by Observation 7, Corollary 5.2.6 holds. ■

As in [84], for a basis B of M , for any $e \in E(M) - B$, we let $C_M(e, B)$ denote the fundamental circuit of e with respect to B . For the given basis $B = \{e_1, e_2, \dots, e_r\}$, define a graph $H = H_B$ with $V(H)$ being the fundamental circuits of e_{r+1}, \dots, e_m , with respect to B , such that two vertices of H are adjacent if and only if the corresponding fundamental circuits have a non-empty intersection. This graph H facilitates our arguments.

Observation 8. *A binary matroid $M = M_2[I_r|D]$ is connected if and only if M does not have any coloop and H_B is connected for any B . Or in another words, each of the following holds.*

- (i) *For any $i \in \{1, 2, \dots, r\}$, there must be a $j \in \{r + 1, \dots, m\}$ such that if $v_j = (x_1, x_2, \dots, x_r)$, then $x_i = 1$.*
- (ii) *If there exist distinct $i, j \in \{r + 1, \dots, m\}$ satisfying $S(v_i) \cap S(v_j) = \emptyset$, then there must be a $t_1, t_2, \dots, t_k \in \{r + 1, \dots, m\} - \{i, j\}$, such that both $S(v_i) \cap S(v_{t_1}) \neq \emptyset$, $S(v_{t_1}) \cap S(v_{t_2}) \neq \emptyset, \dots, S(v_{t_{k-1}}) \cap S(v_{t_k}) \neq \emptyset$, and $S(v_j) \cap S(v_{t_k}) \neq \emptyset$.*

Proof. For sufficiency, we assume the validity of (i)-(ii) to show that M has only one component. Let $H = H_B$ denote the graph defined right before this observation. Condition (ii) indicates that H is connected. Let E_1 denote the component that contains the fundamental circuit of e_{r+1} with respect to the basis B . If $E_1 = E(M)$, then M is connected. Assume to the contrary, that there exists an element $e_t \in E(M) - E_1$.

If $t \in \{r + 1, r + 2, \dots, m\}$, then as H is connected, there exists a sequence of fundamental circuits C^1, C^2, \dots, C^ℓ with respect to B such that $C^1 = C_M(e_{r+1}, B)$ and $C^\ell = C_M(e_t, B)$, and such that $C^i \cap C^{i+1} \neq \emptyset$, for each $i = 1, 2, \dots, \ell - 1$. It follows that for each $i = 1, 2, \dots, \ell - 1$, elements in $C^i \cup C^{i+1}$ are in the same component of M . Thus the elements in C^ℓ , in particular e_t , must be in E_1 , contrary to the assumption that $e_t \in E(M) - E_1$.

Hence we may assume that $t \in \{1, 2, \dots, r\}$. By (i), there must be a $j \in \{r + 1, \dots, m\}$ such that if $v_j = (x_1, x_2, \dots, x_r)$, then $x_t = 1$. This implies that $e_t \in C_M(e_j, B)$. By the connectedness of H , we once again conclude that e_t must be in E_1 , contrary to the assumption that $e_t \in E(M) - E_1$.

For necessity, by definition, M does not have any coloop. We use contradiction to show H_B is connected. Assume M is the minimum connected matroid such that H_B is disconnected for some B . Then H_B has two components, say H_1 and H_2 . Similarly arguing as above, $M(H_1)$ and $M(H_2)$ are connected. Also $E(M(H_1)) \cap E(M(H_2)) = \emptyset$ and $E(M(H_1)) \cup E(M(H_2)) = E(M)$. The contradiction justifies this necessity. ■

Observation 9. *In a binary matroid $M = M_2[I_r|D]$, we denote $D = (d_{ij})$ with $1 \leq i \leq r$ and $r+1 \leq j \leq m$; and let $w_i = (d_{i(r+1)}, d_{i(r+2)}, \dots, d_{im})$ be the i th row of D . Each of the following holds.*

- (i) *If for some $i \in \{1, 2, \dots, r\}$, there is an $i' \in \{r+1, \dots, m\}$ such that if $d_{ij} = 1$ if and only if $j = i'$, then $\{e_i, e_{i'}\} \in \mathcal{C}(M^*)$.*
- (ii) *If there exist distinct $i, j \in \{1, 2, \dots, r\}$ satisfying $w_i = w_j$, then $\{e_i, e_j\} \in \mathcal{C}(M^*)$.*
- (iii) *If there exist distinct $i, j, k \in \{1, 2, \dots, m\}$ such that e_i, e_j, e_k belong to the same serial class of M , then $M/e_i = M_2[I_{r-1}|D_1]$, where D_1 is obtained from D by deleting the i th row of D , is also a simple matroid.*
- (iv) *If there exist distinct $i, j \in \{1, 2, \dots, m\}$ such that e_i, e_j belong to the same serial class of M , then $M/e_i = M_2[I_{r-1}|D_1]$, where D_1 is obtained from D by deleting the i th row of D , is also a connected matroid.*

Proof. The justification of Observation 9 (i) and (ii) follow immediately from the fact that the dual of $M = M_2[I_r|D]$ is $M^* = M_2[D^T|I_{m-r}]$, in which every pair of identical columns form a cocircuit of M . The simpleness and the connectedness of $M/e_i = M_2[I_{r-1}|D_1]$ follow from Observation 6, and from Observation 8, respectively. ■

Definition 5.2.7. *For an integer $h > 0$, we have the following definitions.*

- (i) *Let K_2^h be the loopless graph with 2 vertices and h parallel edges.*
- (ii) *Let K_3P_3 be the loopless graph spanned by a 3-circuit $Z = u_1u_2u_3u_1$ such that $K_3P_3 - E(Z)$ is a path $u_1u_2u_3$. Thus the edge u_1u_3 is the only edge in K_3P_3 not lying in a 2-circuit. For any serial extension of $M(K_3P_3)$, let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(K_3P_3)$.*
- (iii) *Let $Z' = w_1w_2w_3w_4w_1$ denote a 4-circuit. Define C_4M_2 to be the loopless multigraph spanned by Z' such that $C_4M_2 - E(Z')$ is a matching with edges $\{w_1w_2, w_3w_4\}$; and C_4P_4 to be the loopless graph spanned by Z' such that $C_4P_4 - E(Z')$ is a path $w_1w_2w_3w_4$. Thus the edge w_1w_4 is the only edge in C_4P_4 not lying in a 2-circuit. For any serial extension of $M(C_4P_4)$, let $[w_1w_4]$ denote the set of edges obtained by subdividing the edge $w_1w_4 \in E(C_4P_4)$.*
- (iv) *Let L_5 denote the graph with $V(L_5) = \{u_1, u_2, u_3, z_1, z_2\}$ and $E(L_5) = \{u_1u_2, u_2u_3, u_3u_1, z_1u_1, z_1u_2, z_2u_2, z_2u_3\}$. For any serial extension of $M(L_5)$, let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(L_5)$.*

Figure 1. Graphs in Definition 5.2.7.

By definition, both L_5 and C_4M_2 are serial extensions of K_3P_3 . It is routine to verify the observations stated in Proposition 5.2.8 below.

Proposition 5.2.8. *We shall use the notation in Definition 5.2.7. For a given graph G , let $M = M(G)$ denote its cycle matroids.*

- (i) *If $G \in \{K_2^2, K_2^3, K_2^4\}$, and e_0 is any edge in $E(G)$, or if $G = K_3P_3$ and $e_0 \in E(K_3P_3) - \{u_1u_3\}$, then $|\mathcal{C}_{M,e_0}| \leq 3$. If $G = K_3P_3$ and $e_0 = u_1u_3$, then $|\mathcal{C}_{M(K_3P_3),u_1u_3}| \geq 4$.*
- (ii) *If $G \in \{K_2^2, K_2^3, K_2^4, K_3P_3, C_4M_2, K_4\}$, and e_0 is any edge in $E(G)$, or if $G = C_4P_4$ and $e_0 = w_1w_4$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$. If $G = C_4P_4$ and $e_0 \neq w_1w_4$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \geq 4$.*
- (iii) *If G is a member in $\{K_2^4, K_3P_3, C_4M_2\}$, and if G' is obtained from G by adding an edge joining two distinct vertices in G , then for any edge $e_0 \in E(G)$, $|\mathcal{C}_{M,e_0}| \geq 4$.*
- (iv) *If G is a member in $\{K_2^4, K_3P_3, C_4M_2, C_4P_4, K_4\}$, and if G' is obtained from G by adding an edge joining two distinct vertices in G , then for any edge $e_0 \in E(G)$, $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \geq 4$.*
- (v) *If $M \in \{M(K_4), F_7\}$, then for any $e \in E(M)$, $|\mathcal{C}_{M,e}| \geq 4$.*

In the next lemma, we will follow the language of Notation 5.2.5.

Lemma 5.2.9. *Let $r \geq 4$ be an integer and $M = M_2[I_r|D]$ be a connected simple binary matroid where D is an r by 3 matrix. Then M is isomorphic to $M(L_5)$ if each of the following holds.*

- (i) $S(v_{r+1}) \cap S(v_{r+3}) \neq \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) = \emptyset$.
- (ii) For any $\{e_i, e_j\} \in \mathcal{C}(M^*)$, M/e_i is not simple.

Proof. For $j = r+1, r+2, r+3$, denote $v_j = (x_1^j, x_2^j, \dots, x_r^j)^T$. By (i), $S(v_{r+3}) \cap S(v_{r+2}) = \emptyset$, and so without loss of generality, we may assume that for some integers s, s_1, t, t_1 with $0 \leq s_1 \leq s < t \leq t_1 \leq r$, v_{r+1}, v_{r+2} and v_{r+3} satisfy the following:

$$\begin{aligned} x_1^{r+3} &= x_2^{r+3} = \dots x_s^{r+3} = 1 \text{ and } x_j^{r+3} = 0 \text{ if } j > s \text{ with } 2 \leq s \leq r-2, \\ x_t^{r+2} &= x_{t+1}^{r+2} = \dots x_r^{r+2} = 1 \text{ and } x_j^{r+2} = 0 \text{ if } j < t \text{ with } r-1 \leq t \leq r, \\ x_{s_1}^{r+1} &= x_{s_1+1}^{r+1} = \dots x_{t_1}^{r+1} = 1 \text{ and } x_j^{r+1} = 0 \text{ if } j < s_1 \text{ or } j > t_1 \text{ with } 0 \leq s_1 \leq s < t \leq t_1 \leq r. \end{aligned}$$

Note that the assumed inequalities $2 \leq s \leq r-2$ and $r-1 \leq t \leq r$ follow from Observation 6, and the assumed inequalities $s_1 \leq s < t \leq t_1$ follow from Observation 8.

Claim 16. *We have these observations.*

- (a) $0 \leq s_1 \leq s = 2$. (By symmetry, $t = r-1 \leq t_1 \leq r$.)
- (b) $t = s+1$.
- (c) $s = 2, t = 3$ and $r = 4$.

To justify Claim 16, we will use the fact $M^* = M_2[D^T|I_3]$ and Observation 9. If $s \geq 3$, then either $s_1 \geq 3$ and $\{e_1, e_2, e_{r+3}\}$ is contained in a serial class of M , or $s_1 \leq 2$ and $\{e_2, e_3\}$ is contained in a serial class of M . In either case, by Observation 9, M/e_2 is simple, contrary to Lemma 5.2.9 (ii). Hence $s_1 \leq s \leq 2$. By Observation 6, $s = |S(v_{r+3})| \geq 2$ and so $s = 2$, and Claim 16(a) must hold.

If $t \geq s + 2$, then $\{e_{s+1}, e_{r+1}\} \in \mathcal{C}^*(M)$. By Observation 8 and as $s_1 \leq s < t \leq t_1$, it follows by Observation 6 that M/e_2 is simple, contrary to Lemma 5.2.9 (ii). Hence Claim 16(b) must hold.

By Claim 16(a) and (b), we have $s = 2$, $t = 3$ and $r = 4$, and so (c) follows. This proves Claim 16.

As a consequence of Claim 16(c), D must be one of the following matrices:

$$D \in \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\}. \quad (5.5)$$

It is routine to show that for any D in (5.5), $M = M_2[I_4|D]$ is always isomorphic to $M(L_5)$. ■

By Observation 8, if $m = r + 3$, the graph H_B is either a K_3 or a P_3 . This gives us a bit more structural information of M . In the next lemma, we adopt the terms and notation in Definition 5.2.7.

Lemma 5.2.10. *Let M be a binary matroid with $r = r(M) > 0$ and $|E(M)| \geq 2$, and let $e \in E(M)$ be an arbitrary element. For any serial extension of $M(L_5)$, let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(L_5)$. Each of the following holds.*

- (i) *If M is loopless and coloopless with $|E(M)| \geq r(M) + 5$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$.*
- (ii) *If M is connected and simple with $|E(M)| \geq r(M) + 3$, then $|\mathcal{C}_{M,e}| \geq 4$ if and only if M is not isomorphic to a serial extension of $M(L_5)$ with $e \notin [u_1u_3]$.*
- (iii) *If M is connected and simple with $|E(M)| \geq r(M) + 4$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$, unless M is a serial extension of $M(C_4P_4)$ and e is in the serial class obtained from subdividing the only edge in C_4P_4 that is not in a 2-circuit.*
- (iv) *If M is connected and simple with $|E(M)| \geq r(M) + 3$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 3$, unless M is a serial extension of $M(K_3P_3)$ and e is in the serial class obtained from subdividing the only edge in K_3P_3 that not in a 2-circuit.*

Proof. (i) Since M is coloopless, e is not a coloop and so there exists a basis $B \in \mathcal{B}(M)$ such that $e \notin B$. Let $e_1, e_2, e_3, e_4 \in E(M) - (B \cup e)$. Then the fundamental circuits $C_M(e_i, B)$, $1 \leq i \leq 4$, are all in $\mathcal{C}(M) - \mathcal{C}_{M,e}$, and so $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$. This proves (i).

In the proofs for (ii)-(iv), we assume that M is a binary connected simple matroid. Since M is connected, there exists a basis $B \in \mathcal{B}(M)$ such that $e \notin B$. Thus we may assume that for

some r by $(m-r)$ binary matrix D , $M = M_2[I_r|D]$, $E(M) = \{e_1, e_2, \dots, e_m\}$ such that e_i is the label of the i th column vector v_i of $[I_r|D]$ with $B = \{e_1, e_2, \dots, e_r\}$ and $e \in \{e_{r+1}, \dots, e_m\}$.

We are to argue by induction on $r = r(M)$ to prove (ii). Since M is simple and $|E(M)| \geq r+3$, we may assume that $r \geq 3$. If $r = 3$, then since M is simple, it follows by Observation 6 that $6 \leq |E(M)| \leq 7$, and so $M \in \{M(K_4), F_7\}$. Now by Proposition 5.2.8(v), for any $e \in E(M)$, $|\mathcal{C}_{M,e}| \geq 4$. Therefore, we assume that $r \geq 4$ and Lemma 5.2.10 (ii) holds for smaller values of r .

Since L_5 is a serial extension of K_3P_3 , it follows by Proposition 5.2.8(i) that if M is isomorphic to a serial extension of $M(L_5)$ with $e \notin [u_1u_3]$, then $|\mathcal{C}_{M,e}| \leq 3$. It remains to prove the sufficiency of (ii). In the proof for (ii), we may assume that $e = e_m$; and by Observation 8, there must be some j with $r+1 \leq j \leq m-1$ satisfying $S(v_m) \cap S(v_j) \neq \emptyset$. We may assume that $S(v_m) \cap S(v_{r+j}) \neq \emptyset$ for $1 \leq j \leq j_0 < m-r$. If $j_0 \geq 3$, then by Observation 7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+j}) \cup \{e_m, e_{r+j}\}$, $(1 \leq j \leq 3)$ are 4 distinct circuits of M containing e_m . Hence we assume that $j_0 \leq 2$.

(ii-A) Suppose that $j_0 = 2$ and $m-r \geq 4$. Then for any j with $3 \leq j < m-r$, $S(v_{r+j}) \cap S(v_m) = \emptyset$. By Observation 8, we may assume that $S(v_{r+3}) \cap S(v_m) = \emptyset$ and $S(v_{r+1}) \cap S(v_{r+3}) \neq \emptyset$. By Observation 7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+j}) \cup \{e_m, e_{r+j}\}$, $(1 \leq j \leq 2)$ are 3 distinct circuits of M containing e_m . By Corollary 5.2.6, either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. Thus in this case, $|\mathcal{C}_{M,e}| \geq 4$.

(ii-B) Suppose that $j_0 = 1$ and $m-r \geq 4$. Then $S(v_m) \cap S(v_{r+j}) = \emptyset$ for $j = 2, \dots, m-r-1$. By Observation 8, we assume that $S(v_{r+1}) \cap S(v_{r+3}) \neq \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) \neq \emptyset$. By Observation 7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$ are distinct circuits of M containing e_m . By Corollary 5.2.6, either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. To show that $|\mathcal{C}_{M,e}| \geq 4$, we need to find an additional circuit containing e_m .

If $S(v_{r+1}) \cap S(v_{r+2}) \neq \emptyset$, then by Corollary 5.2.6, either $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+2}\} \in \mathcal{C}_{M,e}$. Hence $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, either $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\}$ or $\{e_m, e_{r+1}, e_{r+2}\}$, and either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\}$ or $\{e_m, e_{r+1}, e_{r+3}\}$ are in $\mathcal{C}_{M,e}$, and so $|\mathcal{C}_{M,e}| \geq 4$.

Assume that $S(v_{r+1}) \cap S(v_{r+2}) = \emptyset$ and $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. Then $v_m + v_{r+1} + v_{r+3} = 0$. As $S(v_m) \cap S(v_{r+1}) \neq \emptyset$, $S(v_{r+1}) \cap S(v_{r+3}) \neq \emptyset$ and $S(v_m) \cap S(v_{r+3}) = \emptyset$, we must have $S(v_{r+1}) = S(v_m) \cup S(v_{r+3})$. It follows that $S(v_{r+1}) \cap S(v_{r+2}) \neq \emptyset$ as $S(v_{r+3}) \cap S(v_{r+2}) \subseteq S(v_{r+1}) \cap S(v_{r+2})$; and $v_m + v_{r+1} + v_{r+2} \neq 0$. By Corollary 5.2.6, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\}$ and $\{e_m, e_{r+1}, e_{r+3}\}$ are in $\mathcal{C}_{M,e}$. Thus $|\mathcal{C}_{M,e}| \geq 4$.

Assume that $S(v_{r+1}) \cap S(v_{r+2}) = \emptyset$ and $\{e_m, e_{r+1}, e_{r+3}\} \notin \mathcal{C}_{M,e}$. We are to apply Observe 7 to show that $B(v_{r+1} + v_{r+2} + v_{r+3} + v_m) \cup \{e_{r+1}, e_{r+2}, e_{r+3}, e_m\} \in \mathcal{C}_{M,e}$. Suppose we partition $\{r+1, r+2, r+3, m\}$ into two non-empty subsets J_1 and J_2 with $m \in J_1$. If $r+1 \in J_2$, then $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_m) \cap S(v_{r+1})$; if $J_1 = \{r+1, m\}$, then $S(\sum_{i \in J_1} v_i) \cap$

$S(\sum_{i \in J_2} v_i)$ contains either $S(v_{r+1}) \cap S(v_{r+3})$; if $\{r+1, m\} \subset J_1$ and $|\{r+2, r+3\} \cap J_1| = 1$, then $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_{r+2}) \cap S(v_{r+3})$. In any case, $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i) \neq \emptyset$. It follows by Observation 7 that $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\}$ and $B(v_{r+1} + v_{r+2} + v_{r+3} + v_m) \cup \{e_{r+1}, e_{r+2}, e_{r+3}, e_m\}$ are in $\mathcal{C}_{M,e}$. Thus $|\mathcal{C}_{M,e}| \geq 4$.

(ii-C) Suppose that $j_0 = 1$ and $m-r = 3$. Recall that $S(v_m) \cap S(v_{r+1}) \neq \emptyset$ and $S(v_m) \cap S(v_{r+2}) = \emptyset$. If M has a cocircuit $\{e_i, e_j\}$ such that M/e_i is simple, then by Observation 9, M/e_i is also a connected simple binary matroid with $r(M/e_i) < r(M)$ and $|E(M/e_i)| = r(M/e_i) + 3$. It follows by induction that $|\mathcal{C}_{M/e_i,e}| \geq 4$ if and only if M/e_i is not isomorphic to a serial extension of $M(L_5)$ with $e \in [u_1 u_3]$. By Proposition 5.2.1, and since M is a serial extension of M/e_i , the conclusion of Lemma 5.2.10(ii) must hold. Hence we assume that for any $\{e_i, e_j\} \in \mathcal{C}(M^*)$, M/e_i is not simple. It follows by Lemma 5.2.9 that M is isomorphic to $M(L_5)$. This completes the proof for Lemma 5.2.10(ii).

To justify Lemma 5.2.10(iii) and (iv), we observe that

$$|\mathcal{C}(M(L_5))| \geq 4. \quad (5.6)$$

For a fixed element $e \in E(M)$, if $M - e$ is connected, then Lemma 5.2.10(iii) and (iv) follow by (5.6) and by applying Lemma 5.2.10(ii) to $M - e$. Therefore, we may assume that $M - e$ has connected components M_1, M_2, \dots, M_c with $c \geq 2$ such that

$$|E(M_1)| - r(M_1) \geq |E(M_2)| - r(M_2) \geq \dots \geq |E(M_c)| - r(M_c).$$

Since M is connected, $r(M - e) = r(M)$. Thus $\sum_{i=1}^c |E(M_i)| = |E(M - e)| = |E(M)| - 1$ and $r(M - e) = r(M) = \sum_{i=1}^c r(M_i)$, and so $\sum_{i=1}^c (|E(M_i)| - r(M_i)) = |E(M)| - r(M) - 1$. Note that by matroid rank axioms, if for some i , $|E(M_i)| \geq r(M_i) + 1$, then $E(M_i) \in \mathcal{C}(M)$; and that by matroid circuit axioms, if for some i , $|E(M_i)| \geq r(M_i) + 2$, then $|\mathcal{C}(M_i)| \geq 3$. These, together with $|\mathcal{C}(M) - \mathcal{C}_{M,e}| = |\mathcal{C}(M - e)| = \sum_{i=1}^c |\mathcal{C}(M_i)|$, lead us to the following observations.

(iii-A) If $|E(M)| - r(M) \geq 5$, then $\sum_{i=1}^c (|E(M_i)| - r(M_i)) \geq 4$ and so $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$.

(iii-B) If $|E(M)| - r(M) = 4$, then as $\sum_{i=1}^c (|E(M_i)| - r(M_i)) = 3$, we conclude that either $|E(M_i)| - r(M_i) = 1$ for $i = 1, 2, 3$ and $|E(M_i)| - r(M_i) = 0$ for $i \geq 4$, whence M is isomorphic to a serial extension of $M(C_4 P_4)$ with e being in the serial class obtained from subdividing the only edge in $C_4 P_4$ that is not in a 2-circuit; or $|E(M_1)| - r(M_1) = 2$, $|E(M_2)| - r(M_2) = 1$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 3$, whence $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| + |\mathcal{C}(M_2)| \geq 3 + 1 = 4$; or $|E(M_1)| - r(M_1) = 3$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 2$, whence by applying Lemma 5.2.10(ii) to M_1 and by (5.6), $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| \geq 4$.

(iv) If $|E(M)| - r(M) = 3$, then as $\sum_{i=1}^c (|E(M_i)| - r(M_i)) = 2$, we conclude that either $|E(M_i)| - r(M_i) = 1$ for $i = 1, 2$ and $|E(M_i)| - r(M_i) = 0$ for $i \geq 3$, whence M is isomorphic to a serial extension of $M(K_3 P_3)$ with e being in the serial class obtained from

subdividing the only edge in K_3P_3 that is not in a 2-circuit; or $|E(M_1)| - r(M_1) = 2$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 2$, whence $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| \geq 3$. This proves the lemma. ■

5.2.2 Graphic matroids

We in this subsections study the graphic matroid memberships of \mathcal{F}_1 and \mathcal{F}_2 . Let $G(e_0)$ be a graph with a distinguished edge $e_0 \in E(G)$, and let $M(e_0) = M(G(e_0))$ denote the cycle matroid of G rooted at e_0 . Following [84], a matroid M is **planar** if for some planar graph G , $M = M(G)$ is the cycle matroid of G . The goal of this subsection is to determine all rooted planar matroids $M(e_0)$ such that $|\mathcal{C}_{M,e_0}| \leq 3$, as well as all rooted planar matroids $M(e_0)$ such that $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$.

Definition 5.2.11. Let $M = M(e_0)$ be a connected rooted matroid with $r(M) \geq 1$.

- (i) The **serial reduction** (a **rooted serial reduction**, respectively) of M is a matroid obtained from M by repeatedly taking serial contractions (serial contractions of elements in $M - e_0$, respectively) until the contraction either is isomorphic to $U_{1,2}$ or has no more 2-cocircuit left.
- (ii) A rooted matroid $M(e_0)$ is a **rooted serial extension** of $N(f_0)$ if M is a serial extension of N and e_0 is in the serial class of M that contains f_0 .
- (iii) If $r(M) = 1$ or if $r(M) \geq 2$ and M contains no 2-cocircuits, then M is the serial reduction of itself. In this case, we said that M is **serially reduced**.

Theorem 5.2.12. Let G be a planar graph with $\kappa(G) \geq 2$, and let $M = M(G)$. Each of the following holds.

- (i) For some $e_0 \in E(G)$, $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if the serial reduction of M is isomorphic to $M(H)$, where H is a member in $\{K_2^2, K_2^3, K_2^4\}$ and with e_0 being an arbitrary edge in $E(H)$, or $H = K_3P_3$, with e_0 being any edge of K_3P_3 lying in a 2-circuit.
- (ii) If for some $e_0 \in E(G)$, $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if the serial reduction of M is isomorphic to $M(H)$, where H is a member in $\{K_2^2, K_2^3, K_2^4, K_3P_3, K_4\}$ with e_0 being an arbitrary edge in $E(H)$, or $H = C_4P_4$, with e_0 being the only edge not lying in a 2-circuit.

Proof. By Propositions 5.2.1 and 5.2.8, it suffices to prove the necessity in (i) and (ii). Let M' denote the serial reduction of $M = M(G)$. As a serial contraction in the cycle matroid $M(G)$ amounts to contracting one edge in an edge cut of size 2, we have $M' = M(H)$ is also a cycle matroid of some planar graph H , where either $H = K_2^2$ or H is 3-edge-connected. If $H = K_2^2$, then done. Hence we assume that $H \neq K_2^2$. Hence $\kappa'(H) \geq 3$. Since serial contraction does not reduce connectivity, we assume that $\kappa(H) \geq 2$ as well.

- (i) Suppose that for some $e_0 \in E(H)$, $|\mathcal{C}_{M',e_0}| \leq 3$. By Lemma 5.2.2 and Proposition 5.2.8(v), we may assume that H does not have a K_4 -minor. Let Z_0 be a shortest circuit in H with $e_0 \in Z_0$.

Since Z_0 is shortest, every chord of Z_1 in H is parallel to an edge of Z_0 . Let

$$s = |Z_0|, e_0 = v_s v_1 \text{ and } Z_0 - e_0 = v_1 v_2 \dots v_s \text{ denote the } (v_1, v_s)\text{-path.}$$

If $3 \geq |V(H)| \geq |Z_0| \geq 2$, then by the assumption of $|\mathcal{C}_{M',e_0}| \leq 3$ and by Proposition 5.2.8 (i) and (iii), either $H \in \{K_2^3, K_2^4\}$ with e_0 being any edge of H , or $H = K_3 P_3$ with e_0 being any edge of $K_3 P_3$ lying in a 2-circuit.

Now we assume that $|V(H)| \geq 4$.

Claim 1. $|V(H)| = s$. We may assume that $|V(H)| > s$. Let $V(H) - V(Z_0) = w_1, w_2, \dots, w_t$. Then $t \geq 1$. As $\kappa'(H) \geq 3$ and $\kappa(H) \geq 2$, for each i with $1 \leq i \leq t$, there exist three edge-disjoint paths P_1^i, P_2^i and P_3^i , internally vertex disjoint from $V(Z_0)$, joining w_i to at least two distinct vertices in $V(Z_0)$. Since H is K_4 -minor-free, $|\{z_1^i, z_2^i, z_3^i\}| \leq 2$; since $\kappa(H) \geq 2$, we can choose these path so that $|\{z_1^i, z_2^i, z_3^i\}| \geq 2$. Therefore, we may assume that $z_2^i = z_3^i$. Let P_0 be the (z_1^1, z_2^1) -path in Z_0 that contains e_0 . Since P_1^1, P_2^1 and P_3^1 are edge-disjoint paths, it follows that for each $j \in \{2, 3\}$, there is a circuit $Z^j \subseteq P_0 \cup P_1^1 \cup P_j^1$ containing e_0 .

If $t \geq 2$, then there exists a circuit Z' in H , containing e_0 and using at least one edge in $P_1^2 \cup P_2^2 \cup P_3^2 - (Z^2 \cup Z^3)$. It follows that Z_0, Z', Z^2, Z^3 are 4 circuits in H containing e_0 , contrary to $|\mathcal{C}_{M,e_0}| \leq 3$. Thus we must have $t = 1$. Since $s + t = |V(H)| \geq 4$, we must have $s = 3$, and so there exists a vertex $z \in V(Z_0) - \{z_1^1, z_2^1\}$. As $\kappa'(H) \geq 3$, there must be an edge $e' \in E(H) - (Z_0 \cup Z^2 \cup Z^3)$ incident with z . Since $\kappa(H) \geq 2$, there must be a circuit Z'' containing both e_0 and e' , and so Z_0, Z'', Z^2, Z^3 are 4 circuits in H containing e_0 , contrary to $|\mathcal{C}_{M,e_0}| \leq 3$.

Claim 2. $s \in \{2, 3\}$. If $s \geq 4$. Since $\delta(H) \geq 3$, each v_i , $1 \leq i \leq s - 1$, is incident with an edge e_i in $E(H) - Z_0$. Every e_i should be parallel to an edge of Z_0 , and there are at least two such e'_i 's, contrary to the assumption of $|\mathcal{C}_{M',e_0}| \leq 3$.

(ii) We argue by induction on $|E(H)|$ to show that Theorem 5.2.12 (ii) must hold. If $|E(H)| = 2$, then we must have $H = K_2^2$. We now assume that $|E(H)| > 2$ and Theorem 5.2.12 (ii) holds for graphs with fewer edges. Pick an edge $x \in E(G)$ and $x \neq e_0$. Let $M'' = M(H - x)$. Since $\kappa'(H) \geq 3$, then M'' is connected. By induction, $H - x \in \{K_2^2, K_2^3, K_2^4, K_3 P_3, K_4\}$ with e_0 being an arbitrary edge in $E(H - x)$, or $H - x = C_4 P_4$, with e_0 being the only edge not lying in a 2-circuit. Since $|\mathcal{C}(M(H))| - |\mathcal{C}_{M(H),e_0}| \leq 3$, by some routine checking, H has to be a member in $\{K_2^2, K_2^3\}$. \blacksquare

5.2.3 Binary matroids

Let $\{f, f'\}$ be a 2-circuit of a matroid L and let $M = L - f'$. We denote $L = M^{+,f}$ and call L the **parallel extension of M at f** .

The main purpose of this subsection is to characterize all rooted binary matroids $M(e_0)$ with $|\mathcal{C}_{M,e_0}| \leq 3$, as well as all rooted binary matroids $M(e_0)$ with $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$.

Let G be a connected graph. If X, Y are subsets of $V(G)$, then following the notation of [3], define

$$[X, Y] = \{xy \in E(G) : x \in X \text{ and } y \in Y\}, \text{ and } \partial_G(X) = [X, V(G) - X].$$

Thus $[X, Y]$ is a minimal edge cut if and only if $X \cap Y = \emptyset$ and both $G[X]$ and $G[Y]$ are connected subgraphs of G . Let $v \in V(G)$ be a vertex. Define $E_G(v) = [\{v\}, V(G) - \{v\}]$. Let $M = M(G)$ be the cycle matroid of G . If G is 2-connected, then every edge cut $[X, V(G) - X]$ with both $G[X]$ and $G - X$ being connected is a cocircuit of $M(G)$.

Throughout the rest of this section, we define

$$\mathbb{N} = \{F_7, M^*(K_5), M(K_5), (K_{3,3}), M^*(K_{3,3})\}.$$

By definition, every matroid in $\mathbb{N} \cup \{F_7^*\}$ is serially reduced, and contains K_4 as a minor. The next theorem is well known.

Theorem 5.2.13. (*Kuratowski [43] and Wanger [98], see also Theorem 5.2.5 of [84]*) *A binary matroid M is in $EX(\mathbb{N} \cup \{F_7^*\})$ if and only if $M = M(G)$ is a cycle matroid of a planar graph G .*

Lemma 5.2.14. *Let M be a connected matroid, N be a minor of M and $e_0 \in E(N)$. Each of the following holds.*

- (i) *If $M \in \mathbb{N} \cup \{F_7^*\}$, then $|\mathcal{C}_{M, e_0}| \geq 4$.*
- (ii) *If $M \in \mathbb{N}$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M, e_0}| \geq 4$.*
- (iii) *If a rooted binary matroid $M(e_0)$ contains a rooted minor $N(e_0) \in \mathbb{N} \cup \{F_7^*\}$, then $M(e_0) \notin \mathcal{F}_1$; if a rooted binary matroid $M(e_0)$ contains a rooted minor $N(e_0) \in \mathbb{N}$, then $M(e_0) \notin \mathcal{F}_2$.*

Proof. For any $M \in \mathbb{N} \cup \{F_7^*\}$, we have $|E(M)| - r(M) \geq 3$. Hence Lemma 5.2.10 implies both Lemma 5.2.14(i) and (ii). Lemma 5.2.14(iii) follows from Lemma 5.2.2. ■

Lemma 5.2.15. *If M is a connected matroid and $\{f, f'\} \in \mathcal{C}(M^*)$, then M/f is also connected.*

Proof. Let $G(M)$ denote the circuit graph of M . Then it is known that a coloopless matroid M is connected if and only if $G(M)$ is a connected graph. By a result of Li and Liu [63] (see Lemma 5.3.3(ii) in Section 3), $G(M) = G(M/f)$ and so M/f is connected if and only if M is connected. ■

Proposition 5.2.16. *Define $\mathbb{N}' = \{M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$. Let $r \geq 3$ be an integer and define $\mathcal{F}(r) = \{M : M \text{ is a connected simple binary matroid with } r(M) = r \text{ and } |E(M)| = r(M) + 3\}$. Define $A = [I_r | D]$, where D is an $(0,1)$ -matrix of dimension r by 3. We shall adopt the notation in Notation 5.2.5 and so for $1 \leq i \leq m$, e_i is the label of the i th column vector v_i of $[I_r | D]$. For a fixed matroid $M \in \mathcal{F}(r)$, we have the following observations.*

- (i) $M = M_2[A]$ for some $(0, 1)$ -matrix D with $B = \{e_1, e_2, \dots, e_r\}$ being a basis of M .
- (ii) For any $N \in \mathcal{F}(r)$, N is serially reduced if and only if $r \leq 4$ and D^T does not have a row vector with at most one nonzero entry and does not have two identical columns.
- (iii) $M(K_4)$ and F_7^* are the only serially reduced matroids in $\cup_{r \geq 3} \mathcal{F}(r)$.
- (iv) If $S(v_i) \cap S(v_j) = \emptyset$ holds for some distinct $i, j \in \{r+1, r+2, r+3\}$, then either $\{e_{r+1}, e_{r+2}, e_{r+3}\} \in \mathcal{C}(M)$ or M is not serially reduced.
- (v) Every matroid $M \in \cup_{r \geq 3} \mathcal{F}(r)$ is a serial extension of a matroid in \mathbb{N}' .
- (vi) For any $e \in E(M)$, if $M(e)$ is not a serial extension of $M(K_3P_3)(e_0)$ where e_0 is the only edge in K_3P_3 lying in a single element parallel class, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| = 3$.
- (vii) Let $M \in \mathcal{F}(r)$ and M^+ be a single parallel extension of M . Then for any $e_0 \in E(M^+)$, $|\mathcal{C}(M^+) - \mathcal{C}_{M^+,e_0}| \geq 4$.

To justify (ii), as $N = M_2[I_r|D]$, we have $N^* = M_2[D^T|I_3]$. Since N is connected, N^* is also connected and so N^* is loopless. It follows that N^* does not have a zero column. By definition, N is not serially reduced if and only if N^* has a circuit of size 2, which amounts to that $[D^T|I_3]$ has two identical columns. As $[D^T|I_3]$ is a $(0, 1)$ -matrix of dimension 3 by $r+3$ without a zero column, we observe that $[D^T|I_3]$ does not have two identical columns only if $r \leq 4$ and so (ii) must hold.

We apply (ii) to justify (iii), and assume that M is serially reduced and $|E(M)| = r+3$ with $r \in \{3, 4\}$. By (ii), the matrix D does not have a row with only one nonzero entry, and does not have two identical rows. By Observation 8, we may assume without loss of generality that $1, 2 \in S(v_{r+1})$, and subject to $1, 2 \in S(v_{r+1})$, $|S(v_1)|$ is maximized. If $r = 3$, then

$$D \in \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} \quad (5.7)$$

and so it is routine to show that M is isomorphic to $M(K_4)$. If $r = 4$, then $|E(M)| = r+3 = 7$. Since $[D^T|I_3]$ is a 3 by 7 matrix without an all zero entry column, it follows by definition that $M^* = F_7$, and so $M = F_7^*$.

To justify (iv), we may assume that $S(v_{r+1}) \cap S(v_{r+3}) = \emptyset$. Thus by Observation 8, $S(v_{r+2}) \cap S(v_{r+1}) \neq \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) \neq \emptyset$. If there exists an $i \in S(v_{r+1}) - S(v_{r+2})$, then the i th component of v_{r+1} is the only nonzero entry of the i th row of the matrix D . It follows by Observation 9 that $\{e_i, e_{3+i}\}$ is a 2-cocircuit of M . Similarly, if $S(v_{r+3}) - S(v_{r+2}) \neq \emptyset$ or if $S(v_{r+2}) - (S(v_{r+1}) \cup S(v_{r+3})) \neq \emptyset$, then M contains a 2-cocircuit and so M is not serially reduced. Thus we may assume that $S(v_{r+2}) = S(v_{r+1}) \cup S(v_{r+3})$, whence $v_{r+1} + v_{r+2} + v_{r+3} = 0$ and so $\{e_{r+1}, e_{r+2}, e_{r+3}\} \in \mathcal{C}(M)$. This proves (iv).

We are to justify (v). Let $M \in \cup_{r \geq 3} \mathcal{F}(r)$. By (i), $M = M_2[I_r|D]$. Let M' denote the serial

reduction of M . We argue by induction on $r(M)$ to $M' \in \mathbb{N}'$. By (iii), M is not serially reduced, and so there must be a 2-cocircuit $\{f, f'\} \in \mathcal{C}(M^*)$. If $r = 3$, then M/f is a connected matroid with $r(M/f) = 2$ and $|E(M/f)| - r(M/f) = 3$, which must be the cycle matroid of a graph H with $|V(H)| = 3$. It follows that either $M' = M(K_3P_3) \in \mathbb{N}'$; or H is spanned by a K_3 with 6 edges and a vertex of degree 2, whence $M' = M(K_2^4) \in \mathbb{N}'$. Hence we assume that $r \geq 4$. If there exists a 2-cocircuit $\{f, f'\} \in \mathcal{C}(M^*)$ such that M/f is simple, then as by Lemma 5.2.15, M/f is connected, we have $M/f \in \cup_{r \geq 3} \mathcal{F}(r)$. Thus by induction, the serial reduction of M/f , (and so M'), must be in \mathbb{N}' . Therefore, we assume that

$$r \geq 4 \text{ and, if } \{f, f'\} \in \mathcal{C}(M^*), \text{ then } M/f \text{ is not simple.} \quad (5.8)$$

Then M/f has two parallel elements f', f'' and $(M/f) - f'$ is simple and connected. Also $|E((M/f) - f')| - r((M/f) - f') = 2$. Then $(M/f) - f'$ is a simple connected matroid of corank 2. Hence $(M/f) - f'$ is a serial extension of $M(K_2^3)$ without parallel elements. Therefore M is a serial extension of $M(K_2^4)$ or $M(K_3P_3)$.

To justify (vi), we apply Lemma 5.2.10(iv) to obtain that $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 3$. To see that $|\mathcal{C}(M) - \mathcal{C}_{M,e}| < 4$, we again assume that $B \in \mathcal{B}(M - e) \subset \mathcal{B}(M)$ and so $e \in \{e_{r+1}, e_{r+2}, e_{r+3}\}$. We further assume that $e = e_{r+3}$. For each $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_{r+3}}$, $C - B \neq \emptyset$ and so either $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+1}\}$, or $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+2}\}$ or $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+1}, e_{r+2}\}$. Accordingly, $C \in \{B(v_{r+1}) \cup \{e_{r+1}\}, B(v_{r+2}) \cup \{e_{r+2}\}, B(v_{r+1} + v_{r+2}) \cup \{e_{r+1}, e_{r+2}\}\}$. This proves (vi).

To prove (vii), let $e \in E(M^+) - E(M)$. Then there exists an $e' \in E(M)$ such that $\{e, e'\} \in \mathcal{C}(M^+)$. If $\{e_0, e'_0\} \in \mathcal{C}(M^+)$, then by Lemma 5.2.10(iv), there are three circuits in $\mathcal{C}(M^+ - \{e_0, e'_0\})$. These, together with a circuit using in $\mathcal{C}(M^+ - e_0)$ using e'_0 , implies $|\mathcal{C}(M^+) - \mathcal{C}_{M^+,e_0}| \geq 4$. If $\{e_0, e'_0\} \notin \mathcal{C}(M^+)$, then we may assume that $e' \in E(M - e_0)$ and $\{e, e'\} \in \mathcal{C}(M^+)$. Thus by Lemma 5.2.10(iv), there are three circuits in $\mathcal{C}(M - \{e_0\}) = \mathcal{C}(M^+ - \{e_0, e\})$. These, together with $\{e, e'\} \in \mathcal{C}(M - e_0)$, implies $|\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \geq 4$. This proves (vii).

Definition 5.2.17. Suppose that N is a minor of M such that N is serially reduced. A minor L of M is a **maximum serial extension of N in M** if N is a serial reduction of L with $|E(L)|$ maximized. We similarly define maximum rooted serial extensions in a rooted matroid.

Define $\mathcal{M}_1 = \{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3)\}$ and $\mathcal{M}_2 = \{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3), M(K_4), M(C_4P_4), F_7^*\}$.

Theorem 5.2.18. Let M be a binary matroid. Each of the following holds.

(i) There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_1 - \{M(K_3P_3)\}$ with $e_0 \in E(M)$; or to $M(K_3P_3)$ with e_0 being any edge of K_3P_3 lying in a 2-circuit.

(ii) *There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_2 - \{M(C_4P_4)\}$ with $e_0 \in E(M)$; or to $M(C_4P_4)$ with e_0 being the only edge not lying in a 2-circuit.*

Proof. The sufficiencies of both (i) and (ii) follow from Proposition 5.2.1(ii), Proposition 5.2.8 and Proposition 5.2.16(vi). It remains to show the necessities.

Assume that M is a binary matroid with $|\mathcal{C}_{M,e_0}| \leq 3$. By Lemma 5.2.14, $M(e_0) \in EX(\mathbb{N} \cup \{F_7^*\})$, and so by Theorem 5.2.13, M is isomorphic to the cycle matroid $M(G)$ for a planar graph G . By Theorem 5.2.12, the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\{M(K_2^2), M(K_2^3), M(K_2^4)\}$ with $e_0 \in E(M)$; or to $M(K_3P_3)$ with e_0 being any edge of K_3P_3 lying in a 2-circuit. This proves the necessity of (i).

Assume that M is a binary matroid with $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$. By Lemma 5.2.14, $M(e_0) \in EX(\mathbb{N})$. Suppose that M contains a minor isomorphic to F_7^* . If $M \in \cup_{r \geq 3} \mathcal{F}(r)$, then by Proposition 5.2.16(v), M is a serial extension of a matroid in $\mathbb{N}' = \{M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$. Thus we conclude that if F_7^* is a minor of M , then M is a serial extension of F_7^* . Now assume that $M(e_0) \in EX(\mathbb{N} \cup F_7^*)$. Then by Theorem 5.2.13, M is isomorphic to the cycle matroid $M(G)$ for a planar graph G . By Theorem 5.2.12, the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$ with $e_0 \in E(M)$; or to $M(C_4P_4)$ with e_0 being the only edge not lying in a 2-circuit. This proves the necessity of (ii). ■

5.3 Application to 1-Hamiltonian circuit graphs of matroids

There have been many studies on the properties of graphs arising from matroids. In [96], Tutte defined a graph $C(M)$ of a matroid M . The vertices of $C(M)$ are the circuits of M , where the two vertices in $C(M)$ are adjacent if and only if they are distinct circuits of the same connected line. Tutte [96] showed that a matroid M is connected if and only if $C(M)$ is a connected graph. In [79] and [80], Maurer defined the base graph of a matroid. The vertices are the bases of M and two vertices are adjacent if and only if the symmetric difference of these two bases is of cardinality 2. The graphical properties of the base graph of a matroid are discussed in [79] and [80]. Alspach and Liu [1] studied the properties of paths and circuits in base graphs of matroids. The connectivity of the base graph of matroids is investigated by Liu in [70] and [71]. The graphical properties of the matroid base graphs have also been investigated by many other researchers, as seen in [37], [38], [61], [73], among others.

Li and Liu ([62], [63] and [64]) initiated the investigation of graphical properties of matroid circuits graphs. Let M be a matroid, and let $k > 0$ be an integer. The **circuit graph** $G(M)$ of M has vertex set $V(G(M)) = \mathcal{C}(M)$. Two vertices $Z, Z' \in \mathcal{C}(M)$ are adjacent in $G(M)$ if

and only if $|Z \cap Z'| \geq 1$. For notational convenience, for a circuit $Z \in \mathcal{C}(M)$, we shall use Z to denote both a vertex in $G(M)$ and a circuit (also as a subset of $E(M)$) of M .

In their studies ([62], [63] and [64]), Li and Liu proved that $G(M)$ possesses quite good graphical connectivity properties. A recent study on the connectivity of certain spanning subgraphs of $G(M)$ is done in [104].

Theorem 5.3.1. *Let M be a connected matroid with $|\mathcal{C}(M)| \geq 3$ and rank $r(M)$, and let $G = G(M)$ be the circuit graph of M . Each of the following holds.*

- (i) (Li and Liu, [64]) $\kappa(G) \geq 2(|E(M)| - r(M) - 1)$.
- (ii) (Li and Liu, [62]) G is edge-pancyclic. That is, for any edge $e \in E(G)$ and for any integer ℓ with $3 \leq \ell \leq |V(G)|$, G contains a circuit C_ℓ containing e with length ℓ .
- (iii) (Li and Liu, [63]) For any edge $e \in E(G)$, G has two Hamilton circuits Z' and Z'' such that Z' contains e and Z'' does not contain e .
- (iv) (Liu and Li, [72]) For any distinct vertices $u, v \in V(G)$, and for any integer ℓ with $2 \leq \ell \leq |V(G)| - 1$, G has an (u, v) -path of length ℓ . That is, G is pan-connected. Consequently, G is hamiltonian with $\kappa(G) \geq 3$.

For an integer $s \geq 0$, a graph G is **s -hamiltonian** if for any subset $S \subset V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian. Motivated by Theorem 5.3.1, the main purpose of this section is to investigate the conditions to warrant the circuit graph of a binary matroid to be 1-hamiltonian.

Throughout this section, M denotes a matroid with $|\mathcal{C}(M)| \geq 4$, and $G = G(M)$ denotes the circuit graph of M . The main goal of this section is to prove that the circuit graph of every connected binary matroid M is 1-hamiltonian. The first subsection below is devoted to develop some useful tools for the arguments; and the main result will be proved in the second subsection.

5.3.1 Lemmas

In this section, we will develop some lemmas to be utilized in the arguments of the next subsection, in which the main result of this section will be proved. For two sets X and Y , define the **symmetric difference of X and Y** as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Lemma 5.3.2. *Let M be a loopless matroid with $|E(M)| \geq 2$.*

- (i) (Strong circuit elimination, Page 15 of [84]) Let $C_1, C_2 \in \mathcal{C}(M)$ be distinct circuits. If $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$, then there exists $C_3 \in \mathcal{C}(M)$ such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$.
- (ii) If $|E| \leq 3$, then $M \in \{U_{1,3}, U_{2,3}\}$ and so $|\mathcal{C}(M)| \leq 3$.
- (iii) Suppose that $|E| = 4$. Then $|\mathcal{C}(M)| \geq 4$ if and only if $M \in \{U_{1,4}, U_{2,4}\}$.

Proof. It suffices to assume to prove (ii) and (iii). Let $r = r(M)$. As M is connected and $|E| \geq 2$, M contains at least one circuit and so $1 \leq r \leq \max\{1, |E| - 1\}$.

Assume first that $|E| \leq 3$. If $r = 1$, then $M = U_{1,3}$ and so $|\mathcal{C}(M)| = 3$. If $r = 2$, then $M = U_{2,3}$ and so $|\mathcal{C}(M)| = 1$. This justifies (ii).

To prove (iii), we first observe that if $M \in \{U_{1,4}, U_{2,4}\}$, then $|\mathcal{C}(M)| \geq 4$. Now we assume that $|\mathcal{C}(M)| \geq 4$. If $r = 1$, then $M = U_{1,4}$ and so $|\mathcal{C}(M)| = 6$. If $r = 3$, then $M = U_{3,4}$ and so $|\mathcal{C}(M)| = 1$. Hence we assume that $r = 2$. If M contains no circuit of size 2, then $M = U_{2,4}$ and so $|\mathcal{C}(M)| = 4$. Thus we assume that M has a 3-circuit C . Then M must be a single parallel extension of $U_{2,3}$ and so $|\mathcal{C}(M)| = 3$. \blacksquare

Lemma 5.3.3. *(Li and Liu, [63]) Let M be a matroid, $e \in E(M)$, $V_1 = \mathcal{C}(M - e)$ and $V_2 = \mathcal{C}(M) - \mathcal{C}(M - e)$. Each of the following holds.*

(i) *The circuit graph of $M - e$ is a subgraph of G induced by V_1 , and the subgraph of G induced by V_2 is a complete subgraph of G .*

(ii) *If $\{e', e''\} \in \mathcal{C}(M^*)$, then $G(M) = G(M/e')$.*

(iii) *Suppose that $e \in E(M)$ is an element such that $M - e$ is connected, If $|V_1| \geq 2$, then for any $Z_1 Z_2 \in E(G)$, there exists a 4-circuit $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ in G such that $|E(\Gamma) \cap E(G_1)| \geq 1$, $|E(\Gamma) \cap E(G_2)| \geq 1$ and both $Z_1 Z_2, Z_2 Z_3, Z_3 Z_4 \in E(G)$.*

We need a slightly stronger version of Lemma 5.3.3(iii) for binary matroids, as stated in Lemma 5.3.4 below.

Lemma 5.3.4. *Let M be a connected binary matroid, $G = G(M)$ be the circuit graph of M . For a fixed element $e \in E(M)$, let $V_1 = \mathcal{C}(M - e)$ and $V_2 = \mathcal{C}(M) - \mathcal{C}(M - e)$, and define $G_1 = G[V_1]$ and $G_2 = G[V_2]$. If $M - e$ is connected, and both $|V_1| \geq 3$ and $|V_2| \geq 4$, then for any $Z_0 \in V(G)$ and for any $Z_1 Z_2 \in E(G - Z_0)$, there exists a 4-circuit $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ in $G - Z_0$ such that $|E(\Gamma) \cap E(G_1)| = 1$ and $|E(\Gamma) \cap E(G_2)| = 1$.*

Proof. Let $Z_0 \in V_1$, and $Z_1 Z_2 \in E(G - Z_0)$. We shall show that existence of the desired 4-circuit $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ in $G - Z_0$ according to the different situations of e .

Case 1. $e \in E - (Z_1 \cup Z_2)$.

Then $Z_1 Z_2 \in E(G - Z_0)$, and so there exists an element $e_1 \in Z_1 \cap Z_2$. Since M is connected, both e_1 and e are contained in a circuit $Z_3 \in V_2$. Thus $Z_3 \neq Z_0$ and $Z_1 Z_3, Z_2 Z_3 \in E(G)$. Since $e \in Z_3 - (Z_1 \cup Z_2)$, both $Z_1 \neq Z_3$ and $Z_2 \neq Z_3$.

Assume first that $e \notin Z_0$. Since $Z_1 \neq Z_3$, there exists an $e_2 \in Z_1 - Z_3$. As $Z_1 \in V_1$, $e \neq e_2$. Since M is connected, M has a circuit Z_4 with $e_2, e \in Z_4$. Thus $e \in (Z_3 \cap Z_4) - (Z_1 \cup Z_2)$, $e_1 \in (Z_1 \cap Z_3) - Z_4$ and $e_2 \in (Z_1 \cap Z_4) - Z_3$, and so $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ is a 4-circuit of G with $E(\Gamma) \cap E(G_1) = \{Z_1 Z_2\}$ and $E(\Gamma) \cap E(G_2) = \{Z_3 Z_4\}$. As $Z_1, Z_2 \in V(G - Z_0)$ and as $Z_3, Z_4 \in V_2$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$. Hence $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ is a desired 4-circuit of $G - Z_0$.

Next we assume that $e \in Z_0$. If there exists an element $e_3 \in Z_1 - (Z_0 \cup Z_3)$, then as M is connected, M has a circuit Z_4 with $e, e_3 \in Z_4$. As $e_3 \in Z_4$, $Z_4 \notin \{Z_0, Z_3\}$. Thus

$\Gamma_1 = Z_1 Z_2 Z_3 Z_4 Z_1$ is a 4-circuit of $G - Z_0$ with $E(\Gamma) \cap E(G_1) = \{Z_1 Z_2\}$ and $E(\Gamma) \cap E(G_2) = \{Z_3 Z_4\}$. Therefore, we assume that $Z_1 \subseteq Z_0 \cup Z_3$. As Z_1 is not a proper subset of Z_0 , we have $Z_1 \cap Z_3 \neq \emptyset$. Since M is binary, $Z_1 \triangle Z_3$ is a disjoint union of circuits different from Z_1 and Z_3 . Since $e \in Z_3 - Z_1$, there must be a circuit $Z' \subseteq Z_1 \triangle Z_3$ such that $e \in Z'$. If $Z' \neq Z_0$, then set $Z'_4 = Z'$ and so in this case $Z_1 Z_2 Z_3 Z'_4 Z_1$ is a desired 4-circuit of $G - Z_0$. Thus we assume that $Z' = Z_0$. If Z_0 is a proper subset of $Z_1 \triangle Z_3$, then $Z_1 \triangle Z_3$ contains another circuit Z'' , disjoint from Z_0 and intersecting with both Z_1 and Z_3 . Hence there exists an element $e'_1 \in Z_1 - (Z_0 \cup Z_3)$. In this case, by the connectedness of M , there must be a circuit $Z''_4 \in \mathcal{C}(M)$ such that $e, e'_1 \in Z''_4$. It follows that $Z_1 Z_2 Z_3 Z''_4 Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we conclude that if no desirable 4-circuit exists, then we must have $Z_1 \triangle Z_3 = Z_0$. By the symmetry between Z_1 and Z_2 , we also have $Z_2 \triangle Z_3 = Z_0$, which leads to the contradiction that $Z_1 = Z_0 \triangle Z_3 = Z_2$. This contradiction indicates that we always can find a desirable 4-circuit satisfying the conclusion of the lemma.

Case 2. $e \in Z_1 - Z_2$ or $e \in Z_2 - Z_1$.

By symmetry, we assume that $e \in Z_2 - Z_1$, $e_1 \in Z_1 \cap Z_2$.

Assume first that $e \notin Z_0$. By Lemma 5.3.2(i), M has a circuit $Z_3 \subseteq Z_1 \cup Z_2 - \{e_1\}$ with $e \in Z_3$. Since $e \in Z_3$ and $Z_0 \in V_1$, we have $Z_3 \neq Z_0$. As Z_3 cannot be a proper subset of Z_2 , there must be an element $e_2 \in Z_1 \cap Z_3$. Since $Z_1 \in V_1$, we note that $e_2 \neq e$.

If there exists an element $e_3 \in E(M) - (Z_0 \cup Z_1 \cup e)$, then by the assumption that $M - e$ is connected, there exists a circuit $Z_4 \in \mathcal{C}(M - e)$ with $e_2, e_3 \in Z_4$. In this case, $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ is a 4-circuit of G with $E(\Gamma) \cap E(G_1) = \{Z_1 Z_4\}$ and $E(\Gamma) \cap E(G_2) = \{Z_2 Z_3\}$. As $Z_1, Z_2 \in V(G - Z_0)$, $Z_3 \neq Z_0$ and $e_3 \in Z_4 - Z_0$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$. It follows that in this case $\Gamma = Z_1 Z_2 Z_3 Z_4 Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we may assume that $E(M) = Z_0 \cup Z_1 \cup e$. Since $Z_0 \neq Z_1$, $e \notin Z_0 \cup Z_1$ and since $M - e$ is also binary, $Z_0 \triangle Z_1$ is a disjoint union of circuits. Since $Z_3 \subset E(M) = Z_0 \cup Z_1 \cup e$, $Z_3 \neq e$ and $Z_3 \neq Z_0$, there must be an element $e'_3 \in Z_3 - (Z_0 \cup e)$. Let Z'_4 be a circuit in $Z_0 \triangle Z_1$ with $e'_3 \in Z'_4$. In this case, $\Gamma' = Z_1 Z_2 Z_3 Z'_4 Z_1$ is a 4-circuit of G with $E(\Gamma') \cap E(G_1) = \{Z_1 Z_4\}$ and $E(\Gamma') \cap E(G_2) = \{Z_2 Z_3\}$. As $Z_1, Z_2 \in V(G - Z_0)$, $Z_3 \neq Z_0$ and $e'_3 \in Z_4 - Z_0$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z'_4\}$, and so in this case $\Gamma' = Z_1 Z_2 Z_3 Z'_4 Z_1$ is a desired 4-circuit of $G - Z_0$.

Next we assume that $e \in Z_0$. Since $|V_2| \geq 4$, we may assume that Z_0, Z_1, Z'_1, Z''_1 are different vertices in V_2 . If there is an element $e'_1 \in Z'_1 - (Z_2 \cup \{e\})$, then set $Z_4 = Z'_1$ and, as M is connected, there exists a circuit $Z_3 \in \mathcal{C}(M - e)$ with $e_1, e'_1 \in Z_3$. As $Z_1 Z_4 \in E(G_2)$ and $Z_2 Z_3 \in E(G_1)$, $Z_1 Z_2 Z_3 Z_4 Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we may assume that $Z'_1 \subseteq Z_2 \cup e$. By the symmetry between Z'_1 and Z''_1 , we may also assume that $Z''_1 \subseteq Z_2 \cup e$. This forces that $Z_2 = Z'_1 \triangle Z''_1$. Let Z_3 be a circuit in $Z_1 \triangle Z'_1$. Then $Z_3 \cap Z'_1 \neq \emptyset$ and $Z_3 \cap Z_2 \neq \emptyset$. Hence letting $Z_4 = Z'_1$, once again we have $Z_1 Z_4 \in E(G_2)$ and $Z_2 Z_3 \in E(G_1)$, and so $Z_1 Z_2 Z_3 Z_4 Z_1$ is a desired 4-circuit of $G - Z_0$. This proves Case 2.

Case 3. $e \in Z_1 \cap Z_2$, whence both Z_1 and Z_2 are vertices in G_2 .

Assume first that $e \notin Z_0$. If $Z_0 = Z_1 \triangle Z_2$, then as $Z_1 \neq Z_2$, there must be an element $e_1 \in Z_1 - Z_2$ and an element $e_2 \in Z_2 - Z_1$. As $e_1, e_2 \in E(M - e)$ and as $M - e$ is connected, there exists a circuit $Z_3 \in \mathcal{C}(M - e)$ such that $e_1, e_2 \in Z_3$. Since Z_3 is not a proper subset of Z_0 , we have $Z_3 \neq Z_0$. Since $|V_1| \geq 3$, there must be a $Z \in V_1 - \{Z_0, Z_3\}$. If $e_1 \in Z$, then $\Gamma_1 = Z_1 Z_2 Z_3 Z Z_1$ is a 4-circuit of G with $E(\Gamma_1) \cap E(G_1) = \{Z_3 Z\}$ and $E(\Gamma_1) \cap E(G_2) = \{Z_1 Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3, Z\}$. Hence by symmetry, we may assume that $\{e_1, e_2\} \cap Z = \emptyset$. In this case, we pick $e_3 \in Z - Z_3$. As $M - e$ is connected and as $e_1, e_3 \in E(M - e)$, there must be a $Z_4 \in \mathcal{C}(M - e)$ with $e_1, e_3 \in Z_4$. It follows that $\Gamma_2 = Z_1 Z_2 Z_3 Z_4 Z_1$ is a 4-circuit of G with $E(\Gamma_2) \cap E(G_1) = \{Z_3 Z_4\}$ and $E(\Gamma_2) \cap E(G_2) = \{Z_1 Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$.

Next, we assume that $e \notin Z_0$ and $Z_0 \neq Z_1 \triangle Z_2$. Since M is binary, $Z_1 \triangle Z_2$ contains a circuit Z'_3 such that Z'_3 contains an element $e'_1 \in (Z_1 \triangle Z_2) - Z_0$. As $e'_1 \in Z_1 \triangle Z_2$, we by symmetry may assume that $e'_1 \in Z_1 - Z_2$. Since Z'_3 cannot be a proper subset of Z_1 , there must be an element $e'_2 \in Z'_3 \cap Z_2 - Z_1$. Since $|V_1| \geq 3$, there must be a $Z'' \in V_1 - \{Z_0, Z'_3\}$. If $e'_1 \in Z''$, then $\Gamma_3 = Z_1 Z_2 Z'_3 Z'' Z_1$ is a 4-circuit of G with $E(\Gamma_3) \cap E(G_1) = \{Z'_3 Z''\}$ and $E(\Gamma_3) \cap E(G_2) = \{Z_1 Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z'_3, Z''\}$. Hence by symmetry, we may assume that $\{e_1, e_2\} \cap Z'' = \emptyset$. In this case, we pick $e'_3 \in Z'' - Z'_3$. As $M - e$ is connected and as $e'_1, e'_3 \in E(M - e)$, there must be a $Z'_4 \in \mathcal{C}(M - e)$ with $e'_1, e'_3 \in Z'_4$. It follows that $\Gamma_4 = Z_1 Z_2 Z'_3 Z'_4 Z_1$ is a 4-circuit of G with $E(\Gamma_4) \cap E(G_1) = \{Z'_3 Z'_4\}$ and $E(\Gamma_4) \cap E(G_2) = \{Z_1 Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z'_3, Z'_4\}$.

As the arguments above show that if $e \notin Z_0$, then a desirable 4-circuit always exists, we assume throughout the rest of the proof of this lemma that $e \in Z_0$. Since $e \in Z_1 \cap Z_2$, $Z_1 \neq Z_2$, and as M is binary, $Z_1 \triangle Z_2$ contains a circuit $Z_3 \in V_1$. Since $|V_1| \geq 3$, there exists a circuit $Z' \in V_1 - \{Z_3\}$. Pick $e' \in Z' - Z_3 \subseteq E - \{e\}$. As $Z_3 \subseteq Z_1 \triangle Z_2$, there must be an element $e'' \in Z_3 \cap Z_1$. Since $e \notin Z_3$, $e'' \neq e$. By the connectedness of $M - e$, there exists a circuit $Z_4 \in \mathcal{C}(M - e)$ such that $e', e'' \in Z_4$. Since $Z_1 Z_2 \in E(G_2)$ and $Z_3 Z_4 \in E(G_1)$, it follows that $Z_1 Z_2 Z_3 Z_4 Z_1$ is a desirable 4-circuit. This completes the proof of this case as well as the lemma. ■

An element $e \in E(M)$ of a connected matroid M is **essential** if $M - e$ is not connected. A matroid M is **critically connected** if M is connected and every $e \in E(M)$ is essential.

Theorem 5.3.5. (Murty [83]) *If M is critically connected with $r(M) \geq 2$, then M contains a cocircuit of 2 element.*

Lemma 5.3.6. *If $M \in \{K_2^2, K_2^3, K_2^4, K_3 P_2, C_4 P_3, K_4, F_7^*\}$, then either $G(M)$ has fewer than 4 vertices, or for any $z \in V(G(M))$ and any edge $f \in E(G(M) - z)$, $G(M) - z$ has a hamiltonian circuit containing f .*

Proof. If $M \in \{K_2^2, K_2^3\}$, then $|V(G(M))| \leq 3$. As every pair of distinct circuits of F_7^* or of

$M(K_4)$ must have nonempty intersection. both $G(F_7^*)$ and $G(M(K_4))$ are complete graphs with at least 6 vertices. By definition, if $G(M(K_2^4))$ is the graph obtained from K_6 by deleting perfect matching. Let e be the edge in P_2K_3 not lying in a 2-circuit. Then circuits in P_2K_3 containing e , as vertices in $G(M(P_2K_3))$, induces a K_4 , and so $G(M(P_2K_3))$ is the graph obtained from K_6 by deleting an edge. Likewise, Let e' be the edge in P_3C_4 no lying in a 2-circuit. Then circuits in P_3C_4 containing e' , as vertices in $G(M(P_3C_4))$, induces a K_8 , and so $G(M(P_3C_4))$ is the graph obtained from K_{11} by deleting a 3-circuit. It is routine to show that each of these graphs has the indicated property. ■

Lemma 5.3.7. *If M be a connected serially reduced binary matroid with $|E(M)| - r(M) \leq 2$. Then $M = U_{1,3}$.*

Proof. Let B be a basis of M , let e_1, e_2 be the only two elements in $E(M) - B$, and Z_1, Z_2 be the fundamental circuit of e_1 and e_2 with respect to B , respectively. Then $Z_1 \triangle Z_2 = \{e_1, e_2\}$ is a circuit. Since M is connected, It follows that both $Z_1 = Z_2 \triangle \{e_1, e_2\} = B \cup e_1$ and $Z_2 = Z_1 \triangle \{e_1, e_2\} = B \cup e_2$. As M is serially reduced, M contains no 2-element cocircuits, and so for some element e_3 , we have $B = \{e_3\}$. This shows that $M \cong U_{1,3}$. ■

5.3.2 A result on 1-edge-hamiltonian circuit graphs

If for any vertex subset $S \subset V(G)$ with $|S| \leq 1$ and for any edge $e \in E(G - S)$, $G - S$ has a Hamilton circuit containing e , then G is said to be **1-edge-hamiltonian**. Recall that $M(e_0)$ is a matroid with e_0 being its root.

We prove a slightly stronger result than the statement we made in the beginning of this section, as follows.

Theorem 5.3.8. *Let $M = (E, \mathcal{I})$ be a connected binary matroid with $|\mathcal{C}(M)| \geq 4$, and let $G = G(M)$ be the circuit graph of M . Then G is 1-edge-hamiltonian.*

Proof. By Theorem 5.3.1(ii), it suffices to show that

$$\text{for any } v \in V(G) \text{ and } e \in E(G - v), G - v \text{ has a Hamilton circuit containing } e. \quad (5.9)$$

We argue by induction on $|E|$ to prove (5.9). By Lemma 5.3.2, every matroid $M = (E, \mathcal{I})$ with $|E| \leq 3$ has $|\mathcal{C}(M)| < 4$. By Lemmas 5.3.2 and 5.3.6, (5.9) holds for any connected binary matroid on 4 elements. Hence we assume that $|E| \geq 5$, and (5.9) holds for connected binary matroids with smaller number of elements.

If for some element $e_0 \in E(M)$, $M(e_0)$ is in $\mathcal{F}_1 \cup \mathcal{F}_2$, then by $|E| \geq 5$ and by Lemma 5.3.6, $G(M)$ is 1-edge-hamiltonian. Hence we assume that

$$\text{for any } e_0 \in E(M), M(e_0) \text{ is not in } \mathcal{F}_1 \cup \mathcal{F}_2. \quad (5.10)$$

If M has a 2-cocircuit $\{e', e''\}$, then by Lemma 5.3.3(ii), $G(M) = G(M/e')$, and so by induction, we may assume that

$$M \text{ is serially reduced.} \quad (5.11)$$

Suppose that M is critically connected. Then by Theorem 5.3.5, M has a 2-cocircuit $\{e', e''\}$. By Lemma 5.3.3(ii), $G(M) = G(M/e')$. By induction $G(M/e')$, and so $G(M)$, is 1-edge-hamiltonian. Therefore, we assume that M is not critically connected. By definition, there exists an element $e \in E(M)$ such that $M - e$ is connected. Define $V_1 = \mathcal{C}(M) - \mathcal{C}_{M,e}$, $V_2 = \mathcal{C}_{M,e}$, $G_1 = G[V_1]$ and $G_2 = G[V_2]$. If $|V_1| \geq 4$, then it follows by induction that

$$G_1 = G(M - e) \text{ is 1-edge-hamiltonian.} \quad (5.12)$$

By (5.1), if $|V_1| \leq 3$, then $M(e) \in \mathcal{F}_2$; and if $|V_2| \leq 3$, then $M(e) \in \mathcal{F}_1$. In either case, a contradiction to (5.10) is found. If $|E(M)| - r(M) \leq 2$, then by Lemma 5.3.7, $G(M) \in \{K_1, K_3\}$ and so we may assume $|E(M)| - r(M) \geq 3$. These, together with Lemma 5.3.3(iii) and Lemma 5.3.7, imply that

$$|V_1| \geq 4, |V_2| \geq 4, \kappa(G) \geq 4 \text{ and that } G_2 \text{ is a complete graph.} \quad (5.13)$$

Let $Z_0 \in V(G) = \mathcal{C}(M)$, and an edge $f = Z'Z'' \in E(G - Z_0)$ be given. We shall show that $G - Z_0$ has a Hamilton circuit containing f . By (5.12), G_1 (if $e \in Z_0$) or $G_1 - Z_0$ (if $e \notin Z_0$) has a Hamilton circuit C .

Case 1. $e \notin Z' \cup Z''$.

Then $f = Z'Z'' \in E(G_1)$. By (5.13), $|V_1| \geq 4$ and so there must be a two vertices $Z_1, Z_2 \in V_1 - \{Z', Z''\}$ such that $Z_1Z_2 \in E(C - f)$. By Lemma 5.3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ such that $Z_3Z_4 \in E(G_2)$. By Lemma 5.3.3(i) and by (5.13), $G_2 - Z_0$ (if $e \in Z_0$) or G_2 (if $e \notin Z_0$) is a complete graph on at least 3 vertices, and so $G_2 - Z_0$ contains a spanning (Z_3, Z_4) -path P . It follows that $E(C - Z_1Z_2) \cup E(P) \cup \{Z_2Z_3, Z_1Z_4\}$ induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

Case 2. $e \in Z' - Z''$ or $e \in Z'' - Z'$.

By symmetry, we may assume that $e \in Z'' - Z'$, and so $Z' \in V_1$ and $Z'' \in V_2$. Let $Z_1 = Z'$ and $Z_2 = Z''$. By Lemma 5.3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ such that $Z_2Z_3 \in E(G_2)$. If $Z_1Z_4 \in E(G_1)$. By (5.12), $G_1 - Z_0$ (if $e \notin Z_0$) or G_1 (if $e \in Z_0$) has a Hamilton circuit C_1 with $Z_1Z_4 \in E(C_1)$. As G_2 is a complete graph on at least 3 vertices, $G_2 - Z_0$ (if $e \in Z_0$) or G_2 (if $e \notin Z_0$) contains a spanning (Z_2, Z_3) -path P . It follows that $E(C - Z_1Z_4) \cup E(P) \cup \{Z_1Z_2, Z_3Z_4\}$ induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

Case 3. $e \in Z' \cap Z''$.

Then $f = Z'Z'' \in E(G_2)$. By (5.13), $\kappa(G) \geq 4$, and so $G - \{Z_0, Z', Z''\}$ is connected. Therefore, there must be an edge $Z_1Z'_1 \in E(G - \{Z_0, Z', Z''\})$ such that $Z_1 \in V_1$ and $Z'_1 \in V_2$.

Pick and edge $Z_1Z_2 \in E(C)$. By Lemma 5.3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ with $Z_3, Z_4 \in V_2 - \{Z_0\}$. Assume that $Z'_1 \neq Z_3$ ($Z'_1 = Z_3 \neq Z_4$, respectively). By Lemma 5.3.3(i) and (5.13), G_2 is a complete graph on at least 4 vertices, and so G_2 (if $e \notin Z_0$) or $G_2 - Z_0$ (if $e \in Z_0$) has a spanning (Z'_1, Z_3) -path ((Z'_1, Z_4) -path, respectively) P with $f = Z'Z'' \in E(P)$. It follows that $E(C - Z_1Z_2) \cup E(P) \cup \{Z_1Z'_1, Z_2Z_3\}$ (or $E(C - Z_1Z_2) \cup E(P) \cup \{Z_1Z_4, Z_2Z_3\}$, respectively) induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

As in every cases, $G - Z_0$ always has a Hamilton circuit containing f , the theorem is now proved. ■

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