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Original article

Nonparametric density estimation based on the scaled Laplace transform inversion

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Abstract

New nonparametric procedure for estimating the probability density function of a positive random variable is suggested. Asymptotic expressions of the bias term and Mean Squared Error are derived. By means of graphical illustrations and evaluating the Average of L_2-errors we conducted comparisons of the finite sample performance of proposed estimate with the one based on kernel density method.

Keywords: Laplace transform; Nonparametric estimation; Mean squared error; Kernel density estimation

1. Introduction

In this paper we propose new nonparametric procedure to estimate the density function of a positive random variable that is based on the moment-recovered approach proposed in Mnatsakanov et al. [1].

There are many articles devoted to investigation of the properties of nonparametric density estimators based on the kernel-smoothing technique. A common method for estimating a density function is obtained by using a fixed symmetric kernel density estimate (KDE) with bounded (unbounded) support and choosing the optimal bandwidth (Silverman [2]). The properties of such estimator have been studied by Mielniczuk [3], Zhang [4], among many others. There are series of papers where the asymmetric kernels are used (Bouezmarni and Rolin [5], Chen [6], Mnatsakanov and Ruymgaart [7], and Mnatsakanov and Sarkisian [8]).

Also there are several well-known techniques, e.g. Lejeune and Sarda [9] as well as Nielsen at al. [10] worked on the local linear estimation. Jones [11] and Jones and Forster [12] investigated the performance of the estimates based on boundary kernels, Müller [13] studied the smoothed optimum kernels, and Marron and Ruppert [14] used transformation approach.

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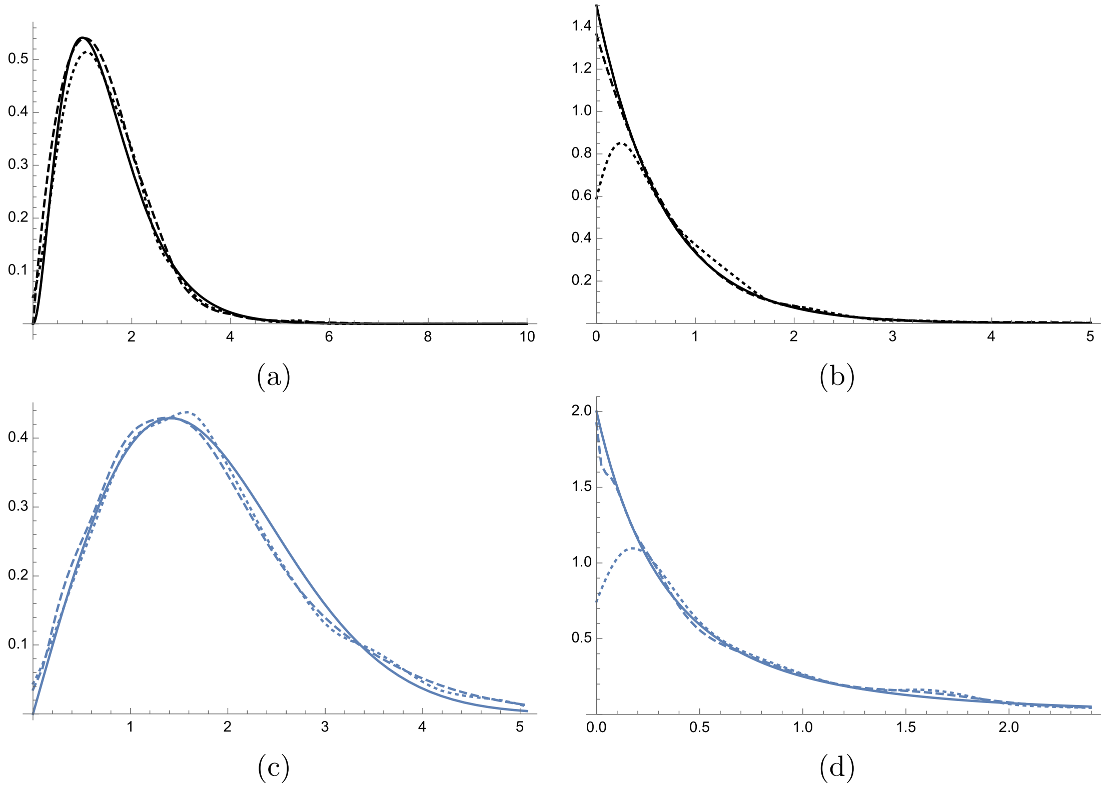

Fig. 1. Estimates $\hat{f}_{\alpha,b}$ (dashed curve) and $\hat{f}_h$ (dotted curve) (a): when $X_i \sim \text{Gamma}(3, 1/2)$ with $\alpha = 120, b = 1.9, n = 500$; (b): when $X_i \sim \text{Exp}(3/2)$ with $\alpha = 120, b = 1.15, n = 500$; (c): when $X_i \sim \text{Weibull}(2, 2)$ with $\alpha = 100, b = 1.23, n = 800$; (d): when $X_i \sim \text{Pareto}(1, 2)$ with $\alpha = 100, b = 1.55, n = 800$.

To reduce the boundary effect for nonnegative data, Chen [6] proposed the estimator based on asymmetric gamma kernels. Jones [11] and Chen [15] showed that the local linear estimator achieves better results than the boundary kernel estimator of Müller [13]. Bouezmarni and Rolin [5] derived the exact asymptotic constants for uniform and L_1-errors of the kernel density estimator defined by asymmetric beta kernels.

Bouezmarni et al. [16] studied another estimator based on the gamma kernels. It was shown that such construction is free of boundary effect and achieves the optimal rate of convergence in terms of integrated mean squared error. Similar properties of the so-called varying kernel density estimates have been established in [7] and [8] as well.

The main goal of the current work is to introduce totally different approach for estimating the density function f of a positive random variable via the values of its scaled Laplace transform. It is based on the empirical counterpart of approximate $f_{\alpha,b}$ suggested in Mnatsakanov et al. [1] (see also (2) in Section 2). The proposed approach provides a unified estimation method that could be applied for estimating the density functions in several indirect models as well, including the right-censored one. The properties of corresponding estimates will be investigated in the forthcoming paper. Simulation study justifies that suggested estimate does not have the edge effect in contrary to those based on symmetric KDE. In particular, when $f(0) > 0$, we recommend the use the estimate defined in (4) instead of traditional KDE (see, for example, plots (b) and (d) in Fig. 1). Besides, it is shown that proposed estimate (in the case of direct model) is reduced to the one based on asymmetric beta kernel density construction.

The rest of the paper is organized as follows. In Section 2, we describe the construction of new nonparametric estimate $\hat{f}_{\alpha,b}$ defined in (4). In Section 3, the finite sample properties of $\hat{f}_{\alpha,b}$ are investigated. In particular, the bias and the Mean Squared Error (MSE) of $\hat{f}_{\alpha,b}$ are derived. Section 4 is devoted to simulation study. Here we evaluated the average L_2-errors of proposed estimate and compared it to the one based on KDE. Several graphs of the estimate are displayed as well. The advantages of the proposed estimates are discussed in Section 5.

2. Some preliminaries and notations

Assume F is an absolutely continuous distribution supported by $\mathbf{R}_+ = [0, \infty)$. Let f be its probability density function (pdf) with respect to the Lebesgue measure on $\mathbf{R}_+$. In this section we introduce the estimate based on approximation of the Laplace transform inversion that recovers f. Denote by X a random variable distributed according to F.

Assume that we are given the sequence $\mathscr{L}(F) = \{\mathscr{L}_t(F), t \in \mathbb{N}_\alpha\}$ of the values of the scaled Laplace transform of F:

$$\mathscr{L}_t(F) = \int_0^\infty e^{-ctx}\, dF(x), \qquad t \in \mathbb{N}_\alpha = \{0, 1, 2, \ldots, \alpha\}. \tag{1}$$

To simplify the notations, assume in (1) that the scaling parameter $c = \ln b$, for some $1 < b \le \exp(1)$. Besides, denote the pdf of a Beta (p, q) distribution by

$$\beta(u, p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} u^{p-1}(1-u)^{q-1}, \quad 0 < u < 1.$$

Here the shape parameters $p, q > 0$, are defined as follows: $p = [\alpha b^{-x}] + 1$ and $q = \alpha - [\alpha b^{-x}] + 1$. Denote also $\beta^*_{\alpha,x}(\cdot) := \beta(\cdot, p, q)$. Consider the following approximation of f introduced in Mnatsakanov et al. [1]. Namely, for each $x \in \mathbf{R}_+$, define $f_{\alpha,b}(x) := (\mathscr{B}_\alpha^{-1}\mathscr{L}(F))(x)$, with

$$(\mathscr{B}_\alpha^{-1}\mathscr{L}(F))(x) = \frac{\ln b\,[\alpha b^{-x}]\,\Gamma(\alpha+2)}{\alpha\,\Gamma([\alpha b^{-x}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \frac{(-1)^m \mathscr{L}_{m+[\alpha b^{-x}]}(F)}{\Gamma(m+1)\,\Gamma(\alpha-[\alpha b^{-x}]-m+1)}. \tag{2}$$

In a general setting, the proposed construction (2) can be applied when the $\sqrt{n}$-consistent estimate of the Laplace transform of F, say, $\tilde{\mathscr{L}}(F) = \{\tilde{\mathscr{L}}_t(F), t \in \mathbb{N}_\alpha\}$ is available, while F is not observed directly. Hence, using $\tilde{\mathscr{L}}(F)$ instead of $\mathscr{L}(F)$ in (2), we arrive at the estimate of f:

$$\tilde{f}_{\alpha,b}(x) = (\mathscr{B}_\alpha^{-1}\tilde{\mathscr{L}}(F))(x), \quad x \in \mathbb{R}_+. \tag{3}$$

In the current work, we apply (2) in direct model, i.e., when the sample of $i.i.d.$ random variables $X_1, \ldots, X_n$ from F is given. In this case, one can consider the empirical Laplace transform $\widehat{\mathscr{L}}(F)$ instead of $\tilde{\mathscr{L}}(F)$ in (3). As a result, the following estimate of f (after multiplying by $\alpha/(\alpha+1)$) is derived:

$$\hat{f}_{\alpha,b}(x) = \frac{\alpha}{\alpha+1}(\mathscr{B}_\alpha^{-1}\widehat{\mathscr{L}}(F))(x), \quad x \in \mathbb{R}_+. \tag{4}$$

Recall that

$$\widehat{\mathscr{L}}_t(F) = \int_{\mathbf{R}_+} e^{-ctx}\, d\widehat{F}_n(x), \tag{5}$$

and $\widehat{F}_n$ is the empirical cdf of the sample $X_1, \ldots, X_n$.

Remark 1. In order to simplify the evaluations of the bias and variance of estimate $\hat{f}_{\alpha,b}$, the normalizing factor in $\alpha/(\alpha+1)$ is used (4). Also note that

$$\widehat{\mathscr{L}}_t(F) = \int_{\mathbf{R}_+} b^{-tx}\, d\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n b^{-tX_i}. \tag{6}$$

Finally, note that given the observations $X_1, \ldots, X_n$ drawn from F, the estimate $\hat{f}_{\alpha,b}$ is reduced to:

$$\hat{f}_{\alpha,b}(x) = \left(\frac{\alpha}{\alpha+1}\right)\frac{[\alpha b^{-x}]\ln(b)\Gamma(\alpha+2)}{\alpha\,\Gamma([\alpha b^{-x}]+1)}\frac{1}{n}\sum_{i=1}^n \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \frac{(-1)^m b^{-(m+[\alpha b^{-x}])X_i}}{m!(\alpha-[\alpha b^{-x}]-m)!}$$

$$= \frac{[\alpha b^{-x}]\ln(b)\Gamma(\alpha+2)}{(\alpha+1)\Gamma([\alpha b^{-x}]+1)}\frac{1}{n}\sum_{i=1}^n \sum_{m=0}^{\alpha-[\alpha b^{-x}]} \frac{(-b^{-X_i})^m (b^{-X_i})^{[\alpha b^{-x}]}}{m!(\alpha-[\alpha b^{-x}]-m)!}$$

$$= \frac{[\alpha b^{-x}] \ln b}{\alpha + 1} \frac{1}{n} \sum_{i=1}^{n} \beta(b^{-X_i}, [\alpha b^{-x}] + 1, \alpha - [\alpha b^{-x}] + 1)$$

$$= \frac{[\alpha b^{-x}] \ln b}{\alpha + 1} \frac{1}{n} \sum_{i=1}^{n} \beta^*_{\alpha,x}(b^{-X_i}), \quad x \in \mathbb{R}_+, \tag{7}$$

with $\beta^*_{\alpha,x}(\cdot)$ defined in Section 2.

It is worth mentioning that construction (3) has a unified form, and can be applied as in direct as well as in indirect models. As a special case, it reduces to the asymmetric beta density kernel estimator (see last line in (7)) when underlying density f is observed directly via the data-sets $\{X_i\}_{i=1}^n$ (cf. with [5,16,15]).

In Section 3 it will be assumed that the following conditions are satisfied:

$$f \in C^{(2)}(\mathbf{R}_+), \quad \sup_{x \in [0,\infty)} |b^{kx} f'(x)| < \infty, \quad k = 1, 2, \quad \text{and} \quad \sup_{x \in [0,\infty)} |b^x f''(x)| < \infty. \tag{8}$$

3. Mean squared errors of $\hat{f}_{\alpha,b}$

In this section we investigate the asymptotic behavior of the estimate $\hat{f}_{\alpha,b}(x)$ defined in (4). In particular, to derive the upper bound for MSE of $\hat{f}_{\alpha,b}(x)$. Recall that

$$\mathrm{MSE}\{\hat{f}_{\alpha,b}(x)\} := E|\hat{f}_{\alpha,b}(x) - f(x)|^2 = var\{\hat{f}_{\alpha,b}(x)\} + \left(\mathrm{Bias}\{\hat{f}_{\alpha,b}(x)\}\right)^2,$$

where $\mathrm{Bias}\{\hat{f}_{\alpha,b}(x)\} = f_{\alpha,b}(x) - f(x)$ and $f_{\alpha,b}(x) := E(\hat{f}_{\alpha,b}(x))$, for each $x \in \mathbb{R}_+$.

Theorem 1. *If conditions (8) are satisfied, then for MSE of $\hat{f}_{\alpha,b}(x)$ we have:*

$$\mathrm{MSE}\{\hat{f}_{\alpha,b}(x)\} \le n^{-4/5}\left[\left(\frac{2\, b^x |f'(x)|}{\ln b} + \frac{b^{2x}|f'(x)|}{2 \ln^2 b} + \frac{b^x |f''(x)|}{2\, \ln b}\right)^2 + \frac{f(x) \ln b}{\sqrt{\pi}\,(b^x - 1)}\right] + o(n^{-4/5}) \tag{9}$$

provided that we choose $\alpha = \alpha(n) \sim n^{2/5}$, $n \to \infty$.

The proof of Theorem 1 is based on investigating the asymptotic behavior of the bias and variance terms of $\mathrm{MSE}\{\hat{f}_{\alpha,b}(x)\}$.

Lemma 2. *(i): $f_{\alpha,b}(x)$ converges uniformly to $f(x)$ as $\alpha \to \infty$, and for each $x > 0$, the absolute value of the bias term of $\hat{f}_{\alpha,b}(x)$ is estimated from above as follows:*

$$|\mathrm{Bias}\{\hat{f}_{\alpha,b}(x)\}| \le \frac{1}{\alpha + 1}\left\{\frac{2\, b^x |f'(x)|}{\ln b} + \frac{b^{2x}|f'(x)|}{2 \ln^2 b} + \frac{b^x |f''(x)|}{2\, \ln b}\right\} + o\left(\frac{1}{\alpha}\right),$$

as $\alpha \to \infty$.

(ii): For each $x > 0$, the asymptotic expression for variance of $\hat{f}_{\alpha,b}(x)$ we have

$$var\{\hat{f}_{\alpha,b}(x)\} = \frac{\sqrt{\alpha}}{n} \frac{f(x) \ln b}{\sqrt{\pi}(b^x - 1)} + o\left(\frac{\sqrt{\alpha}}{n}\right),$$

as $\sqrt{\alpha}/n \to 0$, $\alpha, n \to \infty$.

Proof of Lemma 2. Because $\hat{f}_{\alpha,b} = (\alpha/(\alpha+1))\mathscr{B}_\alpha^{-1}\widehat{\mathscr{L}}(F)$ has representation (7), we have

$$E\hat{f}_{\alpha,b}(x) = \frac{[\alpha b^{-x}] \ln b}{\alpha + 1} \frac{1}{n} \sum_{i=1}^{n} E\beta(b^{-X_i}, [\alpha b^{-x}] + 1, \alpha - [\alpha b^{-x}] + 1)$$

$$= \frac{\ln b\ \Gamma(\alpha + 1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha + [\alpha b^{-x}] + 1)} \int_0^\infty (b^{-u})^{[\alpha b^{-x}]}(1 - b^{-u})^{\alpha - [\alpha b^{-x}]} f(u)du. \tag{10}$$

The change of variable under integral in (10) with $\tau = b^{-u}$ gives

$$E\hat{f}_{\alpha,b}(x) = \frac{\ln b\,\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha+[\alpha b^{-x}]+1)}\int_0^1 \tau^{[\alpha b^{-x}]}(1-\tau)^{\alpha-[\alpha b^{-x}]}\frac{f(-\log_b\tau)}{\tau\ln b}d\tau$$
$$= \frac{\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha+[\alpha b^{-x}]+1)}\int_0^1 \tau^{[\alpha b^{-x}]-1}(1-\tau)^{\alpha-[\alpha b^{-x}]}f(-\log_b\tau)d\tau$$
$$= \int_0^1 \beta(\tau,[\alpha b^{-x}],\alpha-[\alpha b^{-x}]+1)q(\tau)d\tau, \tag{11}$$

where $q = f\circ\phi$ denotes the composition of functions f and $\phi(\tau) = -\log_b(\tau)$. Therefore for the expected value of $\hat{f}_{\alpha,b}$ we have

$$f_{\alpha,b}(x) = E\hat{f}_{\alpha,b}(x) = \int_0^1 \beta_{\alpha,x}(\tau)q(\tau)d\tau, \quad q(\tau) = f(-\log_b\tau). \tag{12}$$

Here, $\beta_{\alpha,x}(\cdot) := \beta(\cdot,[\alpha b^{-x}],\alpha-[\alpha b^{-x}]+1)$. To complete the proof of Lemma 2(i) one can proceed in a similar way as it is done in Mnatsakanov et al. [1].

Namely, let us mention that the first and second derivatives of q with respect to τ can be written as

$$q'(\tau) = (f'\circ\phi)(\tau)\phi'(\tau) \tag{13}$$
$$q''(\tau) = (f''\circ\phi)(\tau)\phi''(\tau) + (f'\circ\phi)(\tau)\phi''(\tau). \tag{14}$$

Evaluation of q' and q'' at $\tau = b^{-x}$ gives: $q'(b^{-x}) = \frac{b^x f'(x)}{\ln b}$ and $q''(b^{-x}) = \frac{b^{2x}f'(x)}{\ln^2 b} + \frac{b^x f''(x)}{\ln b}$. Now, note that the sequence $\{\beta_{\alpha,x}(\cdot),\alpha = 1,2,\dots\}$ represents the sequence of δ-functions with the mean and variance specified as follows:

$$\eta_\alpha := \int_0^1 \tau\beta_{\alpha,x}(\tau)d\tau = \frac{[\alpha b^{-x}]}{\alpha+1} \tag{15}$$
$$\sigma_\alpha^2 := \int_0^1 (\tau-\eta_\alpha)^2\beta_{\alpha,x}(\tau)d\tau = \frac{[\alpha b^{-x}](\alpha-[\alpha b^{-x}]+1)}{(\alpha+1)^2(\alpha+2)} < \frac{1}{\alpha+1}, \tag{16}$$

and

$$|\eta_\alpha - b^{-x}| \le \frac{2}{\alpha+1}. \tag{17}$$

To derive the asymptotic form of the Bias$\{\hat{f}_{\alpha,b}(x)\} = f_{\alpha,b}(x) - f(x)$ let us write

$$f_{\alpha,b}(x) - f(x) = \int_0^1 \beta_{\alpha,x}(\tau)\{q(\tau)-q(b^{-x})\}d\tau. \tag{18}$$

Applying the Taylor expansion of $q(\tau)$ around $\tau = b^{-x}$ we get:

$$\text{Bias}\{\hat{f}_{\alpha,b}(x)\} = \int_0^1 \beta_{\alpha,x}(\tau)\left\{(\tau-b^{-x})q'(b^{-x}) + \frac{1}{2}(\tau-b^{-x})^2q''(\tilde{\tau})\right\}d\tau. \tag{19}$$

Now adding and subtracting η_α from the first and second terms in the right hand side of (19) we obtain the upper bound for the absolute value of Bias$\{\hat{f}_{\alpha,b}(x)\}$:

$$\left|\int_0^1 \beta_{\alpha,x}(\tau)(\tau-\eta_\alpha)q'(b^{-x})d\tau + (\eta_\alpha-b^{-x})q'(b^{-x})\int_0^1 \beta_{\alpha,x}(\tau)d\tau\right.$$
$$\left.+\frac{1}{2}\int_0^1 \beta_{\alpha,x}(\tau)(\tau-\eta_\alpha)^2\,q''(\tilde{\tau})d\tau + \frac{1}{2}(\eta_\alpha-b^{-x})^2\int_0^1 \beta_{\alpha,x}(\tau)q''(\tilde{\tau})d\tau\right|$$
$$\le |\eta_\alpha-b^{-x}||q'(b^{-x})| + \frac{1}{2}\sigma_\alpha^2\,\|q''\| + \frac{1}{2}(\eta_\alpha-b^{-x})^2\,\|q''\|. \tag{20}$$

Now taking into account the bounds for σ_α^2 and $|\eta_\alpha - b^{-x}|$ mentioned in (16) and (17), respectively, we obtain for the bias of $\hat{f}_{\alpha,b}$:

$$|\text{Bias}\{\hat{f}_{\alpha,b}(x)\}| \le \frac{1}{\alpha+1}\left\{\frac{2\,b^x|f'(x)|}{\ln b} + \frac{b^{2x}|f'(x)|}{2\ln^2 b} + \frac{b^x|f''(x)|}{2\,\ln b}\right\} + o(\frac{1}{\alpha}) \tag{21}$$

as $\alpha\to\infty$. The uniform convergence of $f_{\alpha,b}$ to f follows from (21) and the conditions (8).

Now, consider the variance of $\hat{f}_{\alpha,b}$. Taking into account (7) and (12), we obtain:

$$var\{\hat{f}_{\alpha,b}(x)\} = \frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 var\{\beta^*_{\alpha,x}(b^{-X_1})\}$$
$$= \frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 E\beta^{*2}_{\alpha,x}(b^{-X_i}) - \frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 (E\beta^*_{\alpha,x}(b^{-X}))^2. \tag{22}$$

At first let us investigate the asymptotic behavior of the first term in (22). We have

$$\frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 E\beta^{*2}_{\alpha,x}(b^{-X_i}) = \frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 \int_0^\infty \beta^{*2}_{\alpha,x}(b^{-u})f(u)du$$
$$= \frac{1}{n}\left(\frac{\ln b\,\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha-[\alpha b^{-x}]+1)}\right)^2 \int_0^\infty (b^{-u})^{2[\alpha b^{-x}]}(1-b^{-u})^{2\alpha-2[\alpha b^{-x}]} f(u)du$$
$$= \frac{1}{n}\left(\frac{\ln b\,\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha-[\alpha b^{-x}]+1)}\right)^2 \int_0^1 \tau^{2[\alpha b^{-x}]-1}(1-\tau)^{2\alpha-2[\alpha b^{-x}]} q(\tau)\frac{d\tau}{\ln b}$$
$$= \xi_\alpha(x)\int_0^1 \beta(\tau, 2[\alpha b^{-x}], 2\alpha-2[\alpha b^{-x}]+1)q(\tau)d\tau. \tag{23}$$

Here

$$\xi_\alpha(x) = \frac{\ln b}{n}\left(\frac{\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\Gamma(\alpha-[\alpha b^{-x}]+1)}\right)^2 \frac{\Gamma(2[\alpha b^{-x}])\Gamma(2\alpha-2[\alpha b^{-x}]+1)}{\Gamma(2\alpha+1)}.$$

and according to Lemma 2(i):

$$\int_0^1 \beta(\tau, 2[\alpha b^{-x}], 2\alpha-2[\alpha b^{-x}]+1)q(\tau)d\tau \to f(x) \quad \text{as} \quad \alpha\to\infty,$$

uniformly with the rate of $1/\alpha$. The order of magnitude of $\xi_\alpha(x)$ in (23) is specified as follows: for each $x > 0$ we have

$$\xi_\alpha(x) \sim \frac{\ln b}{\sqrt{\pi}}\frac{\alpha^{1/2}}{n}\left(\frac{b^{-x}}{1-b^{-x}}\right)^{1/2}, \quad \alpha, n\to\infty. \tag{24}$$

Hence, for the first term in the right-hand side of (22) we can write:

$$\frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2 E\beta^{*2}_{\alpha,x}(b^{-X_1}) = \frac{\sqrt{\alpha}}{n}\frac{f(x)\,\ln b}{\sqrt{\pi(b^x-1)}} + o\left(\frac{\sqrt{\alpha}}{n}\right), \tag{25}$$

as $\alpha, n\to\infty$. Consider the second term of (22):

$$\frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2(E\beta^*_{\alpha,x}(b^{-X}))^2 = \frac{1}{n}\left\{\frac{[\alpha b^{-x}]\ln b}{\alpha+1}\right\}^2\left(\frac{\Gamma(\alpha+2)}{\Gamma([\alpha b^{-x}]+1)\,\Gamma(\alpha-[\alpha b^{-x}]+1)}\right)^2$$
$$\times\left(\int_0^\infty (b^{-u})^{[\alpha b^{-x}]}(1-b^{-u})^{\alpha-[\alpha b^{-x}]}f(u)du\right)^2$$
$$= \frac{1}{n}\left(\frac{\ln b\,\Gamma(\alpha+1)}{\Gamma([\alpha b^{-x}])\,\Gamma(\alpha-[\alpha b^{-x}]+1)}\right)^2\left(\int_0^1 \tau^{[\alpha b^{-x}]-1}(1-\tau)^{\alpha-[\alpha b^{-x}]}q(\tau)\frac{d\tau}{\ln(b)}\right)^2$$
$$= \frac{1}{n}\left(\int_0^1 \beta_{\alpha,x}(\tau)q(\tau)d\tau\right)^2 = \frac{1}{n}\left(f(x)+O\left(\frac{1}{\alpha}\right)\right)^2, \quad \text{as} \quad \alpha\to\infty. \tag{26}$$

Finally, from (22), (25), and (26) we obtain

$$var\{\hat{f}_{\alpha,b}(x)\} = \frac{\sqrt{\alpha}}{n}\frac{f(x)\ln b}{\sqrt{\pi(b^x-1)}} + o\left(\frac{\sqrt{\alpha}}{n}\right), \tag{27}$$

as $\sqrt{\alpha}/n\to 0$, $\alpha, n\to\infty$. □

Table 1
Records of the average L_2-errors ($\hat{d}_n$), when $X \sim$ Gamma (3, 1/2) and $R = 50$.

b	$n = 300$			$n = 500$			$n = 800$		
	α=80	α=100	α=120	α=80	α=100	α=120	α=80	α=100	α=120
1.05	0.154	0.1153	0.0969	0.1492	0.1213	0.0998	0.1507	0.1209	0.0948
1.15	0.0404	0.0294	0.0227	0.0397	0.0257	0.0179	0.0404	0.0269	0.0196
1.21	0.0235	0.0158	0.0125	0.0219	0.0154	0.0084	0.0244	0.0154	0.0119
1.23	0.0219	0.0140	0.0104	0.0217	0.0134	0.0090	0.0232	0.0147	0.0081
1.25	0.0167	0.0111	0.0091	0.0166	0.0105	0.0092	0.0177	0.0109	0.0127
1.26	0.0147	0.0104	0.0087	0.0175	0.0103	0.0120	0.0158	0.0111	0.0101
1.27	0.0156	00103	0.0082	0.0145	0.0103	0.0104	0.0168	0.0096	0.0119
1.28	0.0144	0.0099	0.0093	0.0144	0.0099	0.0102	0.015	0.0094	0.0117
1.29	0.0139	0.0101	0.0136	0.0159	0.0101	0.0122	0.0158	0.0269	0.0095
1.30	0.0130	0.0984	0.0107	0.0123	0.0098	0.0106	0.0137	0.0095	0.0099
1.50	0.0097	0.0063	0.0061	0.0089	0.0063	0.0064	0.0106	0.0075	0.0059
1.80	0.0079	0.0039	0.0055	0.0073	0.0039	0.0058	0.0068	0.0042	0.0025
1.90	0.0079	0.0049	0.0069	0.0057	0.0049	0.0021	0.0053	0.0037	0.0017
2.0	0.0058	0.0071	0.0052	0.0051	0.0071	0.0039	0.0051	0.0039	0.0030

Proof of Theorem 1. Combining the statements (i) and (ii) of Lemma 2, we obtain:

$$\mathrm{MSE}\{\hat{f}_{\alpha,b}(x)\} \le \frac{1}{(\alpha+1)^2}\left\{\frac{2\,b^x|f'(x)|}{\ln b} + \frac{b^{2x}|f'(x)|}{2\ln^2 b} + \frac{b^x|f''(x)|}{2\,\ln b}\right\}^2 + \frac{\sqrt{\alpha}}{n}\frac{f(x)\,\ln b}{\sqrt{\pi(b^x-1)}} + o(\frac{\sqrt{\alpha}}{n}). \tag{28}$$

Finally, taking $\alpha \sim n^{2/5}$ in (28), we arrive at (9). □

4. Simulation study

In this section we evaluated the average errors using L_2-norm of estimate (4) when the normalizing factor $\alpha/(\alpha+1)$ is removed. The following notation is used:

$$\hat{d}_n = \frac{1}{R}\sum_{r=1}^{R}\left(\frac{1}{m}\sum_{j=1}^{m}(\hat{f}(x_j) - f(x_j))^2\right)^{1/2}. \tag{29}$$

Here by R we denote the number of replications, and $\{x_j,\ j = 1, \ldots, m\}$ represents the partition of support of f. In this section, simulations from three different models (Gamma, Weibull, and shifted Pareto) with three different values of the sample size $n = 300{,}500$, and 800, combined with the number of replications $R = 50$ are conducted.

Now, let us simulate a random sample from $X \sim \text{Gamma}(a, \beta)$. In addition, the Average L_2-errors are computed for different values of α, b, and n, as it is shown in Table 1.

The records from this table specify the optimal values for parameters (α, b) as follows: (100, 1.8), (120, 2.0), and (120, 1.9) for $n = 300{,}500$, and 800, respectively. Corresponding average errors $\hat{d}_n$ are equal to 0.0039, 0.0021, and 0.0017, respectively. Hence, the Average L_2-error is decreasing as a function of sample size n.

Also, we compared $\hat{f}_{\alpha,b}(x)$ with the KDE $\hat{f}_h$ when $X \sim \text{Gamma}(3, 1/2)$ and $X \sim \text{Exp}(2/3)$. Assume the kernel density function K is specified as the Gaussian one and the bandwidth $h = n^{-1/5}\,1.06\,\hat{\sigma}$ (cf. with Silverman [2]). Here $\hat{\sigma}$ represents the standard deviation of the sample $X_1, \ldots, X_n$. In plots (a) and (b) of Fig. 1, the performances of $\hat{f}_{\alpha,b}$ and $\hat{f}_h$ are compared graphically. Two cases when $X_i \sim \text{Gamma}(3, 1/2)$ and $X_i \sim \text{Exp}(\text{rate} = 2/3)$ are considered when $\alpha = 120, b = 1.9$, and $\alpha = 120, b = 1.15$, respectively. In both plots the sample size $n = 500$. In plots (c) and (d) of Fig. 1, the Weibull (2, 2) and shifted Pareto (1, 2), respectively, are considered when $\alpha = 100$ and $n = 800$. We can say that $\hat{f}_{\alpha,b}$ performs better if compared to KDE when $f(0) > 0$ (see plots (b) and (d) in Fig. 1).

Table 2 displays the Average L_2-errors of $\hat{f}_{\alpha,b}$ evaluated for several values of parameter α when $b = 1.23$ and of $\hat{f}_h$ when $X \sim$ Gamma (3, 1/2) and $n = 300{,}500$. Here the number of replications $R = 50$. From this table we conclude that performances of $\hat{f}_{\alpha,b}$ and $\hat{f}_h$ are similar to each other when $X \sim$ Gamma (3, 1/2).

Table 2
Comparison of Average L_2-errors of $\hat{f}_{\alpha,b}$, when $\alpha = 80, 100, 120$, and $b = 1.23$, and of KDE $\hat{f}_h$ when $h = n^{-1/5}\, 1.06\,\hat{\sigma}$. Here $X \sim$ Gamma (3, 1/2) and $R = 50$.

n	$\hat{f}_{80,b}$	$\hat{f}_{100,b}$	$\hat{f}_{120,b}$	$\hat{f}_h$
300	0.01004	0.0097	0.01071	0.01108
500	0.0084	0.0088	0.0083	0.0088
800	0.0083	0.0080	0.0077	0.0081

5. Conclusion

The paper deals with investigation of asymptotic properties of nonparametric density estimate $\hat{f}_{\alpha,b}$ defined in (4). The main advantages of the proposed construction are: (a) it can be used in models where the only available information about the underlying distribution F represents the finite values of the scaled Laplace transform of F; (b) the estimate (3) has a unified form and can be easily implemented in different incomplete models as soon as there exists a consistent estimate of the Laplace transform of f. In the forthcoming paper we are planning to study the properties of corresponding estimates in the models with right-censored and the length-biased observations. Note also that in the case of direct model, the proposed estimate $\hat{f}_{\alpha,b}$ is reduced to the one based on asymmetric beta kernel density construction (see (7)).

It is worth mentioning that the values of $\hat{f}_{\alpha,b}(x)$ became constant for $x > \ln\alpha/\ln b$. That is why it is recommended to choose the values of b closer to 1 (from the right), when one is dealing with distribution having a long tail.

We compared the finite sample performances of $\hat{f}_{\alpha,b}$ with its counterpart based on the kernel density construction $\hat{f}_h$. We found out that it behaves a little bit better in terms of average L_2-errors and is free from the edge effect.

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