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# Soft Γ- Modules

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### Abstract

In this paper, the definitions of soft Γ -module, soft Γ - module homomorphism and soft Γ -exactness are introduced with the aid of the concept of soft set theory introduced by Molodtsov. In the meantime, some of their properties and structural characteristics are investigated and discussed. Thereafter, several illustrative examples are given.

**Keywords**: soft set; soft module; Γ- ring; Γ -module; soft Γ- module; soft Γ -module homomorphism; soft Γ -module isomorphism; soft Γ -exactness.
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# 1 Introduction

In the real world, there are some various uncertainties but classical mathematical tools is not convenient for modeling these. Uncertain and unclear data which are contained by economy, engineering, environmental science, social science, medical science, business administration and many other fields are common. Although many diverse theories such as probability theory, soft set theory, intuitionistic fuzzy soft set theory and rough set theory are known and these present advantageous mathematical approaches for modeling of uncertainties, each of these theories have their inherent diffuculties.

In 1999, Molodstov [1] developed soft set theory which is considered a mathematical tool for working with uncertainties. Since the emergence of soft set theory attracts attention and especially recently works on the soft set theory is progressing rapidly. Maji et al. [2] described some operations on soft sets and these operations are used soft sets of decision making problems. Chen et al. [3] offered a new definition for decrease of parametrerization on soft sets. They made comparasion between this definition and concept of restriction of property in the rough set theory. In theory, Maji et al. [4] worked various operator on soft set. Kong et al. [5] developed definition of parametrerization reduction on soft set. Zou and Xiao suggested some approach of data analysis in case of insuffcent information on soft set. Jiang et al. presented a unique approach of the semantic decision making by means of ontological thinking and ontology-based soft sets.

Besides studies on classic module theory have continued and interesting results have been discovered recently. Macias Diaz et al. [6] studied on modules which are isomorphic to relatively divisible or pure submodules of each other. Abuhlail et al. [7] presented on topological lattices and their applications to module theory. On the other hand, Ameri et al. [8] investigated gamma module and Davvaz et al. [9] studied tensor product of gamma modules.

As for soft module theory, Sun et al. [10] presented the notion of soft set and soft module. Xiang [11] worked soft module theory. T.Shah et al. [12] defined the notion of primary decomposition in a soft ring and soft module, and derived some related properties. Erami et al. [13] gave the concept of a soft MV-module and soft MV- submodule. In these days, there are some studies reletad with soft sets. Ali et al. [14] investigated some new operations in soft set theory and Pei et al. [15] studied from soft sets to information systems. Xiao et al. [16] presented research on synthetically evaluating method for business competitive capacity based on soft set. Aktaş et al. [17] showed soft sets and soft groups and Acar et al. [18] also showed soft sets and soft rings.

The main purpose of this paper is to deal with algebraic structure of $\Gamma-$ module by applying soft set theory. The concept of soft $\Gamma-$ module is introduced, their characterization and algebraic properties are investigated by giving some several examples. In addition to this, soft $\Gamma-$ homomorphism , soft $\Gamma-$ isomorphism and their properties are introduced. After all, we make inferences that images of soft $\Gamma-$ homomorphisms and inverse images of soft $\Gamma-$ homomorphisms are soft $\Gamma-$ homomorphisms. Furthermore soft $\Gamma-$ exactness is investigated and illustrated with a related example.

## 2 Preliminaries

In this section, preliminary informations will be required to soft $\Gamma-$ modules. First of all we give basic concepts of soft set theory.

**Definition 2.1.** *[18] Let $X$ denotes an initial universe set and $E$ is a set of parameters. The power set of $X$ is denoted by $P(X)$. A pair of $(F,E)$ is called a soft set over $X$ if and only if $F$ is a mapping from $E$ into the set of all subsets of $X$, i.e, $F$: $E \to P(X)$.*

**Definition 2.2.** *[18] Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $X$.*

***i)*** *If $A \widetilde{\subseteq} B$ and $F(a) \widetilde{\subseteq} G(a)$ for all $a \in A$ then we say that $(F,A)$ is a soft subset of $(G,B)$, denoted by $(F,A) \widetilde{\subseteq} (G,B)$.*

***ii)*** *If $(F,A)$ is a soft subset of $(G,B)$ and $(G,B)$ is a soft subset of $(F,A)$, then we say that $(F,A)$ is a soft equal to $(G,B)$, denoted by $(F,A) \widetilde{=} (G,B)$ .*

**Example 2.1.** *Let $X = M_2(Z_3)$ denotes an initial universe set, i.e, $2 \times 2$ matrices with $Z_3$ terms and $E = \{\begin{bmatrix}\overline{0} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{1} & \overline{0}\\ \overline{0} & \overline{1}\end{bmatrix}\}$is a set of parameters. Then $F: E \to P(X)$ where $F(\begin{bmatrix}\overline{0} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix}) = \{\begin{bmatrix}\overline{0} & \overline{1}\\ \overline{1} & \overline{1}\end{bmatrix}, \begin{bmatrix}\overline{2} & \overline{1}\\ \overline{0} & \overline{2}\end{bmatrix}\}, F(\begin{bmatrix}\overline{1} & \overline{0}\\ \overline{0} & \overline{1}\end{bmatrix}) = \{\begin{bmatrix}\overline{2} & \overline{0}\\ \overline{0} & \overline{2}\end{bmatrix}\}$. Clearly,$(F,E)$ is called a soft set over $X$.*

**Definition 2.3.** *[18] Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $X$. The intersection of $(F,A)$ and $(G,B)$ is defined as the soft set $(H,C)$ satisfying the following conditions:*

***i)*** *$C = A \cap B$.*

***ii)*** *For all $c \in C$, $H(c) = F(c)$ or $G(c)$ .*

*In this case, we write $(F,A) \widetilde{\cap} (G,B) = (H,C)$ .*

**Definition 2.4.** *[18] Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $X$. The union of $(F,A)$ and $(G,B)$ is defined as the soft set $(H,C)$ satisfying the following conditions:*

***i)*** *$C = A \cup B$.*

***ii)*** *For all* $c \in C$,

$$H(c) = \left\{ \begin{array}{ll} F(c) & \text{if } c \in A - B, \\ G(c) & \text{if } c \in B - A, \\ F(c) \cup G(c) & \text{if } c \in A \cap B. \end{array} \right\}$$

*This is denoted by* $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

**Definition 2.5.** *[18] If* $(F, A)$ *and* $(G, B)$ *are two soft sets over a common universe* $X$*, then* $(F, A)$ $AND$ $(G, B)$ *denoted by* $(F, A) \widetilde{\wedge} (G, B)$ *is defined as* $(F, A) \widetilde{\wedge} (G, B) = (H, C)$*, where* $C = A \times B$ *and* $H(x, y) = F(x) \cap G(y)$*, for all* $(x, y) \in C$.

**Definition 2.6.** *Let* $\{(F_i, A_i) : i \in I\}$ *be a non- empty family soft sets. The* $\wedge-$*intersection of a non-empty family soft sets is defined by* $(\psi, Y) = \widetilde{\wedge}_{i \in I}(F_i, A_i)$ *where* $(\psi, Y)$ *is a soft set,* $Y = \prod\limits_{i \in I} A_i$ *and* $\psi(y) = \cap_{i \in I} F_i(y)$ *for every* $y = (y_i)_{i \in I} \in Y$.

**Definition 2.7.** *[18] If* $(F, A)$ *and* $(G, B)$ *are two soft sets over a common universe* $X$*, then* $(F, A)$ $OR$ $(G, B)$ *denoted by* $(F, A) \widetilde{\vee} (G, B)$ *is defined as* $(F, A) \widetilde{\vee} (G, B) = (H, C)$*, where* $C = A \times B$ *and* $H(x, y) = F(x) \cup G(y)$*, for all* $(x, y) \in C$.

**Definition 2.8.** *Let* $\{(F_i, A_i) : i \in I\}$ *be a non- empty family soft sets. The* $\vee-$*union of a non-empty family soft sets is defined by* $(\psi, Y) = \widetilde{\vee}_{i \in I}(F_i, A_i)$ *where* $(\psi, Y)$ *is a soft set,* $Y = \prod\limits_{i \in I} A_i$ *and* $\psi(y) = \cup_{i \in I} F_i(y)$ *for every* $y = (y_i)_{i \in I} \in Y$.

On the other hand we will introduce modules and soft modules, then we will study some properties and theories of soft modules such as trivial soft module, whole soft module, the concepts of soft submodule and soft module homomorphisms.

**Definition 2.9.** *[10] Let* $R$ *be a ring with identity.* $M$ *is said to be a left* $R$*-module if left scalar multiplication* $\lambda : R \times M \to M$ *via* $(a, x) \mapsto ax$ *satisfying the axioms* $\forall r, r_1, r_2, 1 \in R; m, m_1, m_2 \in M :$

***i)*** $M$ *is an abelian group,*

***ii)*** $r(m_1 + m_2) = rm_1 + rm_2, (r_1 + r_2)m = r_1 m + r_2 m,$

***iii)*** $(r_1 r_2)m = r_1(r_2 m),$

***iv)*** $1m = m.$

*Left* $R-$*module is denoted by* ${}_R M$ *or* $M$ *for short. Similarly we can define right* $R$*- module and denote it by* $M_R$.

**Example 2.2.** *Let* $R = M_2(Z)$ *and* $M = \{\begin{bmatrix} a \\ b \end{bmatrix} |a, b \in Z\}$*. Then* $M$ *is module on* $R$*.*

**Definition 2.10.** *[10] Let* $M$ *be a left* $R$*- module,* $A$ *be a any nonempty set and* $(F, A)$ *is a soft set over* $M$*.* $(F, A)$ *is said to be a soft module over* $M$ *if and only if* $F(x)$ *is submodule over* $M$*, for all* $x \in A$*.*

**Definition 2.11.** *[10] Let* $(F, A)$ *be a soft module over* $M$ *then*

***i)*** $(F, A)$ *is said to be a trivial soft module over* $M$ *if* $F(x) = 0$ *for all* $x \in A$*,where* $0$ *is zero element of* $M$*.*

***ii)*** $(F, A)$ *is said to be an whole soft module over* $M$ *if* $F(x) = M$ *for all* $x \in A$*.*

**Proposition 2.1.** *[10] Let* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$*.*

***1)*** $(F, A) \widetilde{\cap} (G, B)$ *is a soft module over* $M$*.*

***2)*** $(F, A) \widetilde{\cup} (G, B)$ *is a soft module over* $M$ *if* $A \cap B = \emptyset$*.*

**Definition 2.12.** *[10] If* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$*, then* $(F, A) + (G, B)$ *is defined as* $(H, A \times B)$*, where* $H(x, y) = F(x) + G(y)$ *for all* $(x, y) \in A \times B$*.*

**Proposition 2.2.** *[10] Assume that* $(F, A)$ *and* $(G, B)$ *are two soft modules over* $M$*.Then* $(F, A) + (G, B)$ *is soft module over* $M$*.*

**Definition 2.13.** *[10] Suppose that* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$ *and* $N$ *respectively. Then* $(F, A) \times (G, B) = (H, A \times B)$ *is defined as* $H(x, y) = F(x) \times G(y)$ *for all* $(x, y) \in A \times B$*.*

**Proposition 2.3.** *[10] Let* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$ *and* $N$ *respectively. Then* $(F, A) \times (G, B)$ *is soft module over* $M \times N$*.*

**Definition 2.14.** *[10] Let* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$*.Then* $(G, B)$ *is soft submodule of* $(F, A)$ *if*

***i)*** $B \subset A$*,*

***ii)*** $G(x) < F(x), \forall\, x \in B$*.*

*This is denoted by* $(G, B)\widetilde{<}(F, A)$*.*

**Proposition 2.4.** *[10] Let* $(F, A)$ *and* $(G, B)$ *be two soft modules over* $M$*.We say that* $(G, B)$ *is soft submodule of* $(F, A)$ *if* $G(x) \subseteq F(x), \forall x \in A$*.*

**Definition 2.15.** *[10] Assume that* $E = \{e\}$*, where* $e$ *is unit of* $A$*.Then every soft module* $(F, A)$ *over* $M$ *at least have two soft modules* $(F, A)$ *and* $(F, E)$ *called trivial soft submodule.*

**Proposition 2.5.** *[10] Let $(F, A)$ and $(G, B)$ are two soft modules over $M$ and $(G, B)$ is soft submodule of $(F, A)$. If $f : M \to N$ is a homomorphism of module, then $(f(F), A)$ and $(f(G), B)$ are all soft modules over $N$ and $(f(G), B)$ is soft submodule of $(f(F), A)$.*

**Definition 2.16.** *[10] Let $(F, A)$ and $(G, B)$ be two soft modules over $M$ and $N$ respectively, $f : M \to N, g : A \to B$ be two functions. Then we say that $(f, g)$ is a soft homomorphism if the following conditions are satisfied:*

***i)*** *$f : M \to N$ is a homomorphism of module,*

***ii)*** *$g : A \to B$ is a mapping,*

***iii)*** *For all $x \in A$, $f(F(x)) = G(g(x))$.*

*We say that $(F, A)$ is a soft homomorphic to $(G, B)$ which denoted by $(F, A) \tilde{-} (G, B)$. In this definition, if $f$ is an isomorphism from $M$ to $N$ and $g$ is a one-to-one mapping from $A$ onto $B$, then we say that $(F, A)$ is a soft isomorphism and that $(F, A)$ is a soft isomorphic to $(G, B)$, this is denoted by $(F, A) \tilde{\cong} (G, B)$.*

Finally, we will define $\Gamma$- ring and $\Gamma$- module and their homomorphisms which are basic definitions for soft $\Gamma$- module.

**Definition 2.17.** *[8] Let $R$ and $\Gamma$ be additive abelian groups. Then we say that $R$ is a $\Gamma$- ring if there exists a mapping:*

$. : R \times \Gamma \times R \to R$

$(r_1, \gamma, r_2) \to r_1\gamma r_2$

*such that for every $a, b, c \in R$ and $\alpha, \beta \in \Gamma$,the following hold:*

***i)*** $(a + b)\alpha c = a\alpha c + b\alpha c$,

***ii)*** $a(\alpha + \beta)c = a\alpha c + a\beta c$,

***iii)*** $a\alpha(b + c) = a\alpha b + a\alpha c$,

***iv)*** $(a\alpha b)\beta c = a\alpha(b\beta c)$.

**Definition 2.18.** *[8] A subset $A$ of a $\Gamma$- ring $R$ is said to be a right ideal of $R$ if $A$ is an additive subgroup of $R$ and $A\Gamma R \subseteq A$,where $A\Gamma R = \{a\alpha c |\ a \in A, \alpha \in \Gamma, r \in R\}$.*

*A left ideal of $R$ is defined in a similar way. If $A$ is both right and left ideal, we say that $A$ is an ideal of $R$.*

**Definition 2.19.** *[8] If $R$ and $S$ are $\Gamma$- rings, then a pair $(\theta, \varphi)$ of maps from $R$ into $S$ is called a homomorphism from $R$ into $S$ if*

***i)*** $\theta(x + y) = \theta(x) + \theta(y)$,

***ii)*** *$\varphi$ is an isomorphism on $\Gamma$,*

***iii)*** $\theta(x\gamma y) = \theta(x)\varphi(\gamma)\theta(y)$.

**Definition 2.20.** *[8] Let $R$ be a $\Gamma$- ring. A left $\Gamma$- module $R$ is an additive abelian group $M$ together with a mapping $. : R \times \Gamma \times M \to M$ such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma, r, r_1, r_2 \in R$ the following hold:*

***i)*** $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2,$
***ii)*** $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m,$
***iii)*** $r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m,$
***iv)*** $r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m.$
*A right* $\Gamma$ *- module* $R$ *is defined in analogous manner.*

**Example 2.3.** *Let* $R = \{\begin{bmatrix}\overline{k} & \overline{m}\end{bmatrix} | k, m \in Z_2\}$*, i.e,* $1 \times 2$ *matrices and* $\Gamma = \{\begin{bmatrix}\overline{0} \\ \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{1} \\ \overline{0}\end{bmatrix}\} \in Z_2$*, where* $\Gamma$*is* $2 \times 1$ *matrices. Then we say that* $R$ *is a* $\Gamma$*- ring. Similarly,* $R$ *and* $\Gamma$ *are same if we choose* $M = \{\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{1} & \overline{1}\end{bmatrix}\}$*, then* $M$ *is* $\Gamma$ *- module* $R$*.*

**Definition 2.21.** *[8] Presume that* $(M, +)$ *be an* $\Gamma$ *- module* $R$ *. A nonempty subset* $N$ *of* $(M, +)$ *is said to be a left* $\Gamma$ *- submodule* $R$ *of* $M$ *if* $N$ *is a subgroup of* $M$ *and* $R\Gamma N \subseteq N$*,where* $R\Gamma N = \{r\gamma n \,| \gamma \in \Gamma, r \in R, n \in N\}$*, that is for all* $n_1, n_2 \in N$ *and for all* $\gamma \in \Gamma, r \in R; n_1 - n_2 \in N$ *and* $r\gamma n \in N$*. In this case we write* $N \leq M$*.*

**Example 2.4.** *In previous example, let* $N = \{\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}\} \subset M$ *and* $H : N \to P(M)$ *be a set valued function defined by* $H(a) = \{b \in M | R(a, \alpha, b) \Leftrightarrow a\alpha b \in \begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}\}$ *for all* $a \in N$*.*$H$ *is clear that* $H(\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}) = (\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix})$ *is* $\Gamma$ *- submodule* $R$ *of* $M$*..*

**Definition 2.22.** *[8] Let* $M$ *and* $N$ *be arbitrary* $\Gamma$ *- module* $R$ *. A mapping* $f : M \to N$ *is a homomorphism of* $\Gamma$ *- module* $R$ *if for all* $x, y \in M$ *and* $\forall r \in R, \forall \gamma \in \Gamma$ *we have*
***i)*** $f(x + y) = f(x) + f(y),$
***ii)*** $f(r\gamma x) = r\gamma f(x).$
*A homomorphism f is monomorphism if f is one-to-one and f is epimorphism if f is onto. f is called isomorphism if f is both monomorphism and epimorphism. We denote the set of all* $R_\Gamma$*- homomorphisms from* $M$ *into* $N$ *by* $Hom_{R_\Gamma}(M, N)$ *or shortly by* $Hom_{R_\Gamma}(M, N)$*. In particular* $M = N$ *we denote* $Hom(M, M)$ *by* $End(M)$*.*

**Definition 2.23.** *[18] Let* $M$ *be a nonempty set and a* $\Gamma-$*module. The pair* $(F, A)$ *is a soft set over* $M$*. The set* $Supp(F, A) = \{x \in A : F(x) \neq \emptyset\}$ *is called a support of the soft set* $(F, A)$*. The soft set* $(F, A)$ *is non-null if* $Supp(F, A) \neq \emptyset$*.*

# 3 Soft $\Gamma$- Modules

In this section, firstly we will define soft $\Gamma-$ modules, then we will give some operations on this modules.Throughout the section, $M$ is a $\Gamma-$module.

**Definition 3.1.** *Let $(F, A)$ be a non-null soft set over $M$. Then, $(F, A)$ is said to be a soft $\Gamma-$module over $M$ if $F(a)$ is a $\Gamma-$submodule $M$ such that $F : A \to P(M)$, (i.e. $a \to F(a)$) for all $a \in A, y \in Supp(F, A)$.*

**Example 3.1.** *For consider the additively abelian groups $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and $\Gamma = \{\overline{0}, \overline{2}\}$. Let $. : \mathbb{Z}_6 \times \Gamma \times \mathbb{Z}_6 \to \mathbb{Z}_6, (m_1, \Gamma, m_2) = m_1 \Gamma m_2$. Hence $\mathbb{Z}_6$ is a $\Gamma-$ module. Let $A = \mathbb{Z}_6$ and $F : A \to P(M)$ be a set valued function defined by*

$$\begin{aligned} f(\overline{0}) &= f(\overline{2}) = f(\overline{4}) = \mathbb{Z}_6, \\ f(\overline{1}) &= f(\overline{3}) = f(\overline{5}) = \{\overline{0}, \overline{3}\} \end{aligned}$$

*are $\Gamma-$submodule of $\mathbb{Z}_6$. Hence $(F, A, )$ is a soft $\Gamma-$module over $\mathbb{Z}_6$.*

**Example 3.2.** *Let $M$ is a $\Gamma-$ module and $(F, A)$ be a soft set over $M.F : A \to P(M)$ is defined by $F(x) = \{y \in M | \; x\alpha y = 0\}$ for all $x \in A, \alpha \in \Gamma$. It is clear that $(F, A)$ is a soft $\Gamma-$ module.*

**Example 3.3.** *For consider the additively abelian groups*

$$\begin{aligned} M &= R = \{\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{1} & \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{0} & \overline{1}\end{bmatrix}, \begin{bmatrix}\overline{1} & \overline{1}\end{bmatrix}\} \subseteq (\mathbb{Z}_2)_{1\times 2} \\ \textit{and } \Gamma &= \{\begin{bmatrix}\overline{0} \\ \overline{0}\end{bmatrix}, \begin{bmatrix}\overline{1} \\ \overline{0}\end{bmatrix}\} \subseteq (\mathbb{Z}_2)_{2\times 1} \end{aligned}$$

*with addition defined as matrice addition. It is trivial that $R$ is a $\Gamma-$ ring. Also $M$ is a $\Gamma-$ module over $R$. Let $N = \{\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}\} \subseteq M$ and $H : N \to P(M)$ be a set valued function defined by $H(a) = \{b \in M | \; R(a, \alpha, b) \leftrightarrow a\alpha b \in \begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}, \forall \alpha \in \Gamma\}$ for all $a \in N$. It is clear that $H(\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}) = \{\begin{bmatrix}\overline{0} & \overline{0}\end{bmatrix}\}$ are sub $\Gamma-$ module of $M$. Hence $(H, N)$ is soft $\Gamma-$ module of $M$.*

**Theorem 3.1.** *Let $(F, A)$ and $(G, B)$ are two soft $\Gamma-$modules over $M$. Then $(F, A) \widetilde{\cap} (G, B)$ is a soft $\Gamma-$module over $M$ if it is non-null.*

**Proof.** By definition, we have that $(F, A) \widetilde{\cap} (G, B) = (H, C)$ where $H(c) = F(x) \cap G(y)$ for all $c \in C$. We assume that $(H, C)$ is a non-null soft set over $M$. If $c \in Supp(H, C)$, then $H(c) = F(x) \cap G(y) \neq \emptyset$. We know that $(F, A), (G, B)$ are both soft $\Gamma-$module over $M$, and so, the nonempty sets $F(x)$ and $G(y)$ are both $\Gamma-$submodule over $M$. Thus, $H(c)$ is a $\Gamma-$submodule over $M$ for all $c \in Supp(H, C)$. In this position, $(H, C) = (F, A) \widetilde{\cap} (G, B)$ is a soft $\Gamma-$module over $M$. □

**Theorem 3.2.** *Let $(F, A)$ and $(G, B)$ are two soft $\Gamma-$modules over $M$. Then $(F, A) \widetilde{\cup} (G, B)$ is a soft $\Gamma-$module over $M$ if $A \cap B = \emptyset$.*

**Proof.** By definition, we have that $(F,A)\widetilde{\cup}(G,B)=(H,C)$ where $H(c)=F(x)\cap G(y)$ for all $c\in C$. Note first that $(H,C)$ is a non-null owing to the fact that $Supp(H,C)=Supp\,(F,A)\widetilde{\cup}(G,B)$ . Suppose that $c\in Supp(H,C)$.Then $H(c)\neq\emptyset$ so we have $F(x),G(y)\neq\emptyset$. From the hypothesis $A\cap B=\emptyset$, we follow that $H(c)=F(x)\cap G(y)$. On the other hand $F(x)\cap G(y)$ is a soft $\Gamma-$module over $M$, we conclude that $(H,C)$ is a soft $\Gamma-$module over $M$ for all $c\in Supp(H,C)$. Consequently $(F,A)\widetilde{\cup}(G,B)=(H,C)$ is a soft $\Gamma-$module over $M$. □

On the other hand, union of two soft $\Gamma-$ modules is not always soft $\Gamma-$ module. We will explain this situation with following example.

**Example 3.4.** *Let* $M=\mathbb{Z}_6=\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5}\}$ *is a* $M_\Gamma-$*module,* $\Gamma=\{\overline{0},\overline{1}\}$, $A=\mathbb{Z}_2=\{\overline{0},\overline{1}\}$ *and* $B=\mathbb{Z}_3=\{\overline{0},\overline{1},\overline{2}\}$ *such that* $F(\overline{0})=F(\overline{1})=\{\overline{0},\overline{2},\overline{4}\},G(\overline{0})=G(\overline{1})=G(\overline{2})=\{\overline{0},\overline{3}\}$ $A\cap B=\{\overline{0},\overline{1}\}$. *If this condition is hold, then* $(F,A)\widetilde{\cup}(G,B)$ *is not a soft* $\Gamma-$*module over* $M$. *Indeed,* $H(\overline{1})=\{\overline{0},\overline{2},\overline{3},\overline{4}\}\notin P(M)$.

**Definition 3.2.** *If* $(F,A)$ *and* $(G,B)$ *are two soft* $\Gamma-$*modules over* $M$*, then* $(F,A)$ $AND\,(G,B)$ *denoted by* $(F,A)\widetilde{\wedge}(G,B)$ *is defined as* $(F,A)\widetilde{\wedge}(G,B)=(H,C)$*, where* $C=A\times B$ *and* $H\left(x,y\right)=F\left(x\right)\widetilde{\cap}G\left(y\right)$*, for all* $(x,y)\in C$.

**Theorem 3.3.** *Suppose that* $(F,A)$ *and* $(G,B)$ *are two soft* $\Gamma-$*modules over* $M$. *Then* $(F,A)\widetilde{\wedge}(G,B)$ *is soft* $\Gamma-$*module over* $M$ *if it is non-null.*

**Proof.** Using definition, we have that $(F,A)\widetilde{\wedge}(G,B)=(H,C)$ where $C=A\times B$ and $H\left(x,y\right)=F\left(x\right)\widetilde{\cap}G\left(y\right)$, for all $(x,y)\in C$. Then the hypothesis, $(H,C)$ is a non-null soft set over $M$. Since $(H,C)$ is a non-null, $Supp\,(H,C)\neq\emptyset$ and so, for $(x,y)\in Supp\,(H,C)\,,H\left(x,y\right)=F\left(x\right)\widetilde{\cap}G\left(y\right)\neq\emptyset$. We assume that $t_1,t_2\in F\left(x\right)\widetilde{\cap}G\left(y\right)$ . In this position

i) If $t_1,t_2\in F\left(x\right)=\{y:R(x,y)\}$ we have that $xt_1\in A,xt_2\in A$. This implies that $x(t_1+t_2)\in A$.

ii) If $t_1,t_2\in G\left(y\right)=\{y_1:R(y,y_1)\}$ we have that $yt_1\in B$, $yt_2\in B$. This implies that $y(t_1+t_2)\in B$.

Hence $F\left(x\right)\widetilde{\cap}G\left(y\right)$ is a $\Gamma-$ submodule. By the definition of soft $\Gamma-$ module, $(F,A)$ and $(G,B)$ are soft $\Gamma-$modules over $M$. $F\left(x\right),G\left(y\right)$ are also $\Gamma-$ submodule over $M$. Furthermore $H\left(x,y\right)=F\left(x\right)\widetilde{\cap}G\left(y\right)$ is a $\Gamma-$ submodule over $M$ for all $(x,y)\in(H,C)=(F,A)\widetilde{\wedge}(G,B)$ . Hence $(F,A)\widetilde{\wedge}(G,B)$ is soft $\Gamma-$module over $M$. □

**Definition 3.3.** *If* $(F,A)$ *and* $(G,B)$ *are two soft* $\Gamma-$*modules over* $M$*, then* $(F,A)$ $OR\,(G,B)$ *denoted by* $(F,A)\widetilde{\vee}(G,B)$ *is defined as* $(F,A)\widetilde{\vee}(G,B)=(H,C)$*, where* $C=A\times B$ *and* $H\left(x,y\right)=F\left(x\right)\widetilde{\cup}G\left(y\right)$*, for all* $(x,y)\in C$.

**Theorem 3.4.** *Suppose that* $(F,A)$ *and* $(G,B)$ *are two soft* $\Gamma-$*modules over* $M$. *Then* $(F,A)\widetilde{\vee}(G,B)$ *is soft* $\Gamma-$*module over* $M$.

**Proof.** Using definition, we have that $(F,A)\,\widetilde{\vee}\,(G,B) = (H,C)$, where $C = A \times B$ and $H(x,y) = F(x)\,\widetilde{\cup}G(y)$, for all $(x,y) \in C$. Assume that $c \in \mathit{Supp}(H,C)$. Then $H(c) \neq \emptyset$ and so we have that $F(x) \neq \emptyset, G(y) \neq \emptyset$. By assumption, $F(x)\,\widetilde{\cup}G(y)$ is a soft $\Gamma-$ module of $M$ for all $c \in \mathit{Supp}(H,C)$. Consequently $(F,A)\,\widetilde{\vee}\,(G,B) = (H,C)$ is a soft $\Gamma-$module over $M$. □

**Definition 3.4.** *Let $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$. Then $(F,A)\,\widetilde{+}\,(G,B) = (H, A \times B)$ is defined as $H(x,y) = F(x) + G(y)$ for all $(x,y) \in A \times B$.*

**Theorem 3.5.** *Suppose that $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$. Then $(F,A)\,\widetilde{+}\,(G,B)$ is soft $\Gamma-$module over $M$.*

**Proof.** By the definition we write $(F,A)\,\widetilde{+}\,(G,B) = (H, A \times B)$ and $H(x,y) = F(x) + G(y)$ for all $(x,y) \in A \times B$. Let $(x,y) \in \mathit{Supp}(H, A \times B)$.Then, $H(x,y) \neq \emptyset$ and so we have $F(x) \neq \emptyset$, $G(y) \neq \emptyset$. By taking into account, $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$, it follows that $F(x) + G(y)$ is a soft $\Gamma-$module over $M$ for all $(x,y) \in \mathit{Supp}(H, A \times B)$. Hence $(F,A)\,\widetilde{+}\,(G,B)$ is soft $\Gamma-$module over $M$. □

**Definition 3.5.** *Let $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$. Then $(F,A)\,\widetilde{\times}\,(G,B) = (H, A \times B)$ is defined as $H(x,y) = F(x) \times G(y)$ for all $(x,y) \in A \times B$.*

**Theorem 3.6.** *Suppose that $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$. Then $(F,A)\,\widetilde{\times}\,(G,B)$ is soft $\Gamma-$module over $M$.*

**Proof.** By the definition we write $(F,A)\,\widetilde{\times}\,(G,B) = (H, A \times B)$ and $H(x,y) = F(x) \times G(y)$ for all $(x,y) \in A \times B$. Let $(x,y) \in \mathit{Supp}(H, A \times B)$. Then, $H(x,y) \neq \emptyset$ and so we have $F(x) \neq \emptyset$, $G(y) \neq \emptyset$. By taking into account, $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$, it follows that $F(x) \times G(y)$ is a soft $\Gamma-$module over $M$ for all $(x,y) \in \mathit{Supp}(H, A \times B)$. Hence $(F,A)\,\widetilde{\times}\,(G,B)$ is soft $\Gamma-$module over $M$. □

**Definition 3.6.** *Let $(F,A)$ and $(G,B)$ are two soft $\Gamma-$modules over $M$. Then $(G,B)$ is called a soft $\Gamma-$submodule of $(F,A)$ if*

***i)*** $B \subseteq A$,

***ii)*** $\forall b \in \mathit{Supp}(G,B), g(b)$ *is a $\Gamma-$submodule of $F(b)$.*

*This denoted by $(G,B) \subset (F,A)$. From the definition, it is easily deduced that if $(G,B)$ is a soft $\Gamma-$submodule of $(F,A)$, then $\mathit{Supp}(G,B) \subset \mathit{Supp}(F,A)$.*

**Theorem 3.7.** *Let $(F,A)$ and $(G,B)$ be two soft $\Gamma-$modules over $M$ and $(F,A)\widetilde{\subseteq}$ $(G,B)$. Then $(G,B) \subset (F,A)$.*

**Proof.** Straight forward. □

**Corolary 3.1.** *Let* $(F, A)$ *be a soft* $\Gamma-$*module over* $M$ *and* $\{(F_i, A_i) : i \in I\}$ *be a nonempty family of soft* $\Gamma-$*submodules of* $(F, A)$ *.Then,*

***i)*** $\widetilde{\cap}_{i\in I}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $(F, A)$ *if it is non-null.*

***ii)*** $\widetilde{\cup}_{i\in I}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $(F, A)$ *,* $if\ A_i \cap A_j = \emptyset$ *for all* $i, j \in I$ *and if it is non-null.*

***iii)*** *If* $F_i(a_i) \subseteq F_j(a_j)$ *or* $F_j(a_j) \subsetneq F_i(a_i)$ *for all* $i, j \in I, a_i \in A_i$*, then* $\widetilde{\vee}_{i\in I}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $\widetilde{\vee}_{i\in I}\,(F, A)$ *.*

***iv)*** $\widetilde{\wedge}_{i\in I}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $\widetilde{\wedge}_{i\in I}\,(F, A)$ *.*

***v)*** *The cartesian product of the family* $\widetilde{\prod\limits_{i\in I}}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $\widetilde{\prod\limits_{i\in I}}\,(F, A)$ *.*

***vi)*** $\widetilde{\sum}_{i\in I}(F_i, A_i)$ *is a soft* $\Gamma-$*submodule of* $\widetilde{\sum\limits_{i\in I}}\,(F, A)$ *.*

**Proof.** Similar to the proof of Theorems 3.5, 3.6, 3.9, 3.11, 3.13 and 3.15. □

# 4 Soft $\Gamma-$ Module Homomorphism

In this section, firstly we will define trivial and whole soft $\Gamma-$modules over $\Gamma-$module $M$, homomorphism of $\Gamma-$modules and their properties. Moreover we will study soft $\Gamma-$module homomorphism and soft $\Gamma-$module isomorphism. Throughout the section, $M$ is a $\Gamma-$module.

**Definition 4.1.** *Let* $(\rho, A)$ *and* $(\sigma, B)$ *be two soft* $\Gamma-$*modules over* $\Gamma-$*module* $M$ *and* $\Gamma-$*module* $M_1$ *respectively. Let* $f : M \to M_1$ *and* $g : A \to B$ *be two functions. The following conditions:*

***i)*** $f$ *is an epimorphism of* $\Gamma-$*module,*

***ii)*** $g$ *is a surjective mapping,*

***iii)*** $f(\rho(y)) = \sigma(\rho(y))$ *for all* $y \in A$*,*

*were satisfied by the pair* $(f, g)$*, then* $(f, g)$ *is called soft* $\Gamma-$*module homomorphism.*

*If there exists a soft* $\Gamma-$*module homomorphism between* $(\rho, A)$ *and* $(\sigma, B)$*, we say that* $(\rho, A)$ *is soft homomorphic to* $(\sigma, B)$*, and is denoted by* $(\rho, A) \sim (\sigma, B)$*. If there exists a soft* $\Gamma-$*module isomorphism between* $(\rho, A)$ *and* $(\sigma, B)$*, we say that*$(\rho, A)$ *is soft isomorphic to* $(\sigma, B)$*, and is denoted by* $(\rho, A)\widetilde{-}(\sigma, B)$*.*

**Definition 4.2.** *Let* $(F, A)$ *be soft* $\Gamma-$*module over* $M$*.*

***i)*** $(F, A)$ *is called the trivial soft* $\Gamma-$*module over* $M$ *if* $F(a) = \{0\}$ *for all* $a \in A$*.*

***ii)*** *$(F, A)$ is called the whole soft $\Gamma-$module over $M$ if $F(a) = M$ for all $a \in A$.*

**Definition 4.3.** *Let $M$ and $M_1$ be two $\Gamma-$modules and $m : M \to M_1$ a mapping of $\Gamma-$module. If $(F, A)$ and $(G, B)$ are soft sets over $M$ and $M_1$ respectively, then*

***i)*** *$(m(F), A)$ is a soft set over $M_1$ where $m(F) : A \to P(M_1)$, $m(F)(a) = m(F(a))$ for all $a \in A$.*

***ii)*** *$(m^{-1}(G), B)$ is a soft set over $M$ where $m^{-1}(G) : B \to P(M)$, $m^{-1}(G)(b) = m^{-1}(G(b))$ for all $b \in B$.*

**Corolary 4.1.** *Let $m : M \to M_1$ be an onto homomorphism of $\Gamma-$module. Then following statements can be given.*

***i)*** *$(F, A)$ be soft $\Gamma-$module over $M$, then $(m(F), A)$ is a soft $\Gamma-$module over $\Gamma-$module $M_1$.*

***ii)*** *$(G, B)$ be soft $\Gamma-$module over $\Gamma-$module $M_1$, then $(m^{-1}(G), B)$ is a soft $\Gamma-$module over $M$.*

**Proof.** i) Since $(F, A)$ is a soft $\Gamma-$module over $M$, it is clear that $(m(F), A)$ is a non-null soft set over $M_1$. For every $y \in Supp(m(F), A)$ we have $m(F)(y) = m(F(y)) \neq \emptyset$. Hence $m(F(y))$ which is the onto homomorphic image of $\Gamma-$module $F(y)$ is a $\Gamma-$module of $M_1$ for all $y \in Supp(F(m), A)$. That is $(m(F), A)$ is a soft $\Gamma-$module over $\Gamma-$module $M_1$.

ii) It is easy to see that $Supp(m^{-1}(G), B) \subseteq Supp(G, B)$. By this way let $y \in Supp(m^{-1}(G), B)$.Then, $G(y) \neq \emptyset$. Hence $m^{-1}(G(y))$ which is homomorphic inverse image of $\Gamma-$module $G(y)$, is a soft $\Gamma-$module over $M$ for all $y \in B$. □

**Theorem 4.1.** *Let $m : M \to M_1$ be a homomorphism of $\Gamma-$module and $(F, A)$, $(G, B)$ be two soft $\Gamma-$modules over $\Gamma-$module $M$ and $\Gamma-$module $M_1$ respectively. Then following statements can be given.*

***i)*** *If $F(a) =$ ker $(m)$ for all $a \in A$,then $(m(F), A)$ is the trivial soft $\Gamma-$module over $M_1$.*

***ii)*** *If $m$ is onto and $(F, A)$ is whole, then $(m(F), A)$ is the whole soft $\Gamma-$module over $M_1$.*

***iii)*** *If $G(b) = m(M)$ for all $b \in B$,then $(m^{-1}(G), B)$ is the whole soft $\Gamma-$module over $M$.*

***iv)*** *If $m$ is injective and $(G, B)$ is trivial, then $(m^{-1}(G), B)$ is the trivial soft $\Gamma-$module over $M$.*

**Proof.** i) By using $F(a) =$ ker $(m)$ for all $a \in A$. Then $m(F)(a) = m(F(a))$ $= \{0_{M_1}\}$ for all $a \in A$. Hence $(m(F), A)$ is soft $\Gamma-$module over $M_1$.

ii) Suppose that $m$ is onto and $(F, A)$ is whole. Then $F(a) = M$ for all $a \in A$ and so $m(F)(a) = m(F(a)) = m(M) = M_1$ for all $a \in A$. Hence $(m(F), A)$ is whole soft $\Gamma-$module over $M_1$.

iii) If we use hypothesis $G(b) = m(M)$ for all $b \in B$, we can write $m^{-1}(G)(b) = m^{-1}(G(b)) = m^{-1}(m(M)) = M$ for all $b \in B$. It is clear that, $(m^{-1}(G), B)$ is the whole soft $\Gamma-$module over $M$.

iv) Suppose that $m$ is injective and $(G, B)$ is trivial. Then, $G(b) = \{0\}$ for all $b \in B$,so $m^{-1}(G)(b) = m^{-1}(G(b)) = m^{-1}(\{0\}) = \ker m = \{0_M\}$ for all $b \in B$. Consequently, $(m^{-1}(G), B)$ is the trivial soft $\Gamma-$module over $M$. □

**Theorem 4.2.** *Let $m : M \to M_1$ be a homomorphism of $\Gamma-$module and $(F, A)$, $(G, B)$ be two soft $\Gamma-$modules over $M$. If $(G, B)$ is soft $\Gamma-$submodule of $(F, A)$, then $(m(G), B)$ is soft $\Gamma-$submodule of $(m(F), A)$.*

**Proof.** Suppose that $y \in Supp\,(G, B)$. Then $y \in Supp\,(F, A)$ .We know that $B \subseteq A$ and $G(y)$ is a $\Gamma-$submodule $F(y)$ for all $y \in Supp\,(G, B)$. From the expression hypothesis $m$ is a homomorphism, $m(G)(y) = m(G(y))$ is a $\Gamma-$submodule of $m(F)(y) = m(F(y))$ and therefore $(m(G), B)$ is soft $\Gamma-$submodule of $(m(F), A)$. □

**Theorem 4.3.** *Let $m : M \to M_1$ be a homomorphism of $\Gamma-$module and $(F, A)$, $(G, B)$ be two soft $\Gamma-$modules over $M$. If $(G, B)$ is soft $\Gamma-$submodule of $(F, A)$, then $(m^{-1}(G), B)$ is soft $\Gamma-$submodule of $(m^{-1}(F), A)$.*

**Proof.** Let $y \in Supp(m^{-1}(G), B)$. $B \subseteq A$ and $G(y)$ is a $\Gamma-$submodule of $F(y)$ for all $y \in B$. Since $m$ is a homomorphism, $m^{-1}(G)(y) = m^{-1}(G(y))$ is a $\Gamma-$submodule of $m^{-1}(G(y)) = m(G)(y)$ for all $y \in Supp(m^{-1}(G), B)$. Hence $(m^{-1}(G), B)$ is soft $\Gamma-$submodule of $(m^{-1}(F), A)$. □

# 5 Soft $\Gamma-$ Exactness

In this section, we will introduce maximal and minimal soft $\Gamma-$submodules. Then, we will investigate short exact and exact sequence of $\Gamma-$modules. Finally, we will explain soft $\Gamma-$exactness and some their basic theories. Throughout this section $M$ is $\Gamma-$module.

**Definition 5.1.** *Let $(F, A)$ and $(G, B)$ be two soft $\Gamma-$modules over $M$ and $(G, B)$ be soft $\Gamma-$submodule of $(F, A)$ . We say $(G, B)$ is maximal soft $\Gamma-$submodule of $(F, A)$ if $G(x)$ is a maximal $\Gamma-$submodule of $F(x)$ for all $x \in B$. We say $(G, B)$ is minimal soft $\Gamma-$submodule of $(F, A)$ if $G(x)$ is a minimal $\Gamma-$submodule of $F(x)$ for all $x \in B$.*

**Proposition 5.1.** *Let $(F, A)$ be a soft $\Gamma-$module over $M$.*

***i)*** *If $\{(G_i, B_i)\,|i \in I\}$ is a nonempty family of maximal soft $\Gamma-$submodules of $(F, A)$, then $\bigcap_{i\in I}(G_i, B_i)$ is maximal soft $\Gamma-$submodule of $(F, A)$.*

***ii)*** *If $\{(G_i, B_i)\,|i \in I\}$ is a nonempty family of minimal soft $\Gamma-$submodules of $(F, A)$, then $\sum_{i\in I}(G_i, B_i)$ is minimal soft $\Gamma-$submodule of $(F, A)$.*

**Proof.** straight forward. □

**Corolary 5.1.** *Let $(F, A)$ be a soft $\Gamma-$module over $M$ and $f : M \to N$ be a homomorphism if $F(x) = \ker f$ for all $x \in A$, then $(f(F), A)$ is the rivial soft $\Gamma-$module over $N$. Similarly, let $(F, A)$ be an whole soft $\Gamma-$module over $M$ and $f : M \to N$ be an epimorphism, then $(f(F), A)$ is a whole soft $\Gamma-$module over $N$.*

**Definition 5.2.** *The homomorphism sequence of $\Gamma-$modules ... $\to M_{n-1} \to^{f_{n-1}} M_n \to^{f_n} M_{n+1} \to$ ... is called exact sequence of $\Gamma-$modules if Im$f_{n-1} =$ Ker$f_n$ for all $n \in \mathbb{N}$ and we call the exact sequence of $\Gamma-$modules form as $0 \to M_1 \to^f M \to^g M_2 \to 0$ the short exact sequence of $\Gamma-$modules.*

**Proposition 5.2.** *Let $(F, A)$ be a trivial soft $\Gamma-$module over $\Gamma-$module $M_1$ and $(G, B)$ be a whole soft $\Gamma-$module over $\Gamma-$module $M_2$ if $0 \to M_1 \to^f \to M \to^g \to M_2 \to 0$ is a short exact sequence, then $0 \to F(x) \to^{\widetilde{f}} M \to^{\widetilde{g}} \to G(y) \to 0$ is a short exact sequence for all $x \in A, y \in B$.*

**Proof.** $F(x) = 0, \forall x \in A$ since $(F, A)$ is a trivial soft $\Gamma-$module over $\Gamma-$module $M_1$,so $\widetilde{f}$ is a monomorphism. $G(y) = M_2, \forall y \in B$ since $(G, B)$ is a whole soft $\Gamma-$module over $\Gamma-$module $M_2.g : M \to M_2$ is an epimorphism as $0 \to M_1 \to^f \to M \to^g \to M_2 \to 0$ is a short exact sequence, so $\widetilde{g}$ is an epimorphism. □

**Proposition 5.3.** *Let $(F, A)$ be a trivial soft $\Gamma-$module over $\Gamma-$module $M_1$ and $(G, B)$ be a whole soft $\Gamma-$module over $\Gamma-$module $M$ if $0 \to M_1 \to^f \to M \to^g \to M_2 \to 0$ is a short exact sequence, then $0 \to f(F)(x) \to^{\widetilde{f}} M \to^{\widetilde{g}} g(G)(y) \to 0$ is a short exact sequence for all $x \in A, y \in B$.*

**Proof.** $F(x) = 0, \forall x \in A$ since $(F, A)$ is a trivial soft $\Gamma-$module over $\Gamma-$module $M_1$.Ker$f = 0$, so Ker$f = F(x), \forall x \in A$,consequently $(f(F), A)$ is trivial soft $\Gamma-$module over $M$. $(G, B)$ is a whole soft $\Gamma-$module over $M$ and $g : M \to M_2$ is an epimorphism, so $(g(G), B)$ is a whole soft $\Gamma-$module over $M_2$, thus $0 \to f(F)(x) \to^{\widetilde{f}} M \to^{\widetilde{g}} g(G)(y) \to 0$ is a short exact sequence for all $x \in A, y \in B$. □

**Definition 5.3.** *Let $(F, A), (G, B)$ and $(H, C)$ are three soft $\Gamma-$modules over $\Gamma-$modules $M, N$ and $K$ respectively. Then we say soft $\Gamma-$ exactness at $(G, B)$, if the following conditions are satisfied:*

***i)*** $M \rightarrow^{f_1} N \rightarrow^{f_2} K$ *is exact,*

***ii)*** $A \rightarrow^{g_1} B \rightarrow^{g_2} C$ *is exact,*

***iii)*** $f_1(F(x)) = G(g_1(x))$ *for all* $x \in A$,

***iv)*** $f_2(G(x)) = H(g_2(x))$ *for all* $x \in B$,

*which is denoted by* $(F, A) \rightarrow^{(f_1, g_1)} (G, B) \rightarrow^{(f_2, g_2)} (H, C)$.

In this definition, if every $(F_i, A_i), i \in I$ is soft $\Gamma-$ exact, then we say that $(F_i, A_i)_{i \in I}$ is soft $\Gamma-$ exact.

**Proposition 5.4.** *Let $(F, A)$ and $(G, B)$ are two soft $\Gamma-$modules over $\Gamma-$modules $M$ and $N$ respectively. If $(F, A) \rightarrow^{(f,g)} (G, B) \rightarrow 0$ is soft $\Gamma-$ exact, then $(f, g)$ is soft $\Gamma-$ homomorphism. In particular, if $0 \rightarrow (F, A) \rightarrow^{(f,g)} (G, B) \rightarrow 0$ is soft $\Gamma-$ exact, then $(f, g)$ is soft $\Gamma-$isomorphism.*

**Proof.** Since $(F, A) \rightarrow^{(f,g)} (G, B) \rightarrow 0$ is soft $\Gamma-$ exact, we have $M \rightarrow^{f} N \rightarrow 0$ and $A \rightarrow^{g} B \rightarrow 0$ are exact. Thus $f$ and $g$ are epimorphisms, it is clear that $(f, g)$ is homomorphism. If $0 \rightarrow (F, A) \rightarrow^{(f,g)} (G, B) \rightarrow 0$ is soft $\Gamma-$ exact, then $0 \rightarrow M \rightarrow^{f} N \rightarrow 0$ and $0 \rightarrow A \rightarrow^{g} B \rightarrow 0$ are exact. Thus $f$ and $g$ are isomorphisms, it is clear that $(f, g)$ is soft $\Gamma-$isomorphism. □

**Definition 5.4.** *Let $M = 0$ and $A = 0$, then $(F, A) = 0$. We call $(F, A)$ is a zero-soft $\Gamma-$ module.*

**Proposition 5.5.** *Let $(F, A), (G, B)$ and $(H, C)$ are three soft $\Gamma-$modules over $\Gamma-$modules $M, N$ and $K$ respectively. If $(F, A) \rightarrow^{(f_1, g_1)} (G, B) \rightarrow^{(f_2, g_2)} (H, C)$ is soft $\Gamma-$ exact with $f_1, g_1$ epimorphism and $f_2, g_2$ monomorphism, then $(G, B)$ is a zero-soft $\Gamma-$ module.*

**Proof.** Since $(F, A) \rightarrow^{(f_1, g_1)} (G, B) \rightarrow^{(f_2, g_2)} \rightarrow (H, C)$ is soft $\Gamma-$ exact with $f_1, g_1$ epimorphism and $f_2, g_2$ monomorphism, we have $M \rightarrow^{f_1} N \rightarrow^{f_2} K$ and $A \rightarrow^{g_1} B \rightarrow^{g_2} C$, hence $N = 0$ and $B = 0$, it is clear that $(G, B)$ is zero-soft $\Gamma-$ module. □

**Theorem 5.1.** *Let $(F, A)$ and $(H, B)$ are two soft $\Gamma-$modules over $\Gamma-$modules $M$ and $N$ respectively. For any $M \subset N, A \subset B$ and $M \subset H(x)$ where $x \in B$. If $(F, A) \rightarrow^{(f,g)} (H, B)$ is soft $\Gamma-$homomorphism, then $0 \rightarrow (F, A) \rightarrow^{(f,g)} (H, B) \rightarrow^{(f_1, g_1)} (I, B/A) \rightarrow 0$ is soft $\Gamma-$ exact, where $I(x + A) = H(x)/M$ for all $x \in B$.*

**Proof.** We know that $0 \to M \to^f N \to^{f_1} N/M \to 0$ and $0 \to A \to^g B \to^{g_1} B/A \to 0$ are exact. It is clear that $M$ is a $\Gamma-$submodule of $N$, so that $N/M$ is a $\Gamma-$module and $M$ is a $\Gamma-$submodule of $H(x)$ and $H(x)/M$ is always a $\Gamma-$submodule of $N/M$. This shows that $(I, B/A)$ is a soft $\Gamma-$module over $N/M$. For all $x \in B/A$. Define $f_1 : N \to N/M$ by $f_1(n) = n + M$, for all $n \in N$. Meanwhile, we define $g_1 : B \to B/A$ by $g_1(b) = b + A$, for all $b \in B$. Therefore, it gives that

$$f_1(H(x)) = H(x) + M, I(g_1(x)) = I(x + A) = H(x) + M$$

for all $x \in B$, and hence $f_1(H(x)) = I(g_1(x))$.This implies

$$0 \to (F, A) \to^{(f,g)} (H, B) \to^{(f_1,g_1)} (I, B/A) \to 0$$

is soft $\Gamma-$ exact. □

**Theorem 5.2.** *Let* $(F, A_2), (G, A_1)$ *and* $(H, A)$ *are three soft* $\Gamma-$*modules over* $\Gamma-$*modules* $M_2, M_1$ *and* $M$ *respectively. If* $M_1$ *and* $M_2$ *are* $\Gamma-$*submodules of* $M$ *with* $M_2 \subset M_1, A_1$ *and* $A_2$ *are* $\Gamma-$*submodules of* $A$ *with* $A_2 \subset A_1$*, where* $M_1 \subset H(x)$*, for all* $x \in A$ *and* $M_2 \subset G(x)$ *for all* $x \in A_1$*. Then* $0 \to (I, A_1/A_2) \to^{(f_1,g_1)} (J, A/A_1) \to^{(f_2,g_2)} (P, A/A_1) \to 0$ *is soft* $\Gamma-$ *exact, where* $I(x + A_2) = G(x)/M_2$*, for all* $x \in A_1$*,* $J(x + A_2) = H(x)/M_2$*, for all* $x \in A, P(x + A_1) = H(x)/M_1$*, for all* $x \in A$.

**Proof.** Since $M_1$ and $M_2$ are $\Gamma-$submodules of $M$ with $M_2 \subset M_1$, we have a short exact sequence $0 \to M_1/M_2 \to^{f_1} M/M_2 \to^{f_2} M/M_1 \to 0$. Since $A_1$ and $A_2$ are $\Gamma-$submodules of $A$ with $A_2 \subset A_1$, there is a short exact sequence $0 \to A_1/A_2 \to^{g_1} A/A_2 \to^{g_2} A/A_1 \to 0$. It is clear that $M_2$ is a $\Gamma-$submodule of $M_1$, so that $M_1/M_2$ is a $\Gamma-$module. It gives that $G(x)/M_2$ is a $\Gamma-$module for all $x \in A_1$ from $M_2$ is a $\Gamma-$submodule of $G(x)$. However $G(x)/M_2$ is always a $\Gamma-$submodule of $M_1/M_2$. This shows that $(I, A_1/A_2)$ is a soft $\Gamma-$ module over $M_1/M_2$ for all $x \in A_1/A_2$. It is clear that $(J, A/A_2)$ and $(P, A/A_1)$ be a soft $\Gamma-$ module over $M/M_2$ and $M/M_1$ respectively.

Define $f_1 : M_1/M_2 \to M/M_2$ by $f_1(m_1 + M_1) = m + M_2$, for all $m_1 \in M_1$. Meanwhile, we define $g_1 : A_1/A_2 \to A/A_2$ by $g_1(a_1 + A_2) = a + A_2$, for all $a_1 \in A_1$. Therefore, we have $f_1(I(x)) = f_1(G((x)/M_2) = H(x) + M_2, J(g_1(x)) = J(x + A_2) = H(x) + M_2$ for all $x \in A_1/A_2$, so $f_1(I(x)) = J(g_1(x))$ for all $x \in A_1/A_2$.

Define $f_2 : M/M_2 \to M/M_1$ by $f_2(m + M_2) = m + M_1$, for all $m \in M$. Let $g_2 : A/A_2 \to A/A_1$ be defined by $g_2(a + A_2) = a + A_1$, for all $a \in A$. Also, we have $f_2(J(x)) = f_2(H((x)/M_2) = H(x) + M_1$ for all $x \in A/A_2$, so $f_2(J(x)) = P(g_2(x))$ for all $x \in A/A_2$. Hence $0 \to (I, A_1/A_2) \to^{(f_1,g_1)} (J, A/A_1) \to^{(f_2,g_2)} (P, A/A_1) \to 0$ is soft $\Gamma-$ exact. □

**Theorem 5.3.** *Let* $(F_i, A_i), i = 1,2,3,4,5$ *be a soft* $\Gamma-$*module over* $\Gamma-$*module* $M_i, i = 1,2,3,4,5$ *respectively. If* $0 \to (F_1, A_1) \to^{(f_1,g_1)} (F_2, A_2) \to^{(f_2,g_2)} (F_3, A_3) \to 0$ *and* $0 \to (F_3, A_3) \to^{(f_3,g_3)} (F_4, A_4) \to^{(f_4,g_4)} (F_5, A_5) \to 0$ *are soft* $\Gamma-$ *exact. Then* $0 \to (F_1, A_1) \to^{(f_1,g_1)} (F_2, A_2) \to^{(f_3\, f_2, g_3\, g_2)} (F_4, A_4) \to^{(f_4,g_4)} (F_5, A_5) \to 0$ *is soft* $\Gamma-$ *exact.*

**Proof.** Since $0 \to (F_1, A_1) \to^{(f_1,g_1)} (F_2, A_2) \to^{(f_2,g_2)} (F_3, A_3) \to 0$ and $0 \to (F_3, A_3) \to^{(f_3,g_3)} (F_4, A_4) \to^{(f_4,g_4)} (F_5, A_5) \to 0$ are soft $\Gamma-$ exact, we have $0 \to M_1 \to^{f_1} M_2 \to^{f_2} M_3 \to 0$ and $0 \to M_3 \to^{f_3} M_4 \to^{f_4} M_5 \to 0$ are exact. It is clear that $0 \to M_1 \to^{f_1} M_2 \to^{f_3\, f_2} M_4 \to^{f_4} M_5 \to 0$ is exact. Since $0 \to A_1 \to^{g_1} A_2 \to^{g_2} A_3 \to 0$ and $0 \to A_3 \to^{g_3} A_4 \to^{g_4} A_5 \to 0$ are exact. It is clear that $0 \to A_1 \to^{g_1} A_2 \to^{g_3\, g_2} A_4 \to^{g_4} A_5 \to 0$ is exact. Since $f_2(F_2(x)) = F_3(g_2(x))$ for all $x \in A_2$ and $f_3(F_3(x)) = F_4(g_3(x))$ for all $x \in A_3$. We have $f_3 f_2(F_2(x)) = f_3(F_3(g_2(x))) = F_4(g_3 g_2(x))$ for all $x \in A_2$. This implies $0 \to (F_1, A_1) \to^{(f_1,g_1)} (F_2, A_2) \to^{(f_3\, f_2, g_3\, g_2)} (F_4, A_4) \to^{(f_4,g_4)} (F_5, A_5) \to 0$ is soft $\Gamma-$ exactness. □

# 6 Conclusion

In this work the theoretical point of view of soft $\Gamma-$ module is discussed. The work is focused on soft $\Gamma-$ module, soft $\Gamma-$ module homomorphism and soft $\Gamma-$ exactness. By using these concepts, we studied the algebraic properties of soft sets in $\Gamma-$ module structure. One could extend this work by studying other algebraic structures.

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