RATIO MATHEMATICA 27 (2014) 37-47 ISSN:1592-7415

Multivalued linear transformations of hyperspaces

R. Ameri[a], R. A. Borzooei[b], K. Ghadimi[c]

[a] School of Mathematics, Statistic and Computer Sciences, College of Science, University of Tehran,Tehran, Iran

rameri@ut.ac.ir

[b] Department of Mathematics, Shahid Beheshti University, Tehran, Iran,

borzooei@sbu.ac.ir

[c] Department of Mathematics, Payame Noor University, Tehran, Iran,

ghadimi@phd.pnu.ac.ir

Abstract

The purpose of this paper is the study of multivalued linear transformations of hypervector spaces (or hyperspaces) in the sense of Tallini. In this regards first we introduce and study various multivalued linear transformations of hyperspaces and then constitute the categories of hyperspaces with respect the different linear transformations of hyperspaces as the morphisms in these categories. Also, we construct some algebraic hyperoperations on $Hom_K(V, W)$, the set of all multivalued linear transformations from a hyperspace V into hyperspaces W, and obtaine their basic properties. Finally, we construct the fundamental functor F from $\mathcal{HV}_K$, category of hyperspaces over field K into $\mathcal{V}_K$, the category of clasical vector space over K.

Key words: hypervector space, multivalued linear transformation, category,fundamental relation

2000 AMS: 20N20

1 Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [15]. It was later observed that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed ([11], [12], [13],[14] and [20])(for more see [2, 3, 5, 6, 7, 8, 9]).

M.S. Tallini introduced the notion of hyperspaces(hypervector spaces) ([17], [18] and [19]) and studied basic properties of them. R. Ameri and O. Dehghan [2] introduced and studied dimension of hyperspaces and in [16] M. Motameni et. el. studied hypermatrix. R. Ameri in [1] introduced and studied categories of hypermodules. In this paper we introduce and study various types of multivalued linear transformations of hyperspaces. We will proceed by constructing various categories of hyperspaces based on various multilinear linear transformations of hyperspaces. Also, we construct some hyperalgebraic structures on $(Hom_K(V, W)$. Finally, we construct the fundumental functor from category of hyperspaces and multilinear transformations, as morphisms into the category of vectorspces.

2 Preliminaries

The concept of hyperspace, which is a generalization of the concept of ordinary vector space.

Definition 2.1. *Let H be a set. A map $. : H \times H \longrightarrow P_*(H)$ is called hyperoperation or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A.B$ is given by*

$$A.B = \bigcup \{a.b : a \in A \text{ and } b \in B\}.$$

the notations $a.A$ and $A.a$ are used for $\{a\}.A$ and $A.\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.2. *[17] Let K be a field and $(V, +)$ be an abelian group. We define a hyperspace over K to be the quadrupled $(V, +, \circ, K)$, where $\circ$ is a* *mapping*

$$\circ : K \times V \longrightarrow P_*(V),$$

such that the following conditions hold:
(H_1) $\forall\ a \in K,\ \forall\ x, y \in V,\ a \circ (x + y) \subseteq a \circ x + a \circ y,$ *right distributive law,*
(H_2) $\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \subseteq a \circ x + b \circ x,$ *left distributive law,*
(H_3) $\forall\ a, b \in K,\ \forall\ x \in V,\ a \circ (b \circ x) = (ab) \circ x,$ *associative law,*
(H_4) $\forall\ a \in K,\ \forall\ x \in V,\ a \circ (-x) = (-a) \circ x = -(a \circ x),$
(H_5) $\forall\ x \in V,\ x \in 1 \circ x.$

Remark 2.3. *(i) In the right hand side of (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in (H_2).*
(ii) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hyperspaces, respectvely. V is called strongly distributive if it is both strongly left and strongly right distributive.
(iii) The left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ 0_V$, where 0_V is the zero of $(V, +)$, In [17] it is shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$.

Let V be a hyperspace over a field K. $W \subseteq V$ is a subhyperspace of V, if

$$W \neq \emptyset,\ \ W - W \subseteq W,\ \ \forall a \in K,\ \ a \circ W \subseteq W.$$

Example 2.4. *[2] Consider abelian group $(\mathbb{R}^2, +)$. Define hyper-compositions*

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \circ (x, y) = ax \times \mathbb{R} \end{cases}$$

and

$$\begin{cases} \diamond : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \diamond (x, y) = \mathbb{R} \times ay. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ and $(\mathbb{R}^2, +, \diamond, \mathbb{R})$ are a strongly distributive hyperspaces.

Example 2.5. *[2] Let $(V, +, ., K)$ be a classical vector space and P be a subspace of V. Define the hyper-composition*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = a.x + P. \end{cases}$$

Then it is easy to verify that $(V, +, \circ, K)$ is a strongly distributive hyperspace.

Example 2.6. *[?] In $(\mathbb{R}^2, +)$ define the hyper-composition $\circ$ as follows:*

$$\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^2 \ : \ a \circ x = \begin{cases} \text{line } \bar{ox} \ \ if \ x \neq 0_V \\ \{0_V\} \quad if \ x = 0_V, \end{cases}$$

where $0_V = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left, but not right distributive hyperspace.

Proposition 2.7. *[?] Every strongly right distributive hyperspace is strongly left distributive hyperspace. Let $(V, +)$ be an abelian group, Ω a subgroup of V and K a field such that $W = V/\Omega$ is a classical vector space over K. If $p : V \longrightarrow W$ is the canonical projection of $(V, +)$ onto $(W, +)$ and set:*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = p^{-1}(a.p(x)). \end{cases}$$

Then $(V, +, \circ, K)$ is a strongly distributive hyperspace over K. Moreover every strongly distributive hyperspace can be obtained in such a way.

Proposition 2.8. *[?] If $(V, +, \circ, K)$ be a left distributive hyperspace, then for all $a \in K$ and $x \in V$*
1) *$0 \circ x$ is a subgroup of $(V, +)$;*
2) *Ω_V is a subgroup of $(V, +)$;*
3) *$a \circ 0_V = \Omega_V = a \circ \Omega_V$;*
4) *$\Omega_V \subseteq 0 \circ x$;*
5) *$x \in 0 \circ x \Longleftrightarrow 1 \circ x = 0 \circ x \Longleftrightarrow a \circ x = 0 \circ x, \ \forall a \in K$.*

Remark 2.9. *Let $(V, +, \circ, K)$ be a hyperspace and W be a subhyperspace of V. Consider the quotient abelian group $(V/W, +)$. Define the rule*

$$\begin{cases} * : K \times V/W \longrightarrow P_*(V/W) \\ \quad (a, x + W) \longmapsto a \circ x + W. \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ is a hyperspace over K and it is called the quotient hyperspace of V over W.

3 Multivalued linear transformations

Definition 3.1. *Let V and W be two hyperspaces over a field K. A multivalued linear transformation (MLT) $T : V \longrightarrow P_*(W)$ is a mapping such that :*
$\forall x, y \in V, \forall a \in K$
1) $T(x + y) \subseteq T(x) + T(y)$;
2) $T(a \circ x) \subseteq a \circ T(x)$;
3) $T(-a) = -T(a)$.

Remark 3.2. (i) *In Definition 3.1*(1) *and* (2)*, if the equality holds, then T is called a strong multivalued linear transformation $(SMLT)$.*
(ii) *In Definition 3.1, if we consider T as a mapping $T : V \longrightarrow W$, then is it is called a linear transformation. Here we consider only inclusion and equality cases.*
(iii) *If T is a MLT, then $0 \in T(x)$, since $T(x) \neq \emptyset$, so $\exists y \in T(x)$; $0 = y - y \in T(x) - T(x) = T(x) + T(-x) = T(x + (-x)) = T(x - x) = T(0)$.*

Definition 3.3. *[1] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then multivalued kernel and multivalued image of T, denoted by $\overline{Ker}T$ and $\overline{Im}T$, respectively, are defined as follows:*

$$\overline{Ker}T = \{x \in V \mid 0_W \in T(x)\};$$

and

$$\overline{Im}T = \{y \in W \mid y \in T(x) \text{ for some } x \in V\}.$$

Remark 3.4. (i) *Note that $\overline{Ker}T \neq \emptyset$, by Remark 3.2(iii).*

(ii) *For hyperspaces V and W over a field K, by $Hom_K(V, W)$ and $Hom^s_K(V, W)$, we mean the set of all MLT and $SMLT$, respectively and sometimes we use morphism instead multivalued linear transformation, respectively. Also, by $hom_K(V, W)$ and $hom^s_K(V, W)$, we mean the set of all linear transformation LT and strong linear transformation SLT respectively and sometimes we use morphism instead multivalued , respectively.*

In the following we briefly introduced the categories of hyperspaces and study the relationship between monomorphism, epimorphism, isomorphism and*monic, epic* and iso objects in these category.

Definition 3.5. *The category of hyperspaces over a field K denoted by $\mathcal{HV}_K$ is defined as follows:*
1) *The objects of $\mathcal{HV}_K$ are all hyperspaces over K;*

2) *For the objects* V *and* W *of* $\mathcal{HV}_K$*, the set of all morphisms from* V *to* W *denoted by* $Hom_K(V,W)$*, is the set of all* MLT *from* V *to* W*.*
3) *The composition* $ST : V \longrightarrow P_*(W)$ *of morphisms* $T : V \longrightarrow P_*(L)$ *and* $S : L \longrightarrow P_*(W)$ *is defined as follows:*

$$ST(x) = \bigcup_{t \in T(x)} S(t).$$

4) *For any object* V*, the morphism* $1_V : V \longrightarrow P_*(V)$*,* $x \to \{x\}$ *is the identity.* 5) *The category of hyperspaces over a field* K *with (resp.* SLT*)*LT *is denoted by (resp.* $\langle\mathcal{V}_K$*)*$\langle\mathcal{V}_K$*.*

Remark 3.6. *If in Definition 3.5 part* (2) *we replace* $Hom_K(V,W)$ *by* Hom^s_K (V,W)*, the set of all* $SMLT$*, then we will obtain a new category, which it denotes by* $\mathcal{HV}^s_K$*. In fact,* $\mathcal{HV}^s_K \preceq \mathcal{HV}_K$ *(by* $A \preceq B$ *we mean* A *is a subcategory of* B*). Also, denote the category of all vector spaces over a field* K *by* $\mathcal{V}_K$*. Clearly,* $\mathcal{V}_K \preceq \langle\mathcal{V}_K \preceq \langle\mathcal{V}^s_K \preceq \mathcal{HV}^s_K \preceq \mathcal{HV}_K$ *(for more details see [1]).*

Definition 3.7. *Let* $T : V \longrightarrow P_*(W)$ *be a* $SMLT$ *of hyperspaces. We say that* T *is weakly injective if*

$$\forall\ x, y \in V,\ T(x) \cap T(y) \neq \emptyset \Rightarrow x = y.$$

We say that T *is strongly injective if*

$$\forall\ x, y \in V,\ T(x) = T(y) \Rightarrow x = y.$$

Remark 3.8. *Clearly, every weakly injective morphism is also strongly injective. Note that* T *is strongly injective, means that* T *is injective as a function with values in* $P_*(W)$*. In the following example we show that a strongly injective morphism need not to be weakly injective.*

Example 3.9. *Consider the abelain group* $(\mathbb{R},+)$*. Define a mapping*

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R} \longrightarrow P_*(\mathbb{R}) \\ \quad a \circ b = \{-ab, ab\}. \end{cases}$$

Then $(\mathbb{R},+,\circ,\mathbb{R})$ *is a hyperspace. The mapping* $T : \mathbb{R} \longrightarrow P_*(\mathbb{R})$ *defined by* $T(a) = \{0,a\}$ *is a MLT, since* $T(a+b) = \{0, a+b\}$ *and* $T(a)+T(b) = \{0,a\}+\{0,b\} = \{0,a,b,a+b\}$*, so* $T(a+b) \subseteq T(a)+T(b)$*. Also,* $T(a \circ b) = \bigcup_{x \in a \circ b} T(x) = T(-ab) \cup T(ab) = \{0,-ab\} \cup \{0,ab\} = \{-ab,ab\}$ *and* $a \circ T(b) = a \circ \{0,b\} = \bigcup_{x \in \{0,b\}} a \circ x = a \circ 0 \cup a \circ b = \{0\} \cup \{-ab,ab\}$*, hence* $T(a \circ b) = a \circ T(b)$*. Then we have* $-T(a) = \{-x : x \in T(a)\} = T(-a)$*. Clearly,* T *is strongly injective, but it is not weakly injective, as desired.*

Proposition 3.10. *([4]) Let V and W be strongly left distributive hyperspaces such that $|1 \circ x| = 1$ for all $x \in V$. If $T : V \longrightarrow P_*(W)$ is monic in $\mathcal{HV}_K^s$, then T is strongly injective.*

4 Hyperoperations on $Hom_K(V, W)$

Next we proceed to constructs some algebraic hyperstructures on Hom_K (V, W)(resp. $Hom_K^s(V, W)$), the set of all MLT (resp. $SMLT$) as well as on $hom_K(V, W)$ and $hom_K^s(V, W)$), the set of all linear transformation LT (resp. strong linear transformation SLT) respectively and study some basic properties of them.

We start by $hom_K(V, W)$. Defin the operations $\oplus$ and $\odot$ on $hom_K(V, W)$ as follows:

$$T \oplus S(x) = T(x) + S(x); \quad \text{and} \quad (a \circ T)(x) = a \circ T(x).$$

Clearly, in general $T \odot S$ and $a \circ T$ are not members of $hom_K(V, W)$, but they are members of $Hom_K(V, W)$. These shows that the study of multivalued linear transformations are more useful than the linear transformations in a hyperspace. As by Remark4.6 we can consider morphisms in $hom_K(V, W)$ as morphisms of $Hom_K(V, W)$, we will prefer to work by multivalued linear transformations as a general case. Also, for $T \in Hom_K(V, W)$, $-T$ is defined by $-T(x) = -T(x)$. Then the following holds:

Lemma 4.1. *For $S, T \in Hom_K(V, W)$. The following statements are satisfies:*
(i) $(Hom(V, W), \oplus)$ *is a monoid;*
(ii) $0 \in -T \oplus T$;
(iii) $(Hom_K(V, W), \oplus, \odot K)$*, is a quasi vector space(that is a monoid $(M, +)$ by a function $. : K \times V \longrightarrow V$ that satisfies the all axioms of a K-vector space).*

Proof. The proof is strightforward. □

Now we define a hyperoperation $\oplus$, and operation $\odot$ on $Hom_K(V, W)$ as follows:

$$(T \oplus S)(x) = \{U | \ U(x) \subseteq T(x) + S(x)\}.$$

$$(a \odot T)(x) = a \circ T(x).$$

Theorem 4.2. *The following statements are satisfies for every $S, T \in Hom_K$ (V, W) and every $a, b \in K$:*

(i) $(Hom_K(V,W), \oplus)$ *is a commutative hypergroup, with the zero map as identity element ;*

(ii) $(Hom_K(V,W), \oplus)$ *is a commutative hypergroup;*

(iii) $(Hom_K(V,W), \oplus, \odot, K)$ *is a general hypervector space(that is a commutative hypergroup with an scalar identity, together with function* $\odot$: $K \times V \longrightarrow V$ *that satisfies the all axioms of a vector space over field* K*).*

Proof. The proof is routin and omitted. □

5 Fundamental relation of hyperspaces

Let $(V, +, \circ, K)$ be a hypervector space over K. The smallest equivalence relation ε^* on V, such that the quotient V/ε^* is a vector space over K is called the *fundamental relation* of V. T. Vougiouklis in [20] introduced and studied the fundamental relation of H_v-vector space (a general class of hypervector spaces). In the following we characterize the fundamental relation on hypervector spaces (in the sense of Tallini) and study the relationship between V and V/ε^*(for more detailes see [2]). In the following we consider the category $\mathcal{HV}_K$, the category of hyperspaces(with basis) and multivalued linear transformations to construct the fundamental functor from $\mathcal{HV}_K^s$ into $\mathcal{V}_K$, the category of vector spaces over K.

Let $\mathbf{U}$ be the set of all finite linear combinations of elements of V with coefficient in K, that is

$$\mathbf{U} = \left\{ \sum_{i=1}^{n} a_i \circ x_i : a_i \in K \text{ and } x_i \in V \ , n \in \ \mathbb{N} \right\}.$$

Define the relation ε over V by

$$x \varepsilon y \Longleftrightarrow \exists \mathbf{u} \in \mathbf{U} : \{x, y\} \subseteq \mathbf{u}, \quad \forall x, y \in V.$$

Then ε^* is the *transitive closure* of ε. Define addition operation and scalar multiplication on V/ε^* by

$$\begin{cases} \oplus : V/\varepsilon^* \times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ \varepsilon^*(x) \oplus \varepsilon^*(y) = \{\varepsilon^*(t) : t \in \varepsilon^*(x) + \varepsilon^*(y)\}, \end{cases}$$

and
$$\begin{cases} \odot : K \times V/\varepsilon^* \longrightarrow V/\varepsilon^* \\ a \odot \varepsilon^*(x) = \{\varepsilon^*(z) : z \in a \circ \varepsilon^*(x)\}, \end{cases}$$

Lemma 5.1. *([2]) The following statement are satisfied:*

(i) $\varepsilon^*(a \circ x) = \varepsilon^*(y)$ for all $y \in a \circ x$, $\forall a \in K$, $\forall x \in V$, where $\varepsilon^*(a \circ x) = \bigcup\limits_{b \in a \circ x} \varepsilon^*(b)$.

(ii) $\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(x + y)$.

(iii) $\varepsilon^*(\underline{0})$ is the identity element of $(V/\varepsilon^*, \oplus)$.

(iv) $(V/\varepsilon^*, \oplus, \odot, K)$ is a vector space over K.

The vector space $(V/\varepsilon^*, \oplus, \odot, K)$ is called the fundamental vector space of V.

Theorem 5.2. *([2]) Let $(V, +, \circ, K)$ be a hypervector space and $(V/\varepsilon^*, \oplus, \odot, K)$ be the fundamental vector space of V. Then*
$$\dim V = \dim V/\varepsilon^*.$$

Lemma 5.3. *Let V and W be two hypervector spaces and $T : V \longrightarrow W$ be a SM. Then*

(i) $\forall x \in V$, $T(\varepsilon^*(x)) \subseteq \varepsilon^*(T(x))$;

(ii) The map
$$\begin{cases} T^* : V/\varepsilon^* \longrightarrow W/\varepsilon^* \\ T^*(\varepsilon^*(x)) = \varepsilon^*(T(x)) \end{cases}$$
is a linear transformation.

Proof. First note that since $T(x)$ is a nonempty subset of V for every $x \in V$. Then $\varepsilon^*(T(x)) = \bigcup_{y \in T(x)} \varepsilon^*(y) = \varepsilon^*(y), \forall y \in T(x)$. Now since T maps every linear combination of V to a linear combination of W. Then (i) follows. (ii) is strightforward. $\square$

Theorem 5.4. *The mapping $F : HV_K^s \longrightarrow \mathcal{V}_K$ is defined by $F(V) = V/\varepsilon^*$ is a functor. Moreover, the functor F preserves the dimension.*

Proof. The proof is similar to the proof of [2] by some manipulation. $\square$

Corollary 5.5. *Let $T : V \longrightarrow W$ be a morphism in $\mathcal{HV}_K^g$. Then the following diagram is commutative:*
$$\begin{array}{ccc} V & \xrightarrow{T} & P_*(W) \\ \varphi_V \downarrow & & \downarrow \varphi_W \\ V/\varepsilon^* & \xrightarrow{T^*} & W/\varepsilon^* \end{array}$$
where φ_V and φ_W are the canonical projections of V and W, respectively.

Proof. Let $x \in V$. Then

$$\begin{array}{rcl}\varphi_W(T(x)) & = & \varepsilon^*(T(x)) \\ & = & T^*(\varepsilon^*(x)) \\ & = & T^*(\varphi_V(x)) \\ & = & T^*\varphi_V(x).\end{array}$$

□

Acknowledgements
The first author is partially supported by Center of Excellence of Algebraic Hyperstructures and its Applications of Tarbiat Modares University (CEAHA), Tehran, Iran.

References

[1] R. Ameri, *On Categories of Hypergroups and Hypermodules*, Journal of Discrete Mathematical Science and and Cryptography, 6, 2-3 (2003) 121-132.

[2] R. Ameri, O. R. Dehghan, *On Dimension of Hypervector Spaces*, European Journal of Pure and Applied rete Mathematics, 1, 2: 32-50 (2008).

[3] R. Ameri, M. Norouzi, *On multiplication* (m, n)*-hypermodules*, European Journal of Combinatorics, 44 (2015) 153-171.

[4] R. Ameri, K. Ghadimi, *Categories of hyperspaces*, submitted.

[5] R. Ameri, M. Norouzi, (m, n)*-hypermodules*, European Journal of Combinatorics, 44 (2015)

[6] R. Ameri, M. Norouzi, *New Fundamental Relation of Hyperrings*, European Journal of Combinatorics, 34 (2013) 884–891.

[7] R. Ameri, M. Norouzi, *Prime and Primary Hyperideals in Krasner* , European Journal of Combinatorics, 34 (2013) 379–390.

[8] R. Ameri, I. G. Rosenberg, *Congruences of Multialgebras*, Multivalued Logic and Soft Computing, Vol. 15, No. 5-6 (2009) 525-536.

[9] R. Ameri, M. M. Zahedi, *Hyperalgebraic Systems*, Italian Journal of Pure and Applied Mathematics, No. 6 (1999) 21-32.

[10] R. Ameri, M. Norouzi, V. Leoreanu-Fotea, *On prime and primary subhypermodules of* (m, n)*-hypermodules*, European Journal of Combinatorics, 44 (2015) 175190.

[11] P. Corsini, *Prolegomena of Hypergroup Theory*, Second Edition, Aviani Editor, 1993.

[12] P. Corsini, V. Leoreanu-Fotea, *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Dordrecht, Hardbound, 2003.

[13] B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[14] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, 2013.

[15] F. Marty, *Sur une Generalization de la Nontion de Groupe*, 8*th* Congress des Mathematiciens Scandinaves, Stockholm, (1934) 45-49.

[16] M. Motameni, R. Ameri and R. Sadeghi, *Hypermatrix Based on Krasner Hypervector Spaces*, Ratio Mathematica 25 (2013) 77-94.

[17] M. S. Tallini, *Hypervector Spaces, 4th* AHA, World Scientific, Xanthi, Greece, (1990) 167-174.

[18] M. S. Tallini, *Weak Hypervector Spaces and Norms in such Spaces*, Proceedings of the Fifth International Congress on Algebraic Hyperstructures and Applications, Jasi, Rumania, Hadronic Press,(1994) 199-206.

[19] M. S. Tallini, *Dimensions in Multivalued Algebraic Structures*, Italian Journal of Pure and Applied Mathematics, 1 (1997) 51-64.

[20] T. Vougiouklis, *Hyperstructure and Their Representations*, Hadronic Press, Inc., 1994.