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Creation of the concept of zero-point method in teaching mathematics

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Abstract

Pupils learn different calculating algorithms. The effective use of learned algorithms requires creativity in their application to solving diverse tasks. To achieve this goal, it is necessary to create a concept of the calculating algorithm for pupils. The present paper describes a method of creating a zero-point method. The teaching of this method is divided into two stages. In the first stage, the student masters the basic algorithm and becomes familiar with the main ideas of this method, while in the second stage a student learns how to apply this method with some modifications in other types of tasks. In our article, we present the application of a zero-point method in solving quadratic inequalities.

Keywords: concept creation, method of the zero-point, algorithm, pupils' understanding.

2010 AMS subject classification: 97D40.[‡]

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1 Introduction

Recently the education at primary and secondary schools has undergone several reforms. One of the essential features of these reforms has been a reduction of the curriculum of individual subjects and reducing the number of lessons, especially science lessons. The main aim of reducing the curriculum and thus reducing the demands was monitoring the improvements in educational achievements of our students [1]. But PISA 2015 test results say otherwise. Slovak students achieved in 2015, on average, significantly worse results than the OECD average. It is worthy of reflection that our students achieve the best results-the results almost on an average of the best students in the OECD. Another feature of this educational reform is teaching a "playful" way. Pupils should acquire new knowledge and skills not by memorizing and practising, but above all by the playful way. PISA testing in 2015 showed that, in terms of pupils' attitudes to learning, our students declare significantly lower endurance to solve complex problems, lower openness to solve tasks and less belief in their own abilities. It can also negatively be reflected on their results in mathematics. Compared to 2003, many of Slovak students' attitudes to learning significantly deteriorated. 2015 PISA test results are in substantial agreement with the results of the external part of the school leaving examination (maturita). All Slovak students have to pass maturita from Slovak language and literature and a foreign language. Only those students have to pass maturita from mathematics, who choose math as a maturita subject. Nevertheless, over the past three years, the average percentage of school maturita exam in mathematics is always worse than the average percentage of school maturita exam in compulsory subjects. We think that the ideas of school reforms are correct, but it turns out that it is not right to use the same methods to achieve the goals for all subjects.

Mathematics affects almost every area of human life. In the education of our youth, who should be, according to the reference of John Paul II., our hope for the future. Math is challenging in its own way but at the same time can also be beautiful. We think it is necessary to seek such forms and methods of teaching mathematics [2], that we make the beauty of math available to students [3]. In the following lines, we will outline one possible way of teaching mathematics.

2 Two stages of mathematical education

Mathematical education can be divided into two stages. The first, basic stage is the acquisition of basic calculating algorithms. These calculating algorithms are acquired by students, who practice them on the appropriate number of tasks. We can talk about math "drill", without which it is impossible to be a successful solver of mathematical problems. The information-receptive didactic method with a combination of the reproductive method is mainly used in this stage. It is very important that the student acquires the necessary skill of how to use them by repeated use of basic calculating algorithms. The teacher, by the right choice of tasks, ensures that pupils acquire these calculating algorithms at least at the level of understanding, not only at the level of memorization. The second, application stage is the application of the acquired algorithms in different areas of mathematics and other disciplines or in practical everyday life. At this stage, the mathematical "drill" is replaced by mathematical thinking. Based on the assignment a student considers what math knowledge and skills he can use to solve the task. Unlike the first stage, he must learn that the first step of task solution is not to count but to think. Based on a detailed consideration and possible task mathematization the student chooses a suitable calculating algorithm. At this stage, the teacher becomes a moderator of solution and uses a heuristic didactic method. At this stage, in terms of the taxonomy of educational objectives, the level of acquirement of calculating algorithms will be increased for the minimum to the application level. If the teaching is correct, we can say, that at this stage, the students do not learn new calculating algorithms. At this stage, students gain new, mainly theoretical knowledge of mathematics, and also learn how to apply already gained calculating algorithms in a new context. The above-described stages are illustrated on the example of the method of zero points.

3 Method of zero points

Solving of the most mathematical problems includes solving of various equations and inequalities, or their systems. The tasks, where it is necessary to solve equations, inequalities and their systems belong to the declaratory mathematical tasks [4].

Declaratory mathematical tasks are historically the oldest mathematical tasks. When solving these tasks the mathematical concepts and methods. Those are the tasks that require finding, calculating, constructing etc. of all mathematical objects of a particular type, having the desired properties. In each declaratory task, we can define as the frame of considerations some non-empty set ***M*** of mathematical objects, which is a carrier of a particular structure. Using the terms belonging to this structure, it is then possible to express the desired properties of those objects of the set ***M*** that we are looking for. To characterize the elements of the set ***M*** we use propositional form ***V*** (x) which verity domains then create subsets of the set ***M.*** In each determinative task there is a subset ***K*** of set ***M,*** which elements have the characteristics required by task assignment. The task and the objective of the investigator are to determine the set ***P*** by naming of its elements or to operate with already known subsets of the set $\boldsymbol{M}$. We can solve the mathematical declaratory task with the direct and indirect methods.

The direct method of solving means a process by which we determine the set of solutions $\boldsymbol{K}$ so that we work exclusively with sets that belong to the chain of sets inclusions

$$\varnothing \subset \ldots \subset \boldsymbol{K} \subset \ldots \subset \boldsymbol{M},$$

where $\boldsymbol{M}$ is a non-empty set of mathematical objects, among which elements we are looking for the solving of the task. Indirect methods consist in the fact that instead of solving the task that is defined we solve the other task or other tasks (using some direct method) and the results are used to obtain the results of the original task. One of the indirect methods is to switch to subtasks on the same set. We divide the set $\boldsymbol{M}$ to individual subsets and we investigate the specific location of each original task. We will obtain partial solutions to the original task on each of these subsets. The overall result for the task will be obtained by the unification of partial results. Method of zero points can be included precisely into that category of indirect methods (in some literature this method is also called the method of intervals).

The essential feature of the method of zero points is the attempt to divide tasks into several "sub-tasks", solving them on the corresponding subsets - intervals. To deal with this method it is necessary to learn the algorithms of expression modifying, polynomial factorization to the product of the root factors and solving various types of equations [5].

4 Teaching the method of zero points

The teaching of this method is recommended to be realized in three levels.

Level 1: Acquisition of the method

The students meet the method of zero points for the first time when they solve inequalities with an unknown in the denominator. Its basic steps are learned through leading example.

Example 1: On the set **R** solve the inequality $\frac{2x+3}{x-1} < 1$

Solution: Most students have the following knowledge on solving the inequalities: Inequalities are solved using the same equivalent adjustment as the equations. If the inequality is multiplied or divided by a negative number, the sign of inequality is changed to the opposite. On the basis of this knowledge the first step of solving is an attempt to remove a fraction of the assigned inequality, that is, they multiply the inequality by the expression (x-1). Already in the introduction of the model example, students learn another difference between solving the equations and inequalities. Inequalities, unlike equations, cannot be multiplied by the expression of which I cannot clearly decide whether it is positive or negative. If we want students to use the proposed adjustment, it is first necessary to determine for which values of the variable the expression $(x-1)$ is positive and for which negative. Consequently, it is necessary to divide the solving of the inequalities to, in this case, two parts - when the expression $(x-1)$ is positive and the sign of inequality does not change after multiplication, and when the expression $(x-1)$ is negative and the sign of inequality changes to opposite one after multiplication. Basically, the assigned inequality should be tackled twice. We recommend concentrating on the issue of "multiplying inequalities" and pay sufficient attention, because it is needed to change students fixed "definition" of solving the inequalities. The method of zero points does not require multiple solving of the same inequalities and therefore it, is considered to be mora effective method. It can be divided into the following steps:

1. Annulling the right side of the equation:

$$\frac{2x+3}{x-1}-1<0$$

2. Simplifying the expression on the left side of the inequality:

$$\frac{x+4}{x-1}<0 \qquad (1)$$

After these adjustments, we draw the students' attention to the intermediate target of our solutions. We compare the fraction to zero. Therefore, we only need to determine the sign of the expression $\frac{x+4}{x-1}$. Our partial objective is to determine for what value of x it is positive and for what value negative.

3. Determining the zero points:

Zero points are the values of variable x for which numerator and denominator separately on the left side of the inequality takes the zero value. Zero points can be determined based on solving the equation $x+4=0$; $x-1=0$. Zero points are NB: -4; 1.

4. Adjusted numerical axis:

We come to the core of the method. First, we explain the function of zero points. Zero points divided real numbers, in this case, into the three sets - intervals. For each interval is true: The expression $\frac{x+4}{x-1}$ is positive or negative in the whole interval, in other words, it does not change the resulting sign. The adjacent intervals the expression $\frac{x+4}{x-1}$ has different resulting signs. Based on the above it is sufficient, if we want to determine the final sign, to substitute any number belonging to this interval to the expression. If we know the final sign in one of the intervals, we automatically recognize the resulting sign in all intervals as signs alternate. Using that knowledge, we can create a customized numerical axis (Fig. 1):

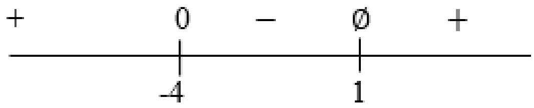

Figure 1.

There are numbers under the axis to be substituted for variable x in the expression; above the axis are values of the expression after substitution. That is, if we substitute any number from the interval (-4, 1) the resulting sign of the expression $\frac{x+4}{x-1}$ is negative, after the substituting $x = -4$, the resulting value of the expression is zero. Symbol $\emptyset$ means that for the value $x = 1$ the expression is not defined.

The adjusted numerical axis can be created as follows. First, on the numerical axis (from the bottom), we mark zero points. (Students often automatically show zero even if it is not the zero point on the numerical axis. There should be **only** zero points on the numerical axis).

We substitute any number different from zero points to the expression on the left side of the inequality. If the zero point is not zero, we substitute number **zero** to the variable. After substituting the number zero to the variable x, the expression $\frac{x+4}{x-1}$ has the value of - 4. Then we write a minus sign above the numerical axis in the part corresponding to the interval, from which we substituted the number zero. The signs in the other intervals will be completed without calculations, whilst complying with the principle of alternating signs. We complete 0 above the zero point “of the numerator“ and the sign $\emptyset$ above the zero point "of the denominator".

5 Determination of results

Those values of variable x for which the expression $\frac{x+4}{x-1}$ acquire negative values will be the solution to the inequality (1). Based on the adjusted numbering axis, the search solution to the assigned inequality is the interval from -4 to 1. Finally, we determine the "brackets" of the final interval. Zero point, above which is symbol $\emptyset$, cannot be the solution, therefore it will be at zero point "of the denominator" always round bracket. If there is the symbol 0 above the zero point, it means, that after its substituting, the resulting value of the expression is zero. However, we are looking for negative values of the expression and therefore the number -4 has a round bracket. The ultimate solution is $x\in$ (-4.1).

After solving the model example we recommend to discuss with students how the solution would change if we solve the inequality

$$\frac{2x+3}{x-1} > 1 \text{ and the inequality } \frac{2x+3}{x-1} \leq 1.$$

Students should be aware, that in both cases, the first four steps will be identical with the model example. In the fifth step, based on the same considerations, the solution of the inequality would be $\frac{2x+3}{x-1} > 1 \;\; x \in$

$(-\infty; -4) \cup (1; \infty)$. The solution of inethe quality $\frac{2x+3}{x-1} \leq 1$ is $x \in \langle -4; 1)$

Level 2: Understanding of the method

The main idea behind the method of zero points can be considered a comparison of the fraction with zero. A student knows that if a numerator and a denominator have the same final sign, so the fraction is positive if they have different sign fraction is negative. The correct application of this idea leads to an understanding of the method of zero points and also to a more efficient using of this method. The correct application of the main idea it is essential to understand the "functioning" of zero points. The zero point for this expression, in principle, divides the set of real numbers (NA) into three subsets. On one of the subsets, it acquires only positive values, on another one just negative. The third subset is only composed of zero point and the expression of the set acquires a value of 0. For example, the expression $x - 5$ has a zero point 5. Then, the expression acquires negative values on the set $M_1 = (-\infty; 5)$, on the set $M_2 = (5; \infty)$ it acquires positive values and on the set $M = \{5\}$ it takes the value 0. Thus, we can simplistically say, that there is a different sign of the expression from the various sides of the zero point. If the expression is in productive form, the zero points of individual members of the product create the zero points of all expression.

Example 2: On the set R solve the inequality $\frac{(x-9)(x+1)^2}{(x-4)(x+5)} > 0$

Solution: Zero points -5; -1; 4; 9.

At first, we draw attention to the expression $(x+1)^2$. This expression acquires for all $x \in$ R non-negative values. Therefore, it has no influence to the final sign of the expression $\frac{(x-9)(x+1)^2}{(x-4)(x+5)}$. The zero point of the expression $(x+1)^2$ can be described as "unnecessary" zero point and it will not be showed on the adjusted numbering axis. (If we showed it there, the theory of alternation marks would not apply.)

To obtain the solution of the inequality we only need to know the final sign of the expression $\frac{(x-9)(x+1)^2}{(x-4)(x+5)}$. Therefore. after substituting, for example $x = 0$, it is not necessary to know the numerical value. At the same time, we know that it is not necessary to substitute to the expression $(x+1)^2$. By applying the above mentioned ideas after substituting $x = 0$ we obtain "a signed" value of the expression: $\frac{-}{-.+}$.

We set the adjusted numbering axis (Fig. 2):

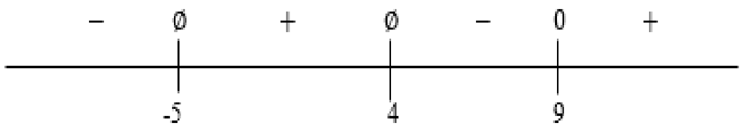

Figure 2.

The solution of the assigned inequality based on the adjusted numbering line and the sign of inequality is $x \in (-5; 4) \cup (9; \infty)$. To obtain the final solution, we must once again pay attention to "needless" zero point. We know that for $= -1$, the expression acquires the resulting value zero on the left side. Therefore, the number -1 does not belong to the solution of our set of ithe nequality. The ultimate solution of the inequality $x \in (-5; -1) \cup (-1; 4) \cup (9; \infty)$.

Level 3: Application of the method

After mastering the basic algorithm and understanding the method of zero points we recommend to focus on the teaching of its application in other types of examples, such as those in which students can penetrate into its mysteries. The closest type of tasks is inequalities in the productive form. The student already knows that there are the same rules for comparison zero to the product as for the comparison of the quotient to zero. Therefore, in solving qualities in productive form, the method of zero points can be used identically as in solving the inequalities in productive form. Quadratic inequality can be seen as inequality in the productive form. In example 3 we show a sample solution.

Example 3: On the set **R** solve the inequality $x^2 + 3x - 4 \geq 0$.

Solution: Quadratic trinomial on the left side of the inequality must be adjusted to the product of the root factors, and therefore we obtain the inequality in the form of productive form

$$(x - 1)(x + 4) \geq 0$$

Zero points are -4; 1. The quadratic trinomial, after substitution $x = 0$, acquires negative value. In fact, zero is not necessary to be substituted, because for $x = 0$ is the final "a signed" value of quadratic trinomial, identical to the sign in front of the absolute member. We set the adjusted numbering axis (Fig. 3):

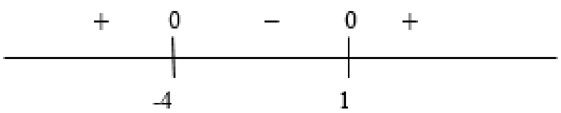

Figure 3.

Based on the sign of inequality, in assigned inequality, we search for which values of unknown x the expression $x^2 + 3x - 4$ acquires positive or zero values. Therefore, the solution is inequality is

$$x \in (-\infty; -4\rangle \cup \langle 1; \infty).$$

If the quadratic equation corresponding to the assigned inequality has less than two real roots, the method of zero points is modified. At this modification, we primarily rely on understanding the "functioning" of zero points.

Example 4: On the set $\boldsymbol{R}$ solve the inequality $x^2 - 4x + 4 \geq 0$.
Solution: The inequality should be adjusted to productive form

$$(x-2)(x-2) \geq 0.$$

The left side of inequality will not be left in this form, because the students would incorrectly use the principle of alternation marks around the zero points. The expression on the left side of the inequality will be written in simplified form, and we receive the inequality

$$(x-2)^2 \geq 0$$

Zero point is 2. Since the expression $(x-2)^2$ is for all $x \in \boldsymbol{R}$ non-negative, number 2 is “unnecessary" zero point. Number 2 is the only zero point and so it is not needed to set the adjusted numbering axis. The solution of the inequality is $x \in \boldsymbol{R}$ and it was discovered when we were considering the zero point.

After solving example 4 we suggest a discussion on solving inequalities:

$$x^2 - 4x + 4 > 0, \qquad x^2 - 4x + 4 \leq 0\ , \qquad x^2 - 4x + 4 < 0.$$

Note: A common mistake at solving the inequality $(x-2)^2 \geq 0$ is the extract of the root of both sides of the inequality, after which students have the wrong inequality $x - 2 \geq 0$. The following consideration can bring them to the fact, that the inequality is incorrect. Both sides of the inequality were non-negative before extracting the root and the left side can also takes negative values. If we want, even after extracting, both sides being non-negative, we must put the left side of inequality to an absolute value($\sqrt{a^2} = |a|$). After correct extracting, we get the inequality with absolute value which can also be solved by the method of zero points.

Example 5: On the set $\boldsymbol{R}$ solve the inequality $x^2 + 2x + 6 < 0$.
Solution: On the set $\boldsymbol{R}$ it is not possible to modify the quadratic trinomial to the product, as the appropriate quadratic equation

$$x^2 + 2x + 6 = 0$$

have no real roots. Based on the understanding of the function of zero points we know, that expression $x^2 + 2x + 6$ has for all x ∈ R a signed value. It is identical with the sign in front of the absolute term. So the expression on the left side of the inequality is for all real numbers positive. The solution of the inequality is $x = \{\}$. Even after solving this inequality we recommend the discussion about solutions for different variants of the sign of inequality.

If we want to see if the students understand the method, they must be able to apply the basic ideas of the method to solving the task. In other words, we understand the method of solving if it developed our mathematical thinking. The following example can be solved by applying the basic ideas of the method of zero points.

Example 6: For which parameter values $a \in R$ is each $x \in$ R the solution of inequality

Solution: The expression $x^2 - 8x + 20$ has no zero points and according to the sign in front of the absolute member we know, that it acquires positive values for all $x \in R$. If all real numbers should be the solution of the assigned inequality, the expression in the denominator of the inequality fraction must be negative for all $x \in R$. Using the basic ideas of the method of zero points, we consider the following. We need the expression $ax^2 + 2(a+1)x + 9a + 4$ "still" negative, and that does not change the final sign, and therefore we cannot have the zero points. That is, the quadratic equation

$$ax^2 + 2(a+1)x + 9a + 4 = 0$$

has no solution. Thus, discriminant has to be negative. This way we get the inequality

$$\frac{x^2 - 8x + 20}{ax^2 + 2(a+1)x + 9a + 4} < 0$$

The solution to this inequality that we solve using the method of zero points is $a \in \left(-\infty; -\frac{1}{2}\right) \cup (1; \infty)$. Now, we secure the final sign will be negative. We know from the method of zero points, that by substituting zero to quadratic trinomial, the final sign is identical with a sign in front of the absolute term. The denominator in the assigned inequality is a quadratic trinomial with parameter. For $a_1 \in \left(-\infty; -\frac{1}{2}\right) \cup (1; \infty)$ has the constant sign for all x ∈ R. If the absolute member is negative, the resulting sign of trinomial will be negative. Therefore we solve the inequality

$$9a + 4 < 0.$$

Its solution is $a_2 \in \left(-\infty; -\frac{4}{9}\right)$. Based on the previous considerations, the parameter a must meet both conditions. The ultimate solution is $a \in a_1 \cap a_2 = \left(-\infty; -\frac{1}{2}\right)$.

Conclusion

The basis for the success of a student in solving mathematical tasks is acquiring the calculating algorithms [6], [7]. To achieve this goal it is necessary to solve, especially alone, the sufficient number of tasks, more or less, of the same type. We believe that the mastery of basic calculating algorithms is necessary but not sufficient condition for student success in dealing with the tasks. It is not enough just to learn the calculating algorithm, it is necessary, after its acquisition, also think about its individual elements. This is the way when the basic ideas, used in the algorithm, occur. The discovering these main ideas of calculating algorithm lead to understanding, as well as acquiring the algorithm at a higher level. The understanding causes the method to be is a powerful tool in students dealing with tasks. It affects his mathematical thinking. The method of zero points is a method that should be understood and not only learned. If a student enters its secrets, it becomes flexible and he will be able to use it in different types of tasks and, as appropriate, be adapted. By understanding the method will become effective tool in the hands of the investigator. The students know that the method of zero points is mainly used to solve inequalities. If the students know the method, it heads their initial ideas, when solving inequality, to adjust the inequality to a productive or quotient form. This fact can be used in teaching solutions to quadratic inequalities. Using the method of zero points the student does not learn new calculating algorithm, but he learns how to apply already acquired knowledge and skills. We think that one of the possible ways to increase the efficiency in mathematical learning is the emphasis on understanding the calculating algorithms and their subsequent application in various areas of mathematics. While we make sure that we choose those tasks, where the main ideas can be applied. This way helps us to create the thought linking of mathematics as a whole and mathematics with other disciplines, e.g. those involving computers into the pedagogical process [8], in the mind of the students. Basically, there is no need to reduce the amount of subject matter, just to organize the mathematical knowledge better in the mind of the students.

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