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Some properties of residual mapping and convexity in $\wedge$-hyperlattices

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Abstract

The aime of this paper is the study of residual mappings and convexity in hyperlattices. To get this point, we study principal down set in hyperlattices and we give some conditions for a mapping between two hyperlattices to be equivalent with a residual maping. Also, we investigate convex subsets in $\wedge$-hyperlattices.

Key words: residuated map, convex, down-set, hyperideal, hyperfilter.

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1 Introduction

Hyperalgebras (multialgebra) are generalization of classical algebras that are introduced by F. Marty in the eighth congress of Scandinavian in 1934 [11].
In [4], Ameri and M. M. Zahedi introduced and studied notion of hyperalgebraic systems. In [2], Ameri and Nozari Studied relationship between the

categories of multialgebra and algebra. C. Pelea and I. Purdea have been proved that complete hyperalgebra can be obtained from a universal algebra and a appropriate congruence on it. Also, Pelea and others studied multialgebra, direct limit, and identities, for more details see [16, 17, 18, 19].

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Theory of hyperlattices introduced by Konstantinidou and J. Mittas in 1977[9]. In [10], G. A. Moghani and A. R. Ashrafi proved that in some cases the set of all subhypergroups G has a hyperlattice structure . In [24], X. L. Xin and X. G. Li studied hyperlattices and quotient hyperlattices. In [5], A. Asokkumar in 2007 proved that under certain conditions, the idempotent elements of a hyperring form a hyperlattice and the orthogonal idempotent elements form a quassi-distributive hyperboolean algebra. In [1], R. Ameri, M. Amiri Bideshki, and A. Borumand Said studied prime hyperfilters (hyperideals) in hyperlattices. Also, they gave some examples of $\wedge$-hyperlattices and dual distributive $\wedge$-hyperlattices.
In section 3, down set and residual maps in hyperlattices are studied and some properties of them are given. In section 4, convex subsets of a hyperlattice and some properties of them are given.

2 Preliminary

In this section we give some results of hyperlattices that we need to develop our paper.

Definition 2.1. [1] Let L be a nonempty set. L is called a $\wedge-$ hyperlattice if

(i) $a \in a \wedge a, a \vee a = a,$

(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a,$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c,$

(iv) $a \in (a \wedge (a \vee b)) \cap (a \vee (a \wedge b))$,

(v) $a \in a \wedge b \Longrightarrow a \vee b = b$,

for all $a, b, c \in L$.
Let $A, B \subseteq L$. Then:

$$A \wedge B = \cup\{a \wedge b | a \in A, b \in B\};$$
$$A \vee B = \{a \vee b | a \in A, b \in B\}.$$

Example 2.2. Let $(L, \vee, \wedge)$ be a lattice and define $a \oplus b = \{x \mid x \leq a \wedge b\}$. Then $(L, \vee, \oplus)$ is a $\wedge-$ hyperlattice.

Definition 2.3. [1] Let L be a $\wedge-$hyperlattice. We say that L is bounded If there exist $0, 1 \in L$, such that $0 \leq x \leq 1$, for all $x \in L$. We say that 0 is the least element of L and 1 is the greatest element of L.

Example 2.4. Let $L = \{0, a, 1\}$, and define $\wedge$-hyper operation and $\vee$-operation on L with tables 3. Then $(L, \wedge, \vee)$ is a bounded $\wedge$-hyperlattice.

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{a, 0\}$	$\{a, 0\}$
1	$\{0\}$	$\{a, 0\}$	L

(a)

$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

(b)

Table 1

Definition 2.5. [1] Let I and F are nonempty subsets of L. Then:

(i) I is called hyperideal if the following conditions hold.

  (a) If $x, y \in I$, then $x \vee y \in I$,

  (b) If $x \in I$ and $a \in L$, such that $a \leq x$, then $a \in I$.

(ii) F is called hyperfilter if the following conditions hold.

  (a) If $x, y \in F$, then $x \wedge y \subseteq F$,

  (b) If $x \in F$ and $a \in L$, such that $x \leq a$, then $a \in F$.

(iii) A hyperideal I is called prime if $x \wedge y \in I$, then $x \in I$ or $y \in I$, for all $x, y \in L$.

(iv) A hyperfilter F is called prime if $x \in F$ or $y \in F$, where $(x \wedge y) \cap F \neq \emptyset$, for all $x, y \in L$.

3 Resedual Mappings in $\wedge$-Hyperlattices

In this section, we are going to introduce down-set and resedual mapping in $\wedge$-hyperlattice. Let L be a $\wedge$-hyperlattice.

Definition 3.1. Let $\emptyset \neq A \subseteq L$. A is called a down-set, if $x \in A$ and $y \leq x$, then $y \in A$.

Example 3.2. every hyperideal of L is a down-set that is called principal down-set.

Example 3.3. Let $L = \{0, a, b, 1\}$. $\wedge$ and $\vee$ are given by Table 2 and 3.

$\wedge$	0	a	b	1
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0\}$	$\{0, a\}$
b	$\{0\}$	$\{0\}$	$\{0, b\}$	$\{0, b\}$
1	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{1\}$

Table 2:

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 3:

$I = \{0, a, b\}$ is a down-set, but it is not a hyperideal. We have $a, b \in I$ and $a \vee b = 1 \notin I$.

Let $x \in L$ and $x^{\downarrow} = \{y \in L | y \in x \wedge y\}$.

Proposition 3.4. *$\forall x \in L, x^{\downarrow}$ is a down set.*

$x^{\downarrow}$ is called a principal down-set.

Proposition 3.5. *Let L be a dual distributive $\wedge$-hyperlattice. Then every principal down-set is a hyperideal.*

Proof. □

If $A \subseteq L$ and $a \vee b \subseteq A$, for all $a, b \in L$, then A is called join-closed.

Corollary 3.6. *Let $I \subseteq L$. Then I is an ideal if and only if I is a down-set and it is a join-closed set.*

Proposition 3.7. *Let L and K be hyperlattice. If $f : L \longrightarrow K$ is a isotone map and $A \subseteq L$ is a down-set, then $f(A)$ is a down-set.*

Proof. Since A is a down-set, there exists $x \in L$ such that $A = x^{\downarrow}$. It is sufficient set $f(A) = f(x)^{\downarrow}$. □

Let L and K be hyperlattices and $f : L \longrightarrow K$ is a mapping. We define two map $f^{\rightarrow}$ and $f^{\leftarrow}$ that $f^{\rightarrow}$ is called direct image map and $f^{\leftarrow}$ is called inverse image map. $f^{\rightarrow} : P(L) \longrightarrow P(K)$ is defined by $f^{\rightarrow}(X) = \{f(x)|x \in L\}$, for all $X \subseteq L$, and $f^{\leftarrow} : P(K) \longrightarrow P(L)$ is defined by $f^{\leftarrow}(Y) = \{x\{\in L|f(x) \in Y\}$ for all $Y \subseteq K$.

Definition 3.8. A mapping $F : L \longrightarrow K$ is called residuated if the inverse image under F of every principal down-set of K is a principal down-set of L.

Example 3.9. Let L be a ∧-hyperlattice and $A \subseteq L$. We define $f^A : P(L) \longrightarrow P(L)$ by $f_A(B) = A \cap B$, for all $B \in P(L)$. Then f^A is a residuated and residual g is given by $g^A(C) = C \cup A'$, where that $A' = L \setminus A$.

Example 3.10. Let L be a ∧-hyperlattice. Mapping $f : P(L) \longrightarrow P(L)$ that is defined by $f(A) = A$, for all $A \in P(L)$, is a residuated mapping.

Theorem 3.11. *Let L and K be two hyperlattices. A mapping $f : L \longrightarrow K$ is a residuated iff f is a is isotone and there exists an isotone mapping $g : K \longrightarrow L$ such that $gof \geq id_L$ and $fog \leq id_K$.*

Proof. For all $x \in L$, $x \in f^{\leftarrow}[f(x)^{\downarrow}]$. If $y \leq x$, then $y \in f^{\leftarrow}[f(x)^{\downarrow}]$. We have: $f(x)^{\downarrow} = \{y|y \leq f(x)\}$ and $f^{\leftarrow}[f(x)^{\downarrow}] = \{t \in L|f(t) \in f(x)^{\downarrow}\}$.
$y \in f^{\leftarrow}[f(x)^{\downarrow}]$, so $f(y) \leq f(x)$. Then f is isotone. By assumption we have $(\forall y \in K)(\exists x \in L)$ such that $f^{\leftarrow}(y^{\downarrow}) = x^{\downarrow}$. Now, for every given $y \in K$, this element x is clearly unique. So we can define a mapping $g : K \longrightarrow L$ by $g(y) = x$. Since $f^{\leftarrow}$ is isotone, it follow that so is g. For this mapping g, we have:

$$g(y) \in g(y)^{\downarrow} = x^{\downarrow} = f^{\leftarrow}(y^{\downarrow}).$$

So, $f[g(y)] \leq y$, for all $y \in K$ and therefore $fog \leq id_K$. Also, $x \in f^{\leftarrow}[f(x)^{\downarrow}] = g[f(x)]^{\downarrow}$, so that $x \leq g[f(x)]$, for all $x \in L$, and therefore $gof \geq id_L$.
Conversely, Since g is isotone, we have:

$$f(x) \leq y \Longrightarrow x \leq g[f(x)].$$

Also, we have:

$$x \leq g(y) \Longrightarrow f(x) \leq f[g(x)] \leq y.$$

It follows from these observations that $f(x) \leq y$ iff $x \leq g(y)$ and therefore $f^{\leftarrow}(y^{\downarrow}) = g(y)^{\downarrow}$. □

Proposition 3.12. *The residual of f is unique.*

Proof. Suppose that g and g' are residual of f. Then we have: $g = id_L og \leq (g'of)og = g'o(fog) \leq g'oid_K = g'$. Similarly, $g' \leq g$, then $g = g'$. □

We shall denote residual of f, by f^+.

Proposition 3.13. *Mapping $f : L \Longrightarrow K$ is residuated iff for every $y \in K$, there exists $g(y) = max f^{\leftarrow}(y^{\downarrow}) = max\{x \in L | f(x) \leq y\}$. Moreover, $f^+ of \geq id_L$ and $fof^+ \leq id_K$.*

Definition 3.14. Let $f : L \longrightarrow K$ be a residuated mapping. Then f is called range closed if $Im(f)$ is a down-set of K.

Example 3.15. Let L be a $\wedge$-hyperlattice with a top element 1. Given $a \in L$, consider the mapping $f_a : L \longrightarrow L$ given by:
$f_a(x) = f_a$ is residuated. Clearly, $Im(f_a)$ is the down-set $a^{\downarrow}$ of L then f_a is a range closed.

Remark 3.16. In Example 3.15, L must have top element 1.

Example 3.17. Let N be the set of natural numbers. We define $\wedge$-hyperoperation and $\vee$ operation by:

$$a \wedge b = \{m \in N | m \leq min\{a, b\}\};$$

$$a \vee b = max\{a, b\}, forall a, b \in N.$$

Then $(L, \wedge, \vee)$ is a $\wedge$-hyperlattice. Consider $f : N \longrightarrow N$ by $f(x) = x$, for all $x \in N$. f is a residated mapping, but it is not range closed.

Theorem 3.18. *Let $f : L \longrightarrow K$ be a residuated mapping. Then $f = f^+$ iff $f^2 = id_L$.*

Proof. $\Longrightarrow$ It is obvious. $\Longleftarrow$ Since f is residuated, then $f^2 = id_L$. By $f^2 = id_L$, we have $fof \leq id_L$ and $fof \geq id_L$. So $f = f^+$. □

Theorem 3.19. *Let L and K be two $\wedge$-hyperlattices and Let L has a top element 1. If $f : L \longrightarrow K$ be a residuated mapping, then the following statements are equivalent.*

(i) *f is range closed.*

(ii) *$forally \in K$ $inf\{y, f(1)\}$ there exists and it equal to $ff^+(y)$.*

Proof. $(i \to ii)$:We have $f^+(y) \leq 1$, for all $y \in L$ and by isotonic f, $ff^+(y) \leq f(1)$. Also $ff^+(y) \leq y$, for all $y \in K$. So $ff^+(y)$ is a lower bound of $f(1)$ and y. We must show that $ff^+(y)$ is the greatest lower bound of $f(1)$ and y. Suppose that $x \in K$ is such that $x \leq y$ and $x \leq f(1)$. By (i), we have $x = f(z)$, for some $z \in L$ and $f(z) \leq y$; Since f^+ is isotone, $f^+f(x) \leq f^+(y)$. We have $z \leq f^+f(x)$, so $z \leq f^+(y)$. By isotonic f, $f(z) \leq ff^+(y)$, Then $x \leq ff^+(y)$. Thus $inf\{y, f(1)\} = ff^+(y)$.
$(ii \to i)$: We claim that $Im(f) = f(1)^{\downarrow}$. We have $x \leq 1$, for all $x \in L$, then $f(x) \leq f(1)$, for all $x \in L$. So $Im(f) \subseteq f(1)^{\downarrow}$. Let $y \in K$ be such that $y \leq f(1)$. Then by (ii), $ff^+(y) = inf\{y, f(1)\} = y$. We Know $ff^+(y) \in Im(f)$, so $y \in Im(f)$. Thus $f(1)^{\downarrow} \subseteq Im(f)$. Therefore $Im(f) = f(1)^{\downarrow}$. □

Proposition 3.20. *Let $f : L \longrightarrow K$ and $g : K \longrightarrow\longrightarrow M$ be residual map. Then gof so is, also $(gof)^+ = f^+og^+$.*

4 Convexity In $\wedge$-hyperlattice

In this section, we are going to introduce convex subsets in $\wedge$-hyperlattices and we are going to give some properties of convex subsets.

Proposition 4.1. *Let $F \subseteq L$. Then F is a hyperfilter of L, if and only if*

(i) *$a, b \in F$ implies that $a \wedge b \in F$.*

(ii) *$\forall a \in F$ and $\forall x \in L$, $a \vee x \in F$.*

Proof. Since F is a filter, $\forall a, b \in F$, $a \wedge b \in F$. We know $a \leq a \vee x$, then $a \vee x \in F$. So (i) and (ii) hold.
Conversely, Let $a \in F$ and $a \leq x$. So, $a \vee x = x$, by (ii) $a \vee x \in F$, then $x \in F$. □

Proposition 4.2. *Every hyperfilter of a $\wedge$-hyperlattice L is a $\wedge$-subhyperlattice.*

Remark 4.3. Converse of the above proposition does not hold. Consider hyperlattice in the Example 3.2. $A = \{0, a\}$ is a subhyperlattice. We have $a \leq 1$ and $1 \notin A$, then A is not a filter.

Remark 4.4. Every hyperideal of L is not a subhyperlattice. Also, every subhyperlattice is not an ideal.

Definition 4.5. Let $\emptyset \neq K \subseteq L$. We say K to be convex subset, if $a, b \in K$ and $c \in L$ such that $a \leq c \leq b$, then $c \in K$.

Example 4.6. Consider hyperlattice L in Example 3.2. Then $A = \{0, a\}$ is a convex subset, but $B = \{0, 1\}$ is not a convex subset. we have $0 \leq a \leq 1$ and $a \notin B$.

Proposition 4.7. *Every hyperideal (hyperfilter) of L is a convex subset of L.*

Remark 4.8. Every convex subset of L is not a hyperideal (a filter). Consider hyperlattice L in Example 3.2. Then $K = \{a, b, 1\}$ is a convex subset, but it is not a hyperideal ($0 \notin K$). Also, K is not a hyperfilter ($a \wedge b = \{0\}$ and $0 \notin K$).

Theorem 4.9. *Let L has a bottom element 0 and let K be a convex subset of L. If K is a chain and $0 \in K$, then K is a hyperideal of L.*

Remark 4.10. In Example 4.9, K must be a chain; also K must contain bottom element 0.

Example 4.11. Let L be hyperlattice in Example3.2

(i) $K_1 = \{a, b, 0\}$ is a convex subset, but it is a not chain(a, b are not comparable). Since $a \vee b \notin K_1$, K_1 is not a hyperideal.

(ii) $K_2 = \{a, b, 1\}$ is a convex subset, but it is not a hyperideal ($0 \notin K_2$).

Example 4.12. Consider hyperlattice L in Example 3.17. Then$K = \{2, 3, 4, ..., 10\}$ is a convex subset; Since K does not has bottom element 1, it is not a hyperideal.

Proposition 4.13. *Every principal down-set of L is a convex subset.*

Theorem 4.14. *Let I be a hyperideal and F be a hyperfiler of L, such that $I \cap F \neq \emptyset$, then $I \cap F$ is a convex sub-hyperlattice if and only if for all $a, b \in I \cap F$, $a \wedge b \subseteq I$.*

Proposition 4.15. *If K_i, $\forall i \in I$ is a convex sub-hyperlattice of L, then $\cap_{i \in I} K_i$ is so.*

Theorem 4.16. *Let K_1 and K_2 be convex sub-hyperlattices of L and let $0 \in K_1 \cap K_2$. Then $K_1 \cup K_2$ is a convex sub-hyperlattice if and only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$.*

Proof. Let $K_1 \cup K_2$ be a convex subhyperlattice, but $K_1 \nsubseteq K_2$ or $K_2 \nsubseteq K_1$. So, there exist $a, b \in L$, such that $a \in K_1 \setminus K_2$ and $K_1 \setminus K_2$. Since $a, b \in K_1 \cup K_2$ and $K_1 \cup K_2$ is a sub-hyperlattice, $a \vee b \in K_1 \cup K_2$; it implies that $a \vee b \in K_1$ or $a \vee b \in K_2$. If $a \vee b \in K_1$, $0 \leq a \leq a \vee b$, then $a \in K_2$, which is a contradiction; if $a \vee b \in K_2$, then we conclude that $b \in K_1$, which is a contradiction.
The converse is obvious. □

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