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Rough Set Theory Applied To Hyper BCK-Algebra

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Abstract

The aim of this paper is to introduce the notions of lower and upper approximation of a subset of a hyper BCK-algebra with respect to a hyper BCK-ideal. We give the notion of rough hyper subalgebra and rough hyper BCK-ideal, too, and we investigate their properties.

Key words: rough set, rough (weak, strong) hyper BCK-ideal, rough hyper subalgebra, regular congruence relation.

MSC 2010: 20N20, 20N25.

1 Introduction

In 1966, Y. Imai and K. Iseki [2] introduced a new notion, called a BCK-algebra. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty [6] at the 8th Congress of Scandinavian Mathematicians. In [3], Y. B. Jun, M. M. Zahedi, X. L. Xin, R. A. Borzooei applied the hyper structures to BCK-algebras and they introduced the notion of hyper BCK-algebra (resp. hyper K-algebra) which is a generalization of BCK-algebra (resp. hyper BCK-algebra). They also introduced the notion of hyper BCK-ideal, weak hyper BCK-ideal, hyper K-ideal and weak hyper K-ideal and gave relations among them. In 1982, Pawlak introduced the concept of rough set (see [7]). Recently Jun [5] applied rough set theory to BCK-algebras. In this paper, we apply the rough set theory to hyper BCK-algebras.

2 Preliminaries

Let U be a universal set. For an equivalence relation Θ on U, the set of elements of U that are related to $x \in U$, is called the *equivalence class* of x and is denoted by $[x]_\Theta$. Moreover, let U/Θ denote the family of all equivalence classes induced on U by Θ. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set $U \backslash X$. A pair (U, Θ) where $U \neq \phi$ and Θ is an equivalence relation on U is called an *approximation space.*
The interpretation in rough set theory is that our knowledge of the objects in U extends only up to membership in the class of Θ and our knowledge about a subset X of U is limited to the class of Θ and their unions. This leads to the following definition.

Definition 2.1. [7] For an approximation space (U, Θ), by a rough approximation in (U, Θ) we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined for every $X \in P(U)$ by $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where

$$\underline{Apr}(X) = \{x \in U | [x]_\Theta \subseteq X\},$$
$$\overline{Apr}(X) = \{x \in U | [x]_\Theta \cap X \neq \phi\}.$$

$\underline{Apr}(X)$ is called a *lower rough approximation* of X in (U, Θ), whereas $\overline{Apr}(X)$ is called an *upper rough approximation* of X in (U, Θ).

Definition 2.2. [7] Given an approximation space (U, Θ), a pair $(A, B) \in P(U) \times P(U)$ is called a *rough set* in (U, Θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.

Definition 2.3. ([7]) Let (U, Θ) be an approximation space and X be a non-empty subset of U.

(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called *definable.*

(ii) If $\underline{Apr}(X) = \phi$, then X is called *empty interior.*

(iii) If $\overline{Apr}(X) = U$, then X is called *empty exterior*.

Let H be a non-empty set endowed with a hyper operation "$\circ$", that is $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) - \{\phi\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 2.4. ([3]) By a *hyper BCK-algebra* we mean a non- empty set H endowed with a hyper operation "$\circ$"and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$"the *hyper order* in H.

Theorem 2.5. ([3]) In any hyper BCK-algebra H, the following hold:

(a1) $0 \circ 0 = \{0\}$,

(a2) $0 \ll x$,

(a3) $x \ll x$,

(a4) $A \ll A$,

(a5) $A \ll 0$ implies $A = \{0\}$,

(a6) $A \subseteq B$ implies $A \ll B$,

(a7) $0 \circ x = \{0\}$,

(a8) $x \circ y \ll x$,

(a9) $x \circ 0 = \{x\}$,

(a10) $y \ll z$ implies $x \circ z \ll x \circ y$,

(a11) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,

(a12) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsets A and B of H.

Definition 2.6. ([3]) Let H be a hyper BCK-algebra and let S be a subset of H containing 0. If S be a hyper BCK-algebra with respect to the hyper operation "$\circ$"on H, we say that S is a *hyper subalgebra* of H.

Theorem 2.7. ([3]) Let S be a non-empty subset of hyper BCK-algebra H. Then S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$, for all $x, y \in S$.

Definition 2.8. ([3]) Let I be a non-empty subset of hyper BCK-algebra H and $0 \in I$.

(i) I is said to be a *hyper BCK-ideal* of H if $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

(ii) I is said to be a *weak hyper BCK-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

(iii) I is called a *strong hyper BCK-ideal* of H if $(x \circ y) \cap I \neq \phi$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Theorem 2.9. ([3]) If H be a hyper BCK-algebra, then

(i) every hyper BCK-ideal of H is a weak hyper BCK-ideal of H.

(ii) every strong hyper BCK-ideal of H is a hyper BCK-ideal of H.

Definition 2.10. ([4]) Let H be a hyper BCK-algebra. A hyper BCK-ideal I of H is called *reflexive* if $x \circ x \subseteq I$ for all $x \in H$.

Definition 2.11. ([1]) Let Θ be an equivalence relation on hyper BCK-algebra H and $A, B \subseteq H$. Then,

(i) $A\Theta B$ means that, there exist $a \in A$ and $b \in B$ such that $a\Theta b$,

(ii) $A\bar{\Theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\Theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\Theta b$,

(iii) Θ is called a *congruence relation* on H, if $x\Theta y$ and $x'\Theta y'$ imply $x \circ x'\bar{\Theta}y \circ y'$ for all $x, y, x', y' \in H$.

(iv) Θ is called a *regular relation* on H, if $x \circ y\Theta\{0\}$ and $y \circ x\Theta\{0\}$ imply $x\Theta y$ for all $x, y \in H$.

Example 2.12. Let $H_1 = \{0, 1, 2\}$, $H_2 = \{0, a, b\}$ and hyper operations "$\circ_1$"and "$\circ_2$"on H_1 and H_2 are defined respectively, as follow:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

$\circ_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, b\}$

Then $(H_1, \circ_1)$ and $(H_2, \circ_2)$ are hyper BCK-algebras. Define the equivalence relation Θ_1 and Θ_2 on H_1 and H_2, respectively, as

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 2), (2, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (0, a), (a, 0)\}.$$

It is easily checked that Θ_1 is a congruence relation on H_1. But Θ_2 is not a congruence relation on H_2, since $b\Theta_2 b$ and $0\Theta_2 a$ but $b \circ 0\bar{\Theta}_2 b \circ a$ is not true.

Example 2.13. Let $(H_1, \circ_1)$ be a hyper BCK-algebra as Example 2.12. Let $H_2 = \{0, a, b, c\}$ and define the hyper operation "$\circ_2$"on H_2 as follow:

$\circ_2$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, c\}$

Then $(H_2, \circ_2)$ is a hyper BCK-algebra. Define the congruence relation Θ_1 and Θ_2 on H_1 and H_2, respectively, by

$$\Theta_1 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\},$$

and

$$\Theta_2 = \{(0, 0), (a, a), (b, b), (c, c), (0, b), (b, 0)\}.$$

It is easily checked that Θ_1 is a regular congruence relation on H_1, but Θ_2 is not a regular relation on H_2, since $a \circ b\Theta_2\{0\}$ and $b \circ a\Theta_2\{0\}$ but $(a, b) \notin \Theta_2$.

Theorem 2.14. ([1]) Let Θ be a regular congruence relation on hyper BCK-algebra H. Then $[0]_\Theta$ is a hyper BCK-ideal of H.

Theorem 2.15. ([1]) Let Θ be a regular congruence relation on $H, I = [0]_\Theta$ and $\frac{H}{I} = \{I_x : x \in H\}$, where $I_x = [x]_\Theta$ for all $x \in H$. Then $\frac{H}{I}$ with hyper operation "$\circ$"and hyper order "$<$"which is defined as follow, is a hyper BCK-algebra which is called *quotient hyper BCK-algebra*,

$$I_x \circ I_y = \{I_z : z \in x \circ y\},$$

and

$$I_x < I_y \Longleftrightarrow I \in I_x \circ I_y.$$

Theorem 2.16. ([1]) Let I be a reflexive hyper BCK-ideal of H and relation Θ on H be defined as follow:

$$x\Theta y \Longleftrightarrow x \circ y \subseteq I \text{ and } y \circ x \subseteq I$$

for all $x, y \in H$. Then Θ is a regular congruence relation on H and $I = [0]_\Theta$.

3 Rough hyper BCK-ideals

Throughout this section H is a hyper BCK-algebra. In this section first we define lower and upper approximation of the subset A of H with respect to hyper BCK-ideal of H and prove some properties. Then we give the definition of (weak, strong) rough hyper BCK-ideals and investigate the relation between them and (weak, strong) hyper BCK-ideals of H.

Definition 3.1. Let Θ be a regular congruence relation on hyper BCK-algebra H, $I = [0]_\Theta$, $I_x = [x]_\Theta$ and A be a non-empty subset of H. Then the sets

$$\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\},$$
$$\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\}.$$

are called *lower and upper approximation* of the set A with respect to the hyper BCK-ideal I, respectively.

Proposition 3.2. For every approximation space (H, Θ) and every subsets $A, B \subseteq H$, we have:

(1) $\underline{Apr}_I(A) \subseteq A \subseteq \overline{Apr}_I(A)$,

(2) $\underline{Apr}_I(\phi) = \phi = \overline{Apr}_I(\phi)$,

(3) $\underline{Apr}_I(H) = H = \overline{Apr}_I(H)$,

(4) if $A \subseteq B$, then $\underline{Apr}_I(A) \subseteq \underline{Apr}_I(B)$ and $\overline{Apr}_I(A) \subseteq \overline{Apr}_I(B)$,

(5) $\underline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A)$,

(6) $\overline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A)$,

(7) $\overline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A)$,

(8) $\underline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A)$,

(9) $\underline{Apr}_I(A) = (\overline{Apr}_I(A^c))^c$,

(10) $\overline{Apr}_I(A) = (\underline{Apr}_I(A^c))^c$,

(11) $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_I(B)$,

(12) $\underline{Apr}_I(A \cap B) = \underline{Apr}_I(A) \cap \underline{Apr}_I(B)$,

(13) $\overline{Apr}_I(A \cup B) = \overline{Apr}_I(A) \cup \overline{Apr}_I(B)$,

(14) $\underline{Apr}_I(A \cup B) \supseteq \underline{Apr}_I(A) \cup \underline{Apr}_I(B)$,

(15) $\underline{Apr}_I(I_x) = H = \overline{Apr}_I(I_x)$ for all $x \in H$.

Proof. (1), (2) and (3) are straightforward.

(4) For any $x \in \underline{Apr}_I(A)$ we have $I_x \subseteq A \subseteq B$ and so $x \in \underline{Apr}_I(B)$. Now, suppose that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$ and so $I_x \cap B \neq \phi$. Hence $x \in \overline{Apr}_I(B)$.

(5) Since $\underline{Apr}_I(A) \subseteq A$, by (4) we have $\underline{Apr}_I(\underline{Apr}_I(A)) \subseteq \underline{Apr}_I(A)$. Now, let $x \in \underline{Apr}_I(A)$. Then $I_x \subseteq A$. Since for any $y \in I_x$, we have $I_x = I_y$, then $I_y \subseteq A$ and so $y \in \underline{Apr}_I(A)$. Therefore, $I_x \subseteq \underline{Apr}_I(A)$ and then we obtain $x \in \underline{Apr}_I(\underline{Apr}_I(A))$.

(6) By (1) and (4), $\overline{Apr}_I(A) \subseteq \overline{Apr}_I(\overline{Apr}_I(A))$. On the other hand, we assume that $x \in \overline{Apr}_I(\overline{Apr}_I(A))$. Then we have $I_x \cap \overline{Apr}_I(A) \neq \phi$ and so there exist $a \in I_x$ and $a \in \overline{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \cap A \neq \phi$ which imply $I_x \cap A \neq \phi$. Therefore, $x \in \overline{Apr}_I(A)$.

(7) By (1), we have $\underline{Apr}_I(A) \subseteq \overline{Apr}_I(\underline{Apr}_I(A))$. Now, let $x \in \overline{Apr}_I(\underline{Apr}_I(A))$. Then $I_x \cap \underline{Apr}_I(A) \neq \phi$ and so there exist $a \in I_x$ and $a \in \underline{Apr}_I(A)$. Hence $I_a = I_x$ and $I_a \subseteq A$ which imply $I_x \subseteq A$. Therefore, $x \in \underline{Apr}_I(A)$.

(8) By (1), we have $\underline{Apr}_I(\overline{Apr}_I(A)) \subseteq \overline{Apr}_I(A)$. Now, we assume that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$. For every $y \in I_x$, we have $I_y = I_x$ and so $I_y \cap A \neq \phi$. Hence $y \in \overline{Apr}_I(A)$ which implies $I_x \subseteq \overline{Apr}_I(A)$. Therefore, $x \in \underline{Apr}_I(\overline{Apr}_I(A))$.

(9) For any subset A of H we have:

$$\begin{aligned}(\overline{Apr}_I(A^c))^c &= \{x \in H : x \notin \overline{Apr}_I(A^c)\}\\ &= \{x \in H : I_x \cap A^c = \phi\}\\ &= \{x \in H : I_x \subseteq A\}\\ &= \{x \in H : x \in \underline{Apr}_I(A)\}\\ &= \underline{Apr}_I(A).\end{aligned}$$

(10) For any subset A of H we have:

$$\begin{aligned}(\underline{Apr}_I(A^c))^c &= \{x \in H : x \notin \underline{Apr}_I(A^c)\}\\ &= \{x \in H : I_x \not\subseteq A^c\}\\ &= \{x \in H : I_x \cap A \neq \phi\}\\ &= \{x \in H : x \in \overline{Apr}_I(A)\}\\ &= \overline{Apr}_I(A).\end{aligned}$$

(11) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by (4), $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A)$ and $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(B)$. Hence $\overline{Apr}_I(A \cap B) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_I(B)$.

(12) For any subset A and B of H we have:

$$\begin{aligned} x \in \underline{Apr}_I(A \cap B) &\Longleftrightarrow I_x \subseteq A \cap B \\ &\Longleftrightarrow I_x \subseteq A \text{ and } I_x \subseteq B \\ &\Longleftrightarrow x \in \underline{Apr}_I(A) \text{ and } x \in \underline{Apr}_I(B) \\ &\Longleftrightarrow x \in \underline{Apr}_I(A) \cap \underline{Apr}_I(B). \end{aligned}$$

(13) For any subset A and B of H we have

$$\begin{aligned} x \in \overline{Apr}_I(A \cup B) &\Longleftrightarrow I_x \cap (A \cup B) \neq \phi \\ &\Longleftrightarrow (I_x \cap A) \cup (I_x \cap B) \neq \phi \\ &\Longleftrightarrow I_x \cap A \neq \phi \text{ or } I_x \cap B \neq \phi \\ &\Longleftrightarrow x \in \overline{Apr}_I(A) \text{ or } x \in \overline{Apr}_I(B) \\ &\Longleftrightarrow x \in \overline{Apr}_I(A) \cup \overline{Apr}_I(B). \end{aligned}$$

(14) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by (4), $\underline{Apr}_I(A) \subseteq \underline{Apr}_I(A \cup B)$ and $\underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \cup B)$, which imply that $\underline{Apr}_I(A) \cup \underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \cup B)$.

(15) The proof is straightforward. □

Corollary 3.3. Let (H, Θ) be an approximation space. Then

(i) for every $A \subseteq H$, $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are definable sets,

(ii) for every $x \in H, I_x$ is definable set.

Proof. (i) By proposition 3.2 (5) and (7), we have $\underline{Apr}_I(\underline{Apr}_I(A)) = \underline{Apr}_I(A) = \overline{Apr}_I(\underline{Apr}_I(A))$. Hence $\underline{Apr}_I(A)$ is a definable set. On the other hand by proposition 3.2 (6) and (8), we have $\overline{Apr}_I(\overline{Apr}_I(A)) = \overline{Apr}_I(A) = \underline{Apr}_I(\overline{Apr}_I(A))$. Therefore $\overline{Apr}_I(A)$ is a definable set.

(ii) By proposition 3.2 (15) the proof is clear. □

Theorem 3.4. Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ be a hyper BCK-ideal of H and A, B are non-empty subsets of H. Then

(i) $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \overline{Apr}_I(A \circ B)$,

(ii) $\underline{Apr}_I(A) \circ \underline{Apr}_I(B) \subseteq \underline{Apr}_I(A \circ B)$.

Proof. (i) Let $z \in \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. Then there exist $a \in \overline{Apr}_I(A)$ and $b \in \overline{Apr}_I(B)$ such that $z \in a \circ b$. Hence $I_a \cap A \neq \phi$ and $I_b \cap B \neq \phi$ and so there exist $c \in I_a \cap A$ and $d \in I_b \cap B$ such that $a\Theta c$ and $b\Theta d$. Since Θ is a congruence relation on H, then we have $a \circ b \bar{\Theta} c \circ d$ and because $z \in a \circ b$, then there exist $y \in c \circ d$ such that $z\Theta y$. Hence $y \in I_z$. On the other hand, $y \in c \circ d \subseteq A \circ B$ which implies $I_z \cap (A \circ B) \neq \phi$ and so $z \in \overline{Apr}_I(A \circ B)$. Therefore $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) \subseteq \overline{Apr}_I(A \circ B)$. Now, suppose that $x \in \overline{Apr}_I(A \circ B)$. Then $I_x \cap (A \circ B) \neq \phi$. Let $z \in I_x \cap (A \circ B)$, then there exist $a \in A$ and $b \in B$ such that $z \in a \circ b$ and $I_x = I_z$. Thus we have $I_z \in I_a \circ I_b$ and so $I_x \in I_a \circ I_b$. Hence $x \in a \circ b \subseteq A \circ B \subseteq \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. Therefore, $\overline{Apr}_I(A \circ B) \subseteq \overline{Apr}_I(A) \circ \overline{Apr}_I(B)$. □

(ii) Let $z \in \underline{Apr}_I(A) \circ \underline{Apr}_I(B)$. Then there exist $a \in \underline{Apr}_I(A)$ and $b \in \underline{Apr}_I(B)$ such that $z \in a \circ b$, $I_a \subseteq A$ and $I_b \subseteq B$. For every $y \in I_z$, we have $I_z = I_y \in I_a \circ I_b$ and so $y \in a \circ b \subseteq A \circ B$. Then $y \in A \circ B$ and so $I_z \subseteq A \circ B$. Therefore $z \in \underline{Apr}_I(A \circ B)$. □

Example 3.5. Let $H = \{0, 1, 2\}$ and define the hyper operation "$\circ$"on H as follow:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Then $(H, \circ)$ is a hyper BCK-algebra. Define the equivalence relation Θ by

$$\Theta = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}.$$

Then Θ is a regular congruence relation on H and so we have:

$$I = [0]_\Theta = \{0, 1\}, I_1 = [1]_\Theta = \{0, 1\}, I_2 = [2]_\Theta = \{2\}.$$

Now, if we let $A = \{1, 2\}$ and $B = \{0, 2\}$, then we have $A \circ B = \{0, 1, 2\}$ and so

$$\begin{aligned}
&\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\} = \{2\},\\
&\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\} = \{0, 1, 2\},\\
&\underline{Apr}_I(B) = \{x \in H | I_x \subseteq B\} = \{2\},\\
&\overline{Apr}_I(B) = \{x \in H | I_x \cap B \neq \phi\} = \{0, 1, 2\},\\
&\underline{Apr}_I(A \circ B) = \{x \in H | I_x \subseteq A \circ B\} = \{0, 1, 2\},\\
&\overline{Apr}_I(A \circ B) = \{x \in H | I_x \cap (A \circ B) \neq \phi\} = \{0, 1, 2\},\\
&\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \{0, 1, 2\},\\
&\underline{Apr}_I(A) \circ \underline{Apr}_I(B) = \{0, 2\}.
\end{aligned}$$

Therefore, we see that $\underline{Apr}_I(A) \circ \underline{Apr}_I(B) \neq \underline{Apr}_I(A \circ B)$ but $\overline{Apr}_I(A) \circ \overline{Apr}_I(B) = \overline{Apr}_I(A \circ B)$.

Definition 3.6. Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ be a hyper BCK-ideal of H and A be a non-empty subset of H. If $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper subalgebra of H, then A is called a *rough hyper subalgebra* of H.

Theorem 3.7. If I be a hyper BCK-ideal and J be a hyper subalgebra of H, then

(i) $\overline{Apr}_I(J)$ is a hyper subalgebra of H.

(ii) If $I \subseteq J$, then $\underline{Apr}_I(J)$ is a hyper subalgebra of H.

Proof. (i) Since $0 \in J \subseteq \overline{Apr}_I(J)$, then $\overline{Apr}_I(J) \neq \phi$. Now, we assume that $x, y \in \overline{Apr}_I(J)$. We must prove that $x \circ y \subseteq \overline{Apr}_I(J)$. Since $I_x \cap J \neq \phi$ and $I_y \cap J \neq \phi$, we can let $t \in I_x \cap J$, $s \in I_y \cap J$ and $z \in x \circ y$. Hence $I_z \in I_x \circ I_y = I_t \circ I_s$ and so $z \in t \circ s \subseteq J$. Thus we have $z \in J$ and $z \in I_z$ and so $I_z \cap J \neq \phi$. Therefore, $z \in \overline{Apr}_I(J)$ and so $x \circ y \subseteq \overline{Apr}_I(J)$.

(ii) Since $I = I_0 \subseteq J$, we have $0 \in \underline{Apr}_I(J) \neq \phi$. Now, suppose that $a, b \in \underline{Apr}_I(J)$. Then $I_a \subseteq J$ and $I_b \subseteq J$. For every $z \in a \circ b$ and every $y \in I_z$, we have $I_z = I_y \in I_a \circ I_b$ and so $y \in a \circ b \subseteq J$. Hence $I_z \subseteq J$, which implies that $z \in \underline{Apr}_I(J)$. Therefore, $a \circ b \subseteq \underline{Apr}_I(J)$. □

Theorem 3.8. Let Θ and Φ be two regular congruence relations on H and $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper BCK-ideals of H such that $I \subseteq J$. Then for any nonempty subset A of H, we have:

(i) $\underline{Apr}_J(A) \subseteq \underline{Apr}_I(A)$,

(ii) $\overline{Apr}_I(A) \subseteq \overline{Apr}_J(A)$.

Proof. (i) First we show that if $I \subseteq J$, then $I_x \subseteq J_x$. Let $y \in I_x$. Then $x\Theta y$. Since Θ is a congruence relation on H and $x\Theta x$, then $x \circ x\bar{\Theta} x \circ y$. Since $0 \in x \circ x$, then there exist $t \in x \circ y$ such that $0\Theta t$ and so $t \in [0]_\Theta = I \subseteq J = [0]_\Phi$. Thus by hypothesis, $t \in [0]_\Phi$ and so $x \circ y\Phi\{0\}$. By the similar way, we can show that $y \circ x\Phi\{0\}$. Since Φ is a regular congruence relation, we get $x\Phi y$ and so $y \in [x]_\Phi = J_x$. Therefore, $I_x \subseteq J_x$. Now, let $x \in \underline{Apr}_J(A)$. Then $J_x \subseteq A$ and so $I_x \subseteq A$ which implies $x \in \underline{Apr}_I(A)$.

(ii) Assume that $x \in \overline{Apr}_I(A)$. Then $I_x \cap A \neq \phi$. Since $I_x \subseteq J_x$, we have $J_x \cap A \neq \phi$. Therefore, $x \in \overline{Apr}_J(A)$. □

Corollary 3.9. Let Θ and Φ are two regular congruence relations on H, $I = [0]_\Theta$, $J = [0]_\Phi$ be two hyper BCK-ideals of hyper BCK-algebra H and A be a non-empty subset of H. Then

(i) $\underline{Apr}_I(A) \cap \underline{Apr}_J(A) \subseteq \underline{Apr}_{I\cap J}(A)$,

(ii) $\overline{Apr}_{I\cap J}(A) \subseteq \overline{Apr}_I(A) \cap \overline{Apr}_J(A)$.

Proof. By theorem 3.8, the proof is clear. □

Definition 3.10. Let Θ be a regular congruence relation on H, $I = [0]_\Theta$ be a hyper BCK-ideal of H, A be a non-empty subset of H and $Apr_I(A) = (\underline{Apr}_I(A), \overline{Apr}_I(A))$ be a rough set in the approximation space (H, Θ). If $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper BCK-ideals (resp, weak, strong) of H, then A is called a *rough hyper BCK-ideal* (resp, weak, strong) of H.

Example 3.11. Let $H = \{0, 1, 2, 3\}$ and hyper operation "$\circ$"on H is defined as follow:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0, 3\}$

Then $(H, \circ, 0)$ is a hyper BCK-algebra. We define the regular congruence relation on H as follow:

$$\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0)\}.$$

So we have:

$$I = I_0 = I_1 = \{0, 1\}, I_2 = \{2\}, I_3 = \{3\}.$$

Now, let $A = \{0, 1, 3\}$ be a subset of H, then

$$\underline{Apr}_I(A) = \{x \in H | I_x \subseteq A\} = \{0, 1, 3\},$$
$$\overline{Apr}_I(A) = \{x \in H | I_x \cap A \neq \phi\} = \{0, 1, 3\}.$$

Easily we give that $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are hyper BCK-ideals. Therefore, A is a rough hyper BCK-ideal of H.

Example 3.12. Let $H = \{0, a, b, c\}$. By the following table $(H, \circ)$ is a hyper BCK-algebra.

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, c\}$

Now, let relation Θ on H is defined as follow:

$$\Theta = \{(0, 0), (a, a), (b, b), (c, c), (0, b), (b, 0), (0, a), (a, 0), (a, b), (b, a)\}.$$

Then,

$$I_0 = I_a = I_b = \{0, a, b\}, I_c = \{c\}.$$

Let $J_1 = \{0, c\}$, $J_2 = \{0, b\}$ and $J_3 = \{c\}$. Then,

$$\underline{Apr}_I(J_1) = \{c\}, \overline{Apr}_I(J_1) = \{0, a, b, c\},$$
$$\underline{Apr}_I(J_2) = \{\}, \overline{Apr}_I(J_2) = \{0, a, b\},$$
$$\underline{Apr}_I(J_3) = \{c\}, \overline{Apr}_I(J_3) = \{c\}.$$

Hence we can see that J_1 is a hyper BCK-ideal of H but $\underline{Apr}_I(J_1)$ is not a hyper BCK-ideal. Moreover J_2 is not a hyper BCK-ideal but $\overline{Apr}_I(J_2)$ is a hyper BCK-ideal of H. In follows, J_3 is not a hyper BCK-ideal and neither $\underline{Apr}_I(J_3)$ nor $\overline{Apr}_I(J_3)$ is a hyper BCK-ideal of H.

Theorem 3.13. Let Θ be a regular congruence relation on H and $I = [0]_\Theta$ be a hyper BCK-ideal of H. Then

(i) If J be a weak hyper BCK-ideal of H containing I, then $\underline{Apr}_I(J)$ is a weak hyper BCK-ideal of H,

(ii) If J be a hyper BCK-ideal of H containing I, then $\underline{Apr}_I(J)$ is a hyper BCK-ideal of H,

(iii) If J be a strong hyper BCK-ideal of H containing I, then $\underline{Apr}_I(J)$ is a strong hyper BCK-ideal of H.

Proof. (i) Since $I = I_0 \subseteq J$, then $0 \in \underline{Apr}_I(J)$. Now, Let $x, y \in H$ be such that $x \circ y \subseteq \underline{Apr}_I(J)$ and $y \in \underline{Apr}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on H, we have $a \circ b \overline{\Theta} x \circ y$ and so for every $z \in a \circ b$, there exist $t \in x \circ y$ such that $z\Theta t$. Since $x \circ y \subseteq \underline{Apr}_I(J)$, we have $t \in \underline{Apr}_I(J)$ and so $I_t = I_z \subseteq J$ which implies $z \in J$. Thus $a \circ b \subseteq J$. On the other hand, $b \in I_y \subseteq J$. Since J is a weak hyper BCK-ideal, we have $a \in J$ and so $I_x \subseteq J$. Hence $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a weak hyper BCK-ideal of H.

(ii) Let $x, y \in H$ be such that $x \circ y \ll \underline{Apr}_I(J)$ and $y \in \underline{Apr}_I(J)$. We must prove that $I_x \subseteq J$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on H, we have $a \circ b \overline{\Theta} x \circ y$ and so for every $z \in a \circ b$, there exist $z' \in x \circ y$ such that $z\Theta z'$. Since $z' \in x \circ y \ll \underline{Apr}_I(J)$, then there exists $t \in \underline{Apr}_I(J) \subseteq J$ such that $z' \ll t$ and so from $z\Theta z'$, we have $I_0 \in I_{z'} \circ I_t = I_z \circ I_t$. Hence $0 \in z \circ t$ and then $z \ll t$. Thus we have proved that for every $z \in a \circ b$, there exist $t \in J$ such that $z \ll t$ which means that $a \circ b \ll J$. On the other hand we have $b \in I_y \subseteq J$. Since J is a hyper BCK-ideal of H, we have $a \in J$. Thus $I_x \subseteq J$ which implies that $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a hyper BCK-ideal of H.

(iii) Suppose that $x, y \in H$ be such that $(x \circ y) \cap \underline{Apr}_I(J) \neq \phi$ and $y \in \underline{Apr}_I(J)$. Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on H, we have $a \circ b \overline{\Theta} x \circ y$. Since $(x \circ y) \cap \underline{Apr}_I(J) \neq \phi$, then there exist $t \in H$ such that $t \in x \circ y$ and $t \in \underline{Apr}_I(J)$. Now, $t \in x \circ y \overline{\Theta} a \circ b$ implies that there exist $z \in a \circ b$ such that $z\Theta t$ and so $I_t = I_z \subseteq J$. Hence $z \in J$ and so $(a \circ b) \cap J \neq \phi$. On the other hand, we have $b \in I_y \subseteq J$. Since J is a strong hyper BCK-ideal of H, then we have $a \in J$ which implies $I_x \subseteq J$ that means $x \in \underline{Apr}_I(J)$. Therefore, $\underline{Apr}_I(J)$ is a strong hyper BCK-ideal of H. □

Theorem 3.14. Suppose that I be a hyper BCK-ideal of H and Θ be a regular congruence relation on H which is defined as follow:

$$x\Theta y \Leftrightarrow x \circ y \subseteq I \text{ and } y \circ x \subseteq I.$$

(i) If J be a weak hyper BCK-ideal of H containing I, then $\overline{Apr}_I(J)$ is a weak hyper BCK-ideal of H,

(ii) If J be a hyper BCK-ideal of H containing I, then $\overline{Apr}_I(J)$ is a hyper BCK-ideal of H,

(iii) If J be a strong hyper BCK-ideal of H containing I, then $\overline{Apr}_I(J)$ is a strong hyper BCK-ideal of H.

Proof. (i) Since $I \subseteq J \subseteq \overline{Apr}_I(J)$, then we have $0 \in \overline{Apr}_I(J)$. Let $x, y \in H$ be such that $x \circ y \subseteq \overline{Apr}_I(J)$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and for every $z \in x \circ y$, we have $z \in \overline{Apr}_I(J)$ which means $I_z \cap J \neq \phi$. Thus there exist $a, b \in H$ such that $a \in I_y \cap J$ and $b \in I_z \cap J$ which imply that $a\Theta y$, $b\Theta z$ and $a, b \in J$. Thus $y \circ a \subseteq I \subseteq J$ and $z \circ b \subseteq I \subseteq J$ and so we get $y, z \in J$, since J is a weak hyper BCK-ideal. Thus we have proved that for any $z \in x \circ y$, we have $z \in J$ and so $x \circ y \subseteq J$. Since J is a weak hyper BCK-ideal and $y \in J$, obviously we have $x \in J$. Since $x \in I_x$, then $I_x \cap J \neq \phi$. Therefore $x \in \overline{Apr}_I(J)$ and so $\overline{Apr}_I(J)$ is a weak hyper BCK-ideal of H.

(ii) Let $x, y \in H$ be such that $x \circ y \ll \overline{Apr}_I(J)$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and for every $z \in x \circ y$, there exist $t \in \overline{Apr}_I(J)$ such that $z \ll t$ and $I_t \cap J \neq \phi$. Thus, there exist $c, d \in H$ such that $c \in I_t \cap J$ and $d \in I_y \cap J$ and so $c\Theta t$, $d\Theta y$ and $c, d \in J$. Hence $t \circ c \subseteq I \subseteq J$ and $y \circ d \subseteq I \subseteq J$. Since J is a hyper BCK-ideal and $c, d \in J$, we have $y, t \in J$. Thus, we have proved that for every $z \in x \circ y$, there exist $t \in J$ such that $z \ll t$ which means that $x \circ y \ll J$ and so from $y \in J$ we get $x \in J$. Consequently, $I_x \cap J \neq \phi$ and so $x \in \overline{Apr}_I(J)$. Therefore, $\overline{Apr}_I(J)$ is a hyper BCK-ideal.

(iii) Let $x, y \in H$ be such that $(x \circ y) \cap \overline{Apr}_I(J) \neq \phi$ and $y \in \overline{Apr}_I(J)$. Then $I_y \cap J \neq \phi$ and so there exist $z \in H$ such that $z \in x \circ y$ and $z \in \overline{Apr}_I(J)$. Hence $I_z \cap J \neq \phi$ and so there exist $c, d \in H$ such that $c \in I_z \cap J$ and $d \in I_y \cap J$. Hence $c\Theta z$ and $d\Theta y$ where $c, d \in J$. Thus we have $z \circ c \subseteq I \subseteq J$ and $y \circ d \subseteq I \subseteq J$. Since J is a strong hyper BCK-ideal and $c, d \in J$, we have $z \in J$ and $y \in J$. Thus we have proved that $(x \circ y) \cap J \neq \phi$ and $y \in J$. Since J is a strong hyper BCK-ideal, we have $x \in J$ and so $I_x \cap J \neq \phi$ which means that $\overline{Apr}_I(J)$ is a strong hyper BCK-ideal of H. □

4 Conclusion

This paper is intend to built up connection between rough sets and hyper BCK-algebras. We have presented a definition of the lower and upper approximation of a subset of a hyper BCK-algebra with respect to a hyper BCK-ideal. This definition and main results are easily extended to other algebraic structures such as hyper K-algebra, hyper I-algebra, etc.

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