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Level subsets and translations of $QFST(G)$

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Abstract

First, we introduce level subsets and translations of $QFST(G)$ and study their properties. Secondly, we prove that union and intersection of two level subsets of $QFST(G)$ are subgroup of G. Also we prove that translations of $QFST(G)$ are also $QFST(G)$. Finally, we define fuzzy image and fuzzy pre-image of translations of $QFST(G)$ under group homomorphisms and anti group homomorphisms and investigate properties of them.

Keywords: Fuzzy algebraic structures, group theory, norms, Q-fuzzy subgroups, normal Q-fuzzy subgroups, homomorphisms.

1 Introduction

The notion of fuzzy set was introduced by Zadeh in 1965. The importance of the introduced notion of fuzzy set was realized by the research worker in all the branches of science and technology and has successfully been exploited. Since Rosenfeld [35], applied the notion of fuzzy sets to algebra and introduced the notion of fuzzy subgroups, many researchers are engaged in extending the concepts of abstract algebra to the broader framework of the fuzzy setting. Reader in [4, 11, 5, 6, 9], will get some definitions and basic results about fuzzy algebras that their properties were carefully studied to a certain extent. The triangular norm, T -norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T -norm. Later, Hohle [7], Alsina et al. [2] introduced the T -norm into fuzzy set theory and suggested that the T-norm be used for the intersection of fuzzy sets. A. Solairaju and R. Nagarajan [36] introduced the notion of Q- fuzzy groups. The author by using norms, investigated some properties of fuzzy algebraic structures [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. In [26] the author defined Q-fuzzy subgroup of G with respect to t-norm T as $QFST(G)$. In Section 2, we recall some definitions and results which will be used later. In Section 3, we introduce the notion a level subset μ_α of $QFST(G)$ for all $\alpha \in [0, 1]$. Next, we investigate conditions that μ_α will be subgroup or normal subgroup of G. Also we show that any subgroup H of a group G can be realized as a level subgroup of $QFST(G)$. Later, we define translations T^μ_α of $QFST(G)$ for all $\alpha \in [0, 1]$ and discuss properties of them. Finally we define fuzzy image and fuzzy pre-image of them under group homomorphisms and anti group homomorphisms such that they will be Q-fuzzy subgroup with respect to t-norm T.

2 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief.

Definition 2.1 *(See [8]) A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that*
(1) if a and b belong to G then ab is also in G (closure),
(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
(3) there is an element $e_G \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),
(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 2.2 *There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e_G.*

Proposition 2.3 *(See [8]) Let G be a group. Let H be a non-empty subset of G. The following are equivalent:*
(1) H is a subgroup of G.
(2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 2.4 *(See [8]) Let H be subgroup of group G. Then we say that H is normal subgroup of G if for all $g \in G$ and $h \in H$ we have that $ghg^{-1} \in H$.*

Definition 2.5 *(See [8]) Let $(G, .), (H, .)$ be any two groups. The function $f : G \to H$ is called a homomorphism if $f(xy) = f(x)f(y)$ and anti-homomorphism if $f(xy) = f(y)f(x))$, for all $x, y \in G$.*

Remark 2.6 *(See [8]) We have the following terminology: monomorphism=injective homomorphism, epimorphism=surjective homomorphism, isomorphism=bijective homomorphism, endomorphism=homomorphism of a group to itself, automorphism=isomorphism of a group with itself.*

Definition 2.7 *(See [10]) Let G be an arbitrary group with a multiplicative binary operation and identity e_G. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $[0, 1]^G$.*

Definition 2.8 *(See [3]) A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:*
(T1) $T(x, 1) = x$ (neutral element)
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity)
(T3) $T(x, y) = T(y, x)$ (commutativity)
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

Recall that T is idempotent t-norm if $T(x, x) = x$ for all $x \in [0, 1]$.

Corollary 2.9 *Let T be a t-norm. Then for all $x \in [0, 1]$*
(1) $T(x, 0) = 0$.
(2) $T(0, 0) = 0$.

Example 2.10 *(1) Standard intersection t-norm*

$$T_m(x,y) = \min\{x,y\}.$$

(2) Bounded sum t-norm

$$T_b(x,y) = \max\{0, x+y-1\}.$$

(3) algebraic product t-norm

$$T_p(x,y) = xy.$$

(4) Drastic t-norm $T_D(x,y) = \begin{cases} y & if x = 1 \\ x & if y = 1 \\ 0 & otherwise. \end{cases}$

(5) Nilpotent minimum t-norm

$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & if x+y > 1 \\ 0 & otherwise. \end{cases}$

(6) Hamacher product t-norm

$T_{H_0}(x,y) = \begin{cases} 0 & if x = y = 0 \\ \frac{xy}{x+y-xy} & otherwise. \end{cases}$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm:

$$T_D(x,y) \leq T(x,y) \leq T_m(x,y)$$

for all $x, y \in [0,1]$.

Lemma 2.11 *(See [1]) Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.12 *(See [26]) Let $(G, .)$ be a group and Q be a non empty set. $\mu \in [0,1]^{G\times Q}$ is said to be a Q-fuzzy subgroup of G with respect to t-norm T if the following conditions are satisfied:*
(1) $\mu(xy,q) \geq T(\mu(x,q), \mu(y,q))$,
(2) $\mu(x^{-1},q) \geq \mu(x,q)$,
for all $x, y \in G$ and $q \in Q$. Throughout this paper the set of all Q-fuzzy subgroup of G with respect to t-norm T will be denoted by $QFST(G)$.

Lemma 2.13 *(See [26]) Let $\mu \in QFST(G)$ and T be idempotent. Then $\mu(e_G,q) \geq \mu(x,q)$ for all $x \in G$ and $q \in Q$.*

Proposition 2.14 *(See [26]) Let T be idempotent. Then $\mu \in QFST(G)$ if and only if $\mu(xy^{-1},q) \geq T(\mu(x,q), \mu(y,q))$ for all $x, y \in G$ and $q \in Q$.*

Definition 2.15 *(See [26]) We say that $\mu \in QFST(G)$ is a normal if $\mu(xyx^{-1},q) = \mu(y,q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NQFST(G)$ the set of all normal Q-fuzzy subgroups of G with respect to t-norm T.*

Definition 2.16 *(See [26]) Let $(G,.),(H,.)$ be any two groups such that $\mu \in [0,1]^{G\times Q}$ and $\nu \in [0,1]^{H\times Q}$. The product of μ and ν, denoted by $\mu \times \nu \in [0,1]^{(G\times H)\times Q}$, is defined as $(\mu\times\nu)((x,y),q) = T(\mu(x,q),\nu(y,q))$ for all $x \in G, y \in H, q \in Q$. Throughout this paper, H denotes an arbitrary group with identity element e_H.*

3 main results

Definition 3.1 *Let $\mu \in QFST(G)$ and $\alpha \in [0,1]$. A level subset of μ corresponding to α is the set*

$$\mu_\alpha = \{x \in G : \mu(x,q) \geq \alpha\}.$$

Proposition 3.2 *Let $\mu \in QFST(G)$ and $\alpha \in [0,1]$ such that $\mu(e_G,q) \geq \alpha$. If T be idempotent t-norm, then μ_α is a subgroup of G.*

Proof 3.3 *Let $x,y \in \mu_\alpha$ then $\mu(x,q) \geq \alpha$ and $\mu(y,q) \geq \alpha$. Then*

$$\mu(xy^{-1},q) \geq T(\mu(x,q),\mu(y^{-1},q)) \geq T(\mu(x,q),\mu(y,q)) \geq T(\alpha,\alpha) = \alpha$$

and so $\mu(xy^{-1},q) \geq \alpha$ which implies that $xy^{-1} \in \mu_\alpha$. Thus Proposition 2.3 gives us that μ_α is a subgroup of G.

Proposition 3.4 *Let $\mu \in QFST(G)$ and be $\alpha_1,\alpha_2 \in [0,1]$ such that $\mu(e_G,q) \geq \alpha_1$ and $\mu(e_G,q) \geq \alpha_2$ with $\alpha_1 > \alpha_2$. Then $\mu_{\alpha_1} = \mu_{\alpha_2}$ iff there is no $x \in G$ such that $\alpha_1 > \mu(x,q) > \alpha_2$.*

Proof 3.5 *Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ and there exists an $x \in G$ such that $\alpha_1 > \mu(x,q) > \alpha_2$. Then $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$ and so $x \in \mu_{\alpha_2}$ but $x \notin \mu_{\alpha_1}$ and this is a contradiction to $\mu_{\alpha_1} = \mu_{\alpha_2}$. Thus there is no $x \in G$ such that $\alpha_1 > \mu(x,q) > \alpha_2$. Conversely, if there is no $x \in G$ such that $\alpha_1 > \mu(x,q) > \alpha_2$, then $\mu_{\alpha_1} = \mu_{\alpha_2}$.*

Proposition 3.6 *Let $\mu \in [0,1]^{G\times Q}$ and μ_α is a subgroup of G for all $\alpha \in [0,1]$ and $\mu(e_G,q) \geq \alpha$. Then $\mu \in QFST(G)$.*

Proof 3.7 *Let $x,y \in G$ and $q \in Q$ with $\mu(x,q) = \alpha_1$ and $\mu(y,q) = \alpha_2$ and then $x \in \mu_{\alpha_1}$ and $y \in \mu_{\alpha_2}$. Now we cinsider the following conditions.*
(1) If $\alpha_1 < \alpha_2$, then $y \in \mu_{\alpha_1}$ and as μ_{α_1} is a subgroup of G so $xy, x^{-1} \in \mu_{\alpha_1}$. Now

$$\mu(xy,q) \geq \alpha_1 = T(\alpha_1,\alpha_2) = T(\mu(x,q),\mu(y,q)).$$

Also

$$\mu(x^{-1},q) \geq \alpha_1 = \mu(x,q).$$

Thus $\mu \in QFST(G)$.
(2) If $\alpha_2 < \alpha_1$, then $x \in \mu_{\alpha_2}$ and as μ_{α_2} is a subgroup of G so $xy, x^{-1} \in \mu_{\alpha_2}$. Now

$$\mu(xy,q) \geq \alpha_2 = T(\alpha_2,\alpha_1) = T(\alpha_1,\alpha_2) = T(\mu(x,q),\mu(y,q)).$$

Also

$$\mu(x^{-1},q) \geq \alpha_2 = \mu(x,q).$$

Thus $\mu \in QFST(G)$.
(3) If $\alpha_2 = \alpha_1$, then it is trivial.

Proposition 3.8 *Let $\mu \in QFST(G)$ and $\alpha_1, \alpha_2 \in [0,1]$. If μ_{α_1} and μ_{α_2} be two subgroups in G, then $\mu_{\alpha_1} \cap \mu_{\alpha_2}$ will be subgroup in G.*

Proof 3.9 *Let $\mu(e_G, q) \geq \alpha_1$ and $\mu(e_G, q) \geq \alpha_2$ and $x \in G, q \in Q$. Then*
(1) If $\alpha_1 < \mu(x,q) < \alpha_2$, then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$ and so $\mu_{\alpha_2} \cap \mu_{\alpha_1} = \mu_{\alpha_2}$ and as μ_{α_2} is a subgroup in G, so $\mu_{\alpha_1} \cap \mu_{\alpha_2}$ will be subgroup in G.
(2) If $\alpha_2 < \mu(x,q) < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$ and $\mu_{\alpha_2} \cap \mu_{\alpha_1} = \mu_{\alpha_1}$ and as μ_{α_1} is a subgroup in G, so $\mu_{\alpha_1} \cap \mu_{\alpha_2}$ will be subgroup in G.
(3) If $\alpha_1 = \alpha_2$, then $\mu_{\alpha_2} = \mu_{\alpha_1}$ and then $\mu_{\alpha_1} \cap \mu_{\alpha_2}$ will be subgroup in G.

Corollary 3.10 *Let $\mu \in QFST(G)$ and $\{\alpha_i\}_{i\in I} \in [0,1]$. If μ_{α_i} be subgroups in G, then $\cap \mu_{\alpha_i}$ will be subgroup in G.*

Proof 3.11 *It is trivial.*

Proposition 3.12 *Let $\mu \in QFST(G)$ and $\alpha_1, \alpha_2 \in [0,1]$. If μ_{α_1} and μ_{α_2} be two subgroups in G, then $\mu_{\alpha_1} \cup \mu_{\alpha_2}$ will be subgroup in G.*

Proof 3.13 *Let $\mu(e_G, q) \geq \alpha_1$ and $\mu(e_G, q) \geq \alpha_2$ and $x \in G, q \in Q$. Then*
(1) If $\alpha_1 < \mu(x,q) < \alpha_2$, then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$ and so $\mu_{\alpha_2} \cup \mu_{\alpha_1} = \mu_{\alpha_1}$ and as μ_{α_1} is a subgroup in G, so $\mu_{\alpha_1} \cup \mu_{\alpha_2}$ will be subgroup in G.
(2) If $\alpha_2 < \mu(x,q) < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$ and $\mu_{\alpha_2} \cup \mu_{\alpha_1} = \mu_{\alpha_2}$ and as μ_{α_2} is a subgroup in G, so $\mu_{\alpha_1} \cup \mu_{\alpha_2}$ will be subgroup in G.
(3) If $\alpha_1 = \alpha_2$, then $\mu_{\alpha_2} = \mu_{\alpha_1}$ and then $\mu_{\alpha_1} \cup \mu_{\alpha_2}$ will be subgroup in G.

Corollary 3.14 *Let $\mu \in QFST(G)$ and $\{\alpha_i\}_{i\in I} \in [0,1]$. If μ_{α_i} be subgroups in G, then $\cup \mu_{\alpha_i}$ will be subgroup in G.*

Proof 3.15 *It is trivial.*

Proposition 3.16 *Let T be idempotent t-norm. Then any subgroup H of a group G can be realized as a level subgroup of $QFST(G)$*

Proof 3.17 *Let $\mu \in [0,1]^{G\times Q}$ defined by*

$$\mu(x,q) = \begin{cases} \alpha & if x \in H \text{ and } q \in Q \text{ and } \; 0 < \alpha < 1 \\ 0 & if x \notin H \text{ and } q \in Q. \end{cases}$$

We show that $\mu \in QFST(G)$. Let $x, y \in G$ and $q \in Q$ and now we consider the following conditions.
(1) If $x, y \in H$, then as H is a subgroup of G so $xy^{-1} \in H$. Thus

$$\mu(x,q) = \mu(y,q) = \mu(xy^{-1},q) = \alpha.$$

Then

$$\mu(xy^{-1},q) = \alpha \geq \alpha = T(\alpha,\alpha) = T(\mu(x,q),\mu(x,q))$$

and from Proposition 2.14 we get that $\mu \in QFST(G)$.
(2) If $x \in H$ and $y \notin H$ then $xy^{-1} \notin H$ and then $\mu(x,q) = \alpha$ and $\mu(y,q) = \mu(xy^{-1},q) = 0$. Thus

$$\mu(xy^{-1},q) = 0 \geq 0 = T(\alpha,0) = T(\mu(x,q),\mu(y,q))$$

and as Proposition 2.14 we obtain that $\mu \in QFST(G)$.
(3) If $x, y \notin H$, *the* $\mu(x, q) = \mu(y, q) = 0$ *and then* xy^{-1} *may or may not belong to* H.
If $xy^{-1} \in H$, *then*

$$\mu(xy^{-1}, q) = \alpha \geq 0 = T(0, 0) = T(\mu(x, q), \mu(y, q))$$

and using Proposition 2.14 gives us $\mu \in QFST(G)$.
If $xy^{-1} \notin H$, *then*

$$\mu(xy^{-1}, q) = 0 \geq 0 = T(0, 0) = T(\mu(x, q), \mu(y, q))$$

and as Proposition 2.14 we will have that $\mu \in QFST(G)$.
Thus in all the cases $\mu \in QFST(G)$.

Proposition 3.18 *If* $\mu \in NQFST(G)$, *then* μ_α *is a normal subgroup of* G *for all* $\alpha \in [0, 1]$ *and* $\mu(e_G, q) \geq \alpha$.

Proof 3.19 *Let* $\mu \in NQFST(G)$ *then from Proposition 3.6 we will have that* μ_α *is a subgroup of* G. *Now let* $x \in \mu_\alpha$ *and* $y \in G$ *and* $q \in Q$ *then* $\mu(x, q) \geq \alpha$. *Thus*

$$\mu(gxg^{-1}, q) = \mu(xgg^{-1}, q) = \mu(xe_G, q) = \mu(x, q) \geq \alpha$$

and then $gxg^{-1} \in \mu_\alpha$ *and thus* μ_α *is a normal subgroup of* G.

Proposition 3.20 *Let* $\mu \in [0, 1]^{G \times Q}$ *and* $\nu \in [0, 1]^{H \times Q}$. *Then* $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ *for all* $\alpha \in [0, 1]$.

Proof 3.21 *Let* $\alpha \in [0, 1]$ *then*

$$(x, y) \in (\mu \times \nu)_\alpha \Longleftrightarrow (\mu \times \nu)((x, y)), q) \geq \alpha \Longleftrightarrow T(\mu(x, q), \nu(y, q)) \geq \alpha$$

$$\Longleftrightarrow \mu(x, q) \geq \alpha \text{ and } \nu(y, q) \geq \alpha \Longleftrightarrow x \in \mu_\alpha \text{ and } y \in \nu_\alpha \Longleftrightarrow (x, y) \in \mu_\alpha \times \nu_\alpha.$$

Thus $(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$.

Definition 3.22 *Let* $\mu \in QFST(G)$ *and* $\alpha \in [0, 1 - \{\mu(x, q) : x \in G, 0 < \mu(x, q) < 1\}]$. *Then*

$$T^\mu_\alpha : G \times Q \to [0, 1]$$

is called a translation of μ *if*

$$T^\mu_\alpha(x, q) = \mu(x, q) + \alpha$$

for all $x \in G$.
Also we say that T^μ_α *is normal if* $T^\mu_\alpha(xyx^{-1}, q) = T^\mu_\alpha(y, q)$ *for all* $x, y \in G$.

Proposition 3.23 *Let* $\mu \in QFST(G)$ *and* T^μ_α *be translation of* μ. *Then*
(1) $T^\mu_\alpha(x^{-1}, q) = T^\mu_\alpha(x, q)$ *for all* $x \in G$ *and* $q \in Q$ *and* $\alpha \in [0, 1]$.
(2) If T *be idempotent t-norm, then* $T^\mu_\alpha(e_G, q) \geq T^\mu_\alpha(x, q)$ *for all* $x \in G$ *and* $q \in Q$ *and* $\alpha \in [0, 1]$.

Proof 3.24 *Let* $x \in G$ *and* $q \in Q$ *and* $\alpha \in [0, 1]$. *Then*
(1)

$$T^\mu_\alpha(x, q) = \mu(x, q) + \alpha = \mu((x^{-1})^{-1}, q) + \alpha \geq \mu(x^{-1}, q) + \alpha$$

$$= T_\alpha^\mu(x^{-1},q) = \mu(x^{-1},q) + \alpha \geq \mu(x,q) + \alpha = T_\alpha^\mu(x,q).$$

Thus $T_\alpha^\mu(x^{-1},q) = T_\alpha^\mu(x,q)$.

(2)

$$T_\alpha^\mu(e_G,q) = \mu(e_G,q) + \alpha = \mu(xx^{-1},q) + \alpha \geq T(\mu(x,q),\mu(x^{-1},q)) + \alpha$$
$$\geq T(\mu(x,q),\mu(x,q)) + \alpha = \mu(x,q) + \alpha = T_\alpha^\mu(x,q)$$

and then $T_\alpha^\mu(e_G,q) \geq T_\alpha^\mu(x,q)$.

Proposition 3.25 *Let* $\mu \in QFST(G)$ *and* T_α^μ *be translation of* μ. *If* T *be idempotent t-norm and* $T_\alpha^\mu(xy^{-1},q) = T_\alpha^\mu(e_G,q)$, *then* $T_\alpha^\mu(x,q) = T_\alpha^\mu(y,q)$ *for all* $x,y \in G$ *and* $q \in Q$ *and* $\alpha \in [0,1]$.

Proof 3.26 *Let* $x,y \in G$ *and* $q \in Q$ *and* $\alpha \in [0,1]$. *Then*

$$T_\alpha^\mu(x,q) = \mu(x,q) + \alpha = \mu(xy^{-1}y,q) + \alpha \geq T(\mu(xy^{-1},q),\mu(y,q)) + \alpha$$
$$= T(\mu(xy^{-1},q) + \alpha, \mu(y,q) + \alpha) = T(T_\alpha^\mu(xy^{-1},q), T_\alpha^\mu(y,q)) = T(T_\alpha^\mu(e_G,q), T_\alpha^\mu(y,q))$$
$$\geq T(T_\alpha^\mu(y,q), T_\alpha^\mu(y,q)) = T_\alpha^\mu(y,q) = \mu(y,q) + \alpha = \mu(yx^{-1}x,q) + \alpha$$
$$\geq T(\mu(yx^{-1},q),\mu(x,q)) + \alpha = T(\mu(yx^{-1},q) + \alpha, \mu(x,q) + \alpha)$$
$$= T(T_\alpha^\mu(yx^{-1},q), T_\alpha^\mu(x,q)) = T(T_\alpha^\mu((xy^{-1})^{-1},q), T_\alpha^\mu(x,q)) = T(T_\alpha^\mu(xy^{-1},q), T_\alpha^\mu(x,q))$$
$$= T(T_\alpha^\mu(e_G,q), T_\alpha^\mu(x,q)) \geq T(T_\alpha^\mu(x,q), T_\alpha^\mu(x,q)) = T_\alpha^\mu(x,q)$$

and thus $T_\alpha^\mu(x,q) = T_\alpha^\mu(y,q)$.

Proposition 3.27 *Let* $\mu \in QFST(G)$ *and* T_α^μ *be translation of* μ. *Then* $T_\alpha^\mu \in QFST(G)$ *for all* $\alpha \in [0,1]$.

Proof 3.28 *Let* $x,y \in G$ *and* $q \in Q$ *and* $\alpha \in [0,1]$. *Then*

(1)

$$T_\alpha^\mu(xy,q) = \mu(xy,q) + \alpha \geq T(\mu(x,q),\mu(y,q)) + \alpha$$
$$= T(\mu(x,q) + \alpha, \mu(y,q) + \alpha) = T(T_\alpha^\mu(x,q), T_\alpha^\mu(y,q)).$$

(2)

$$T_\alpha^\mu(x^{-1},q) = \mu(x^{-1},q) + \alpha \geq \mu(x,q) + \alpha = T_\alpha^\mu(x,q).$$

Then $T_\alpha^\mu \in QFST(G)$.

Proposition 3.29 *Let* $\mu \in QFST(G)$ *and* T_α^μ *be translation of* μ. *If* T *be idempotent t-norm, then* $H = \{x \in G : T_\alpha^\mu(x,q) = T_\alpha^\mu(e_G,q)\}$ *is a subgroup of* G *for all* $\alpha \in [0,1]$.

Proof 3.30 *Let* $x,y \in H$ *and* $q \in Q$ *and* $\alpha \in [0,1]$ *then* $T_\alpha^\mu(x,q) = T_\alpha^\mu(y,q) = T_\alpha^\mu(e_G,q)$. *Now*

$$T_\alpha^\mu(xy^{-1},q) \geq T(T_\alpha^\mu(x,q), T_\alpha^\mu(y^{-1},q)) \geq T(T_\alpha^\mu(x,q), T_\alpha^\mu(y,q))$$
$$= T(T_\alpha^\mu(e_G,q), T_\alpha^\mu(e_G,q)) = T_\alpha^\mu(e_G,q) = T_\alpha^\mu((xy^{-1})(xy^{-1})^{-1},q)$$
$$\geq T(T_\alpha^\mu(xy^{-1},q), T_\alpha^\mu((xy^{-1})^{-1},q)) \geq T(T_\alpha^\mu(xy^{-1},q), T_\alpha^\mu(xy^{-1},q))$$
$$= T_\alpha^\mu(xy^{-1},q)$$

therefore $T_\alpha^\mu(xy^{-1},q) = T_\alpha^\mu(e_G,q)$ *which implies that* $xy^{-1} \in H$ *and Proposition 2.3 gives us that* $H = \{x \in G : T_\alpha^\mu(x,q) = T_\alpha^\mu(e_G,q)\}$ *is a subgroup of* G.

Proposition 3.31 *Let $\mu \in QFST(G)$ and T^{μ}_{α} be translation of μ. If $T^{\mu}_{\alpha}(xy^{-1}, q) = 1$, then $T^{\mu}_{\alpha}(x, q) = T^{\mu}_{\alpha}(y, q)$ for all $x, y \in G$ and $q \in Q$ and $\alpha \in [0, 1]$.*

Proof 3.32 *Let $x, y \in G$ and $q \in Q$ and $\alpha \in [0, 1]$. Then*

$$T^{\mu}_{\alpha}(x, q) = T^{\mu}_{\alpha}(xy^{-1}y, q) \geq T(T^{\mu}_{\alpha}(xy^{-1}, q), T^{\mu}_{\alpha}(y, q)) = T(1, T^{\mu}_{\alpha}(y, q))$$

$$= T^{\mu}_{\alpha}(y, q) = T^{\mu}_{\alpha}(y^{-1}, q) = T^{\mu}_{\alpha}(x^{-1}xy^{-1}, q) \geq T(T^{\mu}_{\alpha}(x^{-1}, q), T^{\mu}_{\alpha}(xy^{-1}, q))$$

$$\geq T(T^{\mu}_{\alpha}(x, q), T^{\mu}_{\alpha}(xy^{-1}, q)) = T(T^{\mu}_{\alpha}(x, q), 1) = T^{\mu}_{\alpha}(x, q).$$

Thus $T^{\mu}_{\alpha}(x, q) = T^{\mu}_{\alpha}(y, q)$.

Definition 3.33 *Let $f : G \to H$ be a group homomorphism such that $\mu \in QFST(G)$ and $\nu \in QFST(H)$. If T^{μ}_{α} be translation of μ and T^{ν}_{α} be translation of ν then fuzzy image $f(T^{\mu}_{\alpha})$ of T^{μ}_{α} under f is defined by*

$$f\,(T^{\mu}_{\alpha}(y, q)) = \begin{cases} \sup\{T^{\mu}_{\alpha}(x, q) \mid (x, q) \in G \times Q, f(x) = y\} & iff^{-1}(y) \neq \emptyset \\ 0 & iff^{-1}(y) = \emptyset \end{cases}$$

and fuzzy pre-image (or fuzzy inverse image) of T^{ν}_{α} under f is

$$f^{-1}(T^{\nu}_{\alpha})(x, q) = T^{\nu}_{\alpha}(f(x), q)$$

for all $(x, q) \in G \times Q$.

Proposition 3.34 *Let f be an epimorphism from group G into group H. If $\mu \in QFST(G)$ and T^{μ}_{α} be translation of μ then $f(T^{\mu}_{\alpha}) \in QFST(H)$.*

Proof 3.35 *Let $h_1, h_2 \in H$ and $q \in Q$. Then*

(1)

$$f(T^{\mu}_{\alpha})(h_1h_2, q) = \sup\{T^{\mu}_{\alpha}(g_1g_2, q) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\}$$

$$\geq \sup\{T(T^{\mu}_{\alpha}(g_1, q), T^{\mu}_{\alpha}(g_2, q)) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\}$$

$$= T(\sup\{T^{\mu}_{\alpha}(g_1, q) \mid g_1 \in G, f(g_1) = h_1\}, \sup\{T^{\mu}_{\alpha}(g_2, q) \mid g_2 \in G, f(g_2) = h_2\})$$

$$= T(f(T^{\mu}_{\alpha})(h_1, q), f(T^{\mu}_{\alpha})(h_2, q)).$$

(2)

$$f(T^{\mu}_{\alpha})(h_1^{-1}, q) = \sup\{T^{\mu}_{\alpha}(g_1^{-1}, q) \mid g_1 \in G, f(g_1^{-1}) = h_1^{-1}\}$$

$$\geq \sup\{T^{\mu}_{\alpha}(g_1, q) \mid g_1 \in G, f(g_1, q) = h_1\} = f(T^{\mu}_{\alpha})(h_1, q).$$

Therefore $f(T^{\mu}_{\alpha}) \in QFST(H)$.

Proposition 3.36 *Let f be a homorphism from group G into group H. If $\nu \in QFST(H)$ and T^{ν}_{α} be translation of ν, then $f^{-1}(T^{\nu}_{\alpha}) \in QFST(G)$.*

Proof 3.37 *Let $g_1, g_2 \in G$ and $q \in Q$. Then*

(1)

$$f^{-1}(T^{\nu}_{\alpha})(g_1g_2, q) = T^{\nu}_{\alpha}(f(g_1g_2), q) = T^{\nu}_{\alpha}(f(g_1)f(g_2), q)$$

$$\geq T(T_\alpha^\nu(f(g_1),q),T_\alpha^\nu(f(g_2),q)) = T(f^{-1}(T_\alpha^\nu)(g_1,q),f^{-1}(T_\alpha^\nu)(g_2,q)).$$

(2)

$$f^{-1}(T_\alpha^\nu)(g_1^{-1},q) = T_\alpha^\nu(f(g_1^{-1}),q) = T_\alpha^\nu(f(g_1)^{-1},q) \geq T_\alpha^\nu(f(g_1),q) = f^{-1}(T_\alpha^\nu)(g_1,q).$$

Then $f^{-1}(T_\alpha^\nu) \in QFST(G)$.

Proposition 3.38 *Let* f *be an anti homorphism from group* G *into group* H*. If* $\nu \in QFST(H)$ *and* T_α^ν *be translation of* ν*, then* $f^{-1}(T_\alpha^\nu) \in QFST(G)$.

Proof 3.39 *Let* $g_1, g_2 \in G$ *and* $q \in Q$*. Then*
(1)

$$f^{-1}(T_\alpha^\nu)(g_1g_2,q) = T_\alpha^\nu(f(g_1g_2),q) = T_\alpha^\nu(f(g_2)f(g_1),q)$$

$$\geq T(T_\alpha^\nu(f(g_2),q),T_\alpha^\nu(f(g_1),q)) = T(T_\alpha^\nu(f(g_1),q),T_\alpha^\nu(f(g_2),q))$$

$$= T(f^{-1}(T_\alpha^\nu)(g_1,q),f^{-1}(T_\alpha^\nu)(g_2,q)).$$

(2)

$$f^{-1}(T_\alpha^\nu)(g_1^{-1},q) = T_\alpha^\nu(f(g_1^{-1}),q) = T_\alpha^\nu(f(g_1)^{-1},q) \geq T_\alpha^\nu(f(g_1),q) = f^{-1}(T_\alpha^\nu)(g_1,q).$$

Thus $f^{-1}(T_\alpha^\nu) \in QFST(G)$.

Proposition 3.40 *Let* f *be an epimorphism from group* G *into group* H*. If* $T_\alpha^\mu \in NQFST(G)$ *and* T_α^μ *be translation of* μ *then* $f(T_\alpha^\mu) \in NQFST(H)$.

Proof 3.41 *As Proposition 3.34 we have* $f(T_\alpha^\mu) \in QFST(H)$*. Let* $x, y \in H$ *and* $q \in Q$*. Since* f *is a surjection,* $f(u) = x$ *for some* $u \in G$*. Then*

$$f(T_\alpha^\mu)(xyx^{-1},q) = \sup\{T_\alpha^\mu(w,q) \mid w \in G, f(w) = xyx^{-1}\}$$

$$= \sup\{T_\alpha^\mu(u^{-1}wu,q) \mid w \in G, f(u^{-1}wu) = y\}$$

$$= \sup\{T_\alpha^\mu(w,q) \mid w \in G, f(w) = y\} = f(T_\alpha^\mu)(y,q).$$

Therefore $f(T_\alpha^\mu) \in NQFST(H)$.

Proposition 3.42 *Let* f *be a homorphism from group* G *into group* H*. If* $T_\alpha^\nu \in NQFST(H)$ *and* T_α^ν *be translation of* ν*, then* $f^{-1}(T_\alpha^\nu) \in NQFST(G)$.

Proof 3.43 *Using Proposition 3.38 implies that that* $f^{-1}(T_\alpha^\nu) \in QFST(G)$*. Now for any* $x, y \in G$ *and* $q \in Q$ *we obtain*

$$f^{-1}(T_\alpha^\nu)(xyx^{-1},q) = T_\alpha^\nu(f(xyx^{-1}),q) = T_\alpha^\nu(f(x)f(y)f(x^{-1}),q)$$

$$= T_\alpha^\nu(f(x)f(y)f^{-1}(x),q) = T_\alpha^\nu(f(y),q) = f^{-1}(T_\alpha^\nu)(y,q).$$

Therefore $f^{-1}(T_\alpha^\nu) \in NQFST(G)$.

4 Conclusions

In this study, we introduce the notion a level subset μ_α of $QFST(G)$ for all $\alpha \in [0,1]$. Next, we investigate conditions that μ_α will be subgroup or normal subgroup of G. Also we show that any subgroup H of a group G can be realized as a level subgroup of $QFST(G)$. Later, we define translations T^μ_α of $QFST(G)$ for all $\alpha \in [0,1]$ and discuss properties of them. Finally we define fuzzy image and fuzzy pre-image of them under group homomorphisms and anti group homomorphisms such that they will be Q-fuzzy subgroup with respect to t-norm T.

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