Identities of Choi-Lee-Srivastava involving the Euler-Mascheroni's constant

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Abstract:

We give an elementary deduction of the Choi-Lee-Srivastava's identities involving the Euler Mascheroni's constant, thus from them is immediate the identity of Wilf.

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1. Introduction

Wilf [1] proposed to prove the identity [2-4]:

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\, e^{\gamma} \prod_{k=1}^{\infty} e^{-\frac{1}{k}} \left(1+\frac{1}{k}+\frac{1}{2\,k^2}\right), \tag{1}$$

where γ =0.5772 1566 4901 5328 6060 … is the Euler-Mascheroni's constant [4, 5].

In Sec. 2 we use properties of the gamma function [4-9] to give an elementary deduction of (1), and in Sec. 3 this process allows generalize (1) to obtain the Choi-Lee-Srivastava's expressions [2]:

$$\cosh(\alpha\,\pi) = \pi\left(\alpha^2+\frac{1}{4}\right) e^{\gamma} \prod_{j=1}^{\infty} e^{-\frac{1}{j}} \left(1+\frac{1}{j}+\frac{\alpha^2+\frac{1}{4}}{j^2}\right), \qquad \alpha \neq \pm\,\frac{i}{2}, \tag{2}$$

which implies (1) for $\alpha = \frac{1}{2}$, and:

$$\sinh(\beta\,\pi) = \beta\,\pi\,(\beta^2+1)\, e^{2\gamma} \prod_{j=1}^{\infty} e^{-\frac{2}{j}} \left(1+\frac{2}{j}+\frac{\beta^2+1}{j^2}\right), \qquad \beta \neq \pm\, i\,. \tag{3}$$

2. Wilf's formula

In [10] we find the following relation involving an infinite product and the gamma function:

$$\prod_{k=m}^{\infty} \frac{(k+z)^2-b}{(k+z)^2-a} = \frac{\Gamma(z+m-\sqrt{a})\,\Gamma(z+m+\sqrt{a})}{\Gamma(z+m-\sqrt{b})\,\Gamma(z+m+\sqrt{b})}, \tag{4}$$

where we can employ $a = -b = \frac{1}{4}$, $m = 1$ and $z = \frac{1}{2}$ to obtain the expression:

$$\prod_{k=1}^{\infty} \frac{(2k+1)^2+1}{(2k+1)^2-1} \equiv \prod_{k=1}^{\infty} \left[1+\frac{1}{2k\,(k+1)}\right] = \frac{1}{\Gamma\left(\frac{3-i}{2}\right)\Gamma\left(\frac{3+i}{2}\right)} = \frac{2}{\Gamma\left(\frac{1-i}{2}\right)\Gamma\left(\frac{1+i}{2}\right)}\,. \tag{5}$$

On the other hand, we know the property [7]:

$$\frac{\pi}{\cosh(\pi\,x)} = \Gamma\left(\frac{1}{2}-i\,x\right)\Gamma\left(\frac{1}{2}+i\,x\right), \tag{6}$$

that we can apply with $x=\frac{1}{2}$ into (5) to deduce the identity:

$$\cosh\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\prod_{k=1}^{\infty}\left[1+\frac{1}{2k\,(k+1)}\right]. \tag{7}$$

Now we observe the relation:

$$\frac{1+\frac{1}{k}+\frac{1}{2k^2}}{1+\frac{1}{k}}=1+\frac{1}{2k\,(k+1)}, \tag{8}$$

then (7) is equivalent to:

$$\cosh\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\,\frac{\prod_{k=1}^{\infty}e^{-\frac{1}{k}}\left(1+\frac{1}{k}+\frac{1}{2k^2}\right)}{\prod_{r=1}^{\infty}e^{-\frac{1}{r}}\left(1+\frac{1}{r}\right)}, \tag{9}$$

but the Newman (1848)-Weierstrass (1856) formula [7]:

$$z\,e^{\gamma z}\prod_{r=1}^{\infty}e^{-\frac{z}{r}}\left(1+\frac{z}{r}\right)=\frac{1}{\Gamma(z)}, \tag{10}$$

with $z=1$ gives the expression:

$$\prod_{r=1}^{\infty}e^{-\frac{1}{r}}\left(1+\frac{1}{r}\right)=e^{-\gamma}, \tag{11}$$

whose application in (9) implies the Wilf's identity (1) [1-4, 11], q.e.d.

3. Choi-Lee-Srivasatava's relations

The process indicated in Sec. 2 permits to prove (2) and (3), in fact, we use (4) with $a=\frac{1}{4}$, $b=-\alpha^2$, $m=1$, $z=\frac{1}{2}$, and (6) for $x=\alpha\neq\pm\frac{i}{2}$:

$$\prod_{k=1}^{\infty}\frac{(2k+1)^2+4\,\alpha^2}{(2k+1)^2-1}=\frac{\cosh(\alpha\,\pi)}{\pi\,(\alpha^2+\frac{1}{4})}, \tag{12}$$

however, we have the property:

$$\frac{(2k+1)^2+4\,\alpha^2}{(2k+1)^2-1}=\frac{1+\frac{1}{k}+\frac{\alpha^2+\frac{1}{4}}{k^2}}{1+\frac{1}{k}}, \tag{13}$$

hence the application of (11) and (13) into (12) implies (2), q.e.d.

Similarly, from (4) for $a=m=z=1$, $b=-\beta^2$, and the companion relation of (6) if $\beta\neq\pm i$:

$$\frac{\pi}{\sinh(\pi\,\beta)}=\beta\,\Gamma(i\beta)\,\Gamma(-i\beta), \tag{14}$$

we deduce the expression:

$$\prod_{k=1}^{\infty}\frac{(k+1)^2+\beta^2}{(k+1)^2-1}=\frac{2}{\beta\,(\beta^2+1)}\,\frac{\sinh(\beta\,\pi)}{\pi}, \tag{15}$$

but we have the decomposition:

$$\frac{(k+1)^2+\beta^2}{(k+1)^2-1} = \frac{1+\frac{2}{k}+\frac{\beta^2+1}{k^2}}{1+\frac{2}{k}} \, , \tag{16}$$

and from (10) with $z = 2$:

$$\prod_{r=1}^{\infty} e^{-\frac{2}{r}} \left(1+\frac{2}{r}\right) = \frac{1}{2}\, e^{-2\gamma} \, , \tag{17}$$

then the use of (16) and (17) into (15) gives (3), q.e.d.

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