Full-Rank Factorization and Moore-Penrose's Inverse

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Abstract

C. C. MacDuffee apparently was the first to point out, in private communications, that a full-rank factorization of a matrix A leads to an explicit formula for its Moore-Penrose's inverse A^{+}. Here we apply this idea of MacDuffee and the Singular Value Decomposition to construct A^{+}.

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1. Introduction

Let's consider a matrix A_{nxm} such that rank A = p, then its full-rank factorization means the existence of matrices F_{nxp} and G_{pxm} with the properties [1]:

$$A = F\,G, \qquad \text{rank}\, F = \text{rank}\, G = p\,; \tag{1}$$

then MacDuffee (1959) constructs the Moore-Penrose's pseudoinverse [2-4] via the expression [1, 5, 6]:

$$A^{+} = G^{T}(\,F^{T}A\,G^{T}\,)^{-1}\,F^{T} = G^{T}(\,G\,G^{T}\,)^{-1}(\,F^{T}\,F\,)^{-1}\,F^{T}, \tag{2}$$

where F^{T} is the corresponding transpose matrix.

In Sec. 2 we employ the Singular Value Decomposition (SVD) of A [7-14] and (2) to construct A^{+}.

2. Full-rank factorization and SVD

For any real matrix A_{nxm}, Lanczos [7, 15] introduces the matrix:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A \\ A^{T} & 0 \end{pmatrix}, \tag{3}$$

and he studies the eigenvalue problem:

$$S\vec{\omega} = \lambda\vec{\omega}, \tag{4}$$

where the proper values are real because S is a real symmetric matrix. Besides:

$$\text{rank}\, A \ \equiv p = \text{Number of positive eigenvalues of } S, \tag{5}$$

such that $1 \leq p \leq \min(n, m)$. Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, \dots, \lambda_p, -\lambda_1, -\lambda_2, \dots, -\lambda_p, 0, 0, \dots, 0, \tag{6}$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of S, named 'essential axes' by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}\begin{matrix} n \\ m \end{matrix}, \tag{7}$$

then (3) and (4) imply the Modified Eigenvalue Problem:

$$A_{nxm}\vec{v}_{mx1} = \lambda\,\vec{u}_{nx1}\,, \qquad A^{T}{}_{mxn}\vec{u}_{nx1} = \lambda\,\vec{v}_{mx1}\,, \tag{8}$$

hence:

$$A^{T}A\vec{v} = \lambda^{2}\vec{v}\,, \qquad AA^{T}\vec{u} = \lambda^{2}\vec{u}\,, \tag{9}$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = (\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p), \qquad V_{mxp} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p), \tag{10}$$

verifying $U^{T}U = V^{T}V = I_{pxp}$ because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk} \, , \tag{11}$$

therefore $\vec{\omega}_j \cdot \vec{\omega}_k = 2\delta_{jk}, \; j,k = 1,2,\dots,p$. Thus, the SVD express [7, 8, 10, 12, 15] that A is the product of three matrices:

$$A_{nxm} = U_{nxp} \, \Lambda_{pxp} \, V^T{}_{pxm} \, , \qquad \Lambda = \mathrm{Diag}\left(\lambda_1, \lambda_2, \dots, \lambda_p\right). \tag{12}$$

This relation tells that in the construction of A we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to A.

The expression (12) is a full-rank factorization of A because it has the structure (1) with:

$$F = U_{nxp} \qquad \text{and} \qquad G = (V_{mxp} \, \Lambda_{pxp})^T, \tag{13}$$

whose substitution into (2) gives the following interesting formula for the Moore-Penrose's inverse [3]:

$$A^+{}_{mxn} = V_{mxp} \, \Lambda_{pxp}^{-1} \, U^T{}_{pxn} \, , \tag{14}$$

which coincides with the natural inverse obtained by Lanczos [7, 15]. The matrix (14) satisfies the relations [1, 3, 16, 17]:

$$A\,A^+A = A, \qquad A^+A\,A^+ = A^+, \qquad (A\,A^+)^T = A\,A^+, \qquad (A^+A)^T = A^+A, \tag{15}$$

which characterize the pseudoinverse of Moore-Penrose. The use of (10) and (12) into (14) implies the following expression for the Lanczos generalized inverse:

$$A^+ = (\vec{t}_1 \;\; \vec{t}_2 \;\cdots\; \vec{t}_n) \, , \qquad \vec{t}_j = \frac{u_1^{(j)}}{\lambda_1} \, \vec{v}_1 + \frac{u_2^{(j)}}{\lambda_2} \, \vec{v}_2 + \cdots + \frac{u_p^{(j)}}{\lambda_p} \, \vec{v}_p \, , \qquad j = 1, \dots, n, \tag{16}$$

where $u_k^{(j)}$ means the j *th-* component of $\vec{u}_k$. Similarly:

$$(A^+)^T = (\vec{r}_1 \;\; \vec{r}_2 \;\cdots\; \vec{r}_m), \qquad \vec{r}_k = \frac{v_1^{(k)}}{\lambda_1} \, \vec{u}_1 + \frac{v_2^{(k)}}{\lambda_2} \, \vec{u}_2 + \cdots + \frac{v_p^{(k)}}{\lambda_p} \, \vec{u}_p \, , \qquad k = 1, \dots, m. \tag{17}$$

MacDuffee proposed [1] to construct A^+ via a full-rank factorization of A, then here we proved that his idea can be applied employing the SVD of the matrix under analysis.

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