Inclusion Properties for Certain Subclasses of Uniformly P-Valent Analytic Functions Involving Linear Operator

R. M. El-Ashwah[1] A. Y. Lashin[2] and A. E. El-Shirbiny[3]

[1,3] Department of Mathematics, Faculty of Science, Damietta University, New Damietta, 34517, Egypt.

[2] Department of Mathematics, Faculty of Science, Mansoura University, 35516, Egypt.

[1] r_elashwah@yahoo.com, [2] aylashin@mans.edueg, [3] amina.ali66@yahoo.com.

Abstract

In this paper, the authors study some inclusion results for new subclasses of β-uniformly p-valent functions in the open unit disc defined by differ-integral operator and some results of certain integral operator are also obtained.

Indexing terms/Keywords: P-Valent Analytic Functions; Uniformly Starlike, Uniformly Convex, Differ-Integral Operator.

Subject Classification: 2010 Mathematics Subject Classification: 30C45.

Language: English

Date of Submission: 2018-02-18

Date of Acceptance: 2018-04-10

Date of Publication: 2018-04-30

Journal: *MATHLAB Journal*

Website: https://purkh.com

Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty} a_n z^{n} \qquad (p\in \mathbb{N}=\{1,2,3,\cdots\}), \tag{1}$$

which are analytic in the open unite disc $U=\{z: z\in \mathbb{C}; |z|<1\}$. We note that $A(1)=A$ the class of univalent analytic functions.

We say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function ω in U with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$ such that $f(z)=g(\omega(z))$ $(z\in U)$. In particular, if the function g is univalent in U the subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$ (see [18,19]).

Definition 1. *[15] A function $f(z)\in A(p)$ is said to be $\beta-$uniformly p-valent starlike functions of order α, if satisfies .*

$$Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq \beta\left|\frac{zf'(z)}{f(z)}-p\right|, \quad (\beta\geq 0, 0\leq\alpha<p, p\in\mathbb{N}).$$

The set of all this functions is denoted by $US_p^*(\alpha,\beta)$.

Definition 2. *[15] A function $f(z)\in A(p)$ is said to be $\beta-$uniformly p-valent convex functions of order α, if satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}\geq \beta\left|\frac{zf''(z)}{f'(z)}+(1-p)\right|, \quad (\beta\geq 0, 0\leq\alpha,<p, p\in\mathbb{N}).$$

The set of all this functions is denoted by $UCV_p(\alpha,\beta)$.

We note that $f\in UCV_p(\alpha,\beta)$ if and only if $\frac{zf'}{p}\in US_p^*(\alpha,\beta)$.

Definition 3. *[1] A function $f(z)\in A(p)$ is said to be $\beta-$uniformly p-valent close-to-convex of order α and type η, if there exist $g(z)\in US_p^*(\beta,\eta)$ such that*

$$Re\left\{\frac{zf'(z)}{g(z)}-\alpha\right\}> \beta\left|\frac{zf'(z)}{g(z)}-p\right|, \quad (0\leq\alpha,\eta<p, \beta\geq 0, p\in\mathbb{N}).$$

The set of all this functions is denoted by $UK_p(\alpha,\beta,\eta)$.

Definition 4. *[1] A function $f\in A(p)$ is said to be $\beta-$uniformly p-valent quasi convex functions of order α and type η if there exist $g(z)\in UCV_p(\beta,\eta)$ such that*

$$Re\left\{\frac{(zf'(z))'}{g'(z)}-\alpha\right\}>\beta\left|\frac{(zf'(z))'}{g'(z)}-p\right| \quad (0\le\alpha,\eta<p,\beta\ge 0,p\in\mathbb{N}).$$

The set of all this functions is denoted by $UK_p^*(\alpha,\beta,\eta)$.

We note that $f\in UK_p^*(\alpha,\beta,\eta)$ if and only if $\frac{zf'(z)}{p}\in UK_p(\alpha,\beta,\eta)$

Also, we note that:

1) $US_1^*(\alpha,\beta)=US^*(\alpha,\beta)$ and $UCV_1(\alpha,\beta)=UCV(\alpha,\beta)$ $(0\le\alpha<1)$ (see [13] and [24]);
2) $US_p^*(\alpha,0)=S_p^*(\alpha)$ $(0\le\alpha<p)$ (see [22] and [23]);
3) $UCV_p(\alpha,0)=C_p(\alpha)$ $(0\le\alpha<p)$ (see [22]);
4) $UK_p(\alpha,\eta,0)=K_p(\alpha,\eta)$ $(0\le\alpha,\eta<p)$ (see [3]);
5) $UK_p^*(\alpha,\eta,0)=K_p^*(\alpha,\eta)$ $(0\le\alpha,\eta<p)$ (see [6]).

Geometric properties: It is known that $f\in US_p^*(\alpha,\beta)$ or $f\in UCV_p(\alpha,\beta)$ with

$$p(z)=\frac{zf'(z)}{f(z)} \text{ or } p(z)=1+\frac{zf''(z)}{f'(z)},$$

respectively, if and only if $p(z)$ takes all the values in the conic domain in the right half plane $R_{p,\alpha,\beta}$ which is given by

$$R_{p,\alpha,\beta}=\left\{w=u+iv\in\mathbb{C}: u-\alpha>\beta\sqrt{(u-p)^2+v^2},\ (\beta\ge 0 \text{ and } 0\le\alpha<p)\right\}.$$

By some computation, $R_{p,\alpha,\beta}$ appears as conic sections that symmetric around the real axis as follows:

$$R_{p,\alpha,\beta}=\begin{cases} u+iv, & \left(\frac{(1-\beta^2)u+(\beta^2p-\alpha)}{\beta(p-\alpha)}\right)^2-\left(\frac{(1-\beta^2)v}{(p-\alpha)\sqrt{1-\beta^2}}\right)>1 \qquad (0<\beta<1), \\ u+iv, & \left(\frac{(\beta^2-1)u+(\beta^2p-\alpha)}{\beta(p-\alpha)}\right)^2+\left(\frac{(\beta^2-1)v}{(p-\alpha)\sqrt{\beta^2-1}}\right)^2<1 \qquad (\beta>1).\end{cases}$$

We note that

$$p(z)\prec Q_{p,\alpha,\beta}(z), \tag{2}$$

with the explicit forms of function $Q_{p,\alpha,\beta}(z)$ given by

$$Q_{p,\alpha,\beta}(z)=\begin{cases} \dfrac{p+(p-2\alpha)z}{1-z} & (\beta=0); \\ p+\dfrac{2(p-\alpha)}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 & (\beta=1); \\ \dfrac{(p-\alpha)}{1-\beta^2}\cos\left\{\dfrac{2}{\pi}\arccos(\beta)i\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right\}-\dfrac{(\beta^2 p-\alpha)}{1-\beta^2} & (0<\beta<1); \\ \dfrac{(p-\alpha)}{\beta^2-1}\sin\left\{\dfrac{\pi}{2K(x)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{x}}}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-x^2t^2}}\right\}+\dfrac{(\beta^2 p-\alpha)}{\beta^2-1} & (\beta>1). \end{cases}$$

where

$$u(z)=\frac{z-\sqrt{x}}{1-\sqrt{x}\,z},\quad x\in(0,1)$$

and $K(x)$ is such that

$$\beta=\cosh\frac{\pi K'(x)}{4K(x)}.$$

By virtue of (2) and the properties of the domains, we have

$$Re(p(z))>Re(Q_{p,\alpha,\beta}(z))>\frac{\beta p+\alpha}{\beta+1}. \tag{3}$$

Making use of the principal of subordination between analytic functions and the definition of $Q_{p,\alpha,\beta}(z)$, we may rewrite the subclasses $US_p^*(\alpha,\beta)$, $UCV_p(\alpha,\beta)$, $UK_p(\alpha,\beta,\eta)$ and $UK_p^*(\alpha,\beta,\eta)$ as the following:

$$US_p^*(\alpha,\beta)=\left\{f\in A(p):\frac{zf'(z)}{f(z)}\prec Q_{p,\alpha,\beta}(z)\right\},$$

$$UCV_p(\alpha,\beta)=\left\{f\in A(p):1+\frac{zf''(z)}{f'(z)}\prec Q_{p,\alpha,\beta}(z)\right\},$$

$$UK_p(\alpha,\beta,\eta)=\left\{f\in A(p):\exists g(z)\in US_p^*(\beta,\eta),\frac{zf'(z)}{g(z)}\prec Q_{p,\alpha,\beta}(z)\right\},$$

$$UK_p^*(\alpha,\beta,\eta)=\left\{ f\in A(p): \exists g(z)\in UCV_p(\beta,\eta), \frac{(zf'(z))'}{g'(z)} \prec Q_{p,\alpha,\beta}(z)\right\}.$$

El-Ashwah and Durbk [7] (see also [4]) introduced the modified an Erdelyi kober type integral operator $I_{p,\mu}^{a,c}: A(p)\to A(p)$ for $\mu>0,\ a,c\in \mathbb{C},\ Re(a)>-\mu p$ and $Re(c-a)>0,$ (see [16]) as follows:

$$I_{p,\mu}^{a,c}f(z)=\frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)\Gamma(c-a)}\int_0^1 (1-t)^{c-a-1}t^{a-1}f\left(zt^{\mu}\right)dt$$

$$=z^p+\frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)}\sum_{n=p+1}^{\infty}\frac{\Gamma(a+n\mu)}{\Gamma(c+n\mu)}a_n z^n,$$

$$I_{p,\mu}^{a,a}f(z)=f(z).$$

It is easy to obtain the following recurrence relations. If $f(z)\in A(p),$ then

$$z\left(I_{p,\mu}^{a,c}f(z)\right)'=\frac{a+\mu p}{\mu}I_{p,\mu}^{a+1,c}f(z)-\frac{a}{\mu}I_{p,\mu}^{a,c}f(z). \tag{4}$$

$$z\left(I_{p,\mu}^{a,c+1}f(z)\right)'=\frac{c+\mu p}{\mu}I_{p,\mu}^{a,c}(a,c,\mu)f(z)-\frac{c}{\mu}I_{p,\mu}^{a,c+1}f(z). \tag{5}$$

The operator $I_{p,\mu}^{a,c}f(z)$ has been extensively studied by many authors with suitable restrictions on the parameters. For examples, see the following:

I. $I_{p,1}^{a-1,c-1}f(z)=L_p(a,c)f(z)\ \left(a,c\in \mathbb{C}/\mathbb{Z}_0^-, \mathbb{Z}_0^-=\{0,-1,-2,\ldots\}\right)$ (see [26] and [27]);

II. $I_{p,1}^{n+p-1,a}f(z)=L_p(n+p,1)f(z)=D^{n+p-1}f(z)\ (n>-p)$ (see [12]);

III. $I_{1,1}^{\beta,\alpha+\beta-\gamma+1}f(z)=R_{\beta}^{\alpha,\gamma}f(z)\ (\gamma>0;\alpha\geq\gamma-1,;\beta>-1)$ (see [2]);

IV. $I_{1,1}^{\beta,\alpha+\beta}f(z)=Q_{\beta}^{\alpha}f(z)\ (\alpha\geq 0;\ \beta>-1)$ (see [14] and [17]);

V. $I_{1,1}^{a-1,c-1}f(z)=L(a,c)f(z)\ \left(a,c\in \mathbb{C}/\mathbb{Z}_0^-, \mathbb{Z}_0^-=\{0,-1,-2,\ldots\}\right)$ (see [9]);

VI. $I_{1,1}^{\mu-1,l}f(z)=I_{l,\mu}f(z)\ (\mu>0;l>-1)$ (see [10]);

VII. $I_{1,1}^{\alpha,0}f(z)=D^{\alpha}f(z)\ (\alpha>-1)$ (see [25]);

VIII. $I_{1,1}^{1,n}f(z)=I^n f(z)\ (n\in\mathbb{N}_0)$ (see [20] and [21]).

Next, using the operator $I_{p,\mu}^{a,c}f(z),$ we introduce the following $\beta-$uniformly subclasses of p-valent functions, for $p\in\mathbb{N},\ 0\leq\alpha,\eta\leq p,\ \beta\geq 0,\ \mu>0\ a,c\in\mathbb{C},\ Re(a)>-\mu p,$ and $Re(c-a)>0,$

Definition 5. *Let* $f(z)\in A(p)$*. Then* $f(z)\in US_p^*(a,c,\mu;\alpha,\beta)$ *if and only if* $I_{p,\mu}^{a,c}f(z)\in US_p^*(\alpha,\beta)$.

Definition 6 . *Let* $f(z)\in A(p)$*. Then* $f(z)\in UCV_p(a,c,\mu;\alpha,\beta)$ *if and only if* $I_{p,\mu}^{a,c}f(z)\in UCV_p(\alpha,\beta)$.

Definition 7. *Let* $f(z)\in A(p)$*. Then* $f(z)\in UK_p(a,c,\mu;\alpha,\beta,\eta)$ *if and only if* $I_{p,\mu}^{a,c}f(z)\in UK_p(\alpha,\beta,\eta)$.

Definition 8. *Let* $f(z)\in A(p)$*. Then* $f(z)\in UK_p^*(a,c,\mu;\alpha,\beta,\eta)$ *if and only if* $I_{p,\mu}^{a,c}f(z)\in UK_p^*(\alpha,\beta,\eta)$.

We not that

$$f(z)\in UCV_p(a,c,\mu;\alpha,\beta)\Leftrightarrow \frac{zf'(z)}{p}\in US_p^*(a,c,\mu;\alpha,\beta)$$

and

$$f(z)\in UK_p^*(a,c,\mu;\alpha,\beta,\eta)\Leftrightarrow \frac{zf'(z)}{p}\in UK_p(a,c,\mu;\alpha,\beta,\eta).$$

In this paper, we investigate several inclusion properties of the subclasses $US_p^*(a,c,\mu;\alpha,\beta)$, $UCV_p(a,c,\mu;\alpha,\beta)$, $UK_p(a,c,\mu;\alpha,\beta,\eta)$, and $UK_p^*(a,c,\mu;\alpha,\beta,\eta)$. Also, some applications involving integral operators are considered associated with the operator $I_{p,\mu}^{a,c}f(z)$.

Inclusion Properties

Unless otherwise mentioned, we shall assume throughout this paper that $0\le\alpha,\eta<p$, $\beta\ge 0$, $\mu>0$, a, $c\in \mathbb{C}$, $Re(a)>-\mu p$ and $Re(c-a)>0$. In order to prove our results, we need the following lemmas.

Lemma 9. *[11] Let* $h(z)$ *be univalent convex in the unit disk* U *with* $Re(\delta h(z)+\zeta)>0(\delta,\zeta\in\mathbb{C})$*. If* $p(z)$ *is analytic in* U *and* $p(0)=h(0)$ *then*

$$p(z)+\frac{zp'(z)}{\delta p(z)+\zeta}\prec h(z)$$

implies $p(z)\prec h(z)$.

Lemma 10. [18] *Let* $h(z)$ *be convex function in* U *and let* $D\ge 0$*. Suppose that* $E(z)$ *is analytic in* U *with* $Re\{E(z)\}\ge D$*. If* g *is analytic in* U *and* $g(0)=h(0)$*, then*

$$Dzg''(z)+E(z)zg'(z)+g(z)\prec h(z)\Rightarrow g(z)\prec h(z).$$

Theorem 11. *Let* $f\in A(p)$ *and* $\frac{Re(a)}{\mu}>-\frac{\beta p+\alpha}{\beta+1}$, $\frac{Re(c)}{\mu}>-\frac{\beta p+\alpha}{\beta+1}$. *Then*

$$US_p^*(a+1,c,\mu;\alpha,\beta)\subset US_p^*(a,c,\mu;\alpha,\beta)\subset US_p^*(a,c+1,\mu;\alpha,\beta).$$

Proof. To prove the first part of this theorem. Let $f\in US_p^*(a+1,c,\mu;\alpha,\beta)$ and set

$$\frac{z\left(I_{p,\mu}^{a,c}f(z)\right)'}{I_{p,\mu}^{a,c}f(z)}=p(z)\ (z\in U),$$

where $p(z)=p+p_1z+p_2z^2+\dots,$ is analytic in U with $p(0)=p$ and $p(z)\neq 0,$ for all $z\in U.$

From $(4),$ we can write

$$\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}=\left(\frac{a+\mu p}{\mu}\right)\frac{I_{p,\mu}^{a+1,c}f(z)}{I_{p,\mu}^{a,c}af(z)}-\frac{a}{\mu},$$

$$\left(\frac{a+\mu p}{\mu}\right)\frac{I_{p,\mu}^{a+1,c}f(z)}{I_{p,\mu}^{a,c}f(z)}=p(z)+\frac{a}{\mu}. \tag{6}$$

By logarithmically differentiating both sides of the equation $(6),$ we get

$$\frac{z(I_{p,\mu}^{a+1,c}f(z))'}{I_{p,\mu}^{a+1,c}f(z)}=p(z)+\frac{zp'(z)}{p(z)+\frac{a}{\mu}}\prec Q_{p,\alpha,\beta}. \tag{7}$$

Since

$$Re\left(Q_{p,\alpha,\beta}+\frac{a}{\mu}\right)>0.$$

Applying Lemma 9 to $(7),$ it follows that

$$p(z)\prec Q_{p,\alpha,\beta}(z),$$

that is $f(z)\in US_p^*(a,c,\mu;\alpha,\beta).$

To prove the second inclusion relationship asserted in Theorem 11, let $f \in US_p^*(a,c,\mu;\alpha,\beta)$ and put

$$\frac{z\left(I_{p,\mu}^{a,c+1} f(z)\right)'}{I_{p,\mu}^{a,c+1} f(z)} = q(z) \quad (z \in U),$$

where the function $q(z)$ is analytic in U with $q(0) = p.$ Then by arguments similar to those detailed above with (5), we have $f \in US_p^*(a,c+1,\mu;\alpha,\beta).$ The proof is completed.

Theorem 12. *Let* $f \in A(p)$ *and* $\frac{Re(a)}{\mu} > -\frac{\beta p+\alpha}{\beta+1}$, $\frac{Re(c)}{\mu} > -\frac{\beta p+\alpha}{\beta+1}$. *Then*

$$UCV_p(a+1,c,\mu;\alpha,\beta) \subset UCV_p(a,c,\mu;\alpha,\beta) \subset UCV_p(a,c+1,\mu;\alpha,\beta).$$

Proof. Applying

$$f(z) \in UCV_p(a+1,c,\mu;\alpha,\beta)$$

$$\Leftrightarrow \frac{zf'(z)}{p} \in US_p^*(a+1,c,\mu;\alpha,\beta)$$

$$\Rightarrow \frac{zf'(z)}{p} \in US_p^*(a,c,\mu;\alpha,\beta)$$

$$\Leftrightarrow f(z) \in UCV_p(a,c,\mu;\alpha,\beta).$$

The second part of the theorem can be proved by using similar arguments, which evidently proves Theorem 12.

Theorem 13. *Let* $f \in A(p)$ *and*

$$\frac{Re(a)}{\mu} > -\frac{\beta p+\eta}{\beta+1}, \frac{Re(c)}{\mu} > -\frac{\beta p+\eta}{\beta+1}.$$

Then

$$UK_p(a+1,c,\mu;\alpha,\beta,\eta) \subset UK_p(a,c,\mu;\alpha,\beta,\eta) \subset UK_p(a,c+1,\mu;\alpha,\beta,\eta).$$

Proof. Let $f \in UK_p(a+1,c,\mu;\alpha,\beta,\eta),$ then, in view of the definition, we write

$$\frac{z\left(I_{p,\mu}^{a+1,c} f(z)\right)'}{\psi(z)} \prec Q_{p,\alpha,\beta} \quad (z \in U), \tag{8}$$

for some $\psi(z) \in US_p^*(\beta,\eta).$ Choose the function $g(z)$ such that $I_{p,\mu}^{a+1,c} g(z) = \psi(z),$ so we have

$$\frac{z\left(I_{p,\mu}^{a+1,c} f(z)\right)'}{I_{p,\mu}^{a+1,c} g(z)} \prec Q_{p,\alpha,\beta} \quad (z \in U).$$

Now, we set

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{I_{p,\mu}^{a,c} g(z)} = p(z) \tag{9}$$

where $p(z) = p + p_1 z + p_2 z^2 + \ldots,$ is analytic in U with $p(0) = p$ and $p(z) \neq 0$ for all $z \in U$. Using identity (4), we have

$$\frac{z\left(I_{p,\mu}^{a+1,c} f(z)\right)'}{I_{p,\mu}^{a+1,c} g(z)} = \frac{I_{p,\mu}^{a+1,c}\left(zf'(z)\right)}{I_{p,\mu}^{a+1,c} g(z)}$$

$$= \frac{z\left(I_{p,\mu}^{a,c}\left(zf'(z)\right)\right)' + \left(\frac{a}{\mu}\right) I_{p,\mu}^{a,c}\left(zf'(z)\right)}{z\left(I_{p,\mu}^{a,c} g(z)\right)' + \left(\frac{a}{\mu}\right) I_{p,\mu}^{a,c} g(z)} \tag{10}$$

$$= \frac{\dfrac{z\left(I_{p,\mu}^{a,c}\left(zf'(z)\right)\right)'}{I_{p,\mu}^{a,c} a g(z)} + \dfrac{\left(\frac{a}{\mu}\right) I_{p,\mu}^{a,c}\left(zf'(z)\right)}{I_{p,\mu}^{a,c} g(z)}}{\dfrac{z\left(I_{p,\mu}^{a,c} g(z)\right)'}{I_{p,\mu}^{a,c} g(z)} + \left(\frac{a}{\mu}\right)}.$$

Since $g(z) \in US_p^*(a+1, c, \mu; \beta, \eta)$ and by Theorem 11, we can write

$$\frac{z\left(I_{p,\mu}^{a,c} g(z)\right)'}{I_{p,\mu}^{a,c} g(z)} = r(z),$$

where $Re\{r(z)\} > \frac{\beta p + \eta}{\beta + 1}$, $(z \in U)$ and

$$\frac{z\left(I_{p,\mu}^{a+1,c}f(z)\right)'}{I_{p,\mu}^{a+1,c}g(z)} = \frac{\dfrac{z\left(I_{p,\mu}^{a,c}\left(zf'(z)\right)\right)'}{I_{p,\mu}^{a,c}g(z)} + \left(\dfrac{a}{\mu}\right)p(z)}{r(z)+\left(\dfrac{a}{\mu}\right)}. \tag{11}$$

From (9), we consider that

$$z\left(I_{p,\mu}^{a,c}f(z)\right)' = I_{p,\mu}^{a,c}g(z)p(z), \tag{12}$$

differentiating both sides of (12) with respect to z and divide by $I_{p,\mu}^{a,c}(a,c,\mu;\beta,\eta)g(z)$, we have

$$\frac{\left[z\left(I_{p,\mu}^{a,c}f(z)\right)'\right]'}{I_{p,\mu}^{a,c}g(z)} = zp'(z)+r(z)p(z). \tag{13}$$

Using (11) and (13), we obtain

$$\frac{z\left(I_{p,\mu}^{a+1,c}f(z)\right)'}{I_{p,\mu}^{a+1,c}g(z)} = \frac{zp'(z)+r(z)p(z)+\left(\dfrac{a}{\mu}\right)p(z)}{r(z)+\left(\dfrac{a}{\mu}\right)} = p(z)+\frac{zp'(z)}{r(z)+\left(\dfrac{a}{\mu}\right)} \tag{14}$$

$$p(z)+\frac{zp'(z)}{r(z)+\left(\dfrac{a}{\mu}\right)} \prec Q_{p,\alpha,\beta}. \tag{15}$$

Putting $D=0$ and $E(z)=\dfrac{1}{r(z)+\left(\dfrac{a}{\mu}\right)}$ in Lemma 10, we obtain

$$Re\{E(z)\} = \frac{1}{\left|r(z)+\left(\dfrac{a}{\mu}\right)\right|^2} Re\left\{r(z)+\left(\dfrac{a}{\mu}\right)\right\} > 0.$$

Therefore, the inequality (15) satisfies the conditions required by Lemma 10. Hence, $p(z) \prec Q_{p,\alpha,\beta}$, so $f \in UK_p(a,c,\mu;\alpha,\beta,\eta)$.

For the second inclusion relationship in Theorem 13, using argument similar to those detailed above, we obtain $UK_p(a,c,\mu;\alpha,\beta,\eta)\subset UK_p(a,c+1,\mu;\alpha,\beta,\eta)$. Thus, we have completed the proof of Theorem 13.

Theorem 14. *Let* $f\in A(p)$ *and*

$$\frac{Re(a)}{\mu}>-\frac{\beta p+\eta}{\beta+1},\frac{Re(c)}{\mu}>-\frac{\beta p+\eta}{\beta+1}.$$

Then

$$UK_p^*(a+1,c,\mu;\alpha,\beta,\eta)\subset UK_p^*(a,c,\mu;\alpha,\beta,\eta)\subset UK_p^*(a,c+1,\mu;\alpha,\beta,\eta). \quad (16)$$

Proof. From the expression

$$f\in UK_p^*(a,c,\mu,\alpha,\beta,\eta)\Leftrightarrow\frac{zf'}{p}\in UK_p(a,c,\mu,\alpha,\beta,\eta),$$

and Theorem 13, we observe that

$$\begin{aligned}&f(z)\in UK_p^*(a+1,c,\mu;\alpha,\beta,\eta)\\&\Leftrightarrow\frac{zf'(z)}{p}\in UK_p(a+1,c,\mu;\beta,\alpha,\eta)\\&\Rightarrow\frac{zf'(z)}{p}\in UK_p(a,c,\mu;\alpha,\beta,\eta)\\&\Leftrightarrow f(z)\in UK_p^*(a,c,\mu;\alpha,\beta,\eta).\end{aligned}$$

The second part of the theorem can be proved by using similar arguments, which evidently proves Theorem 14.

Next, we study the closure properties of generalized Bernardi integral operator [8] defined by Staioh et al. [28] as follows:

$$\pounds_{\gamma,p}f(z)=\frac{\gamma+p}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt\ \ (f\in A(p),\gamma>-p). \quad (17)$$

Theorem 15. *Let* $f\in A(p)$ *and* $\gamma>-\frac{\beta p+\alpha}{\beta+1}$. *If* $f(z)\in US_p^*(a,c,\mu;\alpha,\beta)$, *then*

$$\pounds_{\gamma,p}f(z)\in US_p^*(a,c,\mu;\alpha,\beta).$$

Proof. From (17), we have

$$z\left(I_{p,\mu}^{a,c}\pounds_{\gamma,p}f(z)\right)'=(\gamma+p)I_{p,\mu}^{a,c}f(z)-\gamma I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p}f(z)\right). \quad (18)$$

Setting

$$\frac{z\left(I_{p,\mu}^{a,c}\pounds_{\gamma,p}f(z)\right)'}{I_{p,\mu}^{a,c}\pounds_{\gamma,p}f(z)}=p(z),$$

where $p(z)=p+p_1z+p_2z^2+\ldots$ is analytic in U with $p(0)=p,\ \ p(z)\neq 0$ for all $z\in U$.

From (18), we have

$$(\gamma+p)\frac{I_{p,\mu}^{a,c}f(z)}{I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p}f(z)\right)}=p(z)+\gamma. \tag{19}$$

By logarithmically differentiating both sides of the (19), we have

$$\frac{z\left(I_{p,\mu}^{a,c}f(z)\right)'}{I_{p,\mu}^{a,c}(f(z))}=\frac{z\left(I_{p,\mu}^{a,c}\pounds_{\gamma,p}f(z)\right)'}{I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p}f(z)\right)}+\frac{zp'(z)}{p(z)+\gamma}$$

$$=p(z)+\frac{zp'(z)}{p(z)+\gamma}\prec Q_{p,\alpha,\beta}.$$

Since $Re\left(Q_{p,\alpha,\beta}+\gamma\right)>0,$ then by applying Lemma 9, we obtain the result.

Theorem 16. *Let* $f\in A(p)$ *and* $\gamma>-\dfrac{\beta p+\alpha}{\beta+1}$. *If* $f(z)\in UCV_p(a,c,\mu;\alpha,\beta),$ *then*

$$\pounds_{\gamma,p}f(z)\in UCV_p(a,c,\mu;\alpha,\beta).$$

Proof. Consider the following

$$f(z)\in UCV_p(a,c,\mu;\alpha,\beta)$$

$$\Leftrightarrow\frac{zf'(z)}{p}\in US_p^*(a,c,\mu;\alpha,\beta)\Rightarrow\frac{z\left(\pounds_{\gamma,p}(f(z))\right)'}{p}\in US_p^*(a,c,\mu;\alpha,\beta)$$

$$\Leftrightarrow\pounds_{\gamma,p}f(z)\in UCV_p(a,c,\mu;\alpha,\beta).$$

The proof is completed.

Theorem 17. *Let* $f\in A(p)$ *and* $\gamma>-\dfrac{\beta p+\eta}{\beta+1}$. *If* $f(z)\in UK_p(a,c,\mu;\alpha,\beta,\eta),$ *then*

$$\pounds_{\gamma,p}f(z)\in UK_p(a,c,\mu;\alpha,\beta,\eta).$$

Proof. Let $f(z) \in UK_p(a,c,\mu;\alpha,\beta,\eta)$. Then, in view of the definition, we can write

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{\psi(z)} \prec Q_{p,\alpha,\beta} \quad (z \in U), \tag{20}$$

for some $\psi(z) \in US_p^*(\beta,\eta)$. Choose the function $g(z)$ such that $I_{p,\mu}^{a,c} g(z) = \psi(z)$, so we have

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{I_{p,\mu}^{a,c} g(z)} \prec Q_{p,\alpha,\beta}(z) (z \in U).$$

Now, we set

$$\frac{z\left(I_{p,\mu}^{a,c} \pounds_{\gamma,p} f(z)\right)'}{I_{p,\mu}^{a,c} \pounds_{\gamma,p} g(z)} = h(z). \tag{21}$$

where $h(z) = p + h_1 z + h_2 z^2 + \ldots,$ is analytic in U with $h(0) = p,$ $h(z) \neq 0$ for all $z \in U$.

Also, we have from (18)

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{I_{p,\mu}^{a,c} g(z)} = \frac{\left(I_{p,\mu}^{a,c} zf'(z)\right)}{I_{p,\mu}^{a,c}(g(z))} \tag{22}$$

$$= \frac{z\left(I_{p,\mu}^{a,c} \pounds_{\gamma,p}\left(zf'(z)\right)\right)' + \gamma I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p}\left(zf'(z)\right)\right)}{z\left(I_{p,\mu}^{a,c} \pounds_{\gamma,p} g(z)\right)' + \gamma I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p} g(z)\right)}$$

$$= \frac{\dfrac{z\left(I_{p,\mu}^{a,c} \pounds_{\gamma,p} zf'(z)\right)'}{I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p} g(z)\right)} + \dfrac{\gamma I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p}\left(zf'(z)\right)\right)}{I_{p,\mu}^{a,c}\left(\pounds_{\gamma,,p} g(z)\right)}}{\dfrac{z\left(I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p} g(z)\right)\right)'}{I_{p,\mu}^{a,c}\left(\pounds_{\gamma,p} g(z)\right)} + \gamma}.$$

Since $g(z) \in US_p^*(a,c,\mu;\beta,\eta)$ and by Theorem 15, we have $\pounds_{\gamma,p} g(z) \in US_p^*(a,c,\mu;\beta,\eta)$.

Setting

$$\frac{z\left(I_{p,\mu}^{a,c} \pounds_{\gamma,p} g(z)\right)'}{I_{p,\mu}^{a,c} \pounds_{\gamma,p} g(z)} = H(z).$$

Also, from (21) we obtain

$$z\left(I_{p,\mu}^{a,c}\pounds_{\gamma,p} f(z)\right)' = I_{p,\mu}^{a,c}\pounds_{\gamma,p}\, g(z).(h(z)), \tag{23}$$

differentiating both side of (23) with respect to z, we have

$$\frac{z\left[z\left(I_{p,\mu}^{a,c}\pounds_{\gamma,p} f(z)\right)'\right]'}{I_{p,\mu}^{a,c}\pounds_{\gamma,p}\, g(z)} = zh'(z)+h(z)H(z). \tag{24}$$

Using (22) and (24), we obtain

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{I_{p,\mu}^{a,c} g(z)} = \frac{zh'(z)+h(z)H(z)+\gamma h(z)}{H(z)+\gamma} \tag{25}$$

$$= h(z)+\frac{zh'(z)}{H(z)+\gamma}.$$

This in conjunction with

$$\frac{z\left(I_{p,\mu}^{a,c} f(z)\right)'}{I_{p,\mu}^{a,c} g(z)} \prec Q_{p,\alpha,\beta}(z)\,(z\in U),$$

leads to

$$h(z)+\frac{zh'(z)}{H(z)+\gamma} \prec Q_{p,\alpha,\beta}(z). \tag{26}$$

Putting $D=0$ in Lemma 10 and

$$E(z)=\frac{1}{H(z)+\gamma},$$

we obtain $Re(E(z))>0,$ if $\gamma > -\frac{\beta p+\eta}{\beta+1}$. Therefore, the inequality (26) satisfies the condition required by Lemma 10. Hence, $h(z)\prec Q_{p,\alpha,\beta}(z)$ and the proof is completed.

Theorem 18. *Let* $f \in A(p)$ *and* $\gamma > -\frac{\beta p+\eta}{\beta+1}$. *If* $f(z)\in UK_p^*(a,c,\mu,\alpha,\beta,\eta)$, *then*

$$\pounds_{\gamma,p} f(z)\in UK_p^*(a,c,\mu,\alpha,\beta,\eta).$$

Proof. The proof of Theorem 18 can be derived from the proof of Theorem 17.

Remark 19.

(i) Putting $p = 1,$ *in the above results, we obtain the results obtained by EL-Ashwah and Drbuk [5];*

(ii) By varying some parameters involved in the above subclasses of functions defined, we can obtain several other results.

References

[1] H. A. Al-Kharsani, Multiplier transformations and k-uniformly p-valent starlike functions, General Math., 17 (2009), no. 1, 13–22.

[2] M.K.Aouf, R. M. EL-Ashwah and Abd-Eltawab, Some inclusion relationships of certain subclasses of p-valent functions associated with a family of integral operators, ISRN.Math. Anal. (2003), Art. ID 38470,1-8.

[3] M. K. Aouf, On a class of p-valent close-to-convex functions, Internat. J. Math. Math. Sci., 11 $(1988),$ no. 2, 259-266.

[4] R. M. El-Ashwah and A. H. Hassan, Argument inequalities of certain subclass of multivalent functions defined by using new integral operator, Asian-European J. Math., 09 (2016), 1-6.

[5] R. M. EL-Ashwah and M. E. Drbuk, Inclusion relationship for uniformly certain classes of analytic functions, Int. J. Open Problems Complex Analysis, 7 (2015), no. 1, 1-18.

[6] R. M. El-Ashwah, M.K.Aouf and S.M.El-Deeb, Some inclusion relationship associated with generalized Sirvastava-Attia operator, Int. J. Open Problems Complex Analysis, 5 (2013), no.1, 18-32.

[7] R. M. EL-Ashwah and M. E. Drbuk, Subordination properties of p-valent functions defined by linear operators, British J. of Math and comput. Sci., 4 (2014), no. 21, 3000-3013.

[8] D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429-446.

[9] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM. J. Math. Anal., 15 (1984), 737–745.

[10] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), 432–445.

[11] P.Eenigenburg, S.S.Miller, P. T. Mocanu, and M. O. Reade, On a Briot-Bouquet differential subordination, in General Inequalities, 3 (Oberwolfach,1981), of Internationale Schriftenreihe zur Numerischen Mathematik, Birkhäuser,Basel, Switzerland, 64 (1983), 339-348.

[12] R. M. Goel and N. S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78 (1980), 353–357.

[13] A. W. Goodman, On uniformly starlike functions, J.Math. Anal. Appl., 155 (1991), 364-370.

[14] I. B. Jung, Y. C. Kim and H. M. Srivastava, The hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl. 176 (1993), 138-147.

[15] S. M. Khairnar and M. More ,On subclass of multivalent-uniformly starlike and convex functions defined by a linear operator, IAENG Internat. J. of Appl. Math., 39:3, IJAM_39_06 (2009).

[16] V. Kiryakova, Generalized fractional calculus and applications, Pitman Research Notes in Mathematics Series, 301, John Willey & Sons, Inc. New York; 1994.

[17] J. L. Li, Some properties of two integral operators, Soochow J. Math. 25 (1999), no.1, 91–96.

[18] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 (1981), no. 2, 157–172.

[19] S. S. Miller and P. T. Mocanu, Differential Subordinations Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[20] K. I. Noor, On new classes of integral operators, J. Natur. Geom., 16 (1999), 71–80.

[21] K. I. Noor and M. A. Noor, On integral operators, J. Math. Anal. Appl., 238 (1999), 341–352.

[22] S. Owa, On new classes of p-valent function with negative coefficients, Simon Stevin, 59 (1985), no. 4, 385-402.

[23] D. A. Patil and N. K. Thakare, On convex hulls and extreme points of p-valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S), 27 (1983), 145-160.

[24] F. Ronning, A survey on uniformly convex and uniformly starlike functions, Ann. Univ. Mariae Curie-Sklodowska, 47 (1993), no. 13, 123-134.

[25] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109–115.

[26] H. Saitoh, A linear operator and its application of first order differential subordinations, Math. Japan. 44 (1996), 31–38.

[27] H. Saitoh, On certain subclasses of analytic functions involving a linear operator, in Proceedings of the Second International Workshop, Japan, 1996.

[28] H. Saitoh, S. Owa, T. Sekine, M.Nunokowa and R.Yamakawa, An application of certain integral operator, Appl. Math. letters, 5 (1992), 21-24.