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Integro Cubic B-spline Polynomial for Solving Fractional Differential Problems

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Abstract

In this paper, we use cubic b-spline to construct an approximating polynomial to agree with calculating the control points of the integro cubic b-spline function for solving fractional differential equations for various values, two examples are considered to demonstrate and illustrate the applicability of the method, and to compare the compact results with other known methods, convergence analysis for fractional derivatives of the method is considered.

Keywords: Cubic B- spline, control points, Fractional derivative, Taylor series, Convergence analysis.

1 Introduction

In the last few decades, many physical problems in science and engineering in the fields of mechanics electricity climatology are modelled by fractional differential equations. Fractional calculus are usually difficult to solve analytically so there is a need to obtain an efficient and accuracy approximate solution [1, 2]. For an introduction of fractional derivatives and Taylor series, we refer to [3-5].

Several method have been proposed for solving these equations but most of them have their limitation such as unrealistic assumptions, linearization, low convergence and divergent results. For example, the integro cubic spline methods over the equally spaced knots partition were studied in [6, 7]. The b-spline function is a significant role of numerical analysis and approximate solution as used of cubic b-spline in approximating solutions of boundary value problems in Maria M. and Dambaru B. [8]. Also in [8, 9] presented approximate solutions to linear and nonlinear ordinary differential equations using Bernstein polynomials.

In this paper is present a cubic B-spline method for approximation solution for a Bagely-Torvik fractional differential equation

$$a(t)y'' + b(t)D^{(\alpha)}y + c(t)y = f(xt),\ 0<\alpha<1,\ t\in[a,b] \quad (1)$$

where the function $a(t), b(t), c(t)$ and $f(t)$, are sufficiently smooth real valued functions as[10, 11, 30].

Here we use focus on spline function can used to deal with the new interpolation problems. There have been many works on b-spline function such as used to construct efficient and accurate numerical methods for solving differential equations and well known that the b-spline has been widely used for the numerical solution of boundary value problems in [12, 13], in the last years the interest in b-spline method for solving fractional differential equations is growing and models involving several type of b-spline are widely used in that fields, for an introduction to b-spline approximate solution see instance [13-16] while a survey on tis applications can be found in [17-20].

The aim of paper is study integro cubic b-spline interpolation problems. In section 2, we assume that the control values at the knots are used by apply to solve fractional differential equations. In section3, we describe the use of cubic b-spline for interpolation and approximation and discuss several for error estimation and convergence analysis. In last section, we apply cubic b-spline method to Bagley-Torvik equation and results to show the applications and advantages of approximate solution by graph and tables.

2 Preliminaries and Basic Definitions

Definition 2.1. [25] Suppose that $\alpha > 0$, $x > a, \alpha, a, x \in \mathbb{R}$. Then the Caputo fractional derivative of order $\alpha > 0$ is defined by the following fractional operator.

$$ {}_cD_a^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-a)} \displaystyle\int_a^x (x-\xi)^{n-\alpha-1} \dfrac{a^n}{d\xi^n} f(\xi) d\xi, & n-1 < \alpha < n \in \mathbb{N}; \\ \dfrac{a^n}{dx^n} f(x), & \alpha = n \in \mathbb{N}. \end{cases} $$

Definition 2.2. [24, 25] Riemann (1953) considered power series with non-integer exponents to be extensions of Taylor series and built up generalized derivative for such functions by using the formula

$$ \frac{\partial^q x^p}{\partial x^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q} , $$

this being an obvious generalization of the formula

$$ \frac{\partial^q x^p}{\partial x^q} = p[p-1][p-2] \dots [p-q+1] x^{p-q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q} $$

for p a non-negative integer.

Definition 2.3. [24, 26]

Suppose that $D_a^{k\alpha} f(x) \in C[a,b]$ for $k = 0,1,\dots,n+1$ where $0 < \alpha \leq 1$,then we have the Taylor Series expansion about $x = \tau$

$f(x) = \sum_{i=0}^{n} \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha}$ With $a \leq \xi \leq x$, for all $x \in (a,b]$,

Where $D_a^{k\alpha} = D_a^\alpha . D_a^\alpha \dots D_a^\alpha$ (k times).

Definition 2.4 [27]

The modulus of continuity of a function f continuous on a segment [a, b], $f \in C[a,b]$ is a function $\omega(t) = \omega(f,t)$ defined for $t \in [0, b-a]$ by the relation

$\omega(t) = \omega(f, t{:}\, \delta) = \max\limits_{|t-x|\leq\delta} |f(t) - f(x)|$.

Definition 2.5 [6, 7, 8, 12]

Suppose that us consider a partition Δ_N, on the interval $[a, b]$ is divided into n subinterval using the grids $x_i = a + ih, i = 0, 1, 2, \dots, n,$ where $b = a + nh$.

Given Δ_N, a piecewise polynomial function S on the interval $[a, b]$ is called a spline of degree k if $S \in C^{k-1}[a,b]$ and S is a polynomials of degree at most k on each sub-interval $[x_i, x_{i+1}]$. Let $S_k(\Delta_k)$ denoted the set of all polynomials of degree k

associated with Δ_N. This set is a linear space with respect to Δ_N of dimension $N + k$. Now that we have defined spline functions, we introduced a special kind of spline function called b-spline of degree 3, b-spline
cubic b-spline is defined by

$$B_i^3(x) = \begin{cases} \dfrac{(x-x_i)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)}, & x \in [x_i, x_{i+1}) \\ \dfrac{(x-x_i)(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_{i+1})} + \dfrac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} \\ + \dfrac{(x_{i+4}-x_{i+1})(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})}, & x \in [x_{i+1}, x_{i+2}) \\ \dfrac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-xx_{i+3}-x_{i+2})} \\ + \dfrac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} \\ + \dfrac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+3}-x_{i+2})}, & x \in [x_{i+2}, x_{i+3}) \\ \dfrac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})}, & x \in [x_{i+3}, x_{i+4}) \\ 0, & \text{Otherwise} \end{cases} \quad (2)$$

Note that the cubic b-spline is zero except on the interval $[x_i, x_{i+1}]$. This is true for all b-spline. In fact $B_i^k(x) = 0$ if $x \notin [x_i, x_{i+k+1})$, otherwise $B_i^k(x) > 0$ if $x \in (x_i, x_{i+k+1})$. Since we are only referring to b-spline of degree 3, we write B_i instead of B_i^3 , In our case, we restrict our attention to equally- spaced knots. Therefor after including four addition knots, we assume that $\Delta\, x_{-2} < x_{-1} < x_0 < \cdots < x_{N-1} < x_N < x_{N+1} < x_{N+2}$ is a uniform grid partition using (1) and letting $h = x_{i+1} - x_i$ for any, i=0,1,2,…,n, we defined the uniform cubic b-spline $B_i(x)$ as

$$B_i(x) = \frac{1}{6h^3} \begin{cases} (x-x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}) \\ -3(x-x_{i-1})^3 + 3h(x-x_{i-1}) + 3h^2(x-x_{i-1}) + h^3, & x \in [x_{i-2}, x_{i-1}) \\ -3(x_{i+1}-x)^3 + 3h(x_{i+1}-x)^2 + 3h^2(x_{i+1}-x) + h^3, & x \in [x_{i-2}, x_{i-1}) \\ (x_{i+2}-x)^2, & x \in [x_{i-2}, x_{i-1}) \\ 0, & \text{Otherwise} \end{cases} \quad (3)$$

We list some properties of $B_i(x)$ as follows:

$B_i(x)$ $(i = -1,0,\dots,n+1)$ are linearly independent, and they form the basis splines of $S_3(I), B_i^k(x) = B_i^k(x+h)(i = -1,0,\dots,n; k = 0,1,2)$

$\sum_{i=-1}^{n+1} B_i(x) = 1 \quad (x \in [a,b])$.

Lemma 2.1. For $j = 0,1,2,\ldots,n$, let $y_i = y_i(x)$ for short, we have $I_j = \frac{E-\mathrm{I}}{D} y_i$.

Proof. See [12]

Lemma 2.2. Let $S(\mathrm{x})$ be the integro b-cubic spline polynomial obtained by (2) and (3) for $\mathrm{y(x)}$. for $\mathrm{j} = 0,1,2,\ldots,\mathrm{n}$, we have

$$S_j = \frac{24}{hD}\left(\frac{-E^{-1} - 3I + 3E + E^2}{E^{-1} + 11I + 11E + E^2}\right) y_j \quad (4)$$

$$S''_j = \frac{24}{h^3 D}\left(\frac{-E^{-1} + 3I - 3E + E^2}{E^{-1} + 11I + 11E + E^2}\right) y_j \quad (5)$$

$$S_j^{(1/2)} = \frac{24}{h^{\frac{3}{2}} D}\left(\frac{0.3E^{-1} - 1.391667653I + 1.09E}{E^{-1} + 11I + 11E + E^2}\right) y_j \quad (6)$$

Proof. The integro-cubic spline interpolation problem is start follows

$$\int_{x_j}^{x_{j+1}} S(x)dx = I_j = \int_{x_j}^{x_{j+1}} y(x)dx \qquad (j = 0,1,\ldots,n-1) \quad (7)$$

$$S(x_0) = y_0, S(x_n) = y_n \quad (8)$$

From equation (3), obtain the desired integro cubic b-spline with a known control points as:

$$I_j = \int_{x_j}^{x_{j+1}} \sum_{i=-1}^{n+1} C_i B_i(x)dx \text{ and } I_j = \int_{x_j}^{x_{j+1}} \sum_{i=j-1}^{n+2} C_i B_i(x)dx \quad (9)$$

$$I_j = \int_{x_j}^{x_j+1} C_{j-1}B_{j-1}(x)dx + \int_{x_j}^{x_j+1} C_j B_j(x)dx + \int_{x_j}^{x_j+1} C_{j+1}B_{j+1}(x)dx + \int_{x_j}^{x_j+1} C_{j+2}B_{j+2}(x)dx$$

Using equations (3) and (9), with the boundary conditions of equation (2), we have

$$I_j = \frac{h}{24}C_{j-1} + \frac{33h}{72}C_j + \frac{33h}{72}C_{j+1} + \frac{h}{24}C_{j+2}, \quad I_j = \frac{3h}{72}\left(C_{j-2} + 11C_j + 11C_{j+1} + C_{j+2}\right)$$

$$I_{j-1} = \frac{h}{24}\left(C_{j-2} + 11C_{j-1} + 11C_j + C_{j+1}\right), \quad I_{j+1} = \frac{h}{24}\left(C_j + 11C_{j+1} + 11C_{j+1} + C_{j+3}\right)$$

$$=\frac{h}{24}(S_{j-1}+11s_j+11s_{j+1}+s_{j+2})$$

$$=\frac{h}{24x6}\left(\left(C_{j-2}+4C_{j-1}+C_j\right)+11\left(C_{j-1}+4C_j+C_{j+1}\right)+11\left(C_j+4C_{j+1}+C_{j+2}\right)\right)$$

$$=\frac{1}{6}(I_{j-1}+4I_j+I_{j+1})$$

$$\frac{h}{24}(E^{-1}+11I+11E+E^2)S_j=\frac{1}{6}(I_{j-1}+4I_j+I_{j+1})$$

$$S_j=\frac{4}{h}\left(\frac{E^{-1}+4I+E}{E^{-1}+11I+11E+E^2}\right)I_j \tag{10}$$

Here using the lemma 2.1, we obtain

$$I_j=\sum_{i=0}^{\infty}\frac{h^{i+1}}{(i+1)!}y^{(i)}(xj)=\frac{e^{hD}-1}{D}y(x_j)=\frac{E-\mathrm{I}}{D}y_j$$

$$S_j=\frac{4}{h}\left(\frac{E^{-1}+4I+E}{E^{-1}+11I+11E+E^2}\right)\left(\frac{E-\mathrm{I}}{D}\right)y_i$$

By the same way, we can find the following:

$$S_j^{(1/2)}=\frac{24}{h^{\frac{3}{2}}}\left(\frac{-0.30090111E^{-1}+1.0907665463I}{E^{-1}+11I+11E+E^2}\right)I_j,$$

$$S_j^{(1/2)}=\frac{24}{h^{\frac{3}{2}}}\left(\frac{-0.3E^{-1}+1.09I}{E^{-1}+11I+11E+E^2}\right)\left(\frac{E-\mathrm{I}}{D}\right)y_j$$

$$S_j^{(1/2)}=\frac{24}{h^{\frac{3}{2}}D}\left(\frac{0.3E^{-1}-1.391667653I+1.09E}{E^{-1}+11I+11E+E^2}\right)y_j$$

and finally

$$S''_j=\frac{24}{h^3}\left(\frac{E^{-1}-2I+E}{E^{-1}+11I+11E+E^2}\right)I_j,\ S''_j=\frac{24}{h^3}\left(\frac{E^{-1}-2I+E}{E^{-1}+11I+11E+E^2}\right)\left(\frac{E-\mathrm{I}}{D}\right)y_j$$

$$S''_j=\frac{24}{h^3D}\left(\frac{I-2E+E^2-E^{-1}+2I-E}{E^{-1}+11I+11E+E^2}\right)y_j$$

$$S''_j=\frac{24}{h^3D}\left(\frac{-E^{-1}+3I-3E+E^2}{E^{-1}+11I+11E+E^2}\right)y_j$$

Hence, equations (4)-(6) are obtained.

Theorem 2.3. Let $y(x)$ be a function of class C^{∞} $[a,b]$and $S(x)$ be the integro interpolating b-cubic spline with fractional derivative obtained by (2) and (3) for $j=0,1,2,\dots,n$ we have

$$S(x) = y(x_j) + \frac{11}{240}h^4 y^{(4)} + o(h^5) \quad (11)$$

$$S_j^{(1/2)} = y^{(1/2)}(x_j) + \mu\ h^2 y^{(3/2)} + o(h^3) \quad (12)$$

$$S_j'' = y''(x_j) - \frac{1}{12}h^2 y^{(4)} + o(h^3) \quad (13)$$

where the constant $\mu = -196/739h^{-\frac{1}{2}} + 167/120h^{\frac{1}{2}}y(x_j)$.

Proof. We give a brief proof of (11), the proofs for the others are similar and omitted by (4) and lemma 2.1, we have

$$S_j = \frac{4}{hD}\left(\frac{-E^{-1} - 3I + 3E + E^2}{E^{-1} + 11I + 11E + E^2}\right)y_i$$

Suppose that $E = e^u and\ u = hD$, we get

$$S_j = \frac{4}{hD}\left(\frac{-e^{-u} - 3I + 3e^u + e^{2u}}{e^{-u} + 11I + 11e^u + e^{2u}}\right)$$

$$S_j = \frac{24 + 12u + 8u^2 + 3u^3 + \frac{11}{5}u^4 + \cdots}{24 + 12u + 8u^2 + 3u^3 + \frac{7}{6}u^4 + \cdots} = 1 + \frac{11}{240}u^4 + cu^5 + \cdots$$

where c is a certain constant and the same technique in [12], we obtain

$$S_j = y(x_j) + \frac{11}{240}y^4(x_j)h^4 + o(h^5)$$

$$S(x) = y(x_j) + \frac{11}{240}h^4 y^{(4)} + o(h^5)$$

From equations (6) and (11), with apply lemma 2.1, we have

$$S_j^{(1/2)} = \frac{24}{h^{\frac{3}{2}}D}\left(\frac{0.3E^{-1} - 1.391667653I + 1.09E}{E^{-1} + 11I + 11E + E^2}\right)y_j$$

$$S_j^{(1/2)} = \frac{24}{h^{\frac{3}{2}}D}\left(\frac{-1.0907665463 + 0.78986543634 + 1.3916676563u^2 + 0.78u^3 + 1.39u^4 + 0.789u^5}{24 + 12u + 8u^2 + 3u^2}\right)$$

$$S_j^{(1/2)} = \frac{24D^{(1/2)}}{h^{\frac{3}{2}}D^{(1/2)}.D}\left(\frac{a + bu + cu^2 + du^3}{24 + 12u + 8u^2}\right)y_j,$$

$$S_j^{(1/2)} = \frac{24D^{(1/2)}}{u^{\frac{3}{2}}}\left(\frac{a}{24} + \frac{\left(b - \frac{12a}{24}\right)}{24}\right)u + \cdots)y_j$$

$$S_j^{(1/2)} = \left(au^{-\frac{3}{2}} + b\left(1 - \frac{a}{24}\right)u^{-\frac{1}{2}} + cu^{\frac{1}{2}} + \cdots\right)y_j ,$$

$$S_j^{(1/2)} = ah^{-\frac{3}{2}}D^{-1} + \left(b - \frac{12a}{24}\right)h^{-\frac{1}{2}} + ch^{\frac{1}{2}}y_j + \cdots$$

$$S_j^{(1/2)} = -1.0907665465h^{-\frac{3}{2}}D^{-1} + 0.8255430531h^{-\frac{1}{2}} + 1.3916676563h^{\frac{1}{2}}y_j + \cdots$$

Form definition 2.3 and Taylor series, also using [28, 29], we have

$$S_j^{(1/2)} = y^{(1/2)}(x_j) + \mu h^2\, y^{(3/2)} + o(h^3)$$

where $\mu = -196/739h^{-\frac{1}{2}} + 167/120h^{\frac{1}{2}}y(x_j)$.

To proof of equation (13), the proof for the others are similar and omitted by equation (5) and lemma 2.1, we have

$$E^2 = e^{2u} = 1 + 2u - \frac{4u^2}{2!} + \frac{8u^3}{3!} - \frac{16u^4}{4!} + \frac{32u^5}{5!} + \cdots$$

$$S''_j = \frac{24}{h^3D}\left(\frac{-E^{-1}+3I-3E+E^2}{E^{-1}+11I+11E+E^2}\right)y_j$$

$$S''_j = \frac{u^3+\frac{12}{24}u^4+\frac{30}{120}u^5+\cdots}{24+12u+8u^2+3u^3+\cdots}\left(\frac{24\,D^2}{h^3\,DD^2}\right)y_j$$

$$S''_j = [\frac{1}{h^2u}\frac{u^3\left(24+24u+6u^2+\frac{62}{5}u^3+\cdots\right)}{24+12u+8u^2+3u^3+\cdots}]y_j$$

$$S''_j = \left(\frac{24+12u+6u^2+\cdots}{24+12u+8u^2+\cdots}\right)D^2y_j$$

$$S''_j = (1 + \frac{6-8}{24}u^2 + cu^3 + \cdots)D^2y_j$$

$$S''_j = y''(\mathrm{x_j}) + \frac{(-1)}{12}h^2y^{(4)}(\mathrm{x_j}) + ch^3y^{(5)}(\mathrm{x_j})$$ where c is constant

$$S''_j = y''(\mathrm{x_j}) + \frac{(-1)}{12}h^2y^{(4)}(\mathrm{x_j}) + o(h^3)$$

we obtain

$$S_j'' = y''(x_j) - \frac{h^2}{12}y^{(4)} + o(h^3)$$

Theorem 2.4. Let $y(\mathrm{x})$ be a function of class $\mathrm{C}^\infty[\mathrm{a,b}]$ and $S(\mathrm{x})$ be the integro b-cubic spline polynomial obtained by

$$\int_{x_j}^{x_{j+1}} s(x)dx = I_j = \int_{xj}^{x_{j+1}} y(x)dx \;\; ,j = 0,1,2,n-1 \; and$$

$$S(x_0) = y_0, S(x_1) = y_1, \ldots, S(x_{n-1}) = y_{n-1}, S(x_n) = y_n$$

We have

$$||y^{(\alpha k)}(x) - S^{(\alpha k)}(x)||_\infty = o(h^{3-\alpha k}), \quad k = 0,1,2,3.$$

where $||f(x)||_\infty = \max_{a \le x \le b} |f(x)|$, $0 < \alpha \le 1$.

Proof. First we prove $||y''(x) - S''(x)||_\infty = o(h^2)$. since $S(x)$ is a cubic spline hence $S''(x)$ is a piecewise continuous function over $[a, b]$ with respect to the partition, for $j = 1,2, \ldots, n$. As [12], suppose that $S''(x)$ denoted the restriction of $S''(x)$ over $[x_{j-1}, x_j]$ we have

$$S''(x) = S''(x_{j-1}) \frac{x_j - x}{h} + S''(x_j) \frac{x - x_{j-1}}{h}$$

Now define another linear function $g(x)$ on $[x_{j-1}, x_j]$ as follows

$$g(x) = y''(x_{j-1}) \frac{x_j - x}{h} + y''(x_j) \frac{x - x_{j-1}}{h}$$

Clearly $g(x)$ is a linear interpolation of y''

$$||S''(x) - g(x)||_\infty = \max_{x_{j-1} \le x \le x_j} \left| \left(S''(x_{j-1}) - y''(x_{j-1})\right) \frac{x_j - x}{h} + \left(S''(x_j) - y''(x_j)\right) \frac{x - x_j}{h} \right|$$

$$= o(h^2)$$

$$\begin{aligned} ||S''(x) - y''(x)||_\infty &= ||S''(x) - g(x) + g(x) - y''(x)||_\infty \\ &\le ||S''(x) - g(x)||_\infty + ||g(x) - y''(x)||_\infty \\ &\le o(h^2) + o(h^2) = o(h^2). \end{aligned}$$

and using theorem 4.2 in [12],

$$\begin{aligned} |S(x) - y(x)| &= \int_{x_{j-1}}^{x} S'(x)dt + s(x_{j-1}) - \int_{x_{j-1}}^{x} y'(x)dt + y(x_{j-1}) \\ &= \int_{x_{j-1}}^{x} [S'(x) - y'(x)]dt + [s(x_{j-1}) - y(x_{j-1})] \\ &= o(h^4) + o(h^4) = o(h^4) \end{aligned}$$

And

$$\begin{aligned} ||y^{(\alpha)} - S^{(\alpha)}|| &\le \frac{h^2}{4} h^\alpha \sqrt{\pi} W \alpha \\ &\le \frac{h^2}{4} ||D^2 D^\alpha f|| \end{aligned}$$

$||y^{(\alpha)} - S^{(\alpha)}|| = \frac{h^2}{4} ||D^{(5/2)} f||$, for $\alpha = \frac{1}{2}$, obtain the following

$$\left|g^{(r)}(x)-p^{(r)}_{2m-1}(\alpha)\right| \le \frac{h^r[(x-x_i)(x_{i+1}-x)]^{m-r}G}{r!\,(2m-2r)!}, \qquad G=\max\left|g^{(2m)}(x)\right|$$

$m=1, g=f^{\frac{1}{2}}, p_1=S^{\frac{1}{2}}$, and using [28, 29], we have

$$\left|S^{\left(r+\frac{1}{2}\right)}(x)-y^{\left(r+\frac{1}{2}\right)}(x)\right| \le \frac{h^r[(x-x_i)(x_{i+1}-x)]^{1-r}}{r!\,(2-2r)!}\left\|y^{(3/2)}\right\|_\infty$$

$$r=0 \to \left||S^{(1/2)}(x)-y^{(1/2)}(x)|\right| \le \frac{(x-x_i)(x_{i+1}-x)}{2!}\left\|y^{(3/2)}\right\|_\infty$$

$$\le \frac{h^2}{2!}\left\|y^{(3/2)}\right\|_\infty$$

$$\left\|S^{(1/2)}(x)-y^{(1/2)}(x)\right\| \le \frac{h^2}{2}\left\|y^{(3/2)}\right\|_\infty$$

If $r=1,2,$ respectively, using definition 2.4 and theorem 2.3, then we have

$$\left|S_j-y(x_j)\right| < \frac{11}{240}h^4\omega\left(y^{(4)},x\right)$$

$$|S_j''-y''(x_j)| < \frac{1}{12}h^2\omega\left(y^{(4)},x\right)$$

where $\omega\left(y^{(4)},x\right)$ is the modulus of continuity.

3 Numerical Approximations and Discussion

To verify the applicability of cubic b-Spline to solve fractional Bagley-Torvik differential problem in [9, 10, 11, 30], and to show the our approximate solution approach the nature solution,

$$a(x)y''+b(x)D^{(1/2)}y+c(x)y=f(x) \tag{14}$$

With boundary condition

$$y(a)=\alpha,\ y(b)=\beta \tag{15}$$

where $a(x), b(x), c(x)$ and $f(x)$ are continuous real-valued function on the interval $[a,b]$, we let $y(x)$ be a cubic spline with knots Δ. Then $y(x)$ can be written as a linear combination of $B_i(x)$,

$$y(x) = \sum_{i=-1}^{N+1} C_i B_i(x) \tag{16}$$

Where the constant C_i are determined and $B_i(x)$ are defined in equation (3).

It is necessary that (16) satisfy (14 and 15) at $x=x_i$ where x_i is an interior point that is

$$a(x_i)y''(x_i)+b(x_i)D^{(1/2)}y''(x_i)+c(x_i)y(x_i)=f(x_i)$$

and the boundary conditions are

$$y(a) = \alpha \text{ for } x_0 = a, \qquad y(b) = \beta \text{ for } x_N = b \tag{17}$$

From (16) we obtained

$$y(x_i) = c_{i-1}B_{i-1}(x_i) + C_iB_i(x_i) + c_{i+1}B_{i+1}(x_i) + C_{i+2}B_{i+2}(x_i)$$

$$y^{\frac{1}{2}}(x_i) = C_{i-1}B_{i-1}^{\frac{1}{2}}(x_i) + C_iB_i^{\frac{1}{2}}(x_i) + C_{i+1}B_{i+1}^{\frac{1}{2}}(x_i) + C_{i+2}B_{i+2}^{\frac{1}{2}}(x_i) \tag{18}$$

$$y''(x_i) = C_{i-1}B''_{i-1}(x_i) + C_iB''_i(x_i) + C_{i+1}B''_{i+1}(x_i) + C_{i+2}B''_{i+2}(x_i)$$

There yield

$$a(x_i)\big(C_{i-1}B''_{i-1}(x_i) + C_iB''_i(x_i) + C_{i+1}B''_{i+1}(x_i) + C_{i+2}B''_{i+2}(x_i)\big)$$
$$+b(x_i)\left(C_{i-1}B_{i-1}^{\frac{1}{2}}(x_i) + C_iB_i^{\frac{1}{2}}(x_i) + C_{i+1}B_{i+1}^{\frac{1}{2}}(x_i) + C_{i+2}B_{i+2}^{\frac{1}{2}}(x_i)\right) \tag{19}$$
$$+c(x_i)\big(C_{i-1}B_{i-1}(x_i) + C_iB_i(x_i) + C_{i+1}B_{i+1}(x_i) + C_{i+2}B_{i+2}(x_i)\big) = f(x_i)$$

By the properties of cubic b-spline function, we get the following

$$\begin{array}{lll} B_{i-1(x_i)=\frac{1}{6}} & B_{i-1}^{\frac{1}{2}}(x_i) = \dfrac{-0.30091172}{\sqrt{h}} & B''_{i-1}(x_i) = \dfrac{1}{h^2} \\ B_{i(x_i)=\frac{2}{3}} & B_i^{\frac{1}{2}}(x_i) = \dfrac{1.0907665463}{\sqrt{h}} & B''_{i-1}(x_i) = \dfrac{-2}{h^2} \\ B_{i+1(x_i)=\frac{1}{6}} & B_{i+1}^{\frac{1}{2}}(x_i) = 0 & B''_{i+1}(x_i) = \dfrac{1}{h^2} \\ B_{i+2(x_i)=0} & B_{i+2}^{\frac{1}{2}}(x_i) = 0 & B''_{i+2}(x_i) = 0 \end{array} \tag{20}$$

In (19) and (20) we get

$$C_{i-1}\left(\frac{a(x_i)}{h^2} - \frac{0.30091172b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{6}\right)$$
$$+ C_i\left(\frac{-2a(x_i)}{h^2} + \frac{1,0907665463b(x_i)}{\sqrt{h}} + \frac{2c(x_i)}{3}\right)$$
$$+C_{i+1}\left(\frac{a(x_i)}{h^2} + \frac{c(x_i)}{6}\right) = f(x_i) \tag{21}$$

also for boundary conditions are given below

$$\begin{gathered} B_{-1}(x_0) = B_{N-1}(x_n) = \frac{1}{6} \\ B_0(x_0) = B_N(x_n) = \frac{2}{3} \\ B_1(x_0) = B_{N+1}(x_n) = \frac{1}{6} \\ B_2(x_0) = B_{N+2}(x_n) = 0, \end{gathered} \tag{22}$$

$$C_{-1} = 6\alpha - 4C_0 + C_1 \tag{23}$$

$$C_{N+1} = 6\beta - 4C_N - C_{N-1} \tag{24}$$

where

$$S_i = \frac{a(x_i)}{h^2} - \frac{0.300901172b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{6}$$

$$t_i = \frac{a(x_i)}{h^2} - \frac{1.0907665463b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{6}$$

$$u_i = \frac{a(x_i)}{h^2} + \frac{c(x_i)}{6}$$

$p_1 = -4s_0 + t_0,\ \ p_2 = -s_0 + u_0,\ \ p_3 = s_N + u_N\ and\ p_4 = t_N - 4u_N$ (25)

$$f_1 = f(x_0) - 6s_0\alpha, f_2 = f(x_n) - 6u_n\beta$$

We can write a system of $N + 1$ linear equations in $N + 1$ unknown

$$\begin{pmatrix} p_1 & p_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ s_1 & t_1 & u_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & s_2 & t_2 & u_2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & s_{n-1} & t_{n-1} & u_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & p_3 & p_4 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \\ c_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-1}) \\ f_n \end{pmatrix} \tag{26}$$

The cubic B-spline approximation for the (14) and (15) is obtained using (16), where the constant coefficient C_i satisfy the system defined in (26). We will consider some numerical examples demonstrating the solution using cubic b-spline methods illustrated above. All calculations are implemented with maple programming.

Example 3.1. [10, 11] Consider the fraction boundary value problem:

$$y''(x) + 0.5D^{(1/2)}\ y(x) + y(x) = 3 + x^2\left(\frac{x^{-1/2}}{\Gamma(5/2)} + 1\right) \tag{27}$$

$$y(0) = 1, \qquad y(0.5) = 1.25$$

The exact solution of equation (27) is $y(x) = x^2 + 1$

Using equations (18), (19) and (26) with maple programing, we obtain the following

$$S(x) = \begin{cases} 0.9999986665 + 0.33384000x + 0.3631999x^2 - 0.112000x^3, & 0 \le x \text{ and } x < 0.05 \\ 0.9999963322 + 0.33406003x + 0.3579997x^2 - 0.074666x^3, & 0.05 \le x \text{ and } x < 0.1 \\ 0.999976330 + 0.33466006x + 0.3519996x^2 - 0.054666x^3, & 0.1 \le x \text{ and } x < 0.15 \\ 0.999913337 + 0.33592004x + 0.3435999x^2 - 0.036000x^3, & 0.15 \le x \text{ and } x < 0.2 \\ 0.999827996 + 0.3371998x + 0.3372009x^2 - 0.025334x^3, & 0.2 \le x \text{ and } x < 0.25 \\ 0.99951551 + 0.3409499x + 0.3222009x^2 - 0.005334x^3, & 0.25 \le x \text{ and } x < 0.3 \\ 0.99933546 + 0.3427505x + 0.3161989x^2 + 0.001334x^3, & 0.3 \le x \text{ and } x < 0.35 \\ 0.99853519 + 0.3496101x + 0.2965999x^2 + 0.020000x^3, & 0.35 \le x \text{ and } x < 0.4 \\ 0.99802308 + 0.3534501x + 0.2869999x^2 + 0.028000x^3, & 0.4 \le x \text{ and } x < 0.45 \\ 0.99683518 + 0.3613701x + 0.2694009x^2 + 0.041037x^3, & 0.45 \le x \text{ and } x < 0.5 \end{cases}$$

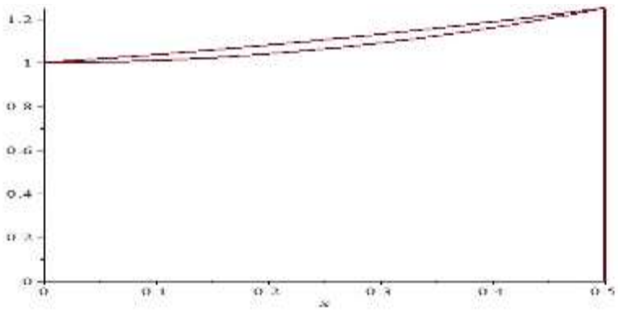

Figure 1: Plots of approximate solution S(x), and y(x) exact solution of the Example 3.1.

$$S^{\frac{1}{2}}(x) = \begin{cases} 0.5464362675x^{\frac{3}{2}} + 0.3766981011\sqrt{x} - 0.2022055467x^{\frac{5}{2}}, & 0 \le x \text{ and } x < 0.05 \\ 0.5386125377x^{\frac{3}{2}} + 0.3769463784\sqrt{x} - 0.1348024942x^{\frac{5}{2}}, & 0.05 \le x \text{ and } x < 0.1 \\ 0.5295853539x^{\frac{3}{2}} + 0.3776234397\sqrt{x} - 0.09869436087x^{\frac{5}{2}}, & 0.1 \le x \text{ and } x < 0.15 \\ 0.5169479586x^{\frac{3}{2}} + 0.3790451749\sqrt{x} - 0.06499464002x^{\frac{5}{2}}, & 0.15 \le x \text{ and } x < 0.2 \\ 0.5073206275x^{\frac{3}{2}} + 0.3804892294\sqrt{x} - 0.04573817251x^{\frac{5}{2}}, & 0.2 \le x \text{ and } x < 0.25 \\ 0.4847530442x^{\frac{3}{2}} + 0.3847207642\sqrt{x} - 0.009630039163x^{\frac{5}{2}}, & 0.25 \le x \text{ and } x < 0.3 \\ 0.4757230019x^{\frac{3}{2}} + 0.3867525237\sqrt{x} + 0.002408412494x^{\frac{5}{2}}, & 0.3 \le x \text{ and } x < 0.35 \\ 0.4462361975x^{\frac{3}{2}} + 0.3944927534\sqrt{x} + 0.03610813334x^{\frac{5}{2}}, & 0.35 \le x \text{ and } x < 0.4 \\ 0.4317929441x^{\frac{3}{2}} + 0.3988257294\sqrt{x} + 0.05055138668x^{\frac{5}{2}}, & 0.4 \le x \text{ and } x < 0.45 \\ 0.4053151508x^{\frac{3}{2}} + 0.4077624924\sqrt{x} + 0.07408847340x^{\frac{5}{2}}, & 0.45 \le x \text{ and } x < 0.5 \end{cases}$$

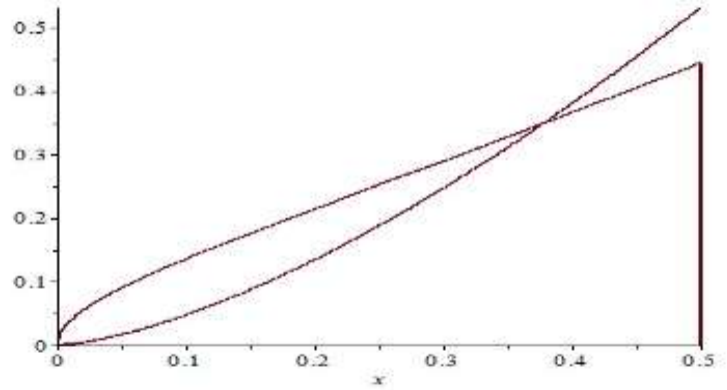

Figure 2: Plots of approximate solution $S^{(1/2)}(x)$ and $y^{(1/2)}$exact solution of the Example 3.1.

Table 1: The maximum error with fractional derivatives for the example 1

x	$e = \|S - y\|$	$e^{(1/2)} = \|S^{(1/2)} - y^{(1/2)}\|$
0.00	0.000001335	0.000000000
0.02	0.006419850	0.050551888
0.07	0.020209123	0.081667288
0.12	0.030709868	0.089793607
0.17	0.037872913	0.086289194
0.23	0.042013639	0.071322238
0.27	0.042055440	0.056474671
0.32	0.039038100	0.032690240
0.37	0.004537612	0.015144138
0.42	0.023773368	0.027735174
0.47	0.009550370	0.063408535
0.50	0.000000080	0.087193755

Example 3.2. [30] Consider the fractional differential equation

$$y''(x) + \frac{1}{2} D^{(1/2)} y(x) + y(x) = 4x^2(5x - 3) + \frac{1}{2} x^{\frac{7}{2}} \left(\frac{120x}{\Gamma\left(\frac{11}{2}\right)} - \frac{24}{\Gamma\left(\frac{9}{2}\right)} \right) + x^5 - x^4$$

$$y(0) = 0$$

$$y(0.5) = -0.03125$$

Exact solution $y(x) = x^4(x - 1)$

Using equations (18), (19) and (26) with maple programing, we obtain the following

$$S(x) = \begin{cases} -1.0\times10^{-14} - 0.00260571x + 0.00004389x^2 - 0.090358215x^3, & 0 \le x \text{ and } x < 0.05 \\ 0.00001857 - 0.00372050x + 0.02233968x^2 - 0.23899678x^3, & 0.05 \le x \text{ and } x < 0.1 \\ 0.00011481 - 0.00660749x + 0.05120959x^2 - 0.33522982x^3, & 0.1 \le x \text{ and } x < 0.15 \\ 0.00026155 - 0.00954225x + 0.07077466x^2 - 0.37870775x^3, & 0.15 \le x \text{ and } x < 0.2 \\ 0.00018802 - 0.00843943x + 0.06526056x^2 - 0.36951758x^3, & 0.2 \le x \text{ and } x < 0.25 \\ -0.00077126 + 0.00307215x + 0.01921419x^2 - 0.30812243x^3, & 0.25 \le x \text{ and } x < 0.3 \\ -0.00381667 + 0.03352616x - 0.0822991x^2 - 0.19532980x^3, & 0.3 \le x \text{ and } x < 0.35 \\ -0.01080790 + 0.09345101x - 0.25351300x^2 - 0.03226900x^3, & 0.35 \le x \text{ and } x < 0.4 \\ -0.02436935 + 0.19516192x - 0.50779028x^2 + 0.17962875x^3, & 0.4 \le x \text{ and } x < 0.45 \\ -0.04797254 + 0.35251649x - 0.85746705x^2 + 0.4386486x^3, & 0.45 \le x \text{ and } x < 0.5 \end{cases}$$

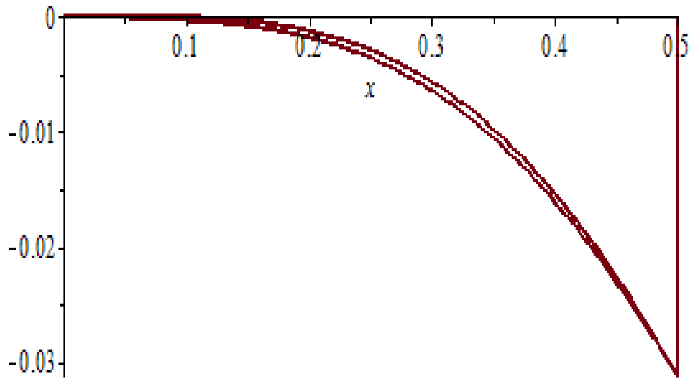

Figure 3: Plots of approximate solution *S*(x), and *y*(x) exact solution of the example 3.2

$$S^{\frac{1}{2}}(x) := \begin{cases} 0.000066045x^{\frac{3}{2}} - 0.00294023\sqrt{x} - 0.16313332x^{\frac{5}{2}}, & 0 \le x \text{ and } x < 0.05 \\ 0.03361018x^{\frac{3}{2}} - 0.00419814\sqrt{x} - 0.43148638x^{\frac{5}{2}}, & 0.05 \le x \text{ and } x < 0.1 \\ 0.07704512x^{\frac{3}{2}} - 0.00745576\sqrt{x} - 0.60522616x^{\frac{5}{2}}, & 0.1 \le x \text{ and } x < 0.15 \\ 0.10648087x^{\frac{3}{2}} - 0.01076728\sqrt{x} - 0.68372150x^{\frac{5}{2}}, & 0.15 \le x \text{ and } x < 0.2 \\ 0.09818488x^{\frac{3}{2}} - 0.00952288\sqrt{x} - 0.66712951x^{\frac{5}{2}}, & 0.2 \le x \text{ and } x < 0.25 \\ 0.02890786x^{\frac{3}{2}} + 0.00346655\sqrt{x} - 0.55628628x^{\frac{5}{2}}, & 0.25 \le x \text{ and } x < 0.3 \\ -0.1238195x^{\frac{3}{2}} + 0.03783022\sqrt{x} - 0.35264972x^{\frac{5}{2}}, & 0.3 \le x \text{ and } x < 0.35 \\ -0.38141171x^{\frac{3}{2}} + 0.10544817\sqrt{x} - 0.0582586x^{\frac{5}{2}}, & 0.35 \le x \text{ and } x < 0.4 \\ -0.76397330x^{\frac{3}{2}} + 0.22021664\sqrt{x} + 0.32430290x^{\frac{5}{2}}, & 0.4 \le x \text{ and } x < 0.45 \\ -1.29006394x^{\frac{3}{2}} + 0.39777225\sqrt{x} + 0.7919391x^{\frac{5}{2}}, & 0.45 \le x \text{ and } x < 0.5 \end{cases}$$

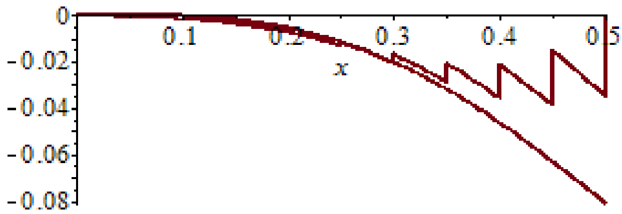

Figure 4: Plots of approximate solution $S^{(1/2)}(x)$ and $y^{(1/2)}$exact solution of the Example 3.2.

Table 2: The maximum error with fractional derivatives for the example 3.2

x	$e = \|S - y\|$	$e^{(1/2)} = \|S^{(1/2)} - y^{(1/2)}\|$
0.00	0.00000000	0.00000000
0.02	0.00005266	0.00042257
0.07	0.00033744	0.00087495
0.12	0.00033746	0.00132868
0.23	0.00064190	0.00169899
0.38	0.00074661	0.01079657
0.42	0.00061938	0.04468813
0.50	0.00000001	0.04621230

4 Conclusions

We used a cubic b-spline to numerically solve some fractional differential equations, and the tested their control points performance on the growth Bagley-Torvik differential equation. A theoretical proof of the convergence of the cubic b-spline method for the fractional derivatives from class arguments. Some numerical examples were included to observed by the tables which are illustrated the error estimation of the method. Finally, the efficient could arise from the control points of the cubic b-spline that is necessary for numerical results.

Conflict of Interests.
There are non-conflicts of interest

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استيفاء دالة تكامل مكعب b– spline لحل مسائل التفاضلية الكسرية

الخلاصة

في هذا البحث، استخدمنا مكعب b–spline لبناء متعدد حدود تقريبي ليطابق حسابات نقاط التحكم لحل دالة b–spline التكاملي المكعب لحل المعادلات التفاضلية الكسرية لقيم مختلفة. قدمنا امثلة عددية لنوضح مدى قابلية الطريقة ومقارنة النتائج مع الطرق المعروفة الأخرى، كذلك تم تحليل تقارب الطريقة المقدمة .