# The Chaotic Properties of the Shift Map and His Conjugacy of Horseshoe Map

**Farah Watan Kamel**
*Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq.*
Farahwatan2@gmail.com
**Iftichar M. T. AL - Shara'a**
*Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq.*
ifticharalshraa@gmail.com

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## Abstract

In our study, we prove some chaotic properties of the shift map $\sigma$ on the symbol space $\sum_2$ and apply the topological conjugacy property of the shift map on the horseshoe map.

**Keywords:** Devaney Chaos; Topological Conjugacy; Topological Mixing; Shift of finite type ; weakly blending; strongly blending .

## 1-Introduction

Chaotic dynamics generally refers to so complicated and seemingly random long term behavior exhibited in dynamical systems that keep simple, straightforward, deterministic laws . This type of dynamics can be seen in dynamical systems as various as electrical circuits, fluid dynamics, oscillating chemical reactions, and motion of planetary bodies .
The smale horseshoe map F is diffeomorphism defined on a square $T$ in the plane . The image of $\mathrm{F}(T)$ is bound to form a horseshoe like shape [1]. From the axioms that map which are topologically conjugate are totally equivalent in terms of their dynamics . Particularly, the horseshoe map is topologically conjugate to the shift map $\sigma$ . Hence the shift map is an exact model for the horseshoe map .

Let X be a compact metric space with no isolated point, $\mathcal{H}:\mathrm{X} \rightarrow \mathrm{X}$ be a map . In [2], Dzul-kifli and Good showed that the set of points with prime period at least $n$ it is dense for each $n$ if $\mathcal{H}$ is Devaney chaotic. In [3] Baloush and Dzul-kifli introduced six various one-step shift of finite type which have totally different dynamics demeanor and clear up the dynamics of each space . [4] showed that Locally Everywhere Onto implies many other chaos properties such as mixing , totally transitive ,and blending.

## 2. Preliminaries

If $\mathcal{H}:\mathrm{X} \rightarrow \mathrm{X}$ be a map on compact metric space with no isolated point, let $\mathrm{p} \in \mathrm{X}$ then $\mathcal{H}(\mathrm{p})$ = the first iterate of p for $\mathcal{H}$. More generally ,if n is any an integer , and $\mathrm{a_n}$ is the n-th iterate of p for $\mathcal{H}$ , then $\mathcal{H}(\mathrm{a_n})$ is the $(\mathrm{n}+1)$ st iterate of p for $\mathcal{H}$. **The orbit** of p it is the set of points $\mathrm{p}, \mathcal{H}(\mathrm{p}), \mathcal{H}^2(\mathrm{p}), \ldots$ , and is symbolize by $\mathrm{orb}(\mathrm{p}) =$

$\{\mathcal{H}^n(p)|n \in N_\circ\}$ such that $N_\circ = N \cup \{0\}$ . A point $p \in X$ it is said **a fixed point** of $\mathcal{H}$ if $\mathcal{H}(p) = p$ , $\mathcal{H}$ is said to be topologically transitive if $\exists\ n > 0$ such that $\mathcal{H}^n(U) \cap V \neq \emptyset$ , where U, V are any two non-empty open subsets of X . If $\exists\ \delta > 0$ for this any $x \in X$ and neighborhood N of x , $\exists\ y \in N$ and $n > 0$ where $d(\mathcal{H}^n(x), \mathcal{H}^n(y)) > \delta$, then $\mathcal{H}$ has sensitive dependence on initial conditions, briefly , we will write (SDIC). If for every pair non-empty open subsets U and V in X , there are a positive integer n such that $\mathcal{H}^k(U) \cap V \neq \emptyset$ for every $k > n$ , then we called that $\mathcal{H}$ is topological mixing . If for any pair of non-empty open sets U and V in X , there exists some $n > 0$ such that $\mathcal{H}^n(U) \cap \mathcal{H}^n(V) \neq \emptyset$ , then we called that $\mathcal{H}$ is weakly blending, , and called strongly blending, if for any pair of non-empty open sets U and V in X , there exists some $n > 0$ where $\mathcal{H}^n(U) \cap \mathcal{H}^n(v)$ contains a non-empty open subset. $\mathcal{H}$ is said to be *locally everywhere onto* if for every open set $U$ of $X$ , there exists a positive integer $n$ such that $\mathcal{H}^n(U) = X$ ,[4] . Let $\mathcal{H}: A \to A$ and $\mathcal{L}: B \to B$ be two continuous map , if there exists a homeomorphism $h: A \to B$ such that $h^\circ\mathcal{H} = \mathcal{L}^\circ h$ then $\mathcal{H}$ and $\mathcal{L}$ are called a topologically conjugate . The homeomorphism $h$ is said to be topological conjugacy between $\mathcal{H}$ and $\mathcal{L}$ . [5]

## 3. On Various Concepts for Topological Dynamical Systems

The set $A_m = \{0,1\}$ . we indicate to $A_m$ as an alphabet and its elements as a symbols . Let $\sum_2$ ($\sum_2^+$) be the set of each bi-infinite sequences (1-sided sequences) with their elements of $\sum_2$ ($\sum_2^+$) ,i.e. every element $\mathcal{S}$ of $\sum_2$ ($\sum_2^+$) is of the form :

$\mathcal{S} = \{\dots, \mathcal{S}_{-i}, \dots, \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_i, \dots\}$, $\mathcal{S}_i \in A_m$, or $\mathcal{S} = \{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_n, \dots\}$, $\mathcal{S}_i \in A_m$ .

Now take another sequence $\overline{\mathcal{S}} \in \sum_2$ , $\overline{\mathcal{S}} = \{\dots, \overline{\mathcal{S}}_{-i}, \dots, \overline{\mathcal{S}}_{-1}, \overline{\mathcal{S}}_0, \overline{\mathcal{S}}_1, \dots, \overline{\mathcal{S}}_i, \dots\}$ , $\overline{\mathcal{S}}_i \in A_m$ ,

or $\overline{\mathcal{S}} = \{\overline{\mathcal{S}}_0, \overline{\mathcal{S}}_1, \dots, \overline{\mathcal{S}}_n, \dots\}$, $\overline{\mathcal{S}}_i \in A_m$ . The metric between $\mathcal{S}$ and $\overline{\mathcal{S}}$ is defined as

$d(\mathcal{S}, \bar{\mathcal{S}}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}}$ , where $i \in \mathbb{Z}$ is the minimal number such that $\mathcal{S}_i \neq \bar{\mathcal{S}}_i$

In case of bi-infinite sequences , or

$d(\mathcal{S}, \bar{\mathcal{S}}) = \sum_{i=0}^{\infty} \frac{1}{2^i}$ , where $i \in \mathbb{N}$ is the minimal number such that $\mathcal{S}_i \neq \bar{\mathcal{S}}_i$

In case of 1-sided sequences . [6 ]

**Definition 3.1 :**

A shift of finite type (SFT) is a shift space $X \subset \sum_2$ which has a finite number of blocks from symbols 0 and 1 such that the blocks do not exist in any element of X . The blocks are called forbidden blocks in X . [3]

**Definition 3.2 :**

A shift of finite type (SFT) is an $M$ −step or have memory $M$ , for some integer $M \geq 1$ , if it can be described by a set of forbidden blocks that have length $M+1$ . [3]

Since we have four possible various blocks of length two i.e. 00,01,10 and 11 , then we have 16 sets of forbidden blocks, as shown in the table(1);

**Table(1)**

**For each $i = \{1, 2, \dots, 16\}, X_i \subseteq \sum_2$ is the one-step SFT with set of forbidden blocks $\mathcal{F}_i$ . [3]**

| | | | |
|---|---|---|---|
| $\mathcal{F}_1 = \emptyset$ | $\mathcal{F}_2 = \{00\}$ | $\mathcal{F}_3 = \{01\}$ | $\mathcal{F}_4 = \{10\}$ |
| $\mathcal{F}_5 = \{11\}$ | $\mathcal{F}_6 = \{00,01\}$ | $\mathcal{F}_7 = \{00,10\}$ | $\mathcal{F}_8 = \{00,11\}$ |
| $\mathcal{F}_9 = \{01,10\}$ | $\mathcal{F}_{10} = \{01,11\}$ | $\mathcal{F}_{11} = \{10,11\}$ | $\mathcal{F}_{12} = \{00,01,10\}$ |
| $\mathcal{F}_{13} = \{00,01,11\}$ | $\mathcal{F}_{14} = \{00,10,11\}$ | $\mathcal{F}_{15} = \{01,10,11\}$ | $\mathcal{F}_{16} = \{00,01,10,11\}$ |

## 4. Some Properties of the Shift Map

In this section we introduce the some results on shift of finite type space

**Theorem 4.1**

Let the one-step SFT $X_i , i = \{6,11,12,15\}$ is finite set then $\sigma: X_i \to X_i$ is stable

Proof :

If $X_i = X_6$ such that the forbidden is $\mathcal{F}_6 = \{00,01\}$ since for every $\mathcal{S} \in X_6$ , $\mathcal{S}_i \neq 0$ for every $i \in \mathbb{N}$ , so $\mathcal{S}_i = 1$ for every $i \in \mathbb{N}$ , so $\mathcal{S}_i = \{\overline{111}\}$ therefore $X_6$ is singleton set , so $\sigma$ has the fixed point $\{\overline{111}\}$ , and the basin of the fixed point is $X_6$ , so $\sigma$ is stable .

In the same way $X_{11} = \{\overline{000}\}, X_{12} = \{\overline{111}\}, X_{15} = \{\overline{000}\}$ . ■

**Theorem 4.2** :

Let the one-step SFT $X_i$ , $i = \{3,4\}$ is infinite set then $\sigma: X_i \to X_i , i = \{3,4\}$ is stable . Proof :

For every $X_i$ , $i = \{3,4\}$ the forbidden of $X_i$ , $i = \{3,4\}$ is $\mathcal{F}_3 = \{01\}$ and $\mathcal{F}_4 = \{10\}$ , so $X_i$ , $i = \{3,4\}$ has two fixed point and do not have any periodic point therefore the periodic point are not dense . Now let $\mathbb{U} = \{\overline{000}\}, \mathbb{V} = \{\overline{111}\}$ two open balls belong to $X_i$ , $i = \{3,4\}$ then for any $\mathcal{S} \in \mathbb{U}$ $\sigma^n(\mathcal{S}) \notin \mathbb{V}$ for all integer $n$ . So $X_i$ , $i = \{3,4\}$ is not transitive and not Deveaney chaotic . Now if take the same open balls $\mathbb{U}$ and $\mathbb{V}$ such that $\sigma^n(\mathbb{U}) \cap \sigma^n(\mathbb{V}) = \emptyset$ for any integer $n$ therefore $X_i$ , $i = \{3,4\}$ is not weakly blending . To prove $X_i$ , $i = \{3,4\}$ is not SDIC let $\mathcal{S} = \{\overline{00}\}$ and $\mathcal{T} = \{1\overline{00}\}$ then $d(\mathcal{S},\mathcal{T}) = 1$ , $\sigma^n(\mathcal{S}) = \{\overline{00}\}$ and $\sigma^n(\mathcal{T}) = \{\overline{00}\}$ so $d\big(\sigma^n(\mathcal{S}), \sigma^n(\mathcal{T})\big) = 0$ . Hence $X_i$ , $i = \{3,4\}$ is not SDIC . ■

**Theorem 4.3 :**

Let the one-step SFT $X_i$ , $i = \{7,9\}$ is finite set on discrete topology then $\sigma: X_i \to X_i$ , $i = \{7,9\}$ have a weakness chaotic .

Proof :

If $X_i = X_7$ and $X_7 = \{\overline{111}, 0\overline{111}\}$ there are only four open balls are $X_7, \emptyset, (\overline{111})$ *and* $(0\overline{111})$ . Because $\sigma(\overline{111}) \cap \sigma(0\overline{111}) = (\overline{111})$ so $X_7$ is strongly blending and weakly blending . Since $(0\overline{111})$ dose not contain any periodic point then the periodic points is not dense. $X_7$ is not transitive because $(0\overline{111}) \notin \sigma^n(\overline{111})$ for all integer $n$ . Now let $\mathcal{S} = (0\overline{11})$ and $\mathcal{T} = (\overline{11})$ then $d(\mathcal{S},\mathcal{T}) = 1$ , $\sigma^n(\mathcal{S}) = (\overline{11})$ , $\sigma^n(\mathcal{T}) = (\overline{11})$ so $d\big(\sigma^n(\mathcal{S}), \sigma^n(\mathcal{T})\big) = 0$ , therefore $\nexists\, \delta > 0$ such that $d\big(\sigma^n(\mathcal{S}), \sigma^n(\mathcal{T})\big) > \delta$ , therefore $X_7$ is not SDIC .

If $X_i = X_9$ and $X_9 = \{\overline{00}, \overline{11}\}$ there are four open balls in $X_9$ are $X_9, \emptyset, (\overline{00}),$ *and* $(\overline{11})$ . The periodic points of $X_9$ are dense since every point of $X_9$ is periodic . to prove it is not transitive , so that let $\mathbb{U} = (\overline{00})$ and , $\mathbb{V} = (\overline{11})$ , then $\sigma^n(\mathbb{U}) = (\overline{00})$ , and $\sigma^n(\mathbb{U}) \cap \mathbb{V} = \emptyset$, for all $n > 0$ . Now let $\delta = 1$, and let $\mathcal{S} = (\overline{00}), \mathcal{T} = (\overline{11})$ . So for each $n > 0$ , $\sigma^n(\mathcal{S}) = \mathcal{S}$ , $\sigma^n(\mathcal{T}) = \mathcal{T}$ , and $d\big(\sigma^n(\mathcal{S}), \sigma^n(\mathcal{T})\big) \geq 1$ , therefore $X_9$ is SDIC .To prove $X_9$ is not weakly blending and not strongly blending , let $\mathbb{U} = (\overline{00})$ and , $\mathbb{V} = (\overline{11})$. Since $\sigma^n(\mathbb{U}) = \mathbb{U}$ and $\sigma^n(\mathbb{V}) = \mathbb{V}$ , then $\sigma^n(\mathbb{U}) \cap \sigma^n(\mathbb{V}) = \emptyset$ , for all $n > 0$ . ■

**Theorem 4.4 :**

The one-step SFT $X_8$ on discrete topology then $\sigma: X_8 \to X_8$ has Devaney chaotic .

Proof:

Since $X_8 = \{\overline{01}, \overline{10}\}$ so it has the only open balls $X_8, \emptyset, (\overline{01})$ *and* $(\overline{10})$. The periodic points of $X_8$ are dense since every point is periodic point . Now let $\mathbb{U} = (\overline{01})$ and , $\mathbb{V} = (\overline{10})$. So for $n > 0$ , it is either $\sigma^n(\mathbb{U}) = \mathbb{U}$ or $\sigma^n(\mathbb{U}) = \mathbb{V}$ . If $\sigma^n(\mathbb{U}) = \mathbb{U}$ then $\sigma^{n+1}(\mathbb{U}) = \mathbb{V}$, and $\sigma^{n+1}(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$, and if $\sigma^n(\mathbb{U}) = \mathbb{V}$ , then $\sigma^n(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$ . therefore $X_8$ is transitive , so that $\sigma$ has Devaney chaotic. ■

**Remark 4.1:** If $X_8$ on any another topology then it is stable .
**Theorem 4.5 :**
The one-step SFT $X_2$ is infinite set then $\sigma: X_2 \to X_2$ has Devaney chaotic, and mixing topological, totally transitive, locally everywhere onto, weakly blending and strongly blending.

Proof:

To prove that the periodic points of $X_2$ are dense , let $\epsilon > 0$ and $\mathcal{S} = (\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \ldots) \in X_2$. Choose $n$ such that $\frac{1}{2^n} < \epsilon$ , now let $\mathcal{T} = (\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2 \ldots)$ be another point such that $\mathcal{S}_i = \mathcal{T}_i$ for $i = 0,1,2, \ldots, n$ . Then $d(\mathcal{S}, \mathcal{T}) < \frac{1}{2^n}$ , therefore the set of periodic point to be dense in $X_2$ we need to structure a periodic point within $\epsilon$ of $\mathcal{S}$ . Let $\mathcal{T} = (\overline{\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \ldots \mathcal{S}_n 1})$ it is obvious that $\mathcal{T}$ is periodic point within $\epsilon$ of $\mathcal{S}$. So the periodic point are dense in $X_2$ .

To show that $X_2$ is locally everywhere onto let $\mathbb{U}$ be nonempty open ball in $X_2$ such that $\mathcal{S} = (\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \ldots \mathcal{S}_n \ldots) \in \mathbb{U}$ , then we have two statuses ; status 1: if $\mathcal{S}_n = 1$, because $(10)$ and $(11)$ are allowed , then $\sigma^n(\mathbb{U}) = X_2$ . Status 2: if $\mathcal{S}_n = 0$ , because $(00)$ is forbidden then $\forall\, \mathcal{S} \in \mathbb{U}, \mathcal{S}_{n+1} = 1$ therefore $\sigma^{n+1}(\mathbb{U}) = X_2$ . because for every open set $\mathcal{S} \subseteq X_2$ there exists a positive integer $n$ such that $\sigma^n(\mathbb{U}) = X_2$ , so that $X_2$ is locally everywhere onto .

Since $X_2$ is locally everywhere onto , therefore it is transitive , topological mixing , totally transitive , weakly blending and strongly blending . And since $X_2$ has dense of periodic point and transitive , then it is SDIC, therefore $X_2$ is Devaney chaotic . ■

## 5. The Horseshoe Map

The horseshoe map will be denoted by F. Its domain is the set $S$ in $\mathbb{R}^2$ collected of the unit square $T = [0,1] \times [0,1]$ , bounded on the left and right by semicircles $B$ and $E$ such that $S$ contains its boundary . The map F shrinks $S$ vertically by a factor of $a < 1/3$ , and expands $S$ horizontally by a factor of $b = 3$ . The result figure is folded by F therefore it fits again inside $S$ , with only the semicircles popeyed to the left of $T$. Thus the range of F looks like a horseshoe when $S$ is partitioned . We can see the effect of F on each member of the partition . Specifically, F sends semicircles $B$ and $E$ in to $B$ and sends the square $T$ into two strips inside $T$ plus a curved strip inside $E$ . [7]

The base interest in the horseshoe map F is to describe its dynamics on the attractor :

$$\Lambda = \{X \in T : F^n(X) \in T \ , \qquad \forall\ n \in \mathbb{Z}\}$$

To make our task easier , we first consider the set

$$\Lambda^{+} = \{X : F^{n}(X) \in T \ , \ \forall \ n \in \mathbb{Z}^{+}\}$$

For the positive orbit of X , $orb^{+}(X)$ . To be in $T$ , X must belong to either $V_0$ or $V_1$ . Now if $F^{n}(X) \in T$ , then obviously $F^{n}(X) \in V_0 \cup V_1$ or $X \in F^{-1}(V_0) \cup F^{-1}(V_1)$ . Now if deduce that $\Lambda^{+}$ is the product of a cantor set with a vertical interval .

Next we take the set

$$\Lambda^{-} = \{X : F^{n}(X) \in T \ , \ \forall \ n \in \mathbb{Z}^{-}\}$$

For the negative orbit of X , $orb^{-}(X)$ . To be in $T$ , X must belong to either $E_0 = F(V_0)$ or $E_1 = F(V_1)$ . Now if $F^{-1}(X) \in E_0 \cup E_1$ , therefore $X \in F(E_0) \cup F(E_1)$ , (see Fig (1)) . [8]

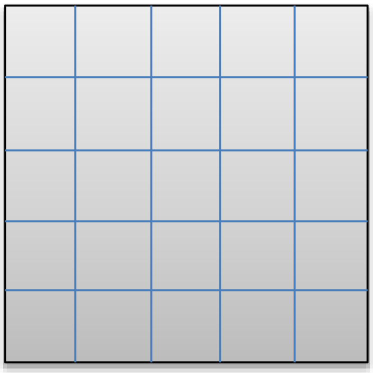


Figure (1)

## 6. Applications of topological conjugacy

Now , we define the topological conjugacy map $h : \sum_2 \to \Lambda$ is defined as follows :

For $\mathcal{S} \in \sum_2$ we let

$$h(\mathcal{S}) = \{\mathcal{S}_0\mathcal{S}_2\mathcal{S}_1 \dots\} , \quad \text{where} \quad \mathcal{S}_n = \begin{cases} H^{n}(\mathcal{S}) \in V_0 & \mathcal{S}_n = 0 \ , \quad n \in \mathbb{Z}^{+} \\ H^{n}(\mathcal{S}) \in V_1 & \mathcal{S}_n = 1 \ , \quad n \in \mathbb{Z}^{+} \end{cases}$$

$$h(\mathcal{S}) = \{\dots\mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}\} , \quad \text{where} \quad \mathcal{S}_n = \begin{cases} H^{n}(\mathcal{S}) \in E_0 & \mathcal{S}_n = 0 \ , \quad n \in \mathbb{Z}^{-} \\ H^{n}(\mathcal{S}) \in E_1 & \mathcal{S}_n = 1 \ , \quad n \in \mathbb{Z}^{-} \end{cases}$$

**Theorem 6.1 :**

Let $h : \sum_2 \to \Lambda$ be a map , then $h$ is a homeomorphism .

Proof :

To prove that $h$ is one-to-one . Let $\mathcal{S}$ and $\mathcal{T}$ are in $\sum_2$ , and $h(\mathcal{S}) = h(\mathcal{T})$ , then $h(\mathcal{S})(h^{-1}(\mathcal{S}))$ and $h(\mathcal{T})$ $(h^{-1}(\mathcal{T}))$ lie on the same vertical ( horizontal) line in $T$ , such that thy have the same forward (backward) sequence . Therefore $\mathcal{S} = \mathcal{T}$ , so that $h$ is one-to-one .

To prove that $h$ is onto , let $J_n = \{V\ in\ C_0 \cup C_1\ :\ h(\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots) = V\}$ and $J_{-n} = \{V\ in\ C_0 \cup C_1\ :\ h(\dots\mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}) = V\}$ then $J_n$ and $J_{-n}$ are closed for all $n$ . Because $\bigcap_{n\geq 0} J_n$ is a single vertical line and $\bigcap_{n<0} J_n$ is single horizontal line in $T$ . It follows that $\bigcap_{-\infty<n<\infty} J_n$ is a unique point $V^*$ . By construction , $h(\mathcal{S}) = V^*$such that $\mathcal{S} = \cdots \mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}, \mathcal{S}_0\mathcal{S}_2\mathcal{S}_1 \dots$ , in $\sum_2$ for all $n$ . so that $h$ is onto .

Therefore we need only to show that $h$ and $h^{-1}$ are continuous , let $\mathcal{S} = \cdots \mathcal{S}_{-2}\mathcal{S}_{-1}, \mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots$ and $\mathcal{T} = \cdots \mathcal{T}_{-2}\mathcal{T}_{-1}, \mathcal{T}_0\mathcal{T}_1\mathcal{T}_2 \dots$ be in $\sum_2$ , with $h(\mathcal{S}) = V$ and $h(\mathcal{T}) = W$ , if $d(\mathcal{S},\mathcal{T}) = \|\mathcal{S} - \mathcal{T}\| = \sum_{|k|=n+1}^{\infty} \frac{|\mathcal{S}_k - \mathcal{T}_k|}{2^{|k|}} \leq \frac{1}{2^{n-1}}$ , then $\mathcal{S}_k = \mathcal{T}_k$ for $k = 0,1,2,\dots,n$ , then $V$ and $W$ lie in the same vertical strip of width $1/_{3^{n+1}}$ , so that $\|V - W\| < 1/_{3^{n+1}}$ . similarly , $\mathcal{S}_k = \mathcal{T}_k$ for $k = -1,-2,\dots,-n$ ,which means that $V$ and $W$ lie in the same horizontal strip at the $n$th stage . So that there is a $\delta_1 > 0$ such that $\|V - W\| < \delta_1$ . Now choose $\delta > 0$ such that $\delta < 1/_{3^{n+1}}$ and $\delta < \delta_1$ it follows that $d(V,W) = \|V - W\| < \delta$ . consequently $h$ is continuous . The proof that $h^{-1}$ is continuous follows by a similar argument . ■

**Proposition 6.2 [9]**

1. $d$ is a metric on $\sum_2$ .
2. If $\mathcal{S}_i = \mathcal{T}_i$ for $i = 0,\dots,k$ , then $d[\mathcal{S},\mathcal{T}] \leq 1/2^k$ .
3. If $d[\mathcal{S},\mathcal{T}] < 1/2^k$ then $\mathcal{S}_i = \mathcal{T}_i$ for $i \leq k$ .

**Theorem 6.3** :

Let $I_1 = V_0 \cap E_0$ , $I_2 = V_1 \cap E_0$ , $I_3 = V_0 \cap E_1$ and $I_4 = V_1 \cap E_1$ , Then $\bigcup_{j=1}^{4} I_j$ is closed and invariant under $h$.

Proof :

It is clear $h(I_j) \subset I_j$ , $j = 1,2,3,4$ so $\bigcup_{j=1}^{4} I_j$ is invariant . To prove that $\bigcup_{j=1}^{4} I_j$ is closed, we need prove that each $I_j$ , $j = 1,2,3,4$ is closed , we suppose that $\mathcal{S} \in I_j$ , $j = 1,2,3,4$ such that $\mathcal{S} = \mathcal{S}_0\ \mathcal{S}_1\ \mathcal{S}_2 \dots$, $\mathcal{S}_i \in \{0,1\}$ for every $i \in \mathbb{N}$ which converge to $\mathcal{T}$. Let $\mathcal{T} \notin I_j$ , $j = 1,2,3,4$ . Since the $\mathcal{S}$ converge to $\mathcal{T}$, there is another integer $k$ such that , if $i > k$ then $d(\mathcal{S},\mathcal{T}) \leq 1/2^{\alpha+1}$. By Proposition [4.3] , this forces $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{\alpha+1}$ to agree with the corresponding entries of $\mathcal{S}_i$ for $i \geq k$ , so that $\mathcal{T}_\alpha \in \{0,1\}$ and $\mathcal{T} \in I_j$ , $j = 1,2,3,4$ , so $\bigcup_{j=1}^{4} I_j$ is closed . ■

**Proposition 6.4 :** The $h(X_1)$ is located in $\bigcup_{j=1}^{4} I_j$ .

Proof: since $X_1$ dose not have any forbidden block , then $h(X_1)$ is located in $\bigcup_{j=1}^{4} I_j$ .

**Proposition 6.5 :** The $h(\mathrm{X}_2)$ is located in $I_2 \cup I_3 \cup I_4$ .

Proof: Let $\mathcal{S} \in \mathrm{X}_2$ such that for every $i \in \mathbb{Z}$ , $\mathcal{S}_i = 0$ , since $\{00\}$ is forbidden block, but 10 is allowed therefore $\mathcal{S}_{i-1} = 1$ . and if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 1 \; or \; 0$ since 11 and 01 are allowed , so that $h(\mathrm{X}_2) \notin I_1$ , then $h(\mathrm{X}_2)$ is located in $I_2 \cup I_3 \cup I_4$ .

**Proposition 6.6 :** The $h(\mathrm{X}_3)$ is located in $I_1 \cup I_3 \cup I_4$ .

Proof: Let $\mathcal{S} \in \mathrm{X}_3$ such that for every $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 1$ for every $i \in \mathbb{Z}$ since 01 are forbidden , and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0 \; or \; 1$ since 00 and 10 are allowed , therefore $h(\mathrm{X}_3) \notin I_2$ and $h(\mathrm{X}_3)$ is located in $I_1 \cup I_3 \cup I_4$ .

**Proposition 6.7** : The $h(\mathrm{X}_4)$ is located in $I_1 \cup I_2 \cup I_4$ .

Proof: Let $\mathcal{S} \in \mathrm{X}_4$ such that for every $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0$ for every $i \in \mathbb{Z}$ since 10 are forbidden , and if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0 \; or \; 1$ since 00 and 01 are allowed, therefore $h(\mathrm{X}_4) \notin I_3$ so $h(\mathrm{X}_4)$ is located in $I_1 \cup I_2 \cup I_4$ .

**Proposition 6.8 :** The $h(\mathrm{X}_5)$ is located in $I_1 \cup I_2 \cup I_3$ .

Proof: Since the only forbidden block is $\{11\}$ so that 01,10 and 00 are allowed , therefore for every $\mathcal{S} \in \mathrm{X}_5$,if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0$ , $i \in \mathbb{Z}$ , therefore $h(\mathrm{X}_5) \notin I_4$ so $h(\mathrm{X}_5)$ is located in $I_1 \cup I_2 \cup I_3$

**Proposition 6.9 :** The $h(\mathrm{X}_6)$ is located in $I_4$ .

Proof: since the forbidden block of $\mathrm{X}_6$ is $\{00,01\}$ then for every $\mathcal{S} \in \mathrm{X}_6$ , $\mathcal{S}_i \neq 0$ for every $i \in \mathbb{N}$ . since 11 is allowed then $\mathcal{S}_i = 1$ for every $i \in \mathbb{Z}$ , therefore $h(\mathrm{X}_6) \notin I_1 \cup I_2 \cup I_3$ and $h(\mathrm{X}_6)$ is located in $I_4$ .

**Proposition 6.10 :** The $h(\mathrm{X}_7)$ is located in $I_2 \cup I_4$ .

Proof: Since $\mathrm{X}_7$ has two forbidden block $\{00\}$and $\{10\}$ then there is $\mathcal{S}_i = 1$ and $\mathcal{S}_{i-1} = 0 \; or \; 1$ , $i \in \mathbb{Z}$ such that $\mathrm{X}_7 = \{\overline{111}, 0\overline{111}\}$ so that $h(\mathrm{X}_7) \notin I_1 \cup I_3$ and $h(\mathrm{X}_7)$ is located in $I_2 \cup I_4$ .

**Proposition 6.11 :** The $h(\mathrm{X}_8)$ is located in $I_2 \cup I_3$ .

Proof: let $\mathcal{S} \in \mathrm{X}_8$ such that for every $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0$ and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 1$ , $i \in \mathbb{Z}$ , since $\{00\}$ and $\{11\}$ are forbidden block , so that $h(\mathrm{X}_8) \notin I_1 \cup I_4$ and $h(\mathrm{X}_8)$ is located in $I_2 \cup I_3$ .

**Proposition 6.12 :** The $h(\mathrm{X}_9)$ is located in $I_1 \cup I_4$ .

Proof: since $\mathrm{X}_9$ has two forbidden block $\{01\}$and $\{10\}$ then let $\mathcal{S} \in \mathrm{X}_9$ such that for every $i \in \mathbb{Z}$ if $\mathcal{S}_i = 1$ than $\mathcal{S}_{i-1} = 1$ and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0$ , so that $h(\mathrm{X}_9) \notin I_2 \cup I_3$ and $h(\mathrm{X}_9)$ is located in $I_1 \cup I_4$ .

**Proposition 6.13 :** The $h(\mathrm{X}_{10})$ is located in $I_1 \cup I_3$ .

Proof: since $\mathrm{X}_{10}$ has two forbidden block $\{01\}$and $\{11\}$ then let $\mathcal{S} \in \mathrm{X}_{10}$ such that for every $i \in \mathbb{Z}$ if $\mathcal{S}_i = 1$ than $\mathcal{S}_{i-1} = 0\ or\ 1$ such that $\mathrm{X}_{10} = \{\overline{000}, 1\overline{000}\}$ , so $h(\mathrm{X}_{10}) \notin I_2 \cup I_4$ and $h(\mathrm{X}_{10})$ is located in $I_1 \cup I_3$ .

**Proposition 6.14 :** The $h(\mathrm{X}_{11})$ is located in $I_1$ .

Proof: since the forbidden block of $\mathrm{X}_{11}$ is $\{10\}$ and $\{11\}$ then for every $i \in \mathbb{Z}$ let $\mathcal{S} \in \mathrm{X}_{11}$ , $\mathcal{S}_i \neq 1$ . since 00 is allowed then $\mathcal{S}_i = 0$ for every $i \in \mathbb{Z}$ , therefore $h(\mathrm{X}_{11}) \notin I_2 \cup I_3 \cup I_4$ and $h(\mathrm{X}_{11})$ is located in $I_1$ .

**Proposition 6.15 :** The $h(\mathrm{X}_{12})$ is located in $I_4$ .

Proof: The forbidden block of $\mathrm{X}_{12}$ is $\{00\}$ , $\{01\}$ and $\{10\}$ so that for every $\mathcal{S} \in \mathrm{X}_{12}$ , $\mathcal{S}_i = 1$ , $i \in \mathbb{Z}$ , therefore $h(\mathrm{X}_{12}) \notin I_1 \cup I_2 \cup I_3$ and so $h(\mathrm{X}_{12})$ is located in $I_4$ .

**Proposition 6.16** The $h(\mathrm{X}_{13})$ , $h(\mathrm{X}_{14})$ and $h(\mathrm{X}_{16})$ are empty .

**Proposition 6.17** : The $h(\mathrm{X}_{15})$ is located in $I_1$ .

Proof: The forbidden block of $\mathrm{X}_{15}$ is $\{01\}$ , $\{10\}$ and $\{11\}$ so that for every $\mathcal{S} \in \mathrm{X}_{15}$ , $\mathcal{S}_i = 0$ , $i \in \mathbb{Z}$ , therefore $h(\mathrm{X}_{15}) \notin I_2 \cup I_3 \cup I_4$ and so $h(\mathrm{X}_{15})$ is located in $I_1$ .

Let $\mathrm{M}_1 = \bigcup_{j=1}^{4} I_j$ , $\mathrm{M}_2 = I_2 \cup I_3 \cup I_4$ , $\mathrm{M}_3 = I_1 \cup I_3 \cup I_4$ , $\mathrm{M}_4 = I_1 \cup I_2 \cup I_4$ , $\mathrm{M}_5 = I_1 \cup I_2 \cup I_3$ , $\mathrm{M}_6 = I_4$ , $\mathrm{M}_7 = I_2 \cup I_4$ , $\mathrm{M}_8 = I_2 \cup I_3$ , $\mathrm{M}_9 = I_1 \cup I_4$ , $\mathrm{M}_{10} = I_1 \cup I_3$ , $\mathrm{M}_{11} = I_1$.

## 7. Some Chaotic Properties of the Shift Map

**Theorem 7.**1 :

Let the map $\sigma: \mathrm{X}_i \to \mathrm{X}_i , i = \{2,5\}$ is $h-$ conjugate to the map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i , i = \{2,5\}$ , and $\sigma$ has chaotic map in sense of Devaney , topologically mixing , totally transitive , weakly blending , strongly blending and locally everywhere onto then so F .

Proof:

To prove the set of periodic points in $\mathrm{M}_i , i = \{2,5\}$ is dense , let $\mathbb{U}$ be any open set of $\mathrm{M}_i , i = \{2,5\}$ and since that $\sigma$ $h-$conjugates F , then $h^{-1}(\mathbb{U})$ is an open set of $\mathrm{X}_i , i = \{2,5\}$ and thus must contain a $p-$periodic point $\mathcal{S} \in \mathrm{X}_i , i = \{2,5\}$ . Since $\mathcal{S} = \sigma^p(\mathcal{S})$ , so that $h(\mathcal{S}) = h\big(\sigma^p(\mathcal{S})\big) = (\mathrm{F})^p\big(h(\mathcal{S})\big)$ . So $h(\mathcal{S})$ is a $p-$periodic point of F . Furthermore , $h(\mathcal{S}) \in h\big(h^{-1}(\mathbb{U})\big) = \mathbb{U}$ , and therefore the set of periodic points are dense in $\mathrm{M}_i , i = \{2,5\}$ . To prove F has locally everywhere onto , let $\mathbb{U}$ be any open set in $\mathrm{M}_i , i = \{2,5\}$ then $h^{-1}(\mathbb{U})$ is an open set of $\mathrm{X}_i , i = \{2,5\}$ .

Since $\sigma$ is locally everywhere onto , there exists a positive integer $n$ such that $\sigma^n\left(h^{-1}(\mathbb{U})\right) = \mathrm{X}_i\,, i = \{2,5\}$ , so that $h\left(\sigma^n\left(h^{-1}(\mathbb{U})\right)\right) = (\mathrm{F})^n\left(h\left(h^{-1}(\mathbb{U})\right)\right) = (\mathrm{F})^n(\mathbb{U})$ . Since $h$ is one to one and onto then $(\mathrm{F})^n(\mathbb{U}) = \mathrm{M}_i\,, i = \{2,5\}$ , So F has locally everywhere onto .Since F has locally everywhere onto then it is transitive ,topologically mixing , totally transitive, weakly blending and strongly blending . Also since Fhas dense periodic point and transitive then it is SDIC and F has Devaney chaotic . ■

**Theorem 7.**2 :

Let the map $\sigma: \mathrm{X}_i \to \mathrm{X}_i\,, i = \{3,4,6,11,12,15\}$ is $h-$ conjugate to the map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i\,, i = \{3,4,6,11\}$ ,so that F is stable .

**Theorem 7.**3 :

Let the map $\sigma: \mathrm{X}_i \to \mathrm{X}_i\,, i = \{7,10\}$ is $h-$ conjugate to the map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i\,, i = \{7,10\}$ , so that F has weakly blending and strongly blending .

Proof:

It is sufficient to prove F has strongly blending . Let $\mathbb{U}$ and $\mathbb{V}$ be two open sets in $\mathrm{M}_i\,, i = \{7,10\}$ . Since $\sigma$ has weakly blending and strongly blending then $h^{-1}(\mathbb{U})$ and $h^{-1}(\mathbb{V})$ are open sets of $\mathrm{X}_i\,, i = \{7,10\}$ and thus $\sigma^n\left(h^{-1}(\mathbb{U})\right) \cap \sigma^n\left(h^{-1}(\mathbb{V})\right)$ contains an open set , so

$$h\left(\sigma^n\left(h^{-1}(\mathbb{U})\right)\right) \cap h\left(\sigma^n\left(h^{-1}(\mathbb{V})\right)\right)$$

$$= (\mathrm{F})^n\left(h\left(h^{-1}(\mathbb{U})\right)\right) \cap (\mathrm{F})^n\left(h\left(h^{-1}(\mathbb{V})\right)\right)$$

$$= (\mathrm{F})^n(\mathbb{U}) \cap (\mathrm{F})^n(\mathbb{V}) \text{ contains an open set also .}$$

Hence F has weakly blending and strongly blending . ■

**Theorem 7.**4 :

Let the map $\sigma: \mathrm{X}_8 \to \mathrm{X}_8$ is $h-$ conjugate to the map $\mathrm{F}: \mathrm{M}_8 \to \mathrm{M}_8$ , than F is chaotic map in sense of Devaney .

Proof:

To prove that F is chaotic , we first prove that it is transitive . Let $\mathbb{U}$ and $\mathbb{V}$ be two open sets in $\mathrm{M}_8$ and suppose that F $h-$ conjugates $\sigma$ , then $h(\mathbb{U})$ and $h(\mathbb{V})$ are open sets in $\mathrm{X}_8$. Since $\sigma$ is transitive , there exists $n \in \mathbb{Z}^+$ such that $\sigma^n\left(h(\mathbb{U})\right) \cap h(\mathbb{V}) \neq \emptyset$ . Hence $h\left((\mathrm{F})^n(\mathbb{U})\right) \cap h(\mathbb{V}) \neq \emptyset$ , so $(\mathrm{F})^n(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$ . Hence F is transitive .

To prove that the set of periodic points are dense in $\mathrm{M}_8$ . , let $\mathbb{U}$ be any open set of $\mathrm{M}_8$ and since that $\sigma$ $h$ −conjugates F , then $h^{-1}(\mathbb{U})$ is an open set of $\mathrm{X}_8$ so there is a $p$ −periodic point $\mathcal{S} \in \mathrm{X}_8$ . Since $\mathcal{S} = \sigma^p(\mathcal{S})$ , so that $h(\mathcal{S}) = h(\sigma^p(\mathcal{S})) = (\mathrm{F})^p(h(\mathcal{S}))$ . So $h(\mathcal{S})$ is a $p$ −periodic point of F . Furthermore , $h(\mathcal{S}) \in h(h^{-1}(\mathbb{U})) = \mathbb{U}$ , and therefore the set of periodic points are dense in $\mathrm{M}_8$ , so that F is Devaney chaotic . ■

**Theorem 7.**5 :

Let the map $\sigma: \mathrm{X}_9 \to \mathrm{X}_9$ is $h -$ conjugate to the map $\mathrm{F}: \mathrm{M}_9 \to \mathrm{M}_9$ , then F has dense periodic points and has SDIC .

Proof:

By the same technique used in the previous proof , so that F has dense periodic points. To prove SDIC , let $\delta > 0$ and $\mathcal{S} \in \mathrm{M}_9$ and N is neighborhood of $\mathcal{S}$ , $\exists$ $\mathcal{T} \in \mathrm{N}$ and suppose that F $h -$ conjugates $\sigma$ , then $h(\mathcal{S}) \in \mathrm{X}_9$ and $h(\mathrm{N})$ is neighborhood of $h(\mathcal{S})$ . Since $\mathrm{X}_9$ is SDIC then for all $n > 0$ , $d\left(\sigma^n(h(\mathcal{S})), \sigma^n(h(\mathcal{T}))\right) > \delta_1$ , hence $d\left(h((\mathrm{F})^n(\mathcal{S})), h((\mathrm{F})^n(\mathcal{T}))\right) > \delta_1$ , $d\left(h^{-1}\left(h((\mathrm{F})^n(\mathcal{S}))\right), h^{-1}\left(h((\mathrm{F})^n(\mathcal{T}))\right)\right) > \delta_1$ . Consequently , $d((\mathrm{F})^n(\mathcal{S}), (\mathrm{F})^n(\mathcal{T})) > \delta$ , so F is SDIC . ■

## Conclusions

- The map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i$ , $i = \{2,5\}$ has chaotic map in sense of Devaney , topologically mixing , totally transitive , weakly blending , strongly blending and locally everywhere onto .
- The map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i$ , $i = \{3,4,6,11\}$ , is stable .
- The map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i$ , $i = \{7,10\}$ has weakly blending and strongly blending .
- The map $\mathrm{F}: \mathrm{M}_8 \to \mathrm{M}_8$ has chaotic in sense of Devaney .
- The map $\mathrm{F}: \mathrm{M}_9 \to \mathrm{M}_9$ , has dense periodic points and has SDIC .

**Conflict of Interests.**
**There are non-conflicts of interest**

## References

[1] S. Smale, "Differential Dynamical Systems" , *Bull. Amer. Math. Soc* , 1967.
[2] S. C. Dzul-Kifli and C. Good, "On Devaney Chaos and Dense Periodic Point: Period 3 and Higher Implies Chaos" , *Mathematical Association of America* , Vol. 122, N. 8, PP: 773-780, 2015.
[3] M. Baloush and S.C. Dzul-Kifli ,"The Dynamics of 1-Step Shifts of Finite Type Over Tow Symbols" , *Indian Journal of Science and Technology* , Vol.9 , N. 46, pp:1-6, 2016 .

[4] M. Baloush and S.C. Dzul-Kifli, "On Some Strong Chaotic Properties of Dynamical Systems" , *American Institute of Physics* , Vol:1830, pp. 1-7, 2017 .
[5] I. Bhaumik and B. S. Choudhury, "The Shift Map and The Symbolic Dynamics and Application of Topological Conjugacy" , *Journal of Physical Sciences* , Vol:13 , pp. 149-160, 2009 .
[6] X. S. Yang ,"Topological Horseshoe and Computer Assisted Verification of Chaotic Dynamics" , *International Journal of Bifurcation and Chaos* , Vol:19 , No.4 , pp. 1127-1145 , 2009.
[7] D. Gulick , *Encounters with Chaos and Fractals* , *2nd ed.* , Taylor & Francis Group, 2012 .
[8] S. N. Elaydi, *Discrete Chaos* , *2nd ed.*, Taylor & Francis Group, 2007 .
[9] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems* , *2nd ed.*, Addison Wesley, 1989.

## الخلاصة

في هذا العمل, درسنا بعض الخصائص الفوضوية المختلفة لفضاء الضرب على دالة التزحيف. اوجدنا ترافقا تبولوجيا بين دالة التزحيف ودالة حدوة الحصان لنقل الخواص الفوضوية المدروسة على فضاء $\sum_2$ .

**الكلمات الدالة :** فوضى ديفيني؛ الترافق التبولوجي ؛ تبولوجي ممزوج ؛ التزحيف في خطوة واحدة ؛خلط ضعيف ؛ خلط بقوة .