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A sequential Monte Carlo approach for the pricing of barrier option under a stochastic volatility model

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In this paper we propose a numerical scheme to estimate the price of a barrier option in a general framework. More precisely, we extend a classical Sequential Monte Carlo approach, developed under the hypothesis of deterministic volatility, to Stochastic Volatility models, in order to improve the efficiency of Standard Monte Carlo techniques in the case of barrier options whose underlying approaches the barriers. The paper concludes with the application of our procedure to two case studies under a SABR model.

keywords: Barrier Options, Stochastic Volatility, Sequential Monte Carlo methods, Bayesian re-sampling techniques, SABR model.

1 Introduction

Barrier options are very common exotic options. This popularity is due to two reasons: they are one of the simplest example of exotic options, which are generally characterized by a high degree of complexity, and they are less expensive than vanilla options. Barrier options differ from vanilla options for the presence of numerical constraints, named *barriers*, which are determined when the contract is concluded and can activate or extinguish the barrier option if the underlying reaches them. On the basis of the action of barriers on the barrier options, we can distinguish two kinds of barrier options: *knock-out* and *knock-in*. A knock-out option is extinguished if the underlying hits the barriers, and in this case its pay-off is null, otherwise its pay-off coincides with the one of a vanilla

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option. A knock-in option is activated if the underlying reaches the barrier, and in this case its pay-off coincides with the one of a vanilla option, otherwise its pay-off is null.

We are interested in pricing barrier options whose underlying is ruled by a continuous stochastic process, and the barriers are represented by two deterministic continuous functions l_t and u_t, defined in a time interval $[0;T]$, with $l_t < u_t \forall t \in [0;T]$ (l_t is named *lower barrier*, u_t is named *upper barrier*) (for a general description of barrier options see Hull, 2003). In no-arbitrage models, the price P of a barrier option is defined as the actualized expectation of the pay-off function under the risk-neutral measure:

$$P = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[h(s_T)1_{I_t}(s_t)_{t\in[0;T]}\right], \tag{1}$$

where s_t is the stochastic process describing the underlying; t denotes the time; T indicates the option maturity; $h(x)$ is the vanilla option pay-off function, i.e. $h(x) = \max(x-k;0)$ if a call, $h(x) = \max(k-x;0)$ if a put, with k as strike price; $1_{I_t}(x)$ denotes the indicator function of I_t, where $I_t =]l_t;u_t[$; r indicates the constant risk-free interest rate; $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation of a random variable under the risk-neutral measure $\mathbb{Q}$.

Differently to vanilla options, the existence of solutions in closed form for the (1) holds only under the strong assumption of the flat structure of market parameters (Merton, 1973; Heynen and Kat, 994b; Rubinstein and Reiner, 1991; Kunitomo and Ikeda, 1992; Broadie et al., 1997). In the more general *Stochastic Volatility Models*, barrier options are generally priced via numerical simulation, based on *lattices rules* (Hull and White, 1993; Kat and Verdonk, 1995), *finite difference schemes* (Dewynne and Wilmott, 1993), *Monte Carlo methods* (Gobet, 2009; Hammersley, 2013; Jackel, 2002; Giles, 2008; Gobet and Menozzi, 2010; Glassermann, 2004).

Monte Carlo (MC) methods are the election methods in multi-assets options. This family of methods finds application in other scientific fields, as statistical parameter estimation (Shamany et al., 2019; Amusa et al., 2019; Algamal, 2018; Félix and Menezes, 2018). A MC procedure for evaluating (1) works as follows. The time interval $[0;T]$ is divided into N subintervals $\{[t_{n-1};t_n]\}_{n=1}^{N}$, with $0 = t_0 < ... < t_N = T$ and, for every $n = 1,...,N$, a set of M numbers $(s_n^{(m)})_{m=1}^{M}$, named *particles*, are extracted from the density of the underlying value in t_n. The expectation in (1) is approximated by the sample mean of the M-dimensional vector $\{h(s_N^{(m)})1_{I_{t_n}}(s_n^{(m)})\}_{m=1}^{M}$, called *paths.* In the case of a knock-out option, the MC approach suffers from a loss of effectiveness when the underlying approaches the barriers (Glasserman and Staum, 2001). In this case, many paths are rejected because they hit the barriers, and this decreases the effective number of paths used in the MC procedure: this implies an increase of the variance, since it depends on the inverse of M.

This problem has been analyzed by many authors. In Baldi et al. (1999) the efficiency of a MC estimator is increased by introducing a correction, represented by the probability that the underlying does not cross the barrier. In Jasra and Del Moral (2011); Deborshee et al. (2017); Shevchenko and Del Moral (2016); Cuomo et al. (2016) this result has been further improved by means of *Sequential Monte Carlo* (MCse) methods. Sequential MC methods have been used in several fields of economics, as portfolio management

(Carmona et al., 2009; Del Moral and Patras, 2011), capital allocation problem (Targino et al., 2015), and in a more general statistical context, as parameter estimation (Hasan et al., 2013; Dey and Maiti, 2012; Prakash, 2013) and classification (McCallum et al., 1998; Rish et al., 2001)[1]. In Jasra and Del Moral (2011); Deborshee et al. (2017); Shevchenko and Del Moral (2016); Cuomo et al. (2016) the MCse works as follows. At every time step, the particles with the lowest probability to be in the barriers interval are replaced by the ones with the highest probability, while the particles which do not satisfy the barriers condition are rejected. MCse schemes have been developed under the Black-Scholes framework, where the volatility of the underlying and the interest rate are supposed to be deterministic.

This paper presents a novel MCse scheme to evaluate the pricing formula (1). The contributions of this paper can be summarized as follows: we construct a MCse estimator for continuous barrier options to solve the problem of high bias and low precision of a MCst estimator under a stochastic volatility model. More precisely, we generalize the approach formulated in Baldi et al. (1999), which holds only under non-stochastic volatility models, by using some more general results illustrated in Baldi et al. (1999), concerning the expression of the probability of a particle of not crossing the barriers. Numerical experiments show the improvement in terms of quality of our MCse approach. Differently to Baldi et al. (1999), who deals with barrier options under deterministic scenarios, we apply our procedure to the case of single constant barrier options by assuming as market model a *SABR model*, introduced by Hagan et al. in 2002 (Hagan et al., 2002, 2015).

The paper is organized as follows. Section 2 recalls MC estimators in deterministic scenarios. Section 3 describes our extension of the MCse approach to volatility stochastic frameworks. In section 4 the results of some numerical experiments show the effectiveness of our method. Finally, in Section 5 we draw some conclusions.

2 MC estimators in deterministic models

In Shevchenko and Del Moral (2016); Jasra and Del Moral (2011) MC estimators for (1) are discussed under non-stochastic volatility models. The underling asset price s_t is assumed to evolve as a Geometric Brownian motion:

$$ds_t = \mu_t s_t dt + \sigma_t s_t dw_t, \tag{2}$$

where μ_t and σ_t are respectively the drift and the volatility of the underlying, which are supposed to be piecewise constant functions of time, and w_t represents a Brownian motion.

Let $N(0,1)$ be a Normal standard random variable, let $0 < t_0 < t_1 < ... < t_N = T$ be a fixed time discretization of $[0;T]$, s_n, σ_n and μ_n respectively the value of the underlying, volatility and drift in t_n, h the time step defined as T/N. In the MC methodology (Shevchenko and Del Moral, 2016; Baldi et al., 1999), the continuous stochastic process

[1]For statistical methods in classification see also Iorio et al. (2018, 2016); Pandolfo et al. (2018).

$(s_t)_{t\in[0;T]}$ is approximated with the N-dimensional discrete stochastic process $(s_n)_{n=1}^N$, deriving from the application of log-normality discretization scheme to (2):

$$s_n = s_{n-1} exp\left[\left(\mu_n - \frac{\sigma_n^2}{2}\right)h + \sigma_n\sqrt{h}z_n\right], \quad z_n \sim \mathcal{N}(0,1). \tag{3}$$

At the $n-$th time step n, a set of M weights

$$\widetilde{\mathbb{G}}_n^{(m)} \;=\; \widetilde{g}_n^{(m)} 1_{I_n}(s_n^{(m)}), \;\; m = 1, ..., M \tag{4}$$

is defined, where

$$\widetilde{g}_n^{(m)} \;=\; \mathbb{P}(s_n^{(m)} \in I_n | s_{n-1}^{(m)}) \tag{5}$$

is obtained from the distribution law of the maximum of a Brownian motion, and may be considered as being the probability of a particle of staying inside the barriers within the time interval $[t_{n-1}, t_n]$. For the cases of (i) *single* ($u_n = l_n = b_n$) and (ii) *double* ($u_n \neq l_n$) barrier option,

(i) $\widetilde{g}_n^{(m)} = 1 - \exp\left(-\frac{2}{\sigma^2 h}\left(\ln(s_n/b_n)\ln(s_{n-1}/b_n)\right)\right)$;

(ii) $\widetilde{g}_n^{(m)} = 1 - \sum_{m=1}^{+\infty}[R_n(\alpha_n m - \gamma_n, x_n)] + R_n(-\alpha_n m + \beta_n, x_n)] +$

$$\begin{aligned} & + \sum_{m=1}^{+\infty}[R_n(\alpha_n m, x_n)] + R_n(-\alpha_n m, x_n)] \\ x_n &= ln\frac{s_n}{s_{n-1}}, \quad \alpha_n = 2\ln\frac{u_n}{l_n}, \quad \beta_n = 2\ln\frac{u_n}{s_{n-1}} \\ \gamma_n &= 2\ln\frac{s_{n-1}}{L_n}, \quad R_n(z,x) = exp\left(-\frac{z(z-2x)}{2\sigma_n^2\delta t}\right). \end{aligned}$$

The weights $\widetilde{\mathbb{G}}_n^{(m)}$ *measure the chance to stay within the barriers during the time interval* $[t_{n-1}; t_n]$ *(Shevchenko and Del Moral, 2016)*. The discrete version of the barrier option (1) is given by

$$P_N = e^{-rT}E^{\mathbb{Q}}\left[h(s_N)\prod_{n=1}^{N}\widetilde{\mathbb{G}}_n^{(m)}\right]. \tag{6}$$

We point out that the hypothesis of deterministic volatility guarantees the existence and uniqueness of the risk-neutral measure.

A Standard Monte Carlo (MCst) scheme (Shevchenko and Del Moral, 2016) to evaluate an estimator Θ of (6) is characterized by the following steps: a) generation of M particles $(s_n^{(m)})_{n=1}^N$ according to (3); b) calculation of the weights $\widetilde{g}_n^{(m)}$ and $\widetilde{\mathbb{G}}_n^{(m)}$ by using the definitions (4−5) and the formula (2); c) evaluation of the estimator Θ, defined as:

$$\Theta = \frac{e^{-rT}}{M} \sum_{m=1}^{M} h(s_N^{(m)}) \prod_{n=0}^{N-1} \widetilde{\mathbb{G}}_n^{(m)}. \tag{7}$$

In Shevchenko and Del Moral (2016) a MCse estimator Λ for the barrier option has been introduced. At every time step n, the random vector $(s_n^{(m)}; \widetilde{\mathbb{G}}_n^{(m)})$ is subject to a re-sampling procedure according to the normalized weights $\widetilde{\mathcal{G}}_n^{(m)}$, given by:

$$\widetilde{\mathcal{G}}_n^{(m)} = \frac{\widetilde{\mathbb{G}}_n^{(m)}}{\sum_{m=1}^{M} \widetilde{\mathbb{G}}_n^{(m)}}. \tag{8}$$

The resampling has a twofold goal: firstly, to discard the particles which are outside the barrier interval; secondly, to replace the particles with the lowest probability of staying in the interval with the ones with the highest probability. The MCse procedure to evaluate Λ can be summarized as follows: a) generation of $(s_n^{(m)})_{m=1}^{M}$ according to (3) at every time step n; b) computation of $\widetilde{g}_n^{(m)}$ and $\widetilde{\mathbb{G}}_n^{(m)}$; c) re-sampling from the discrete random variable $(s_n^{(m)}; \mathbb{G}_n^{(m)})_{m=1}^{M}$ according to $\mathcal{G}_n^{(m)}$ defined in (8); d) evaluation of the following estimator Λ:

$$\Lambda = \frac{e^{-rT}}{M^N} \left(\prod_{n=0}^{N-1} \widetilde{\mathbb{G}}_n^{(m)} \right) \left[\sum_{m=1}^{M} h\left(s_N^{(m)} \right) \right]. \tag{9}$$

In short, in the MCst the evaluation of the multiple integral in (6) is based on the use of a set of particles $s_n^{(m)}$ with Log-Normal distribution, to which the weights $\widetilde{\mathbb{G}}_n^{(m)}$ are associated. In the MCse the particles and the weights are selected as in the MCst, but three further steps are taken: the normalization of the weights $\widetilde{\mathbb{G}}_m^{(m)}$, the implementation re-sampling procedure, reduction of the multiple integral (6) to a chain of simple integrals.

3 MC estimators in stochastic volatility models

In the following we assume that the positive underlying s_t and its volatility σ_t satisfy the following stochastic differential system (SDS)

$$\begin{aligned} ds_t &= \mu_t(s_t)dt + \sigma_t(s_t)dw_t^{(s)}, \qquad (10) \\ d\sigma_t &= \alpha_t(\sigma_t)dt + \beta_t(\sigma_t)dw_t^{(\sigma)} \qquad (11) \\ dw_t^{(s)}dw_t^{(\sigma)} &= \rho dt \quad t \in [0;T], \end{aligned}$$

where $\mu_t(\cdot), \sigma_t(s_t), \alpha_t(\cdot), \beta_t(\cdot)$ are real measurable functions defined in $[0;T] \times \mathbb{R}$, $w_t^{(s)}$ and $w_t^{(\sigma)}$ are Brownian motions with correlation index ρ. Unlike in the case of the Black-Scholes model, the assumption (11) does not guarantee the uniqueness of the risk.neutral measure and, as a consequence, the price P in (1) is not uniquely defined. Therefore our problem is selecting a particular risk-neutral measure $\mathbb{Q}$ from the set $(\mathbb{Q}_n)_{n\mathbb{N}}$ of possible risk-neutral measures consistent with the absence of arbitrage. As claimed in Back (2006), a risk-neutral measure $\mathbb{Q}$ must be chosen on the basis of the preferences and endowments of investors and production possibilities (to simplify the notations, in the (11) and in the following the dependence of α_t and β_t from $\mathbb{Q}$ is omitted).

In this more general scenario the discrete process $(s_n)_{n=1}^N$ is determined by applying the following Euler scheme to (10−11)

$$\begin{aligned} s_n &= s_{n-1} + r_{n-1}h + \sqrt{h}\sigma_{n-1}z_n, \quad z_n \sim \mathcal{N}(0,1), && (12)\\ \sigma_n &= \sigma_{n-1} + \alpha_{n-1}h + \sqrt{h}\beta_{n-1}v_n, \quad v_n \sim \mathcal{N}(0,1), && (13)\\ n &= 1, ..., N. \end{aligned}$$

Baldi et al. (1999) provide a numerical scheme to compute a MCst estimator Π to price barrier options, which is illustrated in Algorithm MCst. In order to implement this procedure, at every time step n they consider the conditional probability of a particle of not crossing the barriers, given the knowledge of the underlying values at the previous states. Here this probability is denoted with the symbol $g_n^{(m)}$, and its expression, for the cases of (i) *single* ($u_n = l_n = b_n$) and (ii) *double* ($u_n \neq l_n$) barrier option, is given respectively by:

(i) $g_n^{(m)} = 1 - \exp\left(-\frac{2}{\sigma^2 h}\left((s_n - b_n)(s_{n-1} - b_n)\right)\right);$

(ii) $g_n^{(m)} = 1 - \sum_{m=1}^{+\infty}[R_n(\alpha_n m - \gamma_n, x_n)] + R_n(-\alpha_n m + \beta_n, x_n)] +$

$$\begin{aligned} &+ \sum_{m=1}^{+\infty}[R_n(\alpha_n m, x_n)] + R_n(-\alpha_n m, x_n)]\\ x_n &= (s_n - s_{n-1}), \quad \alpha_n = 2(u_n - l_n), \quad \beta_n = 2(u_n - s_{n-1}),\\ \gamma_n &= 2(s_{n-1} - l_n), \quad R_n(z, x) = exp\left(-\frac{z(z-2x)}{2\sigma_n^2 \delta t}\right). \end{aligned}$$

In the following The MCst procedure proposed by Baldi is sketched.

Algorithm 1: MCst

---

begin

  for $n = 1$ *to* N **do**

    1): Generate $(s_n^{(m)}; \sigma_n^{(m)})_{m=1}^M$ according to (12−13).

    2): Computation of $g_n^{(m)}$ by using (3).

    3): Evaluation of $\mathbb{G}_n^{(m)}$ from (4).

  4): Evaluation of the following estimator Π:

$$\Pi = \frac{e^{-rT}}{M}\sum_{m=1}^{M}\left(h(s_N^{(m)})\prod_{n=0}^{N-1}\mathbb{G}_n^{(m)}\right). \tag{14}$$

  return Π.

The rest of this section is devoted to the design of a new MCse scheme whose goal is to produce in output an estimator with lower variance, bias and mean squared error than the MCst estimator (14), when the underlying approaches the barriers, which obtained by using a re-sampling technique (properties of re-sampling techniques in MCse methods have been studied in Moral (2004)). Similar to what is done in the MCse approach, a set of normalized weights, to be used in the re-sampling procedure, are defined as follows:

$$\mathcal{G}_n^{(m)} = \frac{\mathbb{G}_n^{(m)}}{\sum_{m=1}^{M} \mathbb{G}_n^{(m)}}. \tag{15}$$

Figure 1 shows the flow chart of our MCse scheme. Our MCse procedure to price barrier options is summarized in Algorithm MCse.

Algorithm 2: MCse

begin
  for $n = 1$ *to* N **do**
    1: Generation of $(s_n^{(m)}; \sigma_n^{(m)})_{m=1}^{M}$ according to (12−13).
    2: Computation of $g_n^{(m)}$ by using (3).
    3: Calculation of $\mathbb{G}^{(m)}$ according to (4).
    4: Determination of the normalized weights $\mathcal{G}_n^{(m)}$, defined in (15).
    5: Re-sampling of $(s_n^{(m)}; \mathbb{G}_n^{(m)})$ according to weights $\mathcal{G}_n^{(m)}$.
  6: Evaluation of Σ, given by

$$\Sigma = \frac{e^{-rT}}{M^N} \left(\prod_{n=0}^{N-1} \mathbb{G}_n^{(m)} \right) \left[\sum_{m=1}^{M} h\left(s_N^{(m)} \right) \right]. \tag{16}$$

  return Σ.

The key issue of Algorithm MCse is the re-sampling step. In literature many re-sampling techniques have been developed, as the *Multinomial Re-sampling* (Efron and Tibshirani, 1994; Efron, 1992), the *Residual Re-sampling* (Liu and Chen, 1998; Whitley, 1994), the *Stratified Re-samping* (Kitagawa, 1996; Douc and Cappé, 2005), the *Systematic Re-sampling* (Carpenter et al., 1999). Multinomial resampling is very easy to implement and has remarkable statistical properties (Douc and Cappé, 2005), which are sketched in the following. Let $N_m, m = 1, ..., M$ be the number of duplicates of the particle $s_n^{(m)}$ for every $n = 1, ..., N$, $\mathcal{F}_n$ the σ-algebra generated by the process $s_{0:n}$; then:

- at every time step n, the conditional variance of the underlying particles can be expressed in a closed form as:

$$V\left[\frac{1}{n}\sum_{i=1}^{n} s_i | \mathcal{F}_n\right] = \frac{1}{M}\left[\sum_{m=1}^{M} \mathbb{G}_n^{(m)} (s_i^{(m)})^2 - \left(\sum_{m=1}^{M} \mathbb{G}_n^{(m)} s_i^{(m)}\right)^2\right];$$

- at every time step n, the asymptotic consistence of the underlying particles is ensured:

$$nV\left[\frac{1}{n}\sum_{i=1}^{n} s_i \middle| \mathcal{F}_n\right] \longrightarrow V[s_i];$$

- The mean and the variance of N_m is given by:

$$E[N_m] = M\mathbb{G}_n^{(m)} \quad V[N_m] = M\mathbb{G}_n^{(m)}(1 - \mathbb{G}_n^{(m)}).$$

At every time step n, the multinomial resampling can be summarized as follows:

(i) an uniform M-dimensional random vector u on $[0;1]$ is generated;

(ii) the quantities $S_i = \sum_{m=1}^{i} \mathcal{G}_n^{(m)}$are computed, where the $\mathcal{G}_n^{(m)}$ are defined in 8;

(iii) the index i is determined such that $S_{i-1} \leq u \leq S_i$;

(iv) $(s_n^{(j)}, \mathbb{G}_n^{(j)}) = (s_n^{(i)}, \mathbb{G}_n^{(i)})$ for $j = 1$ *to* M and i found at the previous step;

In the next section we provide some numerical experiments, in which we show that our MCse estimator has better performances than the MCst one in terms of variance, bias and mean squared error.

4 Numerical Experiments

In Baldi et al. (1999), the MCst algorithm is applied to the case of constant barrier puts under non-stochastic volatility models. In this section we compare MCst to MCse to price a barrier option under a SABR dynamic (Hagan et al., 2002). We assume that the stochastic differential system (10−11) takes the form:

$$ds_t = rs_t dt + \sigma_t s_t^{\beta} dw_t^s \tag{17}$$
$$d\sigma_t = \alpha\sigma_t dw_t^{\sigma}. \tag{18}$$

We consider three data sets (T1 with high volatility−of−volatility, T2 with zero correlation, T3 with low initial volatility and zero correlation), whose parameter values are reported in Table 1. For each data set three different test problems were generated varying the value of b, 2, 1.3 and 0.7 for T1, and 2, 1.7 an 0.7 for T2 and T3, in order to check how the closeness of such parameter to the barrier impacts the algorithms behaviour. Finally, the number of time steps varied from 10 to 80.

Table 1: Test parameters (iV and rP stand for implicit Volatility and real Price, respectively)

	r	s_0	k	σ_0	T	test α	β	ρ	M	b	iV	rP
T1	0.02	0.4	1	0.2	5 years	0.3	1	-0.5	10^3	2	0.1843	0.5067
T1	0.02	0.4	1	0.2	5 years	0.3	1	-0.5	10^3	1.3	0.1843	0.5067
T1	0.02	0.4	1	0.2	5 years	0.3	1	-0.5	10^3	0.7	0.1843	0.4614
T2	0.02	0.5	1	0.5	4 years	0.4	0.5	0	10^3	2	0.6372	0.5675
T2	0.02	0.5	1	0.5	4 years	0.4	0.5	0	10^3	1.7	0.6372	0.5548
T2	0.02	0.5	1	0.5	4 years	0.4	0.5	0	10^3	0.7	0.6372	0.5492
T3	0.02	0.6	1	0.08	3 years	0.4	0.5	0	10^3	2	0.12	0.5234
T3	0.02	0.6	1	0.08	3 years	0.4	0.5	0	10^3	1.7	0.12	0.5121
T3	0.02	0.6	1	0.08	3 years	0.4	0.5	0	10^3	0.7	0.6372	0.5021

For each test problem, MCst was compared to MCse by analyzing the *variance*, the *mean squared error* (MSE) and the bias over 1000 independent runs, computed with respect to P_N, which was evaluated through the following steps: a) calculation of the implicit volatility via the *Hagan Formula* (Hagan et al., 2002); b) computation of the price of the new barrier option with constant volatility set equal to the implicit volatility found at the previous point, via the *Kunimoto-Ikeda Formula* (Kunitomo and Ikeda, 1992).

Tables (2−4) reports the results for the T1 test problems. We observe that the distance of the initial underlying from the barrier has a rather meaningful impact on the algorithms. Table 2 shows that MCse performs just slightly better than MCst when $b = 2$; in this case, at the beginning the option starts with an initial underlying value which is sufficiently far from the barrier. Because of that, the probability that the particles generated at the following steps hit the barrier is quite low, and the resambling strategy in the MCse has a very limited effect with respect to MCst.

The improvement of the MCse respect to the MCst can be also justified as follows. At every step, the MCse makes use of more information than the MCst: in fact, differently to the MCst case, in the MCse we condition on the past underlying and volatility values.

The picture dramatically changes when the initial underlying approaches the barrier ($b = 1.3$, Table 3). In this case, as expected, the behaviour of both algorithms deteriorates, However, the worsening of the results appears much more dramatic for the MCst. Looking at the variance and bias, they appear that MCse is much less erratic in its behaviour with respect to MCst. For the MCst, MSE is about twice (respectively) larger than the corresponding values for the MCse. This lives up our expectations about the validity of the resampling strategy adopted in MCse to tackle the drawback of the loss of effectiveness of MCst when the underlying gets close to the barrier. When the initial underlying is very close to the barrier ($b = .7$, Table 4) the performance deterioration of the two pricing strategies is even more evident, but still MCse seems to clearly outperform MCst.

The number of time steps has not a significative impact on the performance of the MCst estimator thanks to the re-sampling procedure, which lets us substitute the reject paths with the survival ones.

Table 2: Bias, Variance and MSE of MC estimators for b=2€(data set T1)
.

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.0346	0.0332	0.0140	0.0139	0.0152	0.0150
N=20	0.0374	0.0332	0.0161	0.0142	0.0175	0.0153
N=30	-0.0379	-0.0335	0.0173	0.0146	0.0187	0.0157
N=40	0.0383	0.0338	0.0180	0.0149	0.018	0.0160
N=50	-0.0389	-0.0342	0.0184	0.0152	0.0199	0.0164
N=60	0.0392	0.0347	0.0189	0.0157	0.0204	0.0169
N=70	0.0399	0.0351	0.0193	0.0158	0.0209	0.0170
N=80	-0.0405	-0.0355	0.0197	0.0161	0.0213	0.0174

Table 3: Bias, Variance and MSE of MC estimators for b=1.3€(data set T1)

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.0495	0.0440	0.0295	0.0160	0.0320	0.0179
N=20	-0.0509	-0.0453	0.0307	0.0176	0.0333	0.0197
N=30	-0.0515	-0.0459	0.0319	0.0184	0.0346	0.0205
N=40	0.0528	0.0471	0.0331	0.0190	0.0359	0.0212
N=50	0.0542	0.0482	0.0343	0.0199	0.0372	0.0222
N=60	0.0552	0.0496	0.0358	0.0205	0.0388	0.0230
N=70	0.0571	0.0506	0.0370	0.0213	0.0403	0.0239
N=80	0.0579	0.0515	0.0381	0.0246	0.0415	0.0273

Table 4: Bias, Variance and MSE of MC estimators for b=0.7€(data set T1)

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	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.0807	0.0615	0.0629	0.0471	0.0694	0.0509
N=20	-0.0823	-0.0625	0.0648	0.0499	0.0716	0.0538
N=30	0.0846	0.0641	0.0669	0.0503	0.0741	0.0544
N=40	0.0879	0.0678	0.0695	0.0537	0.0772	0.0583
N=50	-0.0901	-0.0696	0.0723	0.0579	0.0804	0.0627
N=60	0.0953	0.0743	0.0786	0.0614	0.0877	0.0669
N=70	-0.1003	-0.0797	0.0849	0.0691	0.0950	0.0755
N=80	-0.1084	-0.0825	0.0953	0.0758	0.1071	0.0826

The same considerations as above apply to numerical experiments for the data set T2, whose results are reported in Tables 5-7. In general, varying the data set parameter we obtained different scenarios (whose results are not reported here for the sake of space), but for which MCse shown to be clearly superior to MCst when the underlying approaches the barrier.

Table 5: Bias, Variance and MSE of MC estimators for b=2€
(data set T2)

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.0446	0.0433	0.0246	0.0238	0.0266	0.0257
N=20	0.0475	0.0439	0.0268	0.0240	0.0291	0.0259
N=30	-0.0480	-0.0450	0.0273	0.0246	0.0296	0.0266
N=40	0.0493	0.0490	0.0280	0.0262	0.0304	0.0286
N=50	-0.0503	-0.0498	0.0284	0.0270	0.0309	0.0295
N=60	0.0525	0.0504	0.0289	0.0286	0.0204	0.0169
N=70	0.0548	0.0531	0.0299	0.0293	0.0329	0.0321
N=80	-0.0561	-0.0555	0.0322	0.0306	0.0353	0.0337

Table 6: Bias, Variance and MSE of MC estimators for b=1.7€(data set T2).

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.2131	0.1398	0.6352	0.4120	0.5811	0.3814
N=20	0.2127	0.1340	0.6214	0.4110	0.5713	0.3490
N=30	0.2105	0.1375	0.6130	0.4108	0.5704	0.3459
N=40	0.2102	0.1372	0.6159	0.4103	0.5700	0.3408
N=50	0.2100	0.1369	0.6160	0.4101	0.5683	0.3401
N=60	0.1984	0.1360	0.6156	0.4100	0.5674	0.3396
N=70	0.1980	0.1359	0.6140	0.4097	0.5638	0.3372
N=80	0.1973	0.1357	0.6139	0.4092	0.5601	0.3321

The considerations relative to the previous cases are also valid for the data set T3, whose results are reported in Tables 8-10.

Table 7: Bias, Variance and MSE of MC estimators for b=0.7€(data set T2)

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.2927	0.1725	0.6739	0.4581	0.7600	0.4879
N=20	-0.2933	-0.1735	0.6758	0.4598	0.7618	0.4899
N=30	0.2962	0.1751	0.6789	0.4613	0.7666	0.4920
N=40	0.2979	0.1878	0.6795	0.4637	0.7682	0.4990
N=50	-0.3011	-0.1802	0.6823	0.4679	0.7730	0.5003
N=60	0.3163	0.1863	0.6896	0.4724	0.7897	0.5071
N=70	-0.3192	-0.1901	0.6959	0.4791	0.7978	0.5152
N=80	-0.3284	-0.1948	0.6981	0.4827	0.8059	0.5207

Table 8: Bias, Variance and MSE of MC estimators for b=2€
(data set T3)

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.0356	0.0342	0.0176	0.0140	0.0189	0.0152
N=20	0.0374	0.0357	0.0183	0.0142	0.0200	0.0155
N=30	-0.0375	-0.0341	0.0193	0.0146	0.0207	0.0158
N=40	0.0382	0.0361	0.0201	0.0155	0.0216	0.0168
N=50	-0.0413	-0.0381	0.0225	0.0176	0.0272	0.0191
N=60	0.0431	0.0400	0.0241	0.0199	0.0280	0.0022
N=70	0.0458	0.0421	0.0276	0.0230	0.0293	0.0248
N=80	-0.0430	-0.0436	0.0301	0.0258	0.0320	0.0278

5 Conclusions

In this paper we have presented a numerical scheme to estimate a barrier option price in Stochastic Volatility Models. Our approach extends a classical MCse procedure, developed under the assumption of deterministic volatility, in order to improve the efficiency of MC methods in the case of an option whose underlying approaches the barriers. At every time step, a set of particles are generated and re-sampled according to their probability of not hitting the barriers; finally, this values are used to estimate the integral in barrier option pricing formula via MC approach. In order to test the validity of our method, our framework has been applied to two single knock-out puts.

Table 9: Bias, Variance and MSE of MC estimators for b=1.7€(data set T3).

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.1812	0.1278	0.5287	0.3410	0.5606	0.3573
N=20	0.1838	0.1281	0.5534	0.3480	0.5872	0.3639
N=30	0.1867	0.1285	0.5731	0.3499	0.6080	0.3664
N=40	0.1893	0.1292	0.5969	0.3518	0.6327	0.3685
N=50	0.1902	0.1299	0.6060	0.3528	0.6422	0.3700
N=60	0.1947	0.1306	0.6086	0.3557	0.6465	0.3728
N=70	0.1988	0.1318	0.6101	0.3585	0.6500	0.3759
N=80	0.2135	0.1327	0.6118	0.3599	0.6574	0.3775

Table 10: Bias, Variance and MSE of MC estimators for b=0.7€(data set T3)

	Bias		Variance		MSE	
Time Steps	Standard	Sequential	Standard	Sequential	Standard	Sequential
N=10	0.1937	0.0736	0.5724	0.3581	0.6099	0.3635
N=20	-0.1938	-0.0740	0.5758	0.3598	0.6173	0.3653
N=30	0.1962	0.0752	0.5788	0.3613	0.6173	0.3670
N=40	0.1979	0.0778	0.5798	0.3637	0.6190	0.3699
N=50	-0.2011	-0.0823	0.5813	0.3679	0.6217	0.3747
N=60	0.2163	0.0853	0.5872	0.3724	0.6340	0.3797
N=70	-0.2192	-0.0914	0.5939	0.3791	0.6420	0.3875
N=80	-0.2284	-0.0956	0.5971	0.3827	0.6493	0.3919

Our experiments provided evidence that the increase of the information, resulting from the conditioning respect to the past underlying and volatility values, and our proposed resampling strategy embedded in the MCst indeed improved in a rather meaningful way towards enhancing the overall pricing strategy robustness.

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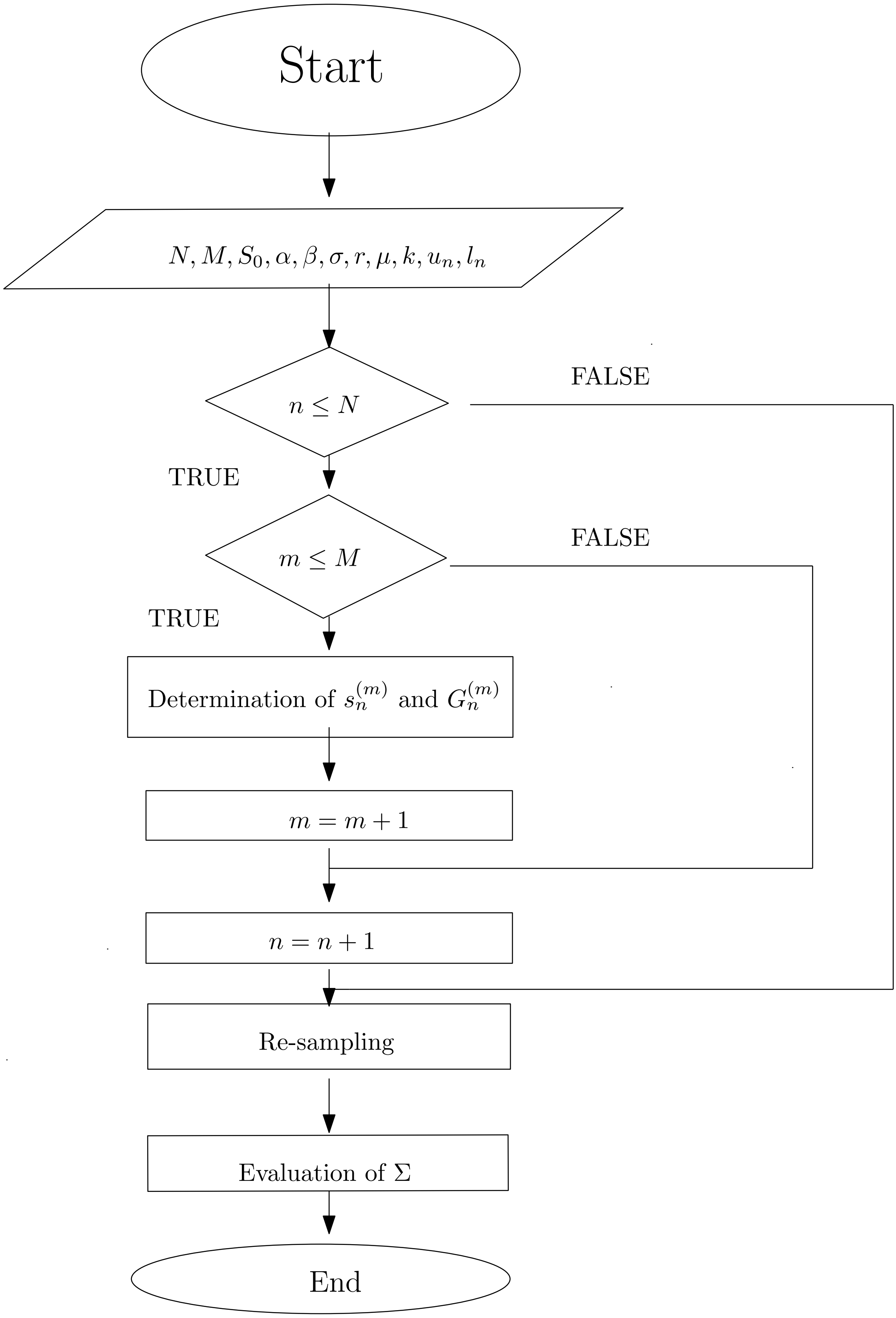

Figure 1: **Flow chart of the MCse procedure**