Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol. 8, No. 2, June 2020, pp.684-692

The new distribution (Topp Leone Marshall Olkin-Weibull) properties with an application

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ABSTRACT

The few standard probability distributions available are insufficient for modelling naturally occurring events. Most especially, normal distribution is not ideal for modelling asymmetrical data. Generalizing a new distribution is an important area in probability theory. It is known that we can improve the performance of the distributions by adding a new parameter. So, in this paper, we introduce Topp Leone Marshall Olkin-Weibull distribution as new method with four parameters based on recently proposed family (Topp Leone Marshall-Olkin of distributions). Accordingly, mathematical properties such as quantile function, moments generating functions, entropy, and order statistics have been investigated. The estimation of four parameters by maximum likelihood method was implemented. Application for real data based on the proposed method can be employed to show the best fitting as compared with the other models.

Keywords: Topp Leone Marshall Olkin-G family, Moment, Maximum likelihood estimation (MLE), Weibull distribution.

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1. Introduction

There are numerous powerful distributions in statistical literature available to use for analyzing data in various scientific fields that require an appropriate model. The modeling and analyzing of data are very important, in which the quality of the outputs of statistical analysis depends heavily on the assumed model. Therefore, it is necessary to find more nearer distributions to get accurate result for this. Some attempts have been made to define new classes of univariate continuous distributions that extended a well-known distribution, by adding some parameters to the baseline model to provide a more flexibility in modeling data.
Herein, Marshall–Olkin –G family is introduced and applied by Marshall and Olkin in 1997 [1], as a method to expand family of distributions. Maxwell et al., in 2019, used this family for generating Inverse Lomax distribution to be applied on Cancer Stem Cell. In 1955, Topp Leone distribution was proposed by Topp and Leone [2]. Recently, several studies are presented based on Topp Leone family variations such as Topp Leone generator distribution [3], Topp Leone G family [4], Topp Leone Burr XII distribution [5] and Topp–Leone Lomax Distribution applied in Airbone Communication Transceiver Dataset. [6]. Abdullah et al., have employed the same family for estimating parameters for extended Burr type X distribution by using conjugate gradient in unconstrained optimization [7,8]. AbuJarad et al., has employed Bayesian Reliability Analysis based on Marshall and Olkin Model for estimation of exponential distribution [8].
On the other hand, Weibull distribution is commonly used for modeling data in practice in many systems with monotone failure rates. To improve the fitting of Weibull distribution, it is necessary to make modified density function with many attempts to extend this distribution. For instance, the exponentiated Weibull [9], the modified Weibull distribution [10], and the extended Weibull compound distribution of Marshal Olkin [11], have been reported in this concern.

A mutation in generating the new family of distribution by using new technique as the Weibull-X family by adding some parameters to the baseline distribution has been testified in [12]. Other researchers have extended the aforementioned family to get a new family with its distribution like exponentiated flexible Weibull extension distribution [13], the beta flexible Weibull distribution [14], the Weibull Burr-X distribution and exponentiated Kumaraswamy exponentiated Weibull [16]. Khaleel et al., generalized Weibull uniform distribution to get a new uniform distribution with bathtub-shaped failure rate [17].

In this paper, a new distribution has been derived and termed as Topp Leone Marshall Olkin- Weibull (TLMO-W) distribution with four parameters, depending on the recently proposed family (Topp Leone Marshall-Olkin of distributions). Mathematical characteristics such as quantile function, moments generating functions, entropy, and order statistics have been examined. The estimation of four parameters by maximum likelihood method was implemented. Application for real data can be employed straightforwardly to show the best fitting as compared with the other models reported in the literature.

2. Topp Leone Marshall Olkin –G family (TLMO-G)

TLMO-G family is recently proposed by the authors of this study. It is compound of two well-known distribution of CDF ($F(x;\alpha,b,\phi)$ and PDF $f(x;\alpha,b,\phi)$,as follows:

$$F(x;\alpha,b,\phi)=\left[1-\left(1-\frac{\alpha\bar{M}(x;\phi)}{1-\bar{\alpha}\ \bar{M}(x;\phi)}\right)^2\right]^b \tag{1}$$

$$f(x;\alpha,b,\phi)=\frac{2b\alpha^2 m(x,\phi)\,\bar{M}(x;\phi)}{\left(1-\bar{\alpha}\bar{M}(x;\phi)\right)^3}\left(1-\left(1-\frac{\alpha\,\bar{M}(x;\phi)}{1-\bar{\alpha}\bar{M}(x;\phi)}\right)^2\right)^{b-1} \tag{2}$$

Where $m(x,\phi)$, and $\bar{M}(x;\phi)=1-M(x;\phi)$ are the PDF and Survival function of baseline distribution, and $\alpha,b>0$ are related to the shape parameters. The survival function is given as:

$$R(x;\alpha,b,\phi)=1-F(x;\alpha,b,\phi)$$

$$R(x;\alpha,b,\phi)=1-\left[1-\left(1-\frac{\alpha\bar{M}(x;\phi)}{1-\bar{\alpha}\ \bar{M}(x;\phi)}\right)^2\right]^b \tag{3}$$

The hazard function or the failure rate is:

$$h(x;\alpha,b,\phi)=\frac{f(x;\alpha,b,\phi)}{R(x;\alpha,b,\phi)} \tag{4}$$

$$h(x;\alpha,b,\phi)=\frac{\dfrac{2b\alpha^2 m(x,\phi)\,\bar{M}(x;\phi)}{\left(1-\bar{\alpha}\bar{M}(x;\phi)\right)^3}\times\left(1-\left(1-\dfrac{\alpha\,\bar{M}(x;\phi)}{1-\bar{\alpha}\,\bar{M}(x;\phi)}\right)^2\right)^{b-1}}{1-\left[1-\left(1-\dfrac{\alpha\bar{M}(x;\phi)}{1-\bar{\alpha}\ \bar{M}(x;\phi)}\right)^2\right]^b} \tag{5}$$

If $\alpha=1$, then we have Topp Leone –G family. But, if $b=1$, we can generate Marshall Olkin–G family.

3. The new distribution Topp Leone Marshall Olkin-Weibull (TLMO-W)

The CDF and PDF functions of the Weibull distribution with two parameters $\lambda,\beta>0$ for a random variable X, is given as following:

$$M(x,\lambda,\beta)=1-e^{-(\lambda x)^\beta},\quad so\quad \bar{M}(x;\lambda,\beta)=e^{-(\lambda x)^\beta} \tag{6}$$

$$m(x,\lambda,\beta)=\beta\lambda^\beta x^{\beta-1}e^{-(\lambda x)^\beta} \tag{7}$$

Let us insert equations 6 and 7 in equations 2 and 1, respectively, we get the CDF and PDF of (TLMO-W) distribution with four parameters respectively as following:

$$F(x;\alpha,b,\lambda,\beta)=\left[1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\ e^{-(\lambda x)^\beta}}\right)^2\right]^b \tag{8}$$

$$f(x;b,\alpha,\lambda,\beta)=\frac{2b\alpha^2\,\beta\lambda^\beta x^{\beta-1}\,e^{-2(\lambda x)^\beta}}{\left(1-\bar{\alpha}\,e^{-(\lambda x)^\beta}\right)^3}\left(1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\,e^{-(\lambda x)^\beta}}\right)^2\right)^{b-1} \tag{9}$$

The survival function of TLMO-W distribution is given by:

$$R(x;b,\alpha,\lambda,\beta)=1-\left(1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\,e^{-(\lambda x)^\beta}}\right)^2\right)^b \tag{10}$$

The hazard rate function (hrf) can be determined by:

$$\tau(x;\alpha,b,\lambda,\beta)=\frac{\frac{2b\alpha^{2}\,\beta\lambda^{\beta}x^{\beta-1}\,e^{-2(\lambda x)^{\beta}}}{\left(1-\bar{\alpha}\,e^{-(\lambda x)^{\beta}}\right)^{3}}\left(1-\left(1-\frac{\alpha\,e^{-(\lambda x)^{\beta}}}{1-\bar{\alpha}\,e^{-(\lambda x)^{\beta}}}\right)^{2}\right)^{b-1}}{1-\left[1-\left(1-\frac{\alpha e^{-(\lambda x)^{\beta}}}{1-\bar{\alpha}\,e^{-(\lambda x)^{\beta}}}\right)^{2}\right]^{b}} \qquad (11)$$

Figure 1 shows some of the possible shapes of PDF of TLMO-W distribution using R software for selected different values of parameters with different shapes such as decreasing, left-skewed, right-skewed, and semi-symmetric features. Figure 2 displays the hazard function of TLMO-W distribution with various shapes using R software. This plot has very flexible shapes such as monotonically decreasing, bathtub shaped, monotonically increasing and upside-down bathtub features, depending on the parameter values.

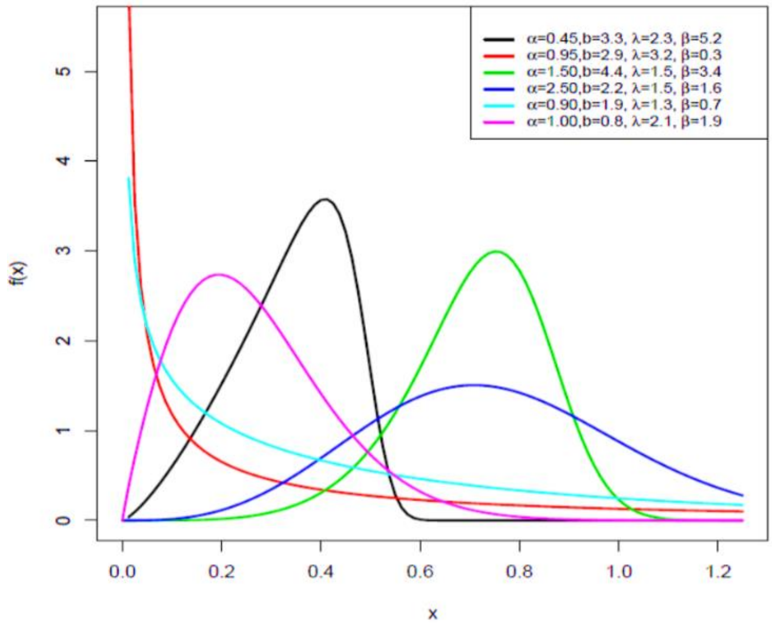

Figure1. Plot of PDF of the TLMO-W distribution with different values of parameters using R software

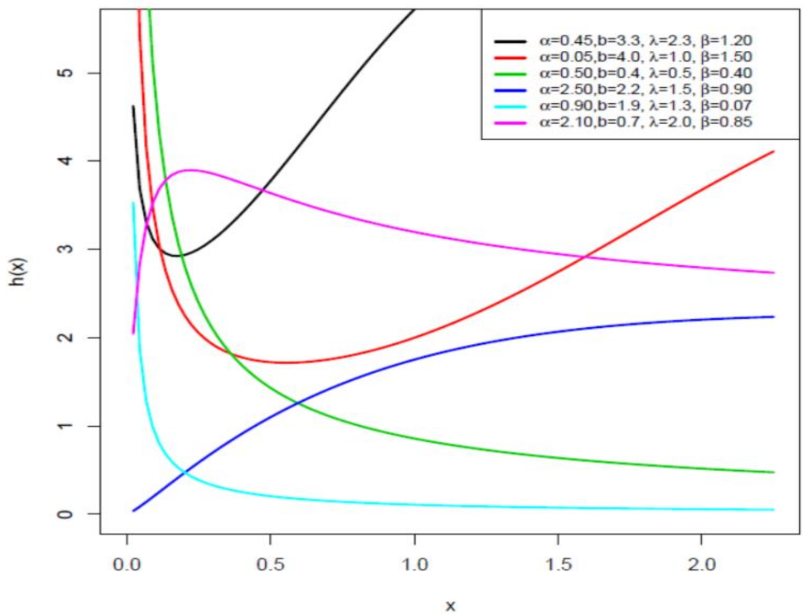

Figure 2. Plot of the hazard function of the TLMO-W distribution with different values of parameters using R software

4. Expansion of functions

Let us consider the generalized binomial expansion as:

$(1-u)^a = \sum_{i=0}^{\infty}(-1)^j \binom{a}{j} u^j$

It can hold for any real non-integer a as $|u| < 1$, we use this formula to simplify the PDF as following:

$$f(x;b,\alpha,\lambda,\beta) = \frac{2b\alpha^2\, \beta\lambda^\beta x^{\beta-1}\, e^{-2(\lambda x)^\beta}}{\left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^3} \left(1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\, e^{-(\lambda x)^\beta}}\right)^2\right)^{b-1}$$

$$\left(1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\, e^{-(\lambda x)^\beta}}\right)^2\right)^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i}\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\, e^{-(\lambda x)^\beta}}\right)^{2i}$$

$$= \sum_{i=j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{2i}{j}\left(\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\, e^{-(\lambda x)^\beta}}\right)^j$$

$$= \sum_{i=j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{2i}{j}\frac{\alpha^j e^{-j(\lambda x)0^\beta}}{\left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^j}$$

$$= \sum_{i=j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{2i}{j}2b\alpha^2\beta\lambda^\beta x^{\beta-1}\,\alpha^j e^{-j(\lambda x)^\beta}\, e^{-2(\lambda x)^\beta}\left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^{-j}$$
$$*\left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^{-3}$$

$$= \sum_{i=j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{2i}{j}2b\alpha^{2+j}\,\beta\lambda^\beta x^{\beta-1}\, e^{-(2+j)(\lambda x)^\beta}\left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^{-(3+j)}$$

$$= \sum_{i=j=k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{2i}{j}\binom{-(3+j)}{k}2b\alpha^{2+j}\beta\lambda^\beta x^{\beta-1}\,(1-\alpha)^k e^{-(2+j+k)(\lambda x)^\beta}$$

$$f(x;\alpha,b,\lambda,\beta) = \sum_{i=j=k=0}^{\infty}\Psi_{i,j,k}\ \ \lambda^\beta x^{\beta-1}\, e^{-(2+j+k)(\lambda x)^\beta} \tag{12}$$

$$\Psi_{i,j,k} = (-1)^{i+j+k}\binom{b-1}{i}\binom{2i}{j}\binom{-(3+j)}{k}2b\alpha^{2+j}\beta\,(1-\alpha)^k\ .$$

The equation (12) is one of the main consequence of this study to construct some statistical and mathematical properties.

5. Statistical properties

In this section, some of the statistical mathematical properties of TLMO-W distribution will be obtained as follows:

5.1. Quantile function

The quantile function **(**QF) is important used function in probability theory, statistical analysis and for simulation. It is obtained by inverting the CDF as follows:

$$Q(u) = F^{-1}(x) = \frac{1}{\lambda}\left(\ln\frac{1-(1-u^{1/b})^{1/2}}{(1-(1-u^{1/b})^{1/2}))\ \ \bar{\alpha}+\alpha}\right) \tag{13}$$

Any quantiles measures of interest can be obtained from (13), by substituting appropriate values for u. If we adjust u =0.5, we can get the median. In the case of u = 0.25, the first quartiles can be obtained. While for u = 0.75, we can get the third quartiles. The well-known measures Bowley's skewness and Moor's kurtosis.can be acquired as another advantage for simulating TLMO-W random variables. If U is continuous and uniform with unit interval $U \sim U(0,1)$, the random variable X is given by:

$$x = \frac{1}{\lambda}\left(-\ln\frac{1-(1-U^{1/b})^{1/2}}{(1-(1-U^{1/b})^{1/2}))\ \ \bar{\alpha}+\alpha}\right)^{\frac{1}{\beta}} \tag{14}$$

5.2. Moments

Moments function is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis. We found that the r^{th} moments of the TLMO-W is as following:

$$\mu_r' = E(x^r) = \frac{1}{\beta}\, \Gamma(\frac{r}{\beta}+1)\, (\tfrac{1}{\lambda})^{r+\beta} \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k}\ (2+j+k)^{-(\frac{r}{\beta}+1)} \qquad (15)$$

Proof: $\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r\, f_{TLMOW}(x;\alpha,b,\lambda,\beta)\, dx$

$$= \int_0^\infty x^r \sum_{i=j=k=0}^{\infty} 2(-1)^{i+j+k}\binom{b-1}{i}\binom{2i}{j}\binom{-(3+j)}{k} b\alpha^{2+j}\beta\lambda^\beta x^{\beta-1}$$

$$* (1-\alpha)^k e^{-(2+j+k)(\lambda x)^\beta} dx$$

$$= \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k} \int_0^\infty x^{\beta-1} e^{-(2+j+k)(\lambda x)^\beta} x^r\, dx$$

$$= \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k} \int_0^\infty x^{r+\beta-1} e^{-\lambda^\beta x^\beta (2+j+k)} dx$$

Where,

$$\Psi_{i,j,k} = 2(-1)^{i+j+k}\binom{b-1}{i}\binom{2i}{j}\binom{-(3+j)}{k} b\alpha^{2+j}\beta\lambda^\beta(1-\alpha)^k\ .$$

Let $y = x^\beta$, $then\ x = y^{\frac{1}{\beta}}, dx = \frac{1}{\beta}\, y^{\frac{1}{\beta}-1}\ dy, 0<y<\infty$

$$= \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k} \int_0^\infty \left(y^{\frac{1}{\beta}}\right)^{r+\beta-1} e^{-\lambda^\beta y(2+j+k)} \frac{1}{\beta}\, y^{\frac{1}{\beta}-1}\ dy$$

$$= \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k} \frac{1}{\beta}\int_0^\infty y^{\frac{r+\beta-1+1-\beta}{\beta}} e^{-\lambda^\beta y(2+j+k)}\, dy$$

$$= \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k} \frac{1}{\beta}\int_0^\infty y^{\frac{r}{\beta}}\ e^{-\lambda^\beta y}\, dy$$

$$= \frac{1}{\beta}\, \Gamma(\frac{r}{\beta}+1)\, (\tfrac{1}{\lambda})^{r+\beta} \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k}\, (2+j+k)^{-(\frac{r}{\beta}+1)}$$

5.3. Moment generating function

The moments generating function (MGF) for the TLMO-W distribution can be define as follows:

$$M_X(t) = \sum_{i=j=k=r=0}^{\infty} \Psi_{i,j,k}\ \frac{t^r}{r!}\ \Gamma\left(\frac{r+1}{\beta}\right)\lambda^{-r}(2+j+k)^{\frac{-r-1}{\beta}} \qquad (16)$$

Proof:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x;\alpha,b,\lambda,\beta)\, dx$$

$$, but\ \ Ee^{tx} = \sum_{r=0}^{\infty} \frac{t^r E\,(x^r)}{r!}\quad ,\ then$$

$$M_X(t) = \sum_{i=j=k=r=0}^{\infty} \Psi_{i,j,k}\, \lambda \int_0^\infty \frac{t^r\,(X^r)}{r!}\ e^{-(2+j+k)(\lambda x)^\beta}\ dx$$

$Let\ y = X^\beta$, $then\ x = y^{\frac{1}{\beta}}, dx = \frac{1}{\beta}\, y^{\frac{1}{\beta}-1}\ dy, 0<y<\infty$

$$M_X(t) = \sum_{i=j=k=r=0}^{\infty} \Psi_{i,j,k}\, \lambda\ \frac{t^r}{r!} \int_0^\infty y^{\frac{r+1}{\beta}-1}\ e^{-(2+j+k)\lambda^\beta y}\ dy$$

$$= \sum_{i=j=k=r=0}^{\infty} \Psi_{i,j,k}\ \frac{t^r}{r!}\ \Gamma(\frac{r+1}{\beta})\, \lambda^{-r}(2+j+k)^{\frac{-r-1}{\beta}}$$

5.4. Entropy

The Renyi entropy of TLMO-W distribution is defined as following:

$$I_R(\delta) = \frac{1}{1-\delta} \log \int_0^\infty f(x;\alpha,b,\lambda,\beta)^\delta \; dx\,, \quad where \quad \delta > 0\,, \delta \neq 1$$

$$I_R(\delta) = \frac{1}{1-\delta} \log \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k}\, \lambda^\delta \beta^\delta \int_0^\infty x^{\delta(\beta-1)}\, e^{-(\lambda x)^{\delta\beta}(2+j+k)} \; dx$$

Where ; $\Psi_{i,j,k} = 2b\alpha^{2+j}(1-\alpha)^k(-1)^{i+j+k} \binom{b-1}{i}\binom{2i}{j}\binom{-(3+j)}{k}$

$$I_R(\delta) = \frac{1}{1-\delta} \log \sum_{i=j=k=0}^{\infty} \Psi_{i,j,k}\, \lambda^{-\delta^2\beta-2\delta\beta+\delta} \beta^{\delta-1} \Gamma(\delta+2)(2+j+k)^{-\delta\beta(\delta-2)} \qquad (17)$$

5.5. Order statistics

The order statistics of random variables from TLMO-W distribution can be given by:

$$f_{i,n}(x) = \frac{n!}{(i-1)!\,(n-i)!} \left[F_{TLMOE}(x)\right]^{i-1}\left[1-F_{TLMO}(x)\right]^{n-i} f_{TLMO}(x)$$

$$f_{i,n}(x) = \sum_{j=0}^{\infty} k\,(-1)^j \binom{n-i}{j} f(x)\left[F_{TLMOE}(x)\right]^{i+j-1} \qquad (18)$$

$$where: \; k = \frac{n!}{(i-1)!\,(n-i)!}$$

$$f(x)\,F(x)^{i+j-1} = 2b\,\alpha^2\beta\,\lambda^\beta\, x^{\beta-1} e^{-2(\lambda x)^\beta} \left(1-\bar{\alpha}\, e^{-(\lambda x)^\beta}\right)^{-3} * \left(1-\left(1-\frac{\alpha e^{-(\lambda x)^\beta}}{1-\bar{\alpha}\, e^{-(\lambda x)^\beta}}\right)^2\right)^{b(i+j)-1}$$

$$= \sum_{i=k=v=0}^{\infty} \sum_{j=0}^{n-i} k\,(-1)^{i+j+k+v} \binom{b(i+j)-1}{i}\binom{2i}{k}\binom{-(3+k)}{v}\binom{n-i}{j} * 2b\,\alpha^{2+k}\beta\,\lambda^\beta\, x^{\beta-1}(1-\alpha)^v e^{-2(\lambda x)^{\beta(k+v+1)}}$$

6. Estimation of parameters

We want to find the maximum likelihood estimators of unknown parameters from complete samples. Let $X_1, X_2 \dots, X_n$ be a random sample of size n from the proposed distribution TLMO-G having the parameters $\phi = (b,\alpha,\lambda,\beta)^T$, the total likelihood function is written as following based on equation (9):

$$L(b,\alpha,\lambda,\beta;x) = \prod_{i=0}^{n} f(x;b,\alpha,\lambda,\beta)$$

$$ln\big(L(b,\alpha,\lambda,\beta;x)\big) = n\,ln(2) + n\,ln(b) + 2n\,ln(\alpha) + n\,ln(\beta) + n\beta\,ln(\lambda) + (b-1)\sum_{i=1}^{n} ln(x_i) - 2\sum_{i=1}^{n}(\lambda x_i)^\beta - 3\sum_{i=1}^{n} ln\left(1-\bar{\alpha}e^{-(\lambda x_i)^\beta}\right) + (b-1)\sum_{i=1}^{n} ln\left(1-\left(1-\frac{\alpha e^{-(\lambda x_i)^\beta}}{\left(1-\bar{\alpha}e^{-(\lambda x_i)^\beta}\right)}\right)^2\right)$$

The (MLEs) can be obtained by equating the derivative of the $ln(L)$ with respect to each parameter to zero.

$$\frac{\partial\, ln\, Lf(x)}{\partial b} = \frac{\partial\, ln\, Lf(x)}{\partial \alpha} = \frac{\partial\, ln\, Lf(x)}{\partial \lambda} = \frac{\partial\, ln\, Lf(x)}{\partial \beta} = 0$$

Note that, these systems of equations cannot be solved analytically and any suitable software can be used to solve them numerically. So, we use (R) software package “AdequecyModel”, for optimizing the value of MLE and estimating the parameters values.

7. Application on real data

We evaluate the performance of the TLMO-W distribution proposed in this paper by application on real data. The data present the possibility of applying the new distribution to the actual data that represents dataset for the ages of 155 patients suffered from breast tumours June to October in 2014. The study was established through Breast Cancer Early Detection Unit in Banha University Hospital in Egypt [18]. The data set is as follows:

46, 32, 50, 46, 44, 42, 69, 31, 25, 29, 40, 42, 24, 17, 35, 48, 49, 50, 60, 26, 36, 56, 65, 48, 66, 44, 45, 30, 28, 40, 40, 50, 41, 39, 36, 63, 40, 42, 45, 31, 48, 36, 18, 24, 35, 30, 40, 48, 50, 60, 52, 47, 50, 49, 38, 30, 52, 52, 12, 48, 50, 45, 50, 50, 50, 53, 55, 38, 40, 42, 42, 32, 40, 50, 58, 48, 32, 45, 42, 36, 30, 28, 38, 54, 90, 80, 60, 45, 40, 50, 50, 40, 50, 50, 50, 60, 39, 34, 28, 18, 60, 50, 20, 40, 50, 38, 38, 42, 50, 40, 36, 38, 38, 50, 50, 31, 59, 40, 42, 38, 40, 38, 50, 50, 50, 40, 65, 38, 40, 38, 58, 35, 60, 90, 48, 58, 45, 35, 38, 32, 35, 38, 34, 43, 40, 35, 54, 60, 33, 35, 36, 43, 40, 45, 56

We have compared the fits of TLMO-W distribution with some chosen distributions because of popularity of their baseline including Beta-Weibull, Kumaraswamy- Weibull, Exponentiated Generalized- Weibull and Weibull-Weibull. The maximum likelihood estimators and the value of criteria (-LL, AIC, CAIC, BIC, HQIC) are listed in Table 1. The computations are performed using the software R. Based on Table 1, we can find that the TLMO-W is the best fit for the tumour data than other distributions. The TLMO-W has achieved the smallest value of criteria.

Table 1. The MLEs, -LL, AIC, CAIC, BIC, HQIC, of the models.

Models	Estimation Parameters				Criteria Values				
	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	-LL	AIC	CAIC	BIC	HQIC
TLMO-W	2.97	33.43	0.09	0.95	600	1208.9	1209.2	1221.1	1213.8
B-W	3.53	1.08	0.02	2.14	603	1214.4	1214.7	1226.6	1219.4
Ku-W	5.61	3.00	0.02	1.44	602	1213.6	1213.9	1225.8	1218.6
EG-W	0.22	7.04	0.10	1.62	604	1217.0	1217.3	1229.2	1222.0
W-W	3.27	4.18	0.07	1.12	610	1228.5	1228.8	1240.7	1233.5

In order to verify which fits data better, we consider Cramer-Von Misses (W*) and Anderson-Darling (A*) statistics to test the goodness-of-fit for the data, as listed Table 2. Overall, we conclude that TLMO-W distribution yields a better fit than the other four distributions.

Table 2. The test criteria W^*, A^*, for the models

Models	Criteria values	
	W*	A*
TLMO-W	0.1513	0.8839
B-W	0.1969	1.2320
Ku-W	0.1903	1.1824
EG-W	0.2325	1.4753
W-W	0.3214	1.9721

The results of the estimated PDF and CDF of the fitted models for the data set are shown in Figures 3-4 using R software, which support the results in tables.

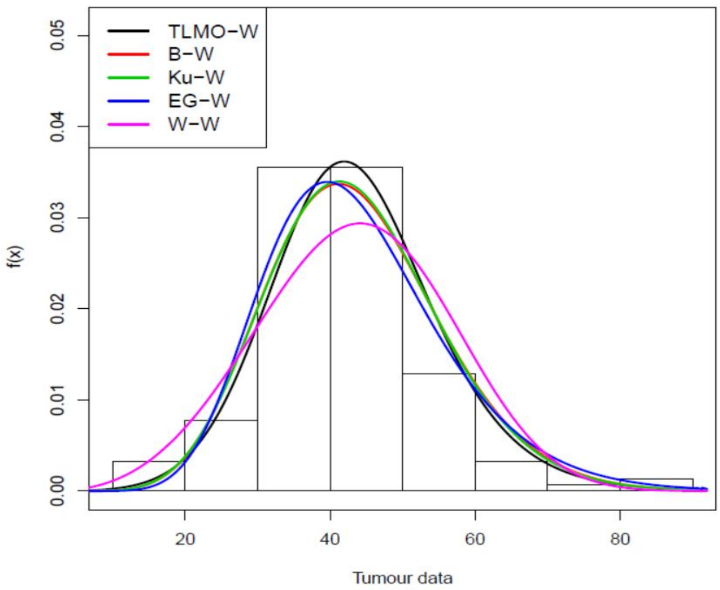

Figure 3. The estimated densities of models for the data using R software

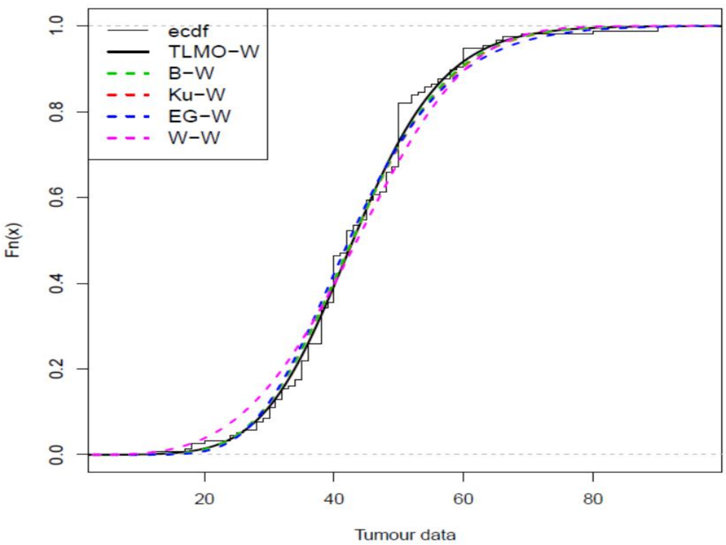

Figure 4. Estimated cumulative distribution functions of models for the data using R software

8. Conclusion

In this paper, a new distribution of TLMO-W has been introduced. The MLE method was used for estimating the four parameters of distribution that has very complex equations to solve. Mathematical properties such as quantile function, moments generating functions, entropy, and order statistics have been investigated. In order to verify which fits data better, we consider Cramer-Von Misses (W*) and Anderson-Darling (A*) statistics to test the goodness-of-fit for the data. The usefulness of this distribution is explained by an application on real data, which gives a better fit than the nearest non-nested models using the well-known statistical criteria. The TLMO-W can be applied in different real-life data based on its practical flexibility and new scalability.

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