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ANALYTICAL SOLUTION OF THE RELATIVISTIC KLEIN-GORDON WAVE EQUATION

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ABSTRACT

In this study, the solution to Klein-Gordon equations with focus on analytical methods is discussed. The analytical methods used in this research are the Variational Iteration Method (VIM) developed by Ji-Huan He, Adomian Decomposition Method (ADM) by Adomian and New Iterative Method (NIM) developed by Daftardar Gejji and Jafari. The modified Adomian Decomposition method by Wazwaz was used to solve the linear inhomogeneous and nonlinear Klein-Gordon equations to accelerate the convergence of the solution and minimizes the size of calculation while still maintaining high accuracy of the analytical solution. All the problems considered yield the exact solutions with few iterations. The solutions obtained were compared with the exact solution and the solutions obtained by other existing methods. The solutions obtained by the three methods yield the same results and all the problems considered show that the Variational Iteration Method, Adomian Decomposition Method and New Iterative Method are very powerful and potent in solving Klein-Gordon equations and can be used to obtain closed form solutions of linear and nonlinear differential equations (ordinary and partial).

Keywords: Klein-Gordon equation, New Iterative Method, Variational iteration Method, Relativistic Wave Equation.

INTRODUCTION

The Klein-Gordon equation has important applications in Plasma Physics together with Zakharov equation describing the interaction of Langmuir wave and the ion acoustic wave in Plasma Physics Batiha (2009). In Astrophysics together with Maxwell equation describing a minimally coupled charged bosom field to a spherically symmetric space time Behzadi (2011). In Biophysics, together with another Klein-Gordon equation describing the long wave limit of a lattice model for one-dimensional nonlinear wave processes in a bi-layer and so on. Furthermore, Klein-Gordon equation coupled with Schrodinger equation (KGS) is introduced and it describes a system of conserved scalar nucleons interacting with neutral scalar mesons coupled through the Yukawa interaction [9] Berna , *et al,* (2013)

Considering the Klein-Gordon equation of the form:

$$u_{tt}(x,t) - u_{xx}(x,t) + gu(x,t) = f(x,t) \qquad (1)$$

Subject to the initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = f(x) \qquad (2)$$

where u is a function of x and t, g is a given nonlinear function and f is a known or given analytical function Berna , *et.al*, (2013)

Because of the importance of Klein-Gordon equation in quantum mechanics, a number of analytical and numerical methods have been proposed to accurately solve various linear and nonlinear Klein-Gordon equations Fadhil , *et.al*, (2013). For example, Adomian Decomposition Method one of the most popular methods has been extensively applied for solving the Klein-Gordon equation Hesameddini , *et.al* (2012). Other methods include Homotopy Perturbation Method Majeed *et.al,* (2017), Taylor Matrix Method Berna, *et.al,* (2013), Reduced Differential Transform Method Yildiray, *et.al* (2011), Differential Transform Method Ravi, *et.al,* (2008), Variational Iteration Method Batiha (2009), Laplace Decomposition Method Rabie Mohammed (2015), Sumudu Decomposition Method, Ramadan *et.al* (2014), Legendre Wavelength Solution, Fukang, *et.al* (2015), Hesameddini, *et.al* (2012), Perturbation Iteration Transform Method Rajni *et.al,* (2016), New Iterative Method Wazwaz Abdul-Majid (2009), Yaseem *et.al* (2012) and Behzadi (2011) used both the Adomian Decomposition Method and Variational Iteration Method (Majeed , *el.al*).

In this study, three (3) of these methods were applied to solve the Klein-Gordon equation, these methods are Variational Iteration Method (VIM), New Iterative Method (NIM) and Adomian Decomposition Method (ADM) with its modification (Rabie Mohammed *el.al*). These methods can be widely used in solving different types of differential equations in Physics, Engineering and Modelling Rajni, *et.al* (2016). They have been proved to be powerful, versatile, which can effectively easily and accurately solve higher order linear and nonlinear differential problems with rapid convergence after a less number of iterations Ramadan, *et.al* (2014).

VARIATIONAL ITERATION METHOD (VIM)

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \tag{3}$$

Where, L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term Ravi *et.al* (2008).

The variational iteration method presents a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\varepsilon)\big(Lu_n(\varepsilon) + N\tilde{u}_n(\varepsilon) - g(\varepsilon)\big)d\varepsilon \tag{4}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$ Wazwaz, *et.al* (2009).

It is obvious now that the main steps of the variational iteration method require first the determination of the Lagrange multiplier $\lambda(\varepsilon)$ that will be identified optimally, Yaseem, *et.al* (2012). Integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\varepsilon)$ Yildiray, *el.al* (2011). In other words, carrying out the integration as follows can yield:

$$\int \lambda(\varepsilon)\, u'_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)u_n(\varepsilon) - \int \lambda'(\varepsilon)\, u_n(\varepsilon)d\varepsilon,$$

$$\int \lambda(\varepsilon)\, u''_n(\varepsilon)d\varepsilon = \lambda(\varepsilon)'u_n(\varepsilon) - \lambda'(\varepsilon)u_n(\varepsilon) + \int \lambda''(\varepsilon)\, u_n(\varepsilon)d\varepsilon \tag{5}$$

Having determined the Lagrange multiplier $\lambda(\varepsilon)$, the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using any selective function u_0.

However, for fast convergence, the function $u_0(x,t)$ should be selected by using the initial conditions as follows:

$$u_0(x,t) = u(x,0) \qquad \text{for first order}$$

$$u_0(x,t) = u(x,0) + tu_t(x,0) \quad \text{for second order} \tag{6}$$

Consequently, the solution

$$u = \lim_{n\to\infty} u_n \tag{7}$$

In other words, equation (5) will give several approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations Wazwaz, (2009).

NEW ITERATIVE METHOD (NIM)

To illustrate the idea of the NIM, we consider the following general functional equation:

$$u = f + N(u) \tag{8}$$

where N is a nonlinear operator and f is a given function. We can find the solution of equation (8) having the series form

$$u = \sum_{i=0}^{\infty} u_i \tag{9}$$

The nonlinear operator N can be decomposed as:

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{10}$$

Substituting equations (9) and (10) into equation (8) gives

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{11}$$

We define the recurrence relation of equation in the following way:

$$u_0 = f$$

$$u_1 = N(u_0)$$
$$u_2 = N(u_0 + u_1) - N(u_0) \quad (12)$$
$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$
$$u_{n+1} = N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1}); \quad n = 1, 2, 3$$

Then

$$u_1 + \cdots + u_{m+1} = N(u_0 + u_1 + \cdots + u_m); \quad m = 1, 2, 3$$

and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{j=0}^{\infty} u_j\right) \quad (13)$$

The m-term approximate solution of (8) is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

ADOMIAN DECOMPOSITION METHOD (ADM)

To give a clear overview of Adomian decomposition method,
Considering the following equation:

$$Lu + Ru = g \quad (14)$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is a linear differential operator, and g is a source term.

Applying the inverse operator (L^{-1}) to both sides of (14) and using the initial condition to obtain

$$u = f - L^{-1}Ru \quad (15)$$

where the function f represents the terms arising from integrating the source term g and noting the prescribed conditions.

The Adomian Decomposition Method assumes that the unknown function u can be expressed by an infinite series of the form

$$u(x,y) = \sum_{n=0}^{\infty} (u_n(\mathrm{x}, \mathrm{y})) \quad (16)$$

or equivalently,

$$u = u_0 + u_1 + u_2 + \ldots\ldots \quad (17)$$

where the components $u_0, u_1, u_2, \cdots$ are usually recurrently determined.
Substituting equation (16) into equation (15) yields

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right)\right) \quad (18)$$

For simplicity, Equation (18) can be re-written as

$$u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_0 + u_1 + u_2 + \cdots\right)\right) \quad (19)$$

To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2, \cdots$, it is important to note that Adomian method suggests that the zeroth component u_0 is
usually defined by the function f described above, that is, by all terms that are not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$u_0 = f$$
$$u_{k+1} = -L^{-1}(R(u_k)) \quad , \quad k \geq 0 \quad (20)$$

or equivalently,

$$u_0 = f$$
$$u_1 = -L^{-1}(R(u_0))$$
$$u_2 = -L^{-1}(R(u_1))$$
$$u_3 = -L^{-1}(R(u_2)) \quad (21)$$

It is clearly seen that the relations (21) reduced the differential equation under consideration into an elegant determination of computable components. Having determined the relations (21), a series form solution is obtained by substituting relations (21) into equations (16).

The approximate solution is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

MODIFIED ADOMIAN DECOMPOSITION METHOD

Modified Adomian decomposition method developed by Wazwaz (2009). The modified decomposition method will further accelerate the convergence of the series solution. It is to be noted that in this study, the modified decomposition method will be applied to linear inhomogeneous and nonlinear Klein-Gordon equations.

The decomposition method admits the use of the recursive relation,

$$u_0 = f,$$
$$u_{k+1} = -L^{-1}(Ru_k), \qquad k \geq 0 \tag{22}$$

the components u_n, $n \geq 0$ is obtained.

The modified decomposition method introduces a slight variation to the recursive relation (22) that will lead to the determination of the components of u in a faster and easier way.

For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2.

In other words, we have

$$f = f_1 + f_2. \tag{23}$$

Using equation (23), we introduce a qualitative change in the formation of the recursive relation (22). To reduce the size of calculations, we identify the zeroth component u_0 by one part of f, namely f_1 or f_2. The other part of f can be added to the component u_1 among other terms. In other words, the modified recursive relation can be identified by

$$u_0 = f_1$$
$$u_1 = f_2 - L^{-1}\big(R(u_0)\big) \tag{24}$$
$$u_{k+1} = -L^{-1}\big(R(u_k)\big)$$

The success of this modification depends only on the choice of f_1 and f_2, and this can be made through trials. Second, if f consists of one term only, the standard decomposition method should be employed in this case.

The approximate solution is given by $u = u_0 + u_1 + u_2 \ldots + u_{m-1}$

NUMERICAL APPLICATIONS

In this section, we use the three methods in solving linear and nonlinear Klein-Gordon equations.

Linear Klein-Gordon Equation

Example 1

Consider the following Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0$$

With the initial conditions

$$u(x,0)=0,\ u_t(x,0) = x \tag{25}$$

The exact solution is given by $u(x,t) = x\,Sin(t)$

Method 1: VIM

The correction functional of equation (25) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} + \tilde{u}_n(x,\varepsilon)\right] d\varepsilon \tag{26}$$

Taking variations on both sides gives

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} + \tilde{u}_n(x,\varepsilon)\right] d\varepsilon \tag{27}$$

$\widetilde{u_n}$ is a restrictive vibration, therefore $\delta\tilde{u}_n = 0$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2}\right] d\varepsilon \tag{28}$$

Simplify by integration by parts

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\lambda(\varepsilon)u_n'(\mathrm{x},\varepsilon) - \delta\lambda'(\varepsilon)u_n(x,\varepsilon) + \int \delta\lambda''(\varepsilon)\, u_n(\mathrm{x},\varepsilon) d\varepsilon \tag{29}$$

Making the correction functional stationary to obtain $\lambda = \varepsilon - t$

Hence, the iterative formula becomes

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)\left(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} + u_n(x,\varepsilon)\right) d\varepsilon, \quad \text{for } n \geq 0 \tag{30}$$

From the initial condition, we have

$u_0(x,t)= u(x,0)+tu_t(x,0) = 0 + xt = xt$

$u_0(x,\varepsilon)= x\,\varepsilon$

Consequently, the successive approximations are obtained

$$u_1(x,t) = xt - \frac{xt^3}{6}$$

$$u_2(x,t) = \mathrm{x}t - \frac{xt^3}{6} + \frac{xt^5}{120} = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} \tag{31}$$

$$u_3(x,t) = \mathrm{x}t - \frac{xt^3}{6} + \frac{xt^5}{120} - \frac{xt^7}{5040} = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}$$

.

.

Hence, the closed form and exact solution is given as

$u(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) = x\sin t$

(32)

Method 2: NIM

Considering Example 1 for New Iterative method Equation (25) is equivalent to the following integral equation

$$u = f + \iint_0^t (u_{xx} - u)dtdt$$

(33)

Set $u_0(x,t) = u(x,0) + tu_t(x,0) = 0 + xt = xt$ and

$$N(u_n) = \iint_0^t (u_{nxx} - u_n)dtdt$$

(34)

The successive approximations are:

$$u_0 = f = xt$$

$$u_1 = -\frac{xt^3}{6} = -\frac{xt^3}{3!}$$

$$u_2 = \frac{xt^5}{120} = \frac{xt^5}{5!}$$

(35)

$$u_3 = \frac{xt^7}{5040} = -\frac{xt^7}{7!}$$

.

Thus the series solution of example 1 is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n = xt - \frac{xt^3}{3!} + \frac{xt^5}{5!} - \frac{xt^7}{7!}$$

$$u(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right) = x\sin t$$

(36)

Method 3: ADM

Applying L_t^{-1} to both sides of (25) and using decomposition series for $u(x,t)$ give

$$u(x,t) = u_0 + L_t^{-1}(u_{xx} - u)$$

(37)

$u_0 = u(x,0) + u_t(x,0) = 0 + t(x) = xt = f$

The recursive relation is

$$u_0(x,t) = xt$$

$$u_{k+1}(x,t) = L_t^{-1}(L_x u_k - u_k) \quad k \geq 0$$

(38)

That in turn gives

$$u_1 = L_t^{-1}(L_x u_0 - u_0) = L_t^{-1}(0 - xt) = -\frac{xt^3}{6} = -\frac{xt^3}{3!}$$

$$u_2 = L_t^{-1}(L_x u_1 - u_1) = L_t^{-1}\left(0 - \left(-\frac{xt^3}{6}\right)\right) = \frac{xt^5}{120} = \frac{xt^5}{5!}$$

(39)

$$u_3 = L_t^{-1}(L_x u_2 - u_2) = L_t^{-1}\left(0 - \left(\frac{xt^5}{120}\right)\right) = -\frac{xt^7}{5040} = -\frac{xt^7}{7!}$$

$$u_4 = \frac{xt^9}{9!}$$

In view of (39) the series solution is given by

$u(x,t) = x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) = x\sin t$

(40)

Example 2

Consider the Klein-Gordon equation

$$u_{tt} - u_{xx} - u = 0$$

With initial conditions

$$u(x,0) = Sin(x) + 1, \quad u_t(x,o) = 0$$

(41)

The exact solution of the equation is

$$u(x,t) = sin(x) + cosh(t)$$

Method 1: VIM

The correction functional for equation (41) is written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} - \tilde{u}_n(x,\varepsilon)\right]d\varepsilon$$

(42)

This yields the stationary conditions,

$$u_n: \quad 1 - \lambda' = 0 \;\Big|\; \varepsilon = t$$

$$u_n': \quad \lambda = 0 \;\Big|\; \varepsilon = t$$

(43)

Solving equation (43), $\lambda = \varepsilon - t$

(44)

Substituting $\lambda = \varepsilon - t$ into equation (42) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)\left(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} - u_n(x,\varepsilon)\right)d\varepsilon, \text{ for n} \geq 0$$

(45)

Considering the given initial values

$$u_0(x,t) = u(x,0) + tu_t(x,0) = Sin(x) + 1$$

(46)

using $u_0(x,t) = Sin(x) + 1$, we obtain the following successive approximations

$$u_1(x,t) = \sin(x) + 1 + \frac{t^2}{2}$$

$$u_2(x,t) = \sin(x) + 1 + \frac{t^2}{2} + \frac{t^4}{24} = \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!}$$
(47)

$$u_3(x,t) = \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!}$$

.

Hence, the series solution of Example 2 is

$$u(x,t) = \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots = \sin x + \cosh t \quad (48)$$

Method 2: NIM

Considering Example 2 for New Iterative method Equation (41) is equivalent to the following integral equation

$$u = f + \iint_0^t (u_{xx} + u)dtdt \quad (49)$$

Set $u_0(x,t) = u(x,0) + tu_t(x,0) = sin(x) + 1$ and

$$N(u_n) = \iint_0^t (u_{nxx} + u_n)dtdt \quad (50)$$

Thus the successive approximations are:

$$u_0 = f = \sin(x) + 1$$

$$u_1 = \frac{t^2}{2} = \frac{t^2}{2!}$$

$$u_2 = \left(\frac{t^2}{2} + \frac{t^4}{24} - \frac{t^2}{2}\right) = \frac{t^4}{24} = \frac{t^4}{4!}$$

(51)

$$u_3 = \frac{t^6}{120} = \frac{t^6}{6!}$$

.

Hence, the series solution of example 2 is

$$u(x,t) = \sum_{n=1}^{\infty} u_n$$

$$= \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots$$

$$u(x,t) = \sin x + \cosh(t)$$

(52)

Method 3: ADM

Writing equation (41) in operator form

$$u(x,t) = u_0 + L_t^{-1}(u_{xx} + u)$$

(53)

Consequently, we set the relation

$$u_0(x,t) = sin(x) + 1$$

$$u_{k+1}(x,t) = L_t^{-1}(L_x u_k + u_k), \quad k \geq 0$$

(54)

This leads to

$$u_0 = sin(x) + 1$$

$$u_1 = L_t^{-1}(L_x u_0 + u_0) = L_t^{-1}(-\sin(x) + \sin(x) + 1) = \frac{t^2}{2!}$$

$$u_2 = L_t^{-1}(L_x u_1 - u_1) = L_t^{-1}\left(0 + \frac{t^2}{2!}\right) = \frac{t^4}{24} = \frac{t^4}{4!} \quad (55)$$

$$u_3 = L_t^{-1}(L_x u_2 - u_2) = L_t^{-1}\left(0 + \frac{t^4}{4!}\right) = \frac{t^6}{720} = \frac{t^6}{6!}$$

Combining the solution in a series form is given by

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots = sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + ..$$

$$u(x,t) = sin(x) + \cosh(t)$$

(56)

Example 3

Consider the inhomogeneous linear Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 2\sin x$$

(57)

With the initial conditions

$u(x,0) = sin\ x, \quad u_t(x,0) = 1$

the exact solution is given by $u(x,t) = sin\ x + sin\ t$

Method 1: VIM

The correction functional for Example 3 is written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} + \tilde{u}_n(x,\varepsilon) - 2sinx\right]d\varepsilon \quad (58)$$

This implies the stationary conditions, which are:

$$1 - \lambda' = 0 \,\big|\, \varepsilon = t$$
$$\lambda = 0 \,\big|\, \varepsilon = t$$

(59)

Solving equation (59), $\lambda = \varepsilon - t$

Substituting $\lambda = \varepsilon - t$ into equation (58) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)\left(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} + u_n(x,\varepsilon) - 2sinx\right)d\varepsilon, \quad \text{for } n \geq 0$$

(60)

Considering the given initial values

$u_0(x,t) = u(x,0) + tu_t(x,0) = Sin(x) + t$

(61)

using $u_0(x,t) = Sin(x) + t$, we obtain the following successive approximations

$$u_1(x,t) = \sin(x) + t - \frac{t^3}{6}$$

$$u_2(x,t) = \sin(x) + t - \frac{t^3}{6} + \frac{t^5}{120} = \sin(x) + t - \frac{t^3}{6} + \frac{t^5}{5!}$$

$$u_3(x,t) = \sin(x) + t - \frac{t^3}{6} + \frac{t^5}{5!} - \frac{t^7}{7!}$$

Hence, the series solution of Example 3 is

$$u(x,t) = \sin(x) + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sin x + \sin t \qquad (62)$$

Method 2: NIM

Equation (57) is equivalent to the following integral equation

$$u = f + \iint_0^t (u_{xx} - u + 2sinx)dtdt \qquad (63)$$

Set $u_0(x,t) = u(x,0) + tu_t(x,0) = sin(x) + t$ and

$$N(u_n) = \iint_0^t (u_{nxx} - u_n + 2sinx)dtdt \qquad (64)$$

Thus the successive approximations are

$$u_0 = f = sin(x) + t$$

$$u_1 = -\frac{t^3}{6} = -\frac{t^3}{3!}$$

$$u_2 = \frac{t^5}{120} = \frac{t^5}{5!}$$

$$u_3 = -\frac{t^7}{5040} = -\frac{t^7}{7!} \qquad (65)$$

Hence, the series solution of example 3 is

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sin(x) + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$u(x,t) = \sin x + sin\ t \qquad (66)$$

Method 3: MADM

Considering Example 3

Example 3 is inhomogeneous so we will use modified Adomian decomposition method

Writing equation (57) in operator form

$$u(x,t) = f + L_t^{-1}(u_{xx} + u) \qquad (67)$$

$$u(x,0) + u_t(x,0) + \iint f(x) = sin(x) + t + t^2 sinx$$

The function $f(x,y)$ consists of two terms, which gives

$$f_1 = \sin x, \quad f_2 = \mathrm{t} + t^2 \sin x$$

The modified recursive relations are

$$u_0(x,t) = f_1$$

$$u_1 = f_2 - L_t^{-1}(L_x u_0 - u_0)$$

$$u_{k+1}(x,t) = L_t^{-1}(L_x u_k - u_k) \quad k \geq 0 \qquad (68)$$

This gives

$$u_0 = sin(x)$$

$$u_1 = \mathrm{t} + t^2 \sin x - t^2 \sin x = t$$

$$u_2 = -\frac{t^3}{6} = \frac{t^3}{3!}$$

$$u_3 = \frac{t^5}{120} = \frac{t^5}{5!} \qquad (69)$$

Hence the solution of example 3 is

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$u(x,t) = sin(x) + t + -\frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$$

$$u(x,t) = sinx + t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\ldots$$

$$u(x,t) = sin(x) + \sin(t) \qquad (70)$$

6.2 Nonlinear Klein-Gordon Equation

Example 4

Consider the following Nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = x^2 t^2 \qquad (71)$$

With the boundary conditions

$u(x,0) = 0, \quad u_t(x,0) = x$

The exact solution is given by $u(x,) = xt$

Method 1: VIM

The correction functional for Example 4 is written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} + \widetilde{u_n^2}(x,\varepsilon) - x^2\varepsilon^2\right] d\varepsilon \qquad (72)$$

From previous calculation of λ, we know that

$$\lambda = \varepsilon - t$$

Substituting $\lambda = \varepsilon - t$ *into equation (72) gives the iteration formula*

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)\left(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} + u_n^2(x,\varepsilon) - x^2\varepsilon^2\right) d\varepsilon, for\ n \geq 0 \qquad (73)$$

Considering the given initial values

$u_0(x,t)= u(x,0)+tu_t(x,0$

$u_0(x,t) = 0 + xt = xt$

using $u_0(x,t) = xt$, we obtain the following successive approximations

$u_1(x,t) = xt$

$u_2(x,t) = xt$

$u_3(x,t) = xt$

.

$u_1 = u_2 = u_3 = \cdots u_n = xt$

(74)

Hence the series solution of Example 4 is

$u(x,t) = xt$

(75)

Method 2: NIM

Consider example 4 using New iterative method Equation (71) is equivalent to the following integral equation

$u = f + \iint_0^t (u_{xx} - u^2 + x^2t^2)dtdt$

(76)

Following the algorithm, the successive approximations are:

$u_0 = f = xt$

$u_1 = 0$

$u_2 = xt - xt = 0$

(77)

$u_3 = xt - xt = 0$

Hence, the series solution of example 4 is

$u(x,t) = \sum_{n=1}^{\infty} u_n = xt + 0 + 0 + 0 ....$

$u(x,t) = xt$

(78)

Method 3: MADM

Considering equation (71) for MADM
Writing equation (71) in operator form

$u_1 = f + L_t^{-1}(L_x u_0 - u_0^2)$

Using the algorithm

$u_0 = f = xt + \iint_0^t (x^2t^2)dtdt$

(79)

$$f = xt + \frac{x^2t^4}{12}$$

For MADM

$f = f_1 + f_2$

$f_1 = xt$ *and* $f_2 = \frac{x^2t^4}{12}$

(80)

The modified recursive relations are

$u_0(x,t) = f_1$

$u_1 = f_2 + L_t^{-1}(L_x u_0 - u_0^2)$

$u_{k+1} = L_t^{-1}\left(u_{k(xx)} - u_k^2\right), \quad k \geq 1$

(81)

This gives

$u_0 = f_1 = xt$

$u_1 = \frac{x^2t^4}{12} + L_t^{-1}(0 - x^2t^2) = (\frac{x^2t^4}{12} - \frac{x^2t^4}{12}) = 0$

$u_2 = L_t^{-1}\left(u_{1(xx)} - u_1^2\right) = 0$

(82)

$u_{k+1} = 0, \quad k \geq 1$

Therefore the solution of example 4 in series form is

$u = u_0 + u_1 + u_2 + u_3 + \cdots .. = xt + 0 + 0 + 0 + \cdots$ $u(x,t) = xt$

(83)

Example 5

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2$

(84)

With the initial conditions

$u(x,0) = 1, \qquad u_t(x,0) = x$

The Exact solution is $u(x,t) = 1 + xt$

Method 1: VIM

The correction functional for example 5 is written as:

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \widetilde{u}_n(x,\varepsilon)}{dx^2} - \widetilde{u_n} + \widetilde{u_n^2}(x,\varepsilon) - x\varepsilon - x^2\varepsilon^2\right]d\varepsilon$ (85)

From previous calculation, λ was found to be

$\lambda = \varepsilon - t$

Substitute $\lambda = \varepsilon - t$ into equation (85) gives the iteration formula

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} - u_n + u_n^2(x,\varepsilon) - x\varepsilon - x^2\varepsilon^2)d\varepsilon,$

for n ≥ 0 (86)

Considering the given initial values

$u_0(x,t) = u(x,0)+tu_t(x,0$

$u_0(x,t) = 1 + xt = xt$

using $u_0(x,t) = 1 + xt$, we obtain the following successive approximations

$u_1(x,t) = 1 + xt$

$u_2(x,t) = 1 + xt$ (87)

$u_3(x,t) = 1 + xt$

Hence the series solution of example 5 is

$u(x,t) = 1 + xt$ (88)

Method 2: NIM

Considering Problem 5

Equation (84) is equivalent to the following integral equation

$u = f + \iint_0^t (u_{xx} + u - u^2 + xt + x^2t^2)dtdt$ (89)

Following the algorithm, the successive approximations are:

$u_0 = f = 1 + xt$

$N(u_0) = u_1 = 0$

$u_2 = N(u_0+u_1) - N(u_0)$

$u_2 = 0$

$u_3 = 0$

Hence, the series solution of example 5 is

$$u(x,t) = \sum_{n=1}^{\infty} u_n = 1 + \mathrm{xt} + 0 + 0 + 0 + + \cdots$$

$u(x,t) = 1 + xt$ (90)

Method 3: MADM

The operator form of equation (84) is

$u = f + L_t^{-1}(u_{xx} + u - u^2)$ (91)

For MADM,

$f = f_1 + f_2$

$f_1 = 1 + xt \quad and\ f_2 = \frac{xt^3}{6} + \frac{x^2t^4}{12}$ (92)

Following the algorithm, the successive approximations are:

$u_0 = f_1 = 1 + xt$

$u_1 = \frac{xt^3}{6} + \frac{x^2t^4}{12} - \frac{xt^3}{6} - \frac{x^2t^4}{12} = 0$ (93)

$u_{k+1} = 0, \quad k \geq 1$

Therefore the solution of example 5 in series form is

$u = u_0 + u_1 + u_2 + u_3 + \cdots .. = 1 + xt + 0 + 0 + 0 + \cdots$

$u(x,t) = 1 + xt$ (94)

Example 6

Given the following nonlinear inhomogeneous Klein-Gordon equation:

$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4$ (95)

With the initial conditions

$u(x,0) = 0, \qquad u_t(x,0) = 0$

Whose exact solution was found to be

$u(x,t) = x^2t^2$

Method 1: VIM

The correction functional for example 6 is written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\varepsilon)\left[\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 \tilde{u}_n(x,\varepsilon)}{dx^2} + \widetilde{u_n^2}(x,\varepsilon) - 2x^2 + 2\varepsilon^2 - x^4\varepsilon^4\right]d\varepsilon \quad (96)$$

From previous calculation, it was found that

$\lambda = \varepsilon - t$

Substitute λ into equation (96) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\varepsilon - t)\left(\frac{\partial^2 u_n(x,\varepsilon)}{\partial \varepsilon^2} - \frac{\partial^2 u_n(x,\varepsilon)}{\partial x^2} + u_n^2(x,\varepsilon) - 2x^2 + 2\varepsilon^2 - x^4\varepsilon^4\right)d\varepsilon$$

for $n \geq 0$ (97)

Following the algorithm, the successive approximations are

$u_0(x,t) = x^2t^2,$

$u_1(x,t) = x^2t^2 + \int_0^t (\varepsilon - t)(0)d\varepsilon = x^2t^2$

$u_2(x,t) = x^2t^2 + \int_0^t (\varepsilon - t)(0)d\varepsilon = x^2t^2$ (98)

$u_3(x,t) = x^2t^2$

$..u_1 = u_2 = u_3 = \cdots . u_n = x^2t^2$

Hence the series solution of example 6 is

$u(x,t) = x^2t^2$ (99)

Method 2: NIM

Equation (95) is equivalent to the following integral equation

$u = f + \iint_0^t (u_{xx} - u^2 - 2t^2 + x^4t^4)dtdt$ (100)

Following the algorithm, the successive approximations are:

$u_0(x,t) = x^2t^2$

$N(u_0) = u_1 = 0$

$u_2 = 0$

(101)

$u_3 = 0$

Hence, the series solution of example 6 is

$$u(x,t) = \sum_{n=1}^{\infty} u_n = x^2t^2 + 0 + 0 + 0 + + \cdots$$

$$u(x,t) = x^2t^2$$

(102)

Method 3: MADM

Considering equation (95)

For MADM,

$f = f_1 + f_2$

$u_0 = f_1 = x^2t^2$ and $f_2 = -\frac{1}{6}t^4 + \frac{1}{30}x^4t^6$

(103)

Following the algorithm, the successive approximations are

$$u_1 = -\frac{1}{6}t^4 + \frac{1}{30}x^4t^6 + \frac{1}{6}t^4 - \frac{1}{30}x^4t^6 = 0$$

$$u_2 = L_t^{-1}\left(u_{1(xx)} - u_1^2\right) = 0$$

$$u_{k+1} = 0, \quad k \geq 1$$

Therefore the solution of example 6 in series form is

$$u = u_0 + u_1 + u_2 + u_3 + \cdots .. = x^2t^2 + 0 + 0 + 0 + \cdots$$

$$u(x,t) = x^2t^2$$

(104)

CONCLUSION

We have obtained the analytical solutions of linear and nonlinear Klein-Gordon equations using the Variational Iteration Method (VIM), the New Iterative Method (NIM) and the Adomian Decomposition Method (ADM). We compared the three particular methods, the Variational Iteration Method (VIM), the New Iterative Method (NIM) and the Adomian Decomposition Method (ADM).

The examples considered show that Variational Iteration Method (VIM), the New Iterative Method (NIM) and the Adomian Decomposition Method (ADM) are very powerful, efficient and effective methods in solving linear, nonlinear, homogeneous and inhomogeneous Klein-Gordon equation.

The obtained results in comparison with other existing results and the exact results admit a remarkable efficiency.

Hence, It was concluded the three methods, Variational Iteration Method (VIM), New Iterative Method (NIM) and Adomian Decomposition Method (ADM) are powerful methods that can be used to solve Klein-Gordon equation efficiently and effectively therefore solving any other linear and nonlinear differential equations of higher order (whether partial or ordinary differential equations) is very easy by using these powerful methods.

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