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Eventually Positive Matrices and Tree Sign Patterns

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Abstract

A matrix $A \in M_n(\mathbb{R})$ is said to be eventually positive if there is a power k such that A^k is entrywise positive, and all subsequent powers are also entrywise positive. Here we provide an expression for the smallest such exponent of a 2-by-2 eventually positive matrix in terms of its entries; we also show that if the graph of an eventually positive matrix is a tree, then the positive part of that matrix must be primitive.

Chapter 1

Introduction

A matrix $A \in M_n(\mathbb{R})$ (that is, an n-by-n matrix whose entries are real numbers) is said to be **positive** if all of its entries are positive, and **nonnegative** if all of its entries are nonnegative. These concepts are also denoted symbolically by $A > 0$ and $A \geq 0$, respectively. We let $\sigma(A)$ denote the **spectrum** of A, or the set of eigenvalues of A, and $\rho(A)$ denote the **spectral radius** of A, or the maximum modulus of an eigenvalue of A.

Perron's theorem establishes properties of positive matrices that have important applications to the modeling of dynamic systems in fields such as economics, demography, and queueing theory, among many others.

Theorem 1.1 (**Perron (1907)** [3])**.** *Let $A \in M_n$ be positive. Then*

1. *$\rho(A) > 0$; i.e. A has at least one nonzero eigenvalue.*

2. *$\rho(A)$ is an algebraically simple eigenvalue of A (i.e. $\rho(A)$ is a root of the characteristic polynomial of A with multiplicity 1).*

3. *There is a unique real vector $x = [x_i]$ with positive entries such that $Ax = \rho(A)x$ and $\sum_{i=1}^{n} x_i = 1$.*

4. *There is a unique real vector $y = [y_i]$ with positive entries such that $y^T A = \rho(A) y^T$ and $x \cdot y = 1$.*

5. *$|\lambda| < \rho(A)$ for every eigenvalue λ of A such that $\lambda \neq \rho(A)$.*

6. *$(\rho(A)^{-1}A)^m \to xy^T$ as $m \to \infty$.*

This theorem guarantees, for instance, that a Markov chain represented by a positive transition matrix A is ergodic and has a well-defined steady-state probability distribution.

The Perron-Frobenius theorem generalizes these results to a class of nonnegative matrices which are known as **irreducible** matrices.

Definition. A matrix A is said to be **reducible** if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

in which B and D are square blocks; A is **irreducible** if no such permutation exists.

Theorem 1.2 (**Perron-Frobenius (1912)** [3])**.** *Let $A \in M_n$ be nonnegative and irreducible. Then*

1. $\rho(A) > 0$
2. $\rho(A)$ *is an algebraically simple eigenvalue of* A
3. *There is a unique real vector* $x = [x_i]$ *with nonnegative entries such that* $Ax = \rho(A)x$ *and* $\sum_{i=1}^{n} x_i = 1$
4. *There is a unique real vector* $y = [y_i]$ *with nonnegative entries such that* $y^T A = \rho(A)y^T$ *and* $x \cdot y = 1$

This is somewhat weaker than Perron's theorem. For instance, an irreducible nonnegative matrix A may have two eigenvalues λ_1 and λ_2 such that $|\lambda_1| = |\lambda_2| = \rho(A)$, as in the case of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

A has two eigenvalues, 1 and -1, both of which have absolute value equal to $\rho(A)$.

It is natural to ask whether there are any nonnegative irreducible matrices that have all of the properties guaranteed by Perron's theorem (usually called the **strong Perron-Frobenius property**) despite not being positive, and, if so, how to characterize such matrices.

Definition. A nonnegative irreducible matrix A is said to be **primitive** if it has a dominant eigenvalue, i.e. if it has only one nonzero eigenvalue of maximum modulus. [3]

If A is an n-by-n primitive matrix, then

1. There is an exponent $k \in \mathbb{N}$ such that $A^k > 0$; the minimum such exponent is called the **index of primitivity** of A.

2. If $A^m > 0$, then $A^{m+1} > 0$.

3. The magnitudes of the entries of A do not affect its primitivity; any individual positive entry can be changed arbitrarily in magnitude (provided it remains positive), and the resulting perturbed matrix will also be primitive.

4. The index of primitivity of A does not depend on the magnitudes of the entries of A.

5. The index of primitivity of A is bounded above by $(n-1)^2+1$, and there is an n-by-n primitive matrix for which this bound is attained.

Example. This matrix is called the Wielandt matrix. Its index of primitivity is $(n-1)^2+1$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Just as the theory of primitive matrices answers the question of how the strong Perron-Frobenius properties generalize to nonnegative matrices, the theory of **eventually positive (EP)** matrices aims to answer how these properties generalize to real matrices with some negative entries, including laying out how eventually positive matrices differ from and are similar to nonnegative primitive matrices.

The study of eventually positive matrices became established with a series of papers in the early to mid-2000's ([4], [5]) attempting to characterize such matrices, and has expanded greatly in the last ten years. One of the aims of this paper is to help provide a general overview of what is already known about eventually positive matrices, with the additional goal of providing new contributions to the existing body of work. In particular, we provide a quantitative analysis of 2-by-2 eventually positive matrices, as well as a characterization of potentially eventually positive sign patterns whose graphs are trees.

Chapter 2

General background

A critical fact is that simple eigenvalues of a matrix are continuous with respect to perturbations of the matrix entries. The Perron-Frobenius theorem, stated in the preceding section, is also crucial; note that in the case of a matrix A with the Perron-Frobenius property, $\rho(A)$ is often referred to as the **Perron root** of A, and a corresponding eigenvector of A a left, or right, **Perron vector** of A.

For a matrix $A = (a_{ij}) \in M_n$, not necessarily symmetric, the **directed graph** of A, denoted by $G(A)$, is the graph with vertices $1, \dots, n$, and an arc from i to j if and only if $a_{ij} \neq 0$. Note that this includes the possibility of $G(A)$ containing self-loops, i.e. edges beginning and ending at the same vertex, but does not allow for multiple arcs beginning and ending at the same vertices. If A is symmetric, then we may consider $G(A)$ to be an **undirected graph** with an edge between vertices i and j if and only if $a_{ij} = a_{ji} \neq 0$. We use the word "edge" to indicate an edge in an undirected graph, and "arc" to indicate an edge in a directed graph.

It will sometimes, but not always, be useful to consider $G(A)$ as a weighted graph, where the weight of the arc from i to j is a_{ij}; weights need not be positive, and indeed we will mostly be concerned here with graphs that have some negative weights.

A **path** in a graph is an ordered list of edges or arcs

$$(i_0, i_1), (i_1, i_2), \dots, (i_{k-1}, i_k).$$

This path is termed a k**-cycle** if it begins and ends at the same vertex (i.e. $i_0 = i_k$), and a **simple** k**-cycle** if no other vertices are visited more than once.

A **tree** is an undirected graph with no simple cycles of length greater than 2. In a tree, every edge is a **cut edge**: an edge whose removal splits the graph into multiple connected components. A directed graph is **strongly connected** if, for any pair of vertices i and j, there exists a path from i to j.

This enables us to give alternate characterizations of irreducibility and primitivity, as follows:

Proposition 2.1. *A matrix A is irreducible if and only if $G(A)$ is strongly connected.*

Proposition 2.2. *Let A be a nonnegative irreducible matrix. A is primitive if and only if the greatest common divisor of the cycle lengths of $G(A)$ is 1.*

Example. The graph of the n-by-n Wielandt matrix consists of an n-cycle and an $(n-1)$-cycle. Since n and $n-1$ are coprime for $n \geq 2$, the Wielandt matrix is primitive.

Definition. A **sign pattern matrix**, or simply a **sign pattern**, is a matrix whose entries are elements of the set $\{+, -, 0\}$.

Sign patterns will be indicated by script letters (e.g. $\mathcal{A}$, $\mathcal{B}$).

Definition. Let $\mathcal{A} = (\alpha_{ij})$ be a sign pattern. $\hat{\mathcal{A}} = (\hat{\alpha}_{ij})$ is a **superpattern** of $\mathcal{A}$ if $\alpha_{ij} = 0$ whenever $\hat{\alpha}_{ij} = 0$, and $\hat{\alpha}_{ij} = \alpha_{ij}$ whenever $\alpha_{ij} \neq 0$. That is to say, $\hat{\mathcal{A}}$ may be obtained from $\mathcal{A}$ by replacing some zero entries with $+$ or $-$. It is also said conversely that $\mathcal{A}$ is a **subpattern** of $\hat{\mathcal{A}}$.

Example.

$$\mathcal{A} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$$

$$\hat{\mathcal{A}} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$\hat{\mathcal{A}}$ is a superpattern of $\mathcal{A}$. $\mathcal{A}$ is a subpattern of $\hat{\mathcal{A}}$.

Example.

$$\mathcal{A} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

Neither $\mathcal{A}$ nor $\mathcal{B}$ is a sub– or superpattern of the other.

The **qualitative class** of a sign pattern $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of matrices $A \in M_n(\mathbb{R})$ such that $\text{sgn}(A_{ij}) = \mathcal{A}_{ij}$; a matrix in $\mathcal{Q}(\mathcal{A})$ is said to be a **realizing matrix** of $\mathcal{A}$.

Example.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

is a realizing matrix of the sign pattern $\mathcal{A}$ from the previous examples.

A sign pattern $\mathcal{A}$ is said to **require** a property if all matrices in $\mathcal{Q}(\mathcal{A})$ have that property, and to **allow** a property if there exists a matrix in $\mathcal{Q}(\mathcal{A})$ that has that property.

Example. It was stated in the previous section that the magnitudes of the positive entries of a primitive matrix do not affect the primitivity of that matrix (meaning that primitivity can be thought of as a property of sign patterns as well as real–valued matrices). That is to say that the sign pattern of a primitive matrix, for example

$$\begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix}$$

requires primitivity.

Every real-valued matrix A can be expressed as the difference of two matrices $P - N$, where $P_{ij} = A_{ij}$ if $A_{ij} \geq 0$, and $N_{ij} = -A_{ij}$ if $A_{ij} \leq 0$; P is the **positive part** of A, and N is the **negative part**. The same holds true for sign patterns; we denote the positive part of the sign pattern $\mathcal{A}$ by $\mathcal{A}^+$, and the negative part by $\mathcal{A}^-$.

Example.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The positive part of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The negative part is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Chapter 3

Theory of eventually positive matrices

Definition. An **eventually positive** (EP) matrix is a matrix $A \in M_n(\mathbb{R})$ such that there is some $k_0 \in \mathbb{N}$ (termed the **power index** of A) where $A^{k_0} > 0$ and $A^k > 0$ for all $k \geq k_0$, and k_0 is the minimal such power.

Definition. A **potentially eventually positive** (PEP) sign pattern is a sign pattern that allows eventual positivity.

Note that, unlike primitive matrices, an EP matrix with some negative entries may have $A^k > 0$ but A^{k+1} non-positive.

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1.25 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2.5 & 1.75 \\ 1.75 & -0.125 \end{bmatrix}$$

Another important difference between EP matrices and primitive matrices is that two matrices with the same sign pattern but different magnitudes of entries may not both be EP.

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A and B have the same sign pattern, and A is EP, but B is not (see §4).

The class of EP matrices is closed under positive scalar multiplication and the addition of positive multiples of the identity matrix. It is also closed under positive diagonal similarity and permutation similarity. When you take the strong Perron-Frobenius property, as we do here, to include the existence of both left and right positive eigenvectors associated with the Perron root, the class of EP matrices is closed under transposition (contrast with [5], which does not require a positive left eigenvector).

Theorem 3.1 ([4]). *Let $A \in M_n(\mathbb{R})$. The following are equivalent:*

1. *A has the strong Perron-Frobenius property*
2. *$A^k > 0$ and $A^{k+1} > 0$ for some $k \in \mathbb{N}$*
3. *A is EP*

It follows from the eigenvalue-eigenvector equation that, if A is EP, then A must have a positive entry in every row and column: if x is a right Perron vector of A and ρ the Perron root of A, then for every i such that $1 \leq i \leq n$, $\sum_{j=1}^{n} a_{ij}x_j = \rho \sum_{j=1}^{n} x_j$, and since the right hand side of the equation is positive, the ith row of A must have some positive entry. Repeating the same argument with a left eigenvector of ρ shows the necessity of a positive entry in every column of A.

Proposition 3.1 ([2]). *If $\mathcal{A}$ is a sign pattern with $\mathcal{A}^+$ primitive, then $\mathcal{A}$ is PEP.*

This follows from continuity of the Perron root and Perron vectors with respect to perturbations of the matrix entries; a realizing matrix need only have negative entries sufficiently small. It was conjectured in [4] that the condition of positive part primitivity was necessary as well as sufficient for a sign pattern to be PEP. That conjecture was disproven in [2], which presents the following matrix:

$$B = \begin{bmatrix} 1.3 & -0.3 & 0 \\ 1.3 & 0 & -0.3 \\ -0.31 & 0.3 & 1.01 \end{bmatrix}$$

B is EP with power index 10, but B^+ is not even irreducible, let alone primitive.

Though the positive part of a sign pattern $\mathcal{A}$ need not be irreducible for $\mathcal{A}$ to be PEP, $\mathcal{A}$ must be irreducible. If $\mathcal{A}$ is reducible, then all powers of $\mathcal{A}$ must also be reducible, and so must have some zero entries.

Many more important lemmas about PEP sign patterns are found in [2]

Lemma 3.1. *If $\mathcal{A}$ is a PEP sign pattern, then so is any superpattern of $\mathcal{A}$. If $\mathcal{A}$ is not PEP, then no subpattern of $\mathcal{A}$ is PEP.*

Lemma 3.2. *If $\mathcal{A}$ is PEP, then so is the sign pattern obtained from $\mathcal{A}$ by replacing all nonpositive diagonal entries with $+$.*

Lemma 3.3. *An n-by-n PEP sign pattern must have at least $n+1$ positive entries.*

A Z **sign pattern** is a sign pattern $\mathcal{A} = (\alpha_{ij})$ where $\alpha_{ij} \neq +$ if $i \neq j$; that is, no off-diagonal entries are $+$.

Lemma 3.4. *An n-by-n Z sign pattern with $n \geq 2$ is not PEP.*

Lemma 3.5. *The block sign pattern $\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$ with $\mathcal{A}_{12}$ nonpositive, $\mathcal{A}_{21}$ nonnegative, and square diagonal blocks is not PEP.*

Let $[+]$ and $[-]$ denote blocks of all $+$ entries and all $-$, respectively.

Lemma 3.6. *The block sign pattern*

$$\mathcal{A}_0 = \begin{bmatrix} [+] & [-] & [+] & \dots \\ [-] & [+] & [-] & \dots \\ [+] & [-] & [+] & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks is not PEP if $\mathcal{A}_0$ has a negative entry. $-\mathcal{A}_0$ is not PEP.

The sign pattern $\mathcal{A}_0$ is called a (block) checkerboard pattern.

Lemma 3.7. *Let $\mathcal{A}$ be a square sign pattern with square diagonal blocks, and $\mathcal{D}$ a diagonal sign pattern that, when partitioned conformally with $\mathcal{A}$, has diagonal blocks $\mathcal{D}_{ii}$ either nonpositive or nonnegative. If $\mathcal{D}\mathcal{A}\mathcal{D} \leq 0$, or $\mathcal{D}\mathcal{A}\mathcal{D} \geq 0$ with $\mathcal{D}$ having at least one $-$, $\mathcal{A}$ is not PEP.*

Lemma 3.7 can be worded alternately as saying that if a sign pattern $\mathcal{A}$ can be made nonnegative by a non-trivial **signature similarity**, that is, a similarity by a diagonal sign pattern containing at least one $-$, then $\mathcal{A}$ is not PEP. The proof in [2] of Lemma 3.7 makes it a corollary of Lemma 3.6, but it is often simpler to show that a sign pattern is signature-similar to a nonnegative sign pattern than it is to show that a sign pattern is a checkerboard pattern.

Example.

$$\mathcal{A} = \left[\begin{array}{c|c|cc|c} + & - & + & + & 0 \\ \hline - & 0 & 0 & 0 & 0 \\ \hline + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & - \\ \hline 0 & 0 & 0 & - & 0 \end{array}\right]$$

$$\mathcal{S} = \left[\begin{array}{c|c|cc|c} - & 0 & 0 & 0 & 0 \\ \hline 0 & + & 0 & 0 & 0 \\ \hline 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ \hline 0 & 0 & 0 & 0 & + \end{array}\right]$$

$\mathcal{A}$ is partitioned such that it has square diagonal blocks; $\mathcal{S}$ has been partitioned conformally with $\mathcal{A}$, and its diagonal blocks have either all negative or all positive diagonal entries, with some blocks being nonpositive.

$$\mathcal{SAS} = \begin{bmatrix} + & + & + & + & 0 \\ + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + \\ 0 & 0 & 0 & + & 0 \end{bmatrix} \geq 0$$

By Lemma 3.7, $\mathcal{A}$ is not PEP.

Lemma 3.8. *Let $\mathcal{A} = (\alpha_{ij})$ be an n-by-n sign pattern with $n \geq 2$. If for all $k \in 1, \ldots, n$, we have:*

1. *$\alpha_{kk} = +$, and*

2. *no off-diagonal entry in row k is $+$ **or** no off-diagonal entry in column k is $+$,*

then $\mathcal{A}$ is not PEP.

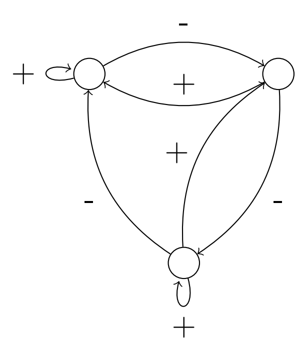
For sign patterns of dimension less than or equal to 3, all PEP sign patterns have been enumerated.

Proposition 3.2 ([2])**.** *If $\mathcal{A}$ is an n-by-n PEP sign pattern with $n \leq 3$, then either $\mathcal{A}^+$ is primitive, or $\mathcal{A}$ is equivalent by permutation similarity to a pattern of the form*

$$\mathcal{B} = \begin{bmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{bmatrix},$$

or to a superpattern of $\mathcal{B}$.

Note that three is the smallest number of vertices a graph that is not a tree may have, and that the smallest PEP sign pattern with positive part not primitive is 3-by-3. If we look at the graph of the sign pattern $\mathcal{B}$, we can see that it has a simple 3-cycle:



This suggests that Conjecture 8.1 in [4], which is false in general, may be revived by restricting the sign patterns under consideration to those whose graphs are trees. The work of Yu et. al. ([6], [7], [8], among others) has established that for sign patterns $\mathcal{A}$ whose graphs are various particular subtypes of trees (stars, paths, certain types of generalized stars), it is necessary as well as sufficient that $\mathcal{A}^+$ be primitive; however, nowhere in the literature reviewed here has this been established for arbitrary trees.

The n-by-n generalization of the sign pattern $\mathcal{B}$ is found in [1]

$$\begin{bmatrix} + & - & - & \cdots & - & 0 \\ + & 0 & 0 & \cdots & 0 & - \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & 0 & \cdots & 0 & - \\ - & + & + & \cdots & + & + \end{bmatrix}$$

and provides another way to construct PEP sign patterns with reducible positive part.

Definition. Let $A = (a_{ij}) \in M_n$, $B \in M_m$. The **Kronecker product** $A \otimes B$ is an mn-by-mn matrix defined as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

Proposition 3.3 ([1])**.** *If A and B are EP matrices, then $A \otimes B$ is EP. If $\mathcal{A}$ and $\mathcal{B}$ are PEP sign patterns, then $\mathcal{A} \otimes \mathcal{B}$ is PEP.*

Chapter 4

The 2-by-2 case

It is already known what sign patterns allow eventual positivity for 2-by-2 matrices [2]. However, beyond general statements that the power index of an EP matrix with some negative entries may be arbitrarily large, there has been little numerical study of EP matrices and their precise power indices. The goal in this section is to provide an expression for the power index of a 2-by-2 EP matrix in terms of its entries.

For A a 2-by-2 matrix, we know that, in order for A to be EP, it must have the form

$$A = \begin{bmatrix} w & x \\ y & -z \end{bmatrix}$$

with $w, x, y > 0$ and $z \geq 0$, up to a permutation similarity. By a positive diagonal similarity and positive scalar multiplication, we can make A symmetric with off-diagonal entries equal to 1:

$$\frac{1}{\sqrt{xy}} \left(\begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{x} \end{bmatrix} \begin{bmatrix} w & x \\ y & -z \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{y}} & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{bmatrix} \right) = \begin{bmatrix} a & 1 \\ 1 & -b \end{bmatrix} = A'$$

For A' to be EP, it is necessary that $a > b$; otherwise $\operatorname{tr}(A') \leq 0$, in which case every odd power of A' would also have non-positive trace, and so have at least one non-positive entry on the main diagonal. We aim to show here that it is also sufficient that $a > b$, as well as to find the power index of A'.

Proposition 4.1. *A' as defined above is EP.*

Proof. It suffices to show that A' satisfies the Perron-Frobenius properties: $\rho(A') \in \sigma(A')$, $\rho(A')$ has algebraic multiplicity 1, and there exist positive vectors x, y such that

$$A'x = \rho(A')x$$

and

$$y^T A' = \rho(A')y^T.$$

The eigenvalues of A' are

$$\frac{a - b \pm \sqrt{(a+b)^2 + 4}}{2}$$

We can see that A' has a positive eigenvalue

$$\lambda_1 = \frac{a - b + \sqrt{(a+b)^2 + 4}}{2}$$

and a negative eigenvalue

$$\lambda_2 = \frac{a - b - \sqrt{(a+b)^2 + 4}}{2}$$

and that $|\lambda_1| > |\lambda_2|$.

The right eigenvector associated with λ_1 can be found by row-reduction of $A' - \lambda_1 I$, and we have $x = [\lambda_1 - a, 1]^T > 0$. Since A' is symmetric, this vector is also a positive left eigenvector associated with λ_1. □

Because A' is EP, A must also be EP if $w > z$.

The next goal is to provide an expression for the power index of A', which will also be the power index of A.

Define the recurrence relations a_n, b_n, c_n such that

$$(A')^n = \begin{bmatrix} a_n & c_n \\ c_n & b_n \end{bmatrix} = \begin{bmatrix} a & 1 \\ 1 & -b \end{bmatrix} \begin{bmatrix} a_{n-1} & c_{n-1} \\ c_{n-1} & b_{n-1} \end{bmatrix}$$

Therefore the recurrences are

$$\begin{aligned} a_n &= a \cdot a_{n-1} + c_{n-1} \\ b_n &= c_{n-1} - b \cdot b_{n-1} \\ c_n &= a \cdot c_{n-1} + b_{n-1} \\ &= a_{n-1} - b \cdot c_{n-1} \end{aligned}$$

By algebraic manipulation, we can express c_n in terms of a_n and b_n:

$$\begin{aligned} a \cdot c_n + b_n &= a_n - b \cdot c_n \\ (a+b) \cdot c_n &= a_n - b_n \\ c_n &= \frac{a_n - b_n}{a+b} \end{aligned}$$

Now we have that

$$a_n = a \cdot a_{n-1} + \frac{a_{n-1} - b_{n-1}}{a+b} = \left(a + \frac{1}{a+b}\right) a_{n-1} - \frac{1}{a+b} \cdot b_{n-1}$$

$$b_n = \frac{a_{n-1} - b_{n-1}}{a+b} - b \cdot b_{n-1} = \frac{1}{a+b} \cdot a_{n-1} - \left(b + \frac{1}{a+b}\right) b_{n-1}$$

and we may now express a_n in terms of b_n and b_{n-1}, and b_n in terms of a_n and a_{n-1}. Then substitute each equation into the other to obtain pure recurrence relations for a_n and b_n.

$$\begin{aligned} a_n &= (a^2 + ab + 1)b_n - \frac{1 - (a^2 + ab + 1)(b^2 + ab + 1)}{a+b} b_{n-1} \\ &\vdots \\ a_n &= (a-b)a_{n-1} + (ab+1)a_{n-2} \end{aligned}$$

$$\begin{aligned} b_n &= (b^2 + ab + 1)a_n - \frac{(a^2 + ab + 1)(b^2 + ab + 1) - 1}{a+b} a_{n-1} \\ &\vdots \\ b_n &= (a-b)b_{n-1} + (ab+1)b_{n-2} \end{aligned}$$

The next step in finding a closed-form equation is to form the characteristic polynomial of the recurrence relation:

$$p(t) = t^2 - (a-b)t - (ab+1)$$

Since the characteristic polynomial of the recurrence relation is the characteristic polynomial of the matrix itself, we already know that its roots are λ_1 and λ_2. Therefore the equations for a_n and b_n will be:

$$a_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n$$

$$b_n = \beta_1\lambda_1^n + \beta_2\lambda_2^n$$

and we can use the initial conditions $a_0 = b_0 = 1$, $a_1 = a$, $b_1 = -b$ to solve for the α_i and β_i.

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \implies \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 - a \\ a - \lambda_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -b \end{bmatrix} \implies \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 - (-b) \\ -b - \lambda_1 \end{bmatrix}$$

Recalling that $\lambda_1 + \lambda_2 = a - b$, we may rewrite β_1 and β_2:

$$\begin{aligned} \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 - (-b) \\ -b - \lambda_1 \end{bmatrix} &= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 - (\lambda_1 + \lambda_2 - a) \\ (\lambda_1 + \lambda_2 - a) - \lambda_1 \end{bmatrix} \\ &= \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} a - \lambda_1 \\ \lambda_2 - a \end{bmatrix} \\ &= \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix} \end{aligned}$$

Finally, we have the closed-form formulas for a_n and b_n:

$$\begin{aligned} a_n &= \frac{a - \lambda_2}{\lambda_1 - \lambda_2}\lambda_1^n + \frac{\lambda_1 - a}{\lambda_1 - \lambda_2}\lambda_2^n \\ b_n &= \frac{\lambda_1 - a}{\lambda_1 - \lambda_2}\lambda_1^n + \frac{a - \lambda_2}{\lambda_1 - \lambda_2}\lambda_2^n \end{aligned}$$

Let $\Delta = (a+b)^2 + 4$, and expand α_1 and α_2:

$$\begin{aligned} \alpha_1 &= \frac{\frac{2a - a + b + \sqrt{\Delta}}{2}}{\frac{2\sqrt{\Delta}}{2}} = \frac{a + b + \sqrt{\Delta}}{2\sqrt{\Delta}} \\ \alpha_2 &= \frac{\frac{a - b - 2a + \sqrt{\Delta}}{2}}{\sqrt{\Delta}} = \frac{\sqrt{\Delta} - (a+b)}{2\sqrt{\Delta}} \end{aligned}$$

Since $\sqrt{\Delta} > a + b$, it is now easier to see that both α_1 and α_2 are positive, and that $\alpha_2 < \alpha_1$. Combined with the facts that $\lambda_2 < 0 < \lambda_1$ and $|\lambda_2| < |\lambda_1|$, we are in a position to show the following:

Theorem 4.1. *For all $n \geq 1$, $a_n > 0$ and $c_n > 0$. Moreover, there exists a least integer N such that, for all $n \geq N$, $b_n > 0$. This N is the power index of A'.*

Proof. By the facts above, $|\alpha_1 \lambda_1^n| > |\alpha_2 \lambda_2^n|$, and so $a_n > 0$ for all n.

We show by induction that $|a_n| > |b_n|$ for all $n \geq 1$.

$$|b_1| = b < a = |a_1|$$

$$|b_2| = b^2 + 1 < a^2 + 1 = |a_2|$$

If, for some $k \in \mathbb{N}$, $|a_{k-1}| > |b_{k-1}|$ and $|a_k| > |b_k|$, then

$$\begin{aligned} |b_{k+1}| &= |(a-b)b_k + (ab+1)b_{k-1}| \\ &\leq (a-b)|b_k| + (ab+1)|b_{k-1}| \\ &< (a-b)a_k + (ab+1)a_{k-1} \\ &= |a_{k+1}| \end{aligned}$$

Therefore, for all $n \geq 1$,

$$c_n = \frac{a_n - b_n}{a+b} > 0$$

Now we turn our attention to b_n. Consider the inequality

$$b_n = \frac{\sqrt{\Delta} - (a+b)}{2\sqrt{\Delta}} \lambda_1^n + \frac{a+b+\sqrt{\Delta}}{2\sqrt{\Delta}} \lambda_2^n > 0$$

If n is even, then both terms are positive, and the statement is trivial. Suppose, therefore, that n is odd.

$$\begin{aligned} \frac{\sqrt{\Delta} - (a+b)}{2\sqrt{\Delta}} \lambda_1^n + \frac{a+b+\sqrt{\Delta}}{2\sqrt{\Delta}} \lambda_2^n &> 0 \\ \left(\sqrt{\Delta} - (a+b)\right) \lambda_1^n + \left(a+b+\sqrt{\Delta}\right) \lambda_2^n &> 0 \\ \left(\sqrt{\Delta} - (a+b)\right) |\lambda_1|^n &> \left(a+b+\sqrt{\Delta}\right) |\lambda_2|^n \\ \left(\frac{|\lambda_1|}{|\lambda_2|}\right)^n &> \frac{\sqrt{\Delta} + a + b}{\sqrt{\Delta} - (a+b)} \\ n &> \frac{\ln(\sqrt{\Delta} + a + b) - \ln(\sqrt{\Delta} - (a+b))}{\ln(\lambda_1) - \ln(-\lambda_2)} \\ n &> \frac{\ln(\sqrt{\Delta} + a + b) - \ln(\sqrt{\Delta} - (a+b))}{\ln(\sqrt{\Delta} + a - b) - \ln(\sqrt{\Delta} - (a-b))} \end{aligned}$$

Call the right-hand side of the inequality L. Then the least positive odd n such that $n > L$ would be

$$n = 1 + \lfloor L \rfloor + \frac{1}{2}\left(1 + (-1)^{1+\lfloor L \rfloor}\right)$$

The power index for A' would then be $n - 1$:

$$N = \lfloor L \rfloor + \frac{1}{2}\left(1 + (-1)^{1+\lfloor L \rfloor}\right)$$

□

Stepping back slightly, remember that A' was obtained from A by positive diagonal similarity and positive scalar multiplication, both of which preserve the positivity of a matrix. That is, if $(A')^k > 0$, then $A^k > 0$, and so the N obtained in Theorem 4.1 is the power index of the original matrix A.

Chapter 5

Tree sign patterns

We call a sign pattern $\mathcal{A}$ whose directed graph has no cycles of length greater than 2 a **tree sign pattern** (TSP); the graph obtained from $G(\mathcal{A})$ by replacing directed arcs with undirected edges will be a tree. In this section, we are concerned with what necessary conditions there are for a TSP to be PEP.

Throughout, assume that $\mathcal{A}$ is a TSP.

Lemma 5.1. *For any off-diagonal entry α_{ij} in $\mathcal{A}$, there exists a permutation similarity $\mathcal{P}$ such that*

$$\mathcal{P}^T\mathcal{A}\mathcal{P} = \left[\begin{array}{cccc|cccc} & & & & 0 & & & \\ & \mathcal{A}_{11} & & & \vdots & & 0 & \\ & & & & 0 & & & \\ & & & & \alpha_{ij} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \alpha_{ji} & & & & \\ & & & 0 & & & & \\ & 0 & & \vdots & & \mathcal{A}_{22} & & \\ & & & 0 & & & & \end{array}\right]$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are both square.

Because all edges in a tree are cut edges, the removal of the i, j edge of the graph gives two connected components G_1 and G_2. Relabel the vertices of $G(\mathcal{A})$ so that the vertices in G_1 come first, ending with vertex i, and then, beginning with vertex j, list the vertices in G_2.

Proposition 5.1. *If $\mathcal{A}$ is not combinatorially symmetric, then $\mathcal{A}$ is not PEP.*

If $\mathcal{A}$ is not combinatorially symmetric, then $\mathcal{A}$ is reducible by Lemma 5.1 (either α_{ij} or α_{ji} is 0), and so is not PEP.

Proposition 5.2. *If $\mathcal{A}$ is PEP, then all nonzero off-diagonal entries are $+$.*

Proof. We will classify the 2-cycles in $G(\mathcal{A})$ based on the weights of their component arcs: the possibilities are $+/+$, $+/-$, and $-/-$.

Case 1: $\alpha_{ij} = -$, $\alpha_{ji} = +$.

In this case, 3.5 applies, and so $\mathcal{A}$ cannot be PEP.

Case 2: $\alpha_{ij} = \alpha_{ji} = -$.

Suppose without loss of generality (by Lemma 3.2) that $\alpha_{ii} = +$ for all $i \in 1, \ldots, n$. Let $\mathcal{P}_i$ be the permutation matrix from Lemma 5.1, and suppose α_{ij} is moved into position $i, i+1$. Then

$$\mathcal{D}_i = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

with the $-I$ block of size i-by-i is a signature matrix such that

$$\mathcal{D}_i \mathcal{P}_i^T \mathcal{A} \mathcal{P}_i \mathcal{D}_i$$

has the i, j and j, i entries positive, with the rest of the entries unchanged in sign. Let $\mathcal{D}'_i = \mathcal{P}_i \mathcal{D}_i$, and $\mathcal{E} = \prod \mathcal{D}'_i$ for all $-/-$ 2-cycles in $G(\mathcal{A})$. $\mathcal{E}$ is a new signature matrix, and it must have at least one $-$ entry, or else the cumulative effect of all the signature similarities by $\mathcal{D}_i$ would be nothing. Since similarity by $\mathcal{E}$ makes each $-/-$ 2-cycle into a $+/+$ 2-cycle, we have that

$$\mathcal{E} \mathcal{A} \mathcal{E} \geq 0,$$

and so, by Lemma 3.7, $\mathcal{A}$ cannot be PEP. □

Proposition 5.3. *If $\mathcal{A}$ is PEP, then at least one diagonal entry is $+$.*

Proof. Suppose $\mathcal{A}$ has all diagonal entries either $-$ or 0, and consider the diagonal entries of $\mathcal{A}^{2k+1}$, $k \in \mathbb{N}$. Based on the mechanics of matrix multiplication, the i, i entry of $\mathcal{A}^{2k+1}$ is the sum, over all paths of length $2k+1$ from vertex i to itself, of the products of the edge weights in each path. Consider an arbitrary path P of length $2k+1$ from i to i. It has odd length, and so it must have an odd number of edges corresponding to self-loops in $G(\mathcal{A})$: all paths not containing self-loops in $G(\mathcal{A})$ necessarily have even length. This means that the product of edge weights in P is $-$, and so the i, i entry in

$\mathcal{A}^{2k+1}$ must be unambiguously $-$ if any paths of length $2k+1$ from i to i exist, and 0 otherwise. This means that any odd power of $\mathcal{A}$ cannot have positive diagonal entries, and so $\mathcal{A}$ is not PEP. $\square$

We are now in a position to show that positive part primitivity is both necessary and sufficient for a TSP to be PEP.

Theorem 5.1. *If $\mathcal{A}$ is a TSP, then $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive.*

Proof. Let $\mathcal{A}$ be a TSP. If $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP by Proposition 3.1.

Conversely, suppose that $\mathcal{A}$ is PEP. Then $\mathcal{A}$ must be irreducible; since all off-diagonal entries of $\mathcal{A}$ must be either $+$ or 0, this means that $\mathcal{A}^+$ must be irreducible as well. $G(\mathcal{A}^+)$ then has cycles of even length, which consist of arcs corresponding to the off-diagonal $+$ entries in $\mathcal{A}^+$, as well as at least one 1-cycle by Proposition 5.3. Therefore the greatest common divisor of the cycle lengths of $G(\mathcal{A}^+)$ must be 1, and so $\mathcal{A}^+$ is primitive. $\square$

Chapter 6

Conclusion

This report has reviewed some useful facts about EP matrices and PEP sign patterns, as well as answering some outstanding questions about such matrices.

For 2-by-2 EP matrices, essentially everything is known: both what sign patterns are possible, and, for a particular matrix, how long it will be until it becomes positive. We now know as well that for any TSP $\mathcal{A}$, regardless of the type of tree, it is necessary as well as sufficient that $\mathcal{A}^+$ be primitive. This raises a natural question:

Question. Are there any other classes of graph for which, if $G(\mathcal{A})$ is part of that class, $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive?

This seems unlikely, since the matrix B from §3 has only one simple cycle longer than 2, and it already does not have primitive positive part.

The quantitative study of larger sign patterns in the vein of §4 is also an area of potential future research, in particular the study of the sign pattern $\mathcal{B}$ from §3. Since its positive part is reducible, there must be a minimum "size" of its negative part to make a realizing matrix of $\mathcal{B}$ EP, and also a maximum "size". This makes the power index of such a matrix not monotone in the magnitudes of its negative entries, unlike the behavior we have seen for a 2-by-2 EP matrix.

Another question relates to the combination of two EP matrices, in a similar way to the observation about the Kronecker product in §3.

Question. Given two EP matrices A and B and two matrices X and Y, where X and Y each have one positive entry and all other entries zero, is the matrix given by

$$\begin{bmatrix} A & X \\ Y & B \end{bmatrix}$$

eventually positive?

This construction may be considered "almost" a direct sum of A and B, minimally perturbed in such a way as to give an irreducible matrix. It seems plausible that the answer to this question is yes, but it appears to be a complex issue worthy of study on its own.

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