http://www.eksakta.ppj.unp.ac.id
DOI : 10.24036/eksakta/vol21-iss01/217
EKSAKTA Vol. 21 No.1 Issue 01/30 April 2020
E-ISSN : 2549-7464, P-ISSN : 1411-3724

*Article*

# A Solution of Nonlinear Boundary Value Problem of System With Rectangular Coefficients

**Badrulfalah*, Iin Irianingsih , Khafsah Joebaedi**

Department of Mathematics, Faculty of Mathematics and Natural Sciences (FMIPA), Universitas Padjadjaran, Bandung, Indonesia

**Article Info**

***Article history :***

Received April 13, 2020
Revised April 20, 2020
Accepted April 22, 2020
Published April 30, 2020

***Keywords :***

nonlinier, boundary value, rectangular coefficients, *Moore-Penrose*

**Abstract.** This paper discusses a nonlinear boundary value problem of system with rectangular coeffisients of the form $A(t)x' + B(t)x = f(t,x)$ with boundary conditions of the form $B_1x(t_0) = a$ and $B_2x(T) = b$ which is $A(t)$ is a real $m \times n$ matrix with $m > n$ whose entries are continuous on $J = [t_0, T]$ and $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$. $B_1$, $B_2$ are nonsingular matrices such that $a$ and $b$ are constant vectors, especially about the proof of the uniqueness of its solution. To prove it, we use *Moore-Penrose* generalized inverse and method of variation of parameters to find its solution. Then we show the uniqueness of it by using fixed point theorem of contraction mapping. As the result, under a certain condition, the boundary value problem has a unique solution.

.



***Corresponding Author :***
Badrulfalah
Department of Mathematics, Faculty of Mathematics and Natural Sciences (FMIPA), Universitas Padjadjaran, Bandung, Indonesia
Email : badrulfalah@unpad.ac.id

## 1. Introduction

One of them is two-points boundary condition[8]. Here we consider the boundary value problem of first-order differential system of the following form

$$A(t)x' + B(t)x = f(t,x) \quad (1.1)$$

with boundary condition of the form

$$B_1x(t_0) = a \text{ and } B_2x(T) = b \quad (1.2)$$

which is $A(t)$ is a real $m \times n$ matrix with $m > n$ whose entries are continuous on $J = [t_0, T]$ and $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$. $B_1$, $B_2$ are nonsingular matrices such that $a$ and $b$ are constant vectors. In general $A(t)$ is nonsingular square matrix and $f$ is a function of one variable $t$ then equation (1.1) can be written as $x' + M(t)x = h(t)$ which is $M(t) = A^{-1}(t)B(t)$; $h(t) = A^{-1}(t)f(t)$. The uniqueness of the solution is found by using the property of uniqueness of *Moore-Penrose* generalized inverse. In [10] it has been proved the uniqueness of solution the boundary value problem (1.1), (1.2) by using Monotone Iterative Technique. Based on [3,8,10], we will give an alternative proving of that problem (1.1), (1.2) by using fixed point theorem of contractive mapping. The objective of this research is to find some conditions in order to the boundary value problem (1.1), (1.2) has a unique solution. As the result is the uniqueness of solution of the boundary value problem (1.1), (1.2) proved.

## 2. Experimental Methods

To prove the uniqueness of solution of the boundary value problem (1.1), (1.2) by using theorem of contraction mapping done in three steps. Firstly, given the solution of the boundary value problem (1.1), (1.2) by using the property of inverse of Moore-Penrose matrix and method of variation of parameters. After that to find some Contractive condition of relevant operator. The last, proving the uniqueness of the solution by using fixed point theorem of contraction mapping.

## 3. Result and Discussion

### 1. Preliminieries

We start with giving some definitions and fixed point theorem.

**Definition 1.1:** Let $V$ be a vector space over the real number. A mapping $x \to \|x\|$ of $V$ into $\mathbb{R}$ is called a norm if and only if it satisfies the following conditions:

$(i)$ $\|x\| \geq 0$ for all $x \in V$,
$(ii)$ If $x \in V$ and $\|x\| = 0$ then $x = 0$,
$(iii)$ $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$,
$(iv)$ $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.[1]

$(V, \|.\|)$ is called a normed space. Moreover it is also a metric space with its metric an induced metric by norm. In this case, $d(x, y) = \|x - y\|$.

**Definition 1.2:** A complete normed space is called a Banach space. [1]

Suppose that $(X, d)$ is a complete metric space and $T: X \to X$ is any function.

**Definition 1.3:** A point $u$ in $X$ is called a fixed point of $X$ if $T(u) = u$. [11]

**Definition 1.4:** The function $T$ is said to satisfy a Lipschitz condition with constant $L$ if $d\big(T(x), T(y)\big) \leq Ld(x, y)$ holds for all $x, y$ in $X$. [11]

**Definition 1.5:** The function $T$ is called a contraction mapping if there exists a positive real number $K < 1$ such that $d\big(T(x), T(y)\big) \leq Kd(x, y)$ for all $x, y$ in $X$. [11]

**Lemma 1.6:** Let $(X, d)$ be a complete metric space and $T: X \to X$ is a contraction mapping then there exists a unique fixed point of $T$ in $X$. [11]

Further given one definition and some lemmas as well as the solution of the boundary value problem (1.1), (1.2) as one of the results in. [10]

**Definition 1.7:** Let $B(t) \in \mathbb{R}^{m\times n}$. $B(t)$ is called in column space of $A(t) \in \mathbb{R}^{m\times n}$ if there exists some matrix $M(t) \in \mathbb{R}^{n\times n}$ a such that
$B(t) = A(t)M(t)$ for every $t$. .[10]

Consider an equation $Ax = b$, (1.3)
which is $A \in \mathbb{R}^{m\times n}$, $m \geq n$.

**Lemma 1.8:** $(a)$ The following statements are equivalent:will give some definitions and

$(i)$ Equation (1.3) is consistent

$(ii)$ $AA^{+}b = b$

$(iii)$ $rank\ A = rank[A{:}b]$, in equal, $b$ is a linear combination of column of $A$.

$(b)$ The solution, if it exists , is unique if and only if $A$ has rank full that is $rank\ A = n$. The solution is given by $x = A^{+}b + (I - A^{+}A)z$, where $z \in \mathbb{R}^n$ arbitrary.
The least square solution of (1.3) is $x = A^{+}b$ with $A^{+} = (A^TA)^{-1}A^T$ where $m > n$.[10]

The following Lemma 1.9 give a boundary value problem that equivalent to (1.1), (1.2).

**Lemma 1.9:** Let $A(t)$ and $B(t)$ are matrices in (1.1). If $A(t)$ and $A^{-1}(t)$ are nonnegative then the system $A(t)x' + B(t)x = f(t,x)$ if and only if $x' + M(t)x = h(t,x)$ which is $h(t,x) = A^{+}(t)f(t,x)[I(t) - A(t)A^{+}(t)]z$ , $z \in \mathbb{R}^n$. [10]

Consider a boundary value problem that equivalent to (1.2) that is $x' + M(t)x = h(t,x)$ which is $h(t,x) = A^{+}(t)f(t,x)[I(t) - A(t)A^{+}(t)]z, z \in \mathbb{R}^n$ (1.4)
with boundary conditions $B_1x(t_0) = a$ dan $B_2x(T) = b$ (1.5)
Observe the homogeneous differential equation related to equation (1.4) that is:

$x' + M(t)x = 0$ . [10] (1.6)

**Lemma 1.10:**$(a)$ Let $X(t)$ is a fundamental matrix for equation (1.6). If $x_p(t)$ is the particular solution of (1.4) then $x(t) = x_p(t) + X(t)c$ (1.7)
is the solution of equation (1.4) for every $c \in V_n[t_0, T]$. where $c \in \mathbb{R}^n$ arbitrary. Any solution of (1.4) has form(1.7). Moreover, the particular solution of (1.4) is given by

$x_p(t) = X(t)\int_{t_0}^{t} X^{-1}(s)h(t,x)ds$ (1.8)

$(b)$ Let $D$ be a characteristic matrix for the homogeneous equation problem

$x' + M(t)x = 0,$ (1.9)

and

$B_1x(t_0) + B_2x(T) = 0$ (1.10)

where is $D = B_1X(t_0) + B_2X(T)$

If $rank\ D = r$. Then its index of compatibility (1.9), (1.10) is $n - r$. .[10]

We give a *Green's* matrix and the assosiated generalized Green's matrix for the boundary value problem (1.4), (1.5) as the following Lemma 1.11.

**Lemma 1.11:** Define a Green's matrix $G(t,s)$ and an associated Generalized Green's matrix $H(t,s)$ as follow:

$$G(t,s) = \begin{cases} X(t)D^{-1}B_1X(t_0)X^{-1}(s), s < t \\ -X(t)D^{-1}B_2X(T)X^{-1}(s), s > t \end{cases} \quad (1.11)$$

and

$$H(t,s) = \begin{cases} X(t)D^{-1}B_1X(t_0)X^{-1}(s)A^{+}(s), s < t \\ -X(t)D^{-1}B_2X(T)X^{-1}(s)A^{+}(s), s > t \end{cases} \quad [9] \quad (1.12)$$

**Lemma 1.12:** Let $X(t)$ be a solution of fundamental matrix for (1.9). If it satisfy:

$(i)$ The homogenous equation (1.9) is incompatible

$(ii)$ Characteristic matrix $D$ for (1.10)
is nonsingular

$(iii)$ rank $\mathrm{B}_1 = \mathrm{n} = $ rank $\mathrm{B}_2$.

Then the boundary value problem (1.4), (1.5) has solution. The solution can be expressed in terms of *Green's* matrix and the assosiated generalized *Green's* matrix

$$x(t) = \int_{t_0}^{T} H(t,s)f\big(s,x(s)\big)\,ds + \int_{t_0}^{T} G(t,s)[I - A^{+}(s)A(s)]zds + X(t)D^{-1}(a+b) \qquad (1.13)$$

where $G(t,s)$ and $H(t,s)$ are as given in (1.11) and (1.12). [10]

Next we denote $C[t_0,T] = \{x | x: [t_0,T] \to \mathbb{R}^n$ continue $\}$.

**Definition 1.13:** The norm of a $n \times n$ matrix $E$ whose components are bounded functions on an interval $J = [t_0,T]$ is $\|E\|_J = sup\{\ \|E(t)\| | t \in J\}$. [6]

A relationship between the norm of an integral of a matrix and the norm of the matrix, as well as the bound of norm of the integral associated Generalized Green's matrix are stated in the Lemma 1.14 and Lemma 1.15 respectifely as follow:

**Lemma 1.14:** Let $E$ is a $n \times n$ matrix. If the components of $E$ are bounded and integrable functions on an interval $J = [t_0,T]$ then $\left\| \int_{t_0}^{T} E(t)dt \right\| \le \|E\| |T - t_0|$ [6]

**Lemma 1.15:** There exists some Constant $P > 0$ such that $\left\| \int_{t_0}^{T} H(t,s) \right\| \le P$

**Lemma 1.16:** $C[t_0,T] = \{x | x: [t_0,T] \to \mathbb{R}^n$ continue $\}$ is a complete metric space.

It is seen that the integral equation (1.13) still contains dependent variable.
Further in the main result, it will be proved that under some conditions, the boundary value problem (1.1), (1.2) has a unique solution

**2. Main Result**

In this part, it will be proved the uniqueness of the solution of the boundary value problem (1.1), (1.2).
It is revealed in the following Theorem 2.1.
**Theorem 2.1:** Let $f(t,x)$ in (1.1) satisfies Lipschitz condition to variable $x$ in the region

$F = \{(t,x) | (t,x) \in [t_0,T] \times C[t_0,T]\}$ with Lipschitz constant $L$ that is:

$\|f(t,x) - f(t,y)\| \le L\|\boldsymbol{x} - \boldsymbol{y}\|$, for every $(t,x),(t,y) \in F$ **.**

If the constant $P$ in Lemma 1.15 satisfies $PL < 1$ then the integral equation (1.13) has a unique solution in $[t_0,T]$.

**Proof:**.
For every $x \in C[t_0,T]$, define $\|x\|_{J=[t_0,T]} = sup\{|\varphi(t)| | t \in J = [t_0,T]\}$

and

$$(Tx)(t) = \int_{t_0}^{T} H(t,s)f\big(s,x(s)\big)\,ds + \int_{t_0}^{T} G(t,s)[I - A^{+}(s)A(s)]z(s)ds + X(t)D^{-1}(a+b)$$

(2.1)

Since the term $X(t)D^{-1}(a+b)$ is continuous function of $t$ and the integrands of each integral also are continuous functions of $t$ over its interval of integration then (2.1) defining an operator $T: C[t_0,T] \to C[t_0,T]$.
Further shown that $T$ is a contraction mapping.
Let $x,y \in C[t_0,T]$ then for all $t \in [t_0,T]$ we have

$$\|(Tx) - (Ty)\| = sup\left\{ \left| \int_{t_0}^{T} H(t,s)\big[f\big(s,x(s)\big) - f\big(s,y(s)\big)\big]ds \right| \middle| t \in [t_0,T] \right\}$$

$$\leq sup\left\{\int_{t_0}^{T}\left|H(t,s)\left[f\big(s,x(s)\big)-f\big(s,y(s)\big)\right]\right|ds\,\middle|\,t\in[t_0,T]\right\}$$
$$\leq sup\left\{\int_{t_0}^{T}|H(t,s)|\left|f\big(s,x(s)\big)-f\big(s,y(s)\big)\right|ds\,\middle|\,t\in[t_0,T]\right\}$$
$$\leq sup\left\{\int_{t_0}^{T}|H(t,s)|L|x(s)-y(s)|ds\,\middle|\,t\in[t_0,T]\right\}$$
$$\leq PL\,sup\{|x(s)-y(s)||s\in[t_0,T]\}$$
$$\leq PL\|x-y\|.$$

Consequently,we get $\|Tx-Ty\| = sup\{|(Tx)(t)-(Ty)(t)||t\in[t_0,T]\}$
$\leq PL\|x-y\|.$

Because of $PL<1$, then based on Definition 1.5 we infer that $T$ is a contraction mapping.

From Lemma 1.15 that $C[t_0,T]$ is a complete space then from Lemma 1.6 $T$ has a unique fixed point in $C[t_0,T]$ that is, there exists a uniquely $x\in C[t_0,T]$ such that $Tx=x$.

So $(Tx)(t)=\int_{t_0}^{T}H(t,s)f\big(s,x(s)\big)\,ds+\int_{t_0}^{T}G(t,s)[I-A^{+}(s)A(s)]z(s)ds+X(t)D^{-1}(a+b)$, for every $t$ in $[t_0,T]$.

Hence $x$ is a unique solution of the boundary value problem (1.4), (1.5) in $[t_0,T]$.

In view of Lemma 1.9, then $x$ is a unique solution of the boundary value problem (1.1), (1.2) in $[t_0,T]$.

## 4. Conclusion

In this paper we have the conclusion that nonlinear boundary value problem of system with rectangular coefficients has a unique solution provide the multiplication of the norm of integral associated Generalized Green's matrix and constant Lipschitz of related function is in open unit interval.

## References


[1] A. L. Brown and A. Page.1970. *Element of Functional analysis*. Van Nostrand Reinhold Company. London.

[2] A. S. Delf. 1982. Advanced Matrix Theory for Scientists and Engineers. Abacus Press. London.

[3] Badrulfalah. 1992. Masalah Syarat Batas Dua Titik. Thesis Program S2 Matematika ITB. Tidak Dipublikasikan. Bandung. ITB.

[4] Boyce, E., William and Diprima, C., Richard.1992. Elementary Differential Equations and Boundary Value Problem (5nd Edition). Canada: John Willey & Sons. Inc.

[5] Braun, Martin. 1982. Differential Equqtions and their Applications (3nd Edition). New York. Heidelberg Berlin: Springer-Verlag

[6] Cole, R. H. 1968. Theory of Ordinary Differential Equations. Appleton-Century-Croft, New York.

[7] Deo, S. G. , V. Raghavendra. 1987. Ordinary Differential Equations And Stability Theory. Tata McGraw-Hill Publ. Co. Ltd., New Delhi.

[8] Murti, K. N., P. V. S. Lakshmi. 1990. On Two- Point Boundary Value Problems. Journal of Mathematical analysis And Applications 153, 217-225.

[9] P. Lancaster and M. Tismenetsky. 1985. The Theory of Matrices. Academic Press. New York

[10] Uvah J. A. & Vatsala A. S. 1990. Monotone Iterative Technique for Nonlinear Boundary-Value Problem of First Order Differential System with Rectangulatr Coeffisients. Journal of Mathematical analysis and Applications, 482-49.

[11] Vasile I. Istratescu. 1979. Fixed Point Theory. An Introduction. D. Reidel Publishing Company.Dordrecht: Holland/Boston: U.S.A./London: England.