Article

Convergence Analysis from the Solution of Riccati's Fractional Differential Equation by Using Polynomial Least Squares Method

Dian Nuryani [1*], Endang Rusyaman [2] and Betty Subartini [3]

Department of Mathematics, Faculty of Mathematics and Natural Science (FMIPA), Universitas Padjadjaran, Bandung, Indonesia

Article Info

Abstract. Riccati's Fractional Differential Equation (RFDE) has become a topic of study for researchers because RFDE can model variety of phenomenon in science such as random processes, optimal control and diffusion problems. Phenomena that can be modeled in a mathematical form can make it easier for humans to analyze several things from that phenomenon. RFDE generally does not have an exact solution, therefore a numerical approach solution is needed, one of the methods that gives good accuracy to the actual or exact solution is Polynomial Least Squares, where the errors calculated based on mean absolute percentage error (MAPE) produce a percentage below 1%. In addition, the convergence of a sequence from approximate solutions indicates that the sequence will converge to a solution
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Corresponding Author :
Dian Nuryani
Department of Mathematics, Faculty of Mathematics and Natural Science (FMIPA),
Universitas Padjadjaran, Bandung, Indonesia
Email : dian16006@mail.unpad.ac.id

1. Introduction

Fractional Differential Equations (FDE) have become the main topic for researchers, because FDE can model a phenomena, in various fields such as field of engineering, physics, and economics [9]. In more detail, phenomena such as viscoelasticity, fluid mechanics, dynamics problems, economic problems, population development and so on can be modeled with a mathematical equation that makes it easy for humans to analyze such a phenomenon. Riccati's Fractional Differential Equation (RFDE) was first introduced by Count Jacopo Francesco Riccati in 1724. RFDE is a nonlinear form of a fractional differential equation. Reid [10] explains that RFDE can be applied to model phenomenon such as random processes, optimal control and diffusion problems.

In general, Fractional Differential Equations do not have exact solutions. Therefore several methods both analytically and numerically are needed. Several studies have been carried out beforehand to find out the approximate solution from RFDE including, El-Tawil used the Adomian Decomposition

Method [4], Tan and Abbasbandy used the Homoopy Analysis Method [12], F. Geng used Variational Iteration Method [6], Biazar and Eslami used the Differential Transformation Method [2], Jafari used B-spline Operational Matrix Method [7]. In this paper, a numerical approach will be used to find solutions of RFDE using the Polynomial Least Square Method.

2. Research Method

In this research, there are steps to follow. Firstly, author reviewing various materials from textbook, scientific articles, and internet that support the research. Then research start with finding solution to the general form of Riccati's Fractional Differential Equation using Polynomial Least Squares Method, afterwards constructing sequence of approximate solutions from RFDE order (α_k) that converge to α using polynomial least squares. Then analyze the convergence from that sequences of functions based on graphs and numerical result. Last, analyze the errors based on mean absolute percentage error (MAPE) between approximate solution order-α and exact solution that already known. Click here to enter text.

3. Results and Discussion
3.1 Preliminaries

We consider the following riccati's fractional differential equation order-α and $t \in [0,1]$,

$$D^\alpha y(t) = Ay^2(t) + By(t) + C, \tag{1}$$

with initial condition $y(0) = c_j$. Where D^α denote fractional derivative Caputo orde-α and A, B, C are real numbers.

We will give the definitions and theorem that are used to solve (1), as follows

Definition 1 (Kisela,[8])
Fractional derivative Caputo is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) \, dt, \tag{2}$$

where $\alpha > 0, x > 0$ and $n \in \mathbb{N}$.

Theorem 2 (Kimeu, [8])
Let $f(x) = x^m$ for $m \geq 0$,

$$D^\alpha f(x) = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \tag{3}$$

where $0 < \alpha < 1$.

Definition 3 (Bartle et.al, [1])
Let $A \subseteq \mathbb{R}$, and for all $n \in \mathbb{N}$ there is function $f_n: A \to \mathbb{R}$. (f_n) is sequence of function on A to $\mathbb{R}$, or it can be written $(f_n(x))$ as a sequence of function for all $x \in A$.

Definition 4 (Bartle et.al, [1])
Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f: A_0 \to \mathbb{R}$. Sequence (f_n) converges on A_0 to f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case f the limit on A_0 of the sequence (f_n). When such a function f exists, we say that the sequence (f_n) is convergent on A_0, or that (f_n) converges pointwise on A_0.

Definition 5 (Sungil et.al, [11])
MAPE is the average of absolute percentage errors (APE), Let A_t the actual data and F_t forecast values at data point t. MAPE is defined by

$$MAPE = \frac{100\%}{N} \sum_{t=1}^{N} \left| \frac{A_t - F_t}{A_t} \right|. \tag{4}$$

3.2 Solution of general form RFDE

Based on the Polynomial Least Squares Method, the approximate solution of RFDE on (1) will be in the form of a polynomial degree n,

$$\tilde{y}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n = \sum_{i=0}^{n} a_i t^i. \tag{5}$$

Select the initial condition $\tilde{y}(t) = 0$ then substitute to (5) so the approximate solution becomes,

$$\tilde{y}(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n = \sum_{i=1}^{n} a_i t^i. \tag{6}$$

Afterwards, we will defined an operator denotes O with $y(t)$ as the solution of (1),

$$O\big(y(t)\big) = D^{\alpha} y(t) - A y^2(t) - B y(t) - C \tag{7}$$

the value of operator O in (7) is zero because it satisfies (4). Then we defined an operator T with $\tilde{y}(t)$ as the approximate solution of (4),

$$T\big(\tilde{y}(t)\big) = D^{\alpha} \tilde{y}(t) - A \tilde{y}^2(t) - B \tilde{y}(t) - C \tag{8}$$

the value of the operator T in (8) is close to zero but not exactly zero, because $\tilde{y}(t)$ is the approximate solution. Furthermore, this T operator is called a residual or remainder operator because the result is an error. Then, substitute (6) to (8) we obtained,

$$T\left(\sum_{i=1}^{n} a_i t^i\right) = \sum_{i=1}^{n} a_i f_i(\alpha) t^{i-\alpha} - A\left(\sum_{i=1}^{n} a_i^2 t^{2i} + 2\sum_{i<j} a_i a_j t^{i+j}\right) - B\left(\sum_{i=1}^{n} a_i t^i\right) - C \tag{9}$$

where $f_i = \dfrac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}$.

After that, based on the polynomial least squares, the remainder operator T will be squared,

$$\left[T\left(\sum_{i=1}^{n} a_i t^i\right)\right]^2 = \sum_{i=1}^{n} a_i^2 f_i^2(\alpha) t^{2(i-\alpha)} + 2\sum_{i<j} a_i a_j f_i(\alpha) f_j(\alpha) t^{i+j-2\alpha}$$

$$+ A^2 \left[\sum_{i=1}^{n} a_i^2 t^{2i} + 2\sum_{i<j} a_i a_j t^{i+j}\right]^2$$

$$+ B^2 \sum_{i=1}^{n} a_i^2 t^{2i} + 2B^2 \sum_{i<j} a_i a_j t^{i+j} + C^2 - 2A\left(\sum_{i=1}^{n} a_i f_i(\alpha) t^{i-\alpha}\right)\left(\sum_{i=1}^{n} a_i^2 t^{2i}\right)$$

$$-4A\left(\sum_{i=1}^{n} a_i f_i(\alpha) t^{i-\alpha}\right)\left(\sum_{i<j} a_i a_j t^{i+j}\right) - 2B\left(\sum_{i=1}^{n} a_i f_i(\alpha) t^{i-\alpha}\right)\left(\sum_{i=1}^{n} a_i t^i\right) \tag{10}$$

$$- 2C\left(\sum_{i=1}^{n} a_i f_i(\alpha) t^{i-\alpha}\right)$$

$$+2AB\left(\sum_{i=1}^{n} a_i^2 t^{2i}\right)\left(\sum_{i=1}^{n} a_i t^i\right) + 4AB\left(\sum_{i<j} a_i a_j t^{i+j}\right)\left(\sum_{i=1}^{n} a_i t^i\right) + 2AC\left(\sum_{i=1}^{n} a_i^2 t^{2i}\right)$$

$$+4AC\left(\sum_{i<j} a_i a_j t^{i+j}\right) + 2BC\left(\sum_{i=1}^{n} a_i t^i\right).$$

After the T operator is squared, the T operator is integrated to variable t at interval $[0,1]$. Named the result of the integral is the function J in which it has n variables $a_1, a_2, \ldots, a_n$ so that it can be written as a

function $J(a_1, a_2, \ldots, a_n)$. According to Constantin and Bota [3], to find the minimum value of this function $J(a_1, a_2, \ldots, a_n)$, we compute the stationary points as the solution of the system nonlinear as follows,

$$\frac{\partial J(a_1, a_2, \ldots, a_n)}{\partial a_i} = 0, \qquad i = 1, 2, \ldots, n. \tag{11}$$

Because (11) is a system of nonlinear equations, solving it gives the parameter value a_i. In this paper, we used software maple as helper to solve the system of nonlinear equations. Last, by substituting a_i to (6), the approximate solution from RFDE using the Polynomial Least Squares Method is,

$$\tilde{y}(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n = \sum_{i=1}^{n} a_i t^i.$$

3.3 Application and Numerical Result
3.3.1 Convergence Analysis
Given the RFDE with $A = 1, B = 0$, and $C = 1$ as follows,

$$D^\alpha y(t) = y^2(t) + 1 \tag{12}$$

With initial condition $y(0) = 0$. According to Eldien et.al [5] the exact solution of (12) is $y(t) = \tan t$.

Next we will be analyzing the convergence from sequence of approximate solutions with order $(\alpha_k) = \left(\frac{k}{k+1}\right)$ converges to $\alpha = 1$ where the sequence of functions will also converge to a solution function with $\alpha = 1$.

Let $g_k(t)$ is an approximate solution calculated using polynomial least squares with order $(\alpha_k) = \left(\frac{k}{k+1}\right)$,

Table 1. Numerical Solutions of Degree 5 for order $(\alpha_k) = \left(\frac{k}{k+1}\right)$

k	α_k	$g_k(t)$
1	$\frac{1}{2}$	$-4{,}010746531\, t + 48{,}15812676\, t^2 - 140.6481519\, t^3 + 159{,}6162510\, t^4 - 61.49787541\, t^5$
2	$\frac{2}{3}$	$2{,}659998977\, t - 11{,}88715592\, t^2 + 35{,}43217019\, t^3 - 45{,}58111856\, t^4 + 22{,}00948126\, t^5$
3	$\frac{3}{4}$	$2{,}423909683\, t - 9{,}362561466\, t^2 + 28{,}09488557\, t^3 - 35{,}92478673\, t^4 + 17{,}46957088\, t^5$
4	$\frac{4}{5}$	$2{,}139890650\, t - 6{,}709464878\, t^2 + 19{,}97284516\, t^3 - 25{,}30201711\, t^4 + 12{,}47843262\, t^5$
5	$\frac{5}{6}$	$1{,}937302527\, t - 5{,}065665887\, t^2 + 14{,}85164477\, t^3 - 18{,}47843648\, t^4 + 9{,}168048335\, t^5$
$\vdots$		
100	$\frac{100}{101}$	$1{,}058475851\, t - 0{,}4015196940\, t^2 + 1{,}701387429\, t^3 - 2{,}005741602\, t^4 + 1{,}235314368\, t^5$
$\vdots$		
500	$\frac{500}{501}$	$1{,}026343966\, t - 0{,}2834312919\, t^2 + 1{,}426622115\, t^3 - 1{,}706072595\, t^4 + 1{,}099197406\, t^5$
$\vdots$		
10^6	$\frac{10^6}{10^6 + 1}$	$1{,}018361874\, t - 0{,}2544764904\, t^2 + 1{,}359786824\, t^3 - 1{,}633640228\, t^4 + 1{,}066416180\, t^5$
$\vdots$		
	$\alpha = 1$	$g(t) = 1{,}018360742\, t - 0{,}2544679292\, t^2 + 1{,}359762068\, t^3 - 1{,}633611638\, t^4 + 1{,}066404538\, t^5.$

From table 1 it can be seen that the approximate solution functions with the order $(\alpha_k) = \left(\frac{k}{k+1}\right)$ converges to a certain value in each term.

The coefficient of t converges to the value of $1{,}018360742$,

the coefficient t^2 converges to the value of $-0{,}2544679292$,

the coefficient t^3 converges to the value of $1{,}359762068$,

the coefficient t^4 converges to the value of $-1{,}633611638$, and

the coefficient t^5 converges to the value $1{,}066404538$.

In this case, the approximate solution functions are proven to be convergent to the solution function with $\alpha = 1$. So, it can be concluded that $\left(g_k(t)\right)$ converges to $g(t)$.

To make it easier to see the convergence of the approximate solution functions, the following picture is presented,
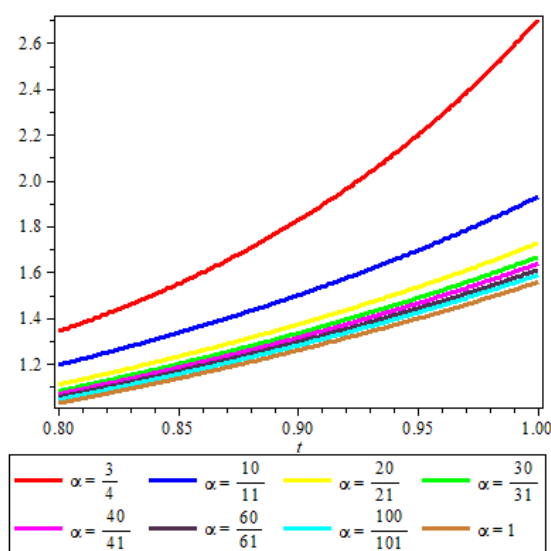

Figure 1. Graph of the approximate solutions with order $(\alpha_k) = \left(\frac{k}{k+1}\right)$ at $t \in [0,8,1]$

From Figure 1 it can be seen that the approximate solution graphs with order $\alpha_k = \frac{k}{k+1}$ approach the graph coloured-brown, where the graph is the approximate solution for $\alpha = 1$. In conclusion, the sequence of solution functions with order $(\alpha_k) = \left(\frac{k}{k+1}\right)$ that converging to $\alpha = 1$ shows that the sequence of solution functions will also converge to the solution function with $\alpha = 1$.

3.3.2 Error Analysis

In this section, the errors between approximate solution with $\alpha = 1$ and the exact solution will be calculated using the mean absolute percentage error (MAPE) at $t \in [0,1]$. The exact solution for equation (12) is $y(t) = \tan t$ (Eldien et.al, [5]), and the approximate solution for $\alpha = 1$ according to the result in previous subsection is,

$\tilde{y}(t) = 1{,}018360742\, t - 0{,}2544679292\, t^2 + 1{,}359762068\, t^3 - 1{,}633611638\, t^4 + 1{,}066404538\, t^5.$

Taking up to 10 decimal, the APE calculation is presented in the following table,

Table 2. APE for RFDE with $A = 1, B = 0$, and $C = 1$

t	$y(t)$	$\widetilde{y}(t)$	APE
0,1	0,1003346721	0,1004984599	0,163241449 %
0,2	0,2027100355	0,2020989986	0,301433967 %
0,3	0,3093362496	0,3086787936	0,212537665 %
0,4	0,4227932187	0,4227537250	0,009341144 %
0,5	0,5463024898	0,5467580616	0,083391857 %
0,6	0,6841368083	0,6843241460	0,027383065 %
0,7	0,8422883805	0,8415620798	0,086229449 %
0,8	1,0296385571	1,0283394098	0,126175076 %
0,9	1,2601582176	1,2595608127	0,047407133 %
1	1,5574077247	1,5564477808	0,061637286 %
MAPE			0,111877809 %

The resulting MAPE value is around 0,11%. Therefore, the approximate solution $\widetilde{y}(t)$ approaches the exact solution $y(t)$. Next, to see the error rate at the interval $t \in [0,1, 1]$ the APE graph is presented as follows,
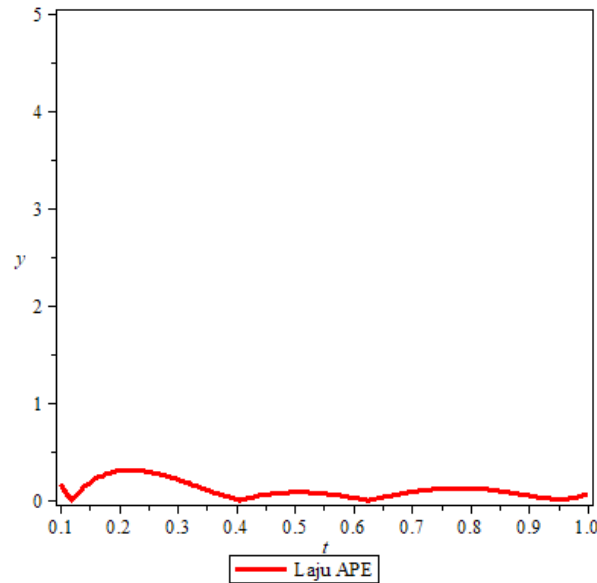

Figure 2. APE rate for RFDE with $A = 1, B = 0, C = 1$

From Figure 2 it can be seen that the rate of APE will decrease with respect of t. Then, a comparison graph between exact solution and approximate solution at $t \in [0,1]$ is also shown as follows,
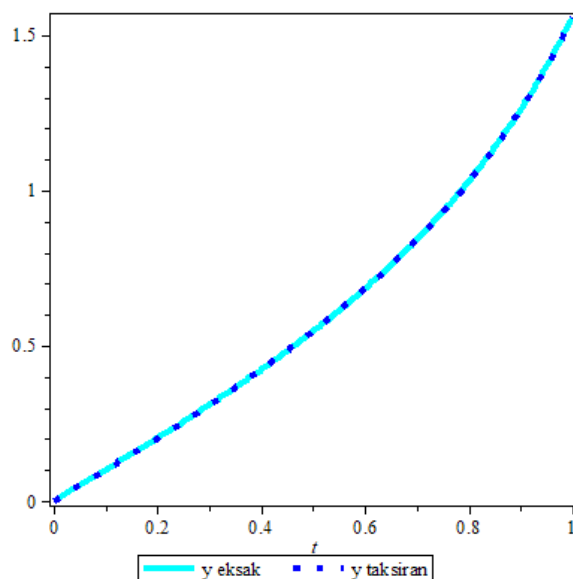
Figure 3. Comparison graph between exact solutions and approximate solution for RFDE with
$A = 1, B = 0, C = 1$

From Figure 3 it can be seen that the approximate solution $\tilde{y}(t)$ coincides with the exact solution $y(t)$ which indicates that the error is very small between the two functions.

4. Conclusion

Based on the results in the previous chapter, it can be concluded that to find solutions to the Riccati's Fractional Differential Equation (RFDE) numerically, the polynomial least squares method can be used. In addition, the convergence of the sequences of the approximate solutions order (α_k) that converging to α indicates that the sequence of approximate solutions will also converge to a solution function with $\alpha = 1$, and it is proven that the method produces good accuracy of the exact solution by producing an error based on the mean absolute percentage error (MAPE) below 1%

References

[1]   Bartle, Robert G., Sherbert, Donald R. 2011. *Introduction to Real Analysis (4th ed.).* Hoboken, NJ: John Wiley & Sons.

[2]   Biazar, Jafar., Eslami, Mahdi. 2010. Differential Transform Method for Quadratic Riccati Differential Equation. *International Journal of Nonlinear Science*, volume 9, 444-447.

[3]   Bota, Constantin., Caruntu, Bogdan. 2017. Analytical Approximate Solutions for Quadratic Riccati Differential Equation of Fractional Order using Polynomial Least Squares Method. *Chaos, Solitons and Fractals*, 1-7.

[4]   El-Tawil, M. A., Bahnasawi, A. A., Abdel-Naby, A. 2004. Solving Riccati differential equation using Adomian's decomposition method. *Applied Mathematics and Computation*, volume 157, Issue 2, 503-514.

[5]   Eldien, S. S., Machado, J. A. T., Wang, Y., Aldraiweesh, A.A. 2018. An Algorithm for the Approximate Solution of the Fractional Riccati Differential Equation. *International Journal of Nonlinear Sciences and Numerical Simulation*.

[6]   Geng, Fazhan. 2010. A Modified Variational Iteration Method for Solving Riccati Differential Equations, *Computer & Mathematics with Applications*, volume 60, 1868-1872.

[7]   Jafari, H., Tajadodi, H., Baleanu, D. 2015. A Numerical Approach for Fractional Order Riccati Differential Equation using B-spline Operational Matrix. *Fractional Calculus & Applied Analysis*, volume 18, 387-399.

[8]   Kisela, Tomas. 2008. *Fractional Differential Equations and Their Applications*. Thesis. Brno: Diploma of Mechanical Engineering Institute of Mathematics, Brno University of Technology.

[9]   Rusyaman, E., Parmikanti, K., Chaerani, D., Asefan, Irianingsih, I. 2018. The convergence of the order sequence and the solution function sequence on fractional partial differential equation. *Journal of Physics*: Conf. Series 983.

[10] Reid, William T. 1972. *Riccati Diferential Equations*. New York: Academic Press.

[11] Sungil, Kim., Heeyoung, Kim. 2016. A new metric of absolute percentage error for intermittent demand forecasts. *International Journal of Forecasting*, volume 32, 669-679.

[12] Tan, Yue., Abbasbandy, Saeid. 2008. Homotopy Analysis Method for Quadratic Riccati Differential Equation. *Communications in Nonlinear Science and Numerical Simulation*, volume 13, 539-546.