

Investment timing and capital structure under liquidity risk[☆]

Huamao Wang^a, Qing Xu^b, Jinqiang Yang^b

^a*Kent Business School, University of Kent, Canterbury, UK.*

^b*School of Finance, Shanghai University of Finance and Economics, Shanghai, China.*

Abstract

Deterioration in debt market liquidity reduces debt values and affects firms' decisions. Considering such risk, we develop an investment timing model and obtain analytic solutions. We carry out a comprehensive analysis in optimal financing, default and investment strategies, and stockholder-bondholder conflicts. Our model explains stylized facts and replicates empirical findings in credit spreads. We obtain six new insights for decision makers. We propose a “new trade-off theory” for optimal capital structure, a new tax effect, and new explanations of “debt conservatism puzzle” and “zero-leverage puzzle”. Failure in recognizing liquidity risk results in substan-

[☆]The authors are grateful to the two anonymous reviewers and the editors for their helpful comments and advice. The authors are responsible for any errors that may remain. The first author acknowledges support from Kent Faculty Research Fund. The paper is completed when the second author visited University of Kent. The second author acknowledges support from Postgraduate Students Innovation Foundation of Shanghai University of Finance and Economics (# CXJJ-2013-325). The third author acknowledges support from Natural Science Foundation of China (# 71202007), New Century Excellent Talents in University (# NCET-13-0895), Innovation Program of Shanghai Municipal Education Commission (# 13ZS050) and “Chen Guang” Project of Shanghai Municipal Education Commission and Shanghai Education Development Foundation (# 12CG44).

Email addresses: h.wang@kent.ac.uk (Huamao Wang), xuqing1228@163.com (Qing Xu), yang.jinqiang@mail.shufe.edu.cn (Jinqiang Yang)

tially over-leveraging, early bankruptcy or investment, overpriced options, and undervalued coupons and credit spreads. In addition, agency costs are surprisingly small for a high liquidity risk or a low project risk. Interestingly, the risk shifting incentive and debt overhang problem decrease with liquidity risk under moderate tax rates while they increase under high tax rates.

Keywords: Real options, Capital structure, Liquidity risk, Low-leverage puzzle, Stockholder-bondholder conflicts

JEL: G11, G13, G32

1. Introduction

Liquidity dries up in financial markets from time to time especially under severe market distress. The recent financial crisis 2007-2008 and the European debt crisis since 2009 demonstrate the importance of liquidity risk management. A lesson learned from these crises is that incorporating liquidity risk to decision making brings values and benefits to corporate managers.

Specifically, deterioration in corporate debt market liquidity triggers severe financing obstacles, which further influences firms' optimal decisions in investment and capital structure. Particularly, the effects of illiquidity on credit spreads have been documented by empirical studies. Longstaff et al. (2005) observe that the nondefault component of credit spreads is strongly related to both bond-specific illiquidity and market liquidity. Bao et al. (2011) show that bond illiquidity explains a substantial part of credit spreads.

Prior researches examine liquidity risk usually in an empirical perspective on asset pricing without a theoretical insight into its implications for a firm's optimal policy. To fill this gap, we study the optimal decision problem of

a firm whose debt incurs liquidity shocks in the secondary debt market. Liquidity risk reduces the debt value and imposes interactive effects with other factors on decision policies. Our work contributes to the literature in three interesting aspects.

First, we solve a theoretical model within a framework of optimal investment timing and structural credit risk incorporating the effects of debt market liquidity. We represent shocks in debt market liquidity as Poisson jumps following the way in He and Xiong (2012). We adopt this modeling approach since it captures the random arrivals of liquidity shocks and it provides an analytically tractable channel of quantifying the effects of liquidity risk. We derive analytic solutions to the optimal coupon and investment threshold determined endogenously, which is more complicated than a liquid market. Second, we comprehensively illustrate the optimal policies in financing, default, investment and their implications to stockholder-bondholder conflicts. Third, we achieve a number of novel insights to the firm’s policies for its decision makers. Our theoretical results not only provide new predictions for empirical studies but also are consistent with existing empirical results.

We highlight six insights related to practice. First, a “new trade-off theory” balances tax benefits, bankruptcy costs, and the new liquidity-risk cost preventing debt issuance. Second, together with liquidity risk, the tax rate has a new effect causing particularly low retained earnings. Third, we provide new alternative explanations for the “debt conservatism puzzle”¹ and

¹The “debt conservatism puzzle” describes that although a firm could take larger tax benefits by raising debt issuance until the marginal tax benefit begins to decline, many firms with light expected financial distress use debt conservatively (Bolton et al., 2014a).

the “zero-leverage puzzle”². The liquidity-risk cost forces the firm to issue low or even zero debt and leverage for reducing default risk and postponing bankruptcy. Meanwhile, it results in low option value and late investment.

Fourth, high coupons are unavoidable as debtholders demand liquidity risk premium, which raises credit spreads and may double credit spreads. These situations are aggravated by the new effect of tax mentioned before. Fifth, exercising the investment option for maximizing equity value rather than firm value leads to agency costs. Interestingly, they are small for a high liquidity risk or a low project risk. The effects of project risk on agency costs are contrary to the literature (e.g. Mauer and Sarkar (2005)) due to liquidity risk. Finally, under moderate tax rates, a rise in tax rate or a fall in liquidity risk increases equityholders’ disincentives to replenish equity as well as their moral hazard incentives to increase equity value at the expense of debtholders. For high tax rates, they are strong under high liquidity risk due to large coupons.

We incorporate liquidity risk into a real options framework to study its impact on investment decisions. A representative study on real options is Miao and Wang (2007) who examine four models with a lump-sum/cash-flow payoff where risk can/cannot be hedged against by the market portfolio. Hackbarth and Johnson (2015) enrich the investment-based asset pricing implications by setting up a neoclassical model with repeated investment and disinvestment. Song et al. (2014) find both idiosyncratic- and estimation-

²The “zero-leverage puzzle” refers to the highly persistent phenomenon found by Strebulaev and Yang (2013) that an average 10.2% of large public nonfinancial US firms choose zero leverage and nearly 22% take merely 5% or less book leverage ratio from 1962 to 2009.

risk-induced saving demands lead to late investment and losses.

We assume that the firm can raise enough equity and the effect of internal cash liquidity is negligible. Therefore, this article addresses the importance of external market liquidity for a firm's decisions without interference from internal liquidity. As justified by Almeida et al. (2009) and He and Xiong (2012), though retaining more internal liquidity holdings can alleviate the exposure to market liquidity risk, firms cannot eliminate the impact from market liquidity shocks since internal liquidity is almost limited in reality. The representative studies on the role of internal liquidity in firm strategies are Boyle and Guthrie (2003), Gryglewicz (2011), Bolton et al. (2014a), and Bolton et al. (2014b). We provide an ex post heuristic discussion of internal liquidity in Section 3.5.2 and leave the rigorous treatment of models with internal liquidity to our next research. Instead, we present the robustness of our findings to cyclical market liquidity by extending our model to this case in Section 3.8.³

Section 2 presents the model and optimal solution. Section 3 explores implications of debt liquidity risk for the firm's various decisions and agency conflicts. Section 4 concludes. The Appendix gathers technical derivations.

2. Model setup and solution

We first describe the investment optimization problem of a firm whose debt is influenced by market liquidity shocks. Then, we provide the solution.

³We are grateful to an anonymous reviewer's comments on internal liquidity and cyclical market liquidity.

2.1. *Financing and default strategies*

Fundamental. A firm has a perpetual option to invest in a project by paying a fixed investment cost I and receiving cash flow X_t following the dynamics:

$$dX_t = \mu X_t dt + \sigma X_t dZ_t, \quad X_0 = x > 0, \quad (1)$$

where μ is the drift rate⁴, σ is the volatility (*project risk*), and $(Z_t)_{t \geq 0}$ is a standard Brownian motion under the risk-neutral probability measure Q .

Without loss of generality, the current time is zero instead of t and the current cash flow level is $X_0 = x$. The unlevered firm value $V(x)$ depends on the cash flow after tax:

$$V(x) = \mathbb{E} \left[\int_0^\infty e^{-rs} (1 - \pi) X_s ds \right] = (1 - \pi) \frac{x}{r - \mu}, \quad (2)$$

where \mathbb{E} is the expectation operator under risk-neutral measure Q upon time $t = 0$ and π is a corporate tax rate.

Security valuation. The firm finances its investment and operation by issuing equity and debt. We assume it pays a perpetual non-negative coupon $C > 0$ per unit time to debtholders.⁵

After investment, equityholders can choose an endogenous default time to maximize their equity values. Upon bankruptcy, bondholders recover a proportion of unlevered firm value at $(1 - \alpha)V(x_b)$ after deducting costs $\alpha V(x_b)$, where α is the bankruptcy cost rate.

⁴We assume $r > \mu > 0$ to keep the firm value convergent and finite, where r is the risk-free rate.

⁵For the zero coupon case, we briefly give out the threshold and security valuation at the appendix B.

Following risk-neutral pricing theory (e.g., Dixit and Pindyck (1994)), we denote by $w_1 < 0$ and $w_2 > 1$ the two roots of the quadratic equation $\frac{1}{2}\sigma^2w(w-1) + \mu w - r = 0$, then

$$w_{1,2} = \frac{1}{2} - \frac{\mu}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \quad (3)$$

Define $\tau_d = \inf\{t \geq 0 | x \leq x_b\}$ and $\tau_u = \inf\{t \geq 0 | x \geq x^*\}$ by stopping times for optimal default and investment, where x_b and x^* are the default boundary and investment threshold. Then the present values of a security that claims one unit of account at τ_d and τ_u are

$$\mathbb{E}[e^{-r\tau_d}] = \left(\frac{x}{x_b}\right)^{w_1} \quad \text{and} \quad \mathbb{E}[e^{-r\tau_u}] = \left(\frac{x}{x^*}\right)^{w_2}. \quad (4)$$

Therefore, using the standard arguments we have the equity value $E(x)$:

$$\begin{aligned} E(x) &= \sup_{\tau_d \geq 0} \mathbb{E} \left[\int_0^{\tau_d} e^{-rs} (X_s - C)(1 - \pi) ds \right] \\ &= (1 - \pi) \left(\frac{x}{r - \mu} - \frac{C}{r} \right) + (1 - \pi) \left(\frac{C}{r} - \frac{x_b}{r - \mu} \right) \left(\frac{x}{x_b} \right)^{w_1}, \end{aligned} \quad (5)$$

where the endogenous default boundary x_b optimizing the equity value is

$$x_b = \frac{w_1 (r - \mu) C}{r (w_1 - 1)}. \quad (6)$$

In Equation (5), the first term reflects the value of a forever firm without bankruptcy and the second one is the value loss once the firm is bankrupt.

What's more, we formulate debt market illiquidity as He and Xiong (2012). Liquidity deterioration shocks are modeled as exogenous Poisson jumps reducing the debt value by a transaction cost rate k . Denoting by ξ

the jump intensity indicating *liquidity risk*, we obtain the debt value $D(x)$:⁶

$$\begin{aligned} D(x) &= \mathbb{E} \left\{ \int_0^{\tau_d} [e^{-rs} C ds - kD(X_{s-}) dN_{s-}] + e^{-r\tau_d} (1 - \alpha) V(x_b) \right\} \\ &= \frac{C}{r + \xi k} + \left((1 - \alpha) V(x_b) - \frac{C}{r + \xi k} \right) \left(\frac{x}{x_b} \right)^\gamma. \end{aligned} \quad (7)$$

where dN_s denotes a Poisson process and $\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2(r + \xi k)}{\sigma^2}}$. The first term of Equation (7) is the perpetual debt value without default and the second term represents the debt value loss when the firm defaults.

2.1.1. Investment strategy

Denoting F the firm's option value, we formulate its problem of interacting investment timing and optimal capital structure as the *objective function*:

$$J(x, C) = \sup_{C \geq 0} F(x; C) = \sup_{C \geq 0, \tau_u \geq 0} \mathbb{E}\{e^{-r\tau_u} [E(X_{\tau_u}; C) + D(X_{\tau_u}; C) - I]\}.$$

The investment threshold x^* is determined by the smooth pasting condition:

$$\left. \frac{\partial F}{\partial x} \right|_{x=x^*} = \left. \frac{\partial E}{\partial x} \right|_{x=x^*} + \left. \frac{\partial D}{\partial x} \right|_{x=x^*}. \quad (8)$$

We summarize our analytic results on the firm's optimal decisions in investment, default, and financing under debt liquidity risk in a proposition.

Proposition 2.1. *Under debt liquidity risk, the firm's real option value is*

$$F(x; C) = [E(x^*; C) + D(x^*; C) - I] \left(\frac{x}{x^*} \right)^{w_2}, \text{ for a given } x < x^*. \quad (9)$$

⁶He and Xiong (2012) focus on the valuation of the maturity debt with a rollover debt structure and their solution is the same as ours.

The investment threshold indicated by x^* is a solution of the equation:

$$0 = (w_2 - 1) \left(\frac{x^*}{r - \mu} \right) + (w_2 - w_1) \left(\frac{C}{r(1 - w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} + \frac{w_2 - \gamma}{1 - \pi} H \left(\frac{x^*}{x_b} \right)^\gamma + \frac{w_2}{1 - \pi} \left[\frac{C}{r + \xi k} - \frac{(1 - \pi)C}{r} - I \right], \quad (10)$$

where $H := (1 - \alpha)V(x_b) - C/(r + \xi k)$. The default strategy is x_b in (6). The financing strategy comprises the optimal capital structure of equity $E(x^*; C^*)$ and debt $D(x^*; C^*)$ given by (5) and (7) with the optimal coupon C^* solving

$$\frac{\partial F}{\partial C} = \left[\frac{\partial E(x^*)}{\partial C} + \frac{\partial D(x^*)}{\partial C} \right] \left(\frac{x}{x^*} \right)^{w_2} - \frac{w_2 F(x^*)}{x^*} \left(\frac{x}{x^*} \right)^{w_2} \frac{dx^*}{dC} = 0, \quad (11)$$

$$\frac{dx^*}{dC} = \frac{\frac{w_1 - w_2}{r} \left(\frac{x^*}{x_b} \right)^{w_1} + \frac{(\gamma - w_2)(1 - \gamma)}{(1 - \pi)C} H \left(\frac{x^*}{x_b} \right)^\gamma + \frac{w_2}{1 - \pi} \left(\frac{1 - \pi}{r} - \frac{1}{r + \xi k} \right)}{\frac{w_2 - 1}{r - \mu} + \frac{(w_2 - \gamma)\gamma}{(1 - \pi)x^*} H \left(\frac{x^*}{x_b} \right)^\gamma + \frac{w_1(w_2 - w_1)C}{r(1 - w_1)x^*} \left(\frac{x^*}{x_b} \right)^{w_1}}. \quad (12)$$

See Appendix A for derivation. We present three remarks to liquidity risk.

Remark 2.2. *The above proposition shows that liquidity risk impacts the firm's various decisions through the optimal coupon C^* . The expected liquidity risk modifies the optimal coupon, which changes the retained cash flow of the firm. Then these changes affect the optimal capital structure and further disturb the investment decision. The underlying cash flow $x_{\tau_u} = x^*$ at the exercising time τ_u in turn determines the values of equity and debt. Besides, the optimal coupon C^* can be zero as a zero-leverage firm, which reflects the effect of liquidity risk shock. $C^* \geq 0$ assures the existence of optimal coupon.*

Remark 2.3. *He and Xiong (2012) show that a liquidity shock makes firms incur rollover losses when replacing maturing bonds. We consider the effects of liquidity shocks on firms' decisions under a long term (perpetual) debt structure. Studying long-term debt under liquidity risk in our framework is*

interesting in its own ways. First, long term corporate bonds are also traded on the secondary debt market and their values are affected by liquidity shocks as well. Hence, they also put bond investors at risk in an illiquid bond market. Second, we consider the firm's optimal investment that is financing by issuing equity and perpetual debt. The firm anticipates the effects of liquidity risk on perpetual debt and makes the optimal decision. Therefore, liquidity shocks influence the firm's decisions through the optimal coupon as discussed in Remark 2.2. Third, He and Xiong (2012) focus on liquidity premium and default premium but do not consider investment timing and optimal coupon. We build a model of investment timing and optimal capital structure with perpetual debt. This practical debt structure is widely used in the literature of dynamic corporate finance for analytic tractability (e.g. Leland, 1994; Mauer and Sarkar, 2005). The setting allows us to complement the literature by providing a comprehensive investigation into liquidity risk, project risk, tax, default, investment, and agency conflicts. Forth, previous studies on market micro-structure document that market liquidity degenerates due to a number of reasons such as changes in insider trading and the bid-ask spreads, see the following Remark 2.4. In addition, we extend our model to the short-term debt structure in Section 3.7 to explore the effects of debt maturity. We find that the results with perpetual debt are robust under short-term debt structure with different maturities.

Remark 2.4. *Market liquidity fluctuates with market dynamics, which affects the valuations of both short-term and long-term debts.⁷ Kyle (1985) s-*

⁷We appreciate an anonymous reviewer's comments on relating liquidity shocks to

tudies sequential / continuous auction equilibrium and finds that the adverse selection of insiders makes the market less liquid. Market makers also reduce market liquidity when responding to insider trading. Thus, the measure of market liquidity is positively related to the amount of noise trading while negatively related to the amount of private information. Specifically, Glosten and Milgrom (1985) show that a positive bid-ask spread characterizing market liquidity exists due to the presence of insiders. The spread increases with the insiders' information and it is possible that markets close down entirely because of large bid-ask spread. In short, perpetual debt also experiences liquidity shocks when market circumstances change, e.g. more inside trades and large bid-ask spreads.

3. Model implications: financing, default, investment, and conflicts

3.1. Baseline parameters

We take the baseline parameter values equivalent to those in Mauer and Sarkar (2005): risk-free rate $r = 5\%$, growth rate $\mu = 3\%$, volatility $\sigma = 25\%$, initial cash flow $x_0 = 1$, investment cost $I = 20$, bankruptcy cost rate $\alpha = 35\%$, and corporate tax rate $\pi = 30\%$. For the debt liquidity risk, we set the transaction cost rate $k = 1.0\%$ (for BB-rated bonds) and liquidity shock intensity $\xi = 1$ as He and Xiong (2012). These values exclude the uninteresting cases of an immediate default or option exercise. In addition, all of our comparative statics demonstrate a comprehensive robustness and sensitivity analysis of interesting quantities with respect to key parameters.

trades, bid-ask spreads. A more comprehensive model of insider trading, bid-ask spreads, investment, and optimal financing is beyond the scope of this paper.

3.2. Optimal financing decisions

When a liquidity shock arrives at the debt market, it reduces the firm's bond value due to the trading cost caused by the shock. The firm realizes such possible liquidity deterioration and takes the optimal financing strategy different from that without liquidity risk, in order to maximize the firm value by finding a trade-off of benefits and costs. Figs. 1 to 3 reveal our findings on the joint effects of debt liquidity risk (captured by ξ), project risk (σ), tax, and bankruptcy costs on the optimal financing policies in terms of debt value, coupon payment, and leverage. These new insights are not discussed in the previous studies (e.g. Mauer and Sarkar, 2005; He and Xiong, 2012).

3.2.1. Optimal debt and coupon under debt liquidity risk

Fig. 1 shows that there are three interactive forces affecting the firm's optimal financing decision. First, the values of debt and coupon generally rise with the project risk σ due to the standard *option effect*.⁸ Similarly, Bolton et al. (2014a) find that a financially constrained firm should increase debt with volatility as the firm adds cash holdings in responses to an increase in volatility, which reduces debt servicing costs.

Second, the liquidity shock intensity ξ generally decreases the optimal debt and coupon because of the *liquidity-risk effect*: a higher liquidity risk makes the firm more vulnerable to liquidation and hence it issues less debt and coupon to reduce the chance of financial distress. Particularly, for the project with a volatility lower than 0.18, the debt value with the high liq-

⁸The standard option effect means that given a certain lump-sum option payoff, a rise in volatility increases the option value and delays investment (e.g. Miao and Wang, 2007).

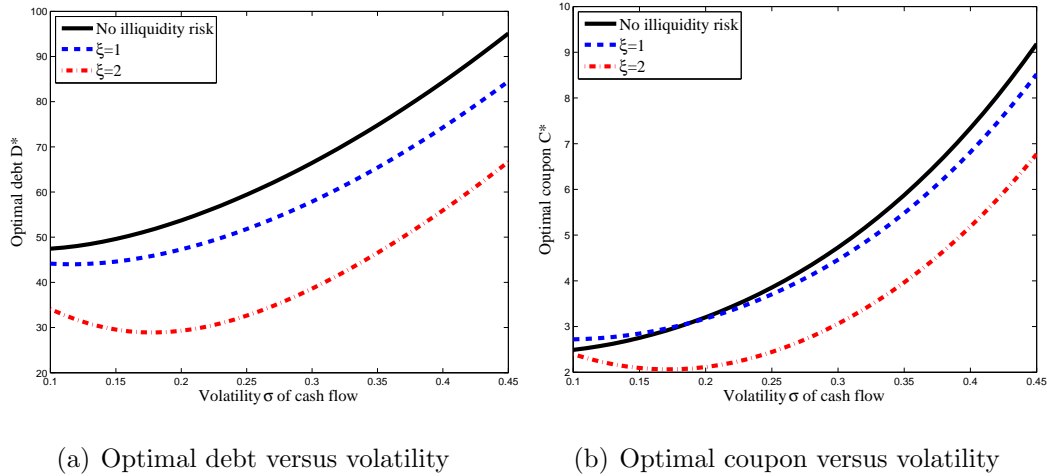


Fig. 1. The figure depicts (a) optimal debt D^* ; and (b) optimal coupon C^* against the volatility σ for three levels of liquidity shock intensity ξ .

liquidity risk at $\xi = 2$ declines with the volatility since the liquidity-risk effect dominates. Similarly, a precautionary demand for cash liquidity in Bolton et al. (2014a) forces financially constrained firms to limit their debt.

Third, debtholders demand a higher coupon in anticipation of debt liquidity deterioration, which is referred to as the *liquidity-risk-premium effect*. When the project risk $\sigma < 0.18$, the optimal coupon with the low liquidity risk $\xi = 1$ is slightly higher than that without liquidity risk as shown in Fig. 1(b). Although a lower debt value with $\xi = 1$ in Fig. 1(a) seems to indicate a lower coupon, the liquidity-risk-premium effect dominates in the case with low project risk and it instead leads to a higher coupon.⁹

⁹The U-shape of optimal coupon against volatility in Leland (1994) is based on a static level of cash flow. Nevertheless, the financing decision here depends on different investment levels of cash flows. The curve without liquidity risk ($\xi = 0$) does not display a U-shape.

To sum up, our first novel insight is that a firm should choose low value of debt considering the debt liquidity risk, while it should note that coupon payments might have to be high due to the liquidity risk premium, especially when the project risk is low. Otherwise, the firm might be over optimistic to make wrong decisions in issuing large debt, or the debtholders might not receive sufficient liquidity risk premium as compensation.

3.2.2. *New trade-off among illiquidity, tax, and bankruptcy*

The classic trade-off theory of capital structure states that a firm optimally chooses the proportions of debt and equity by balancing the tax shield benefits of debt and bankruptcy costs. In addition to these, we identify another factor in the trade-off due to debt liquidity shock: the *liquidity-risk cost*, and we refer to this new insight as a “new trade-off theory”.

Fig. 2 exhibits that the firm suffering debt liquidity risk does not issue any debt at low tax rates because the liquidity-risk cost (along with bankruptcy costs) dominates tax benefits. When the tax rate is sufficiently high and tax benefits dominate, the firm issues debt. A higher liquidity risk induces a later debt issuance. In addition, the tax rate increases debt and coupon. High tax rates even narrow the differences in debt values under different levels of liquidity risk due to the overwhelming tax benefits.

By contrast, Fig. 2(b) shows that the optimal coupon payments with liquidity risk overpass that without liquidity risk after moderate values of tax rate. Furthermore, a high tax rate leads to a particularly large coupon

It becomes a U-shape only if the liquidity risk is sufficiently high ($\xi = 2$), which implies that the U-shape is due to liquidity risk here.

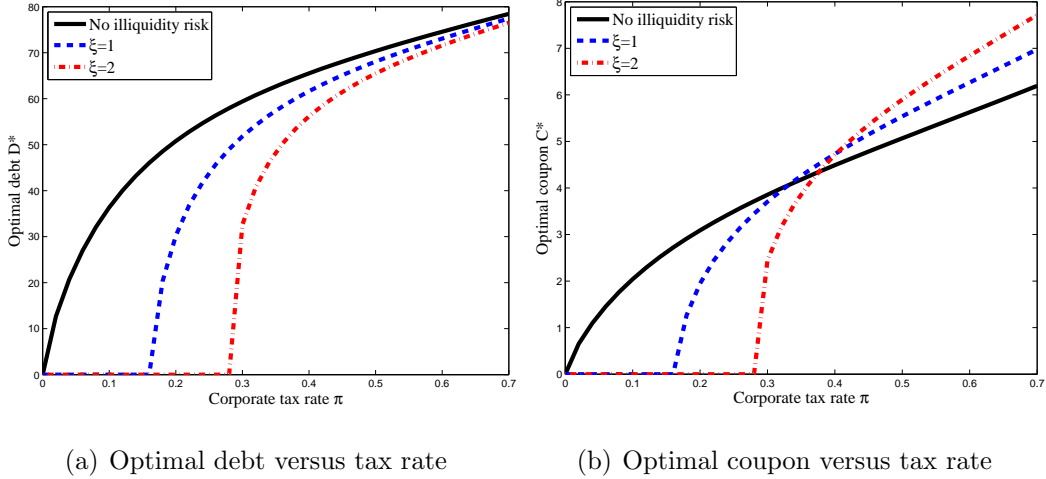


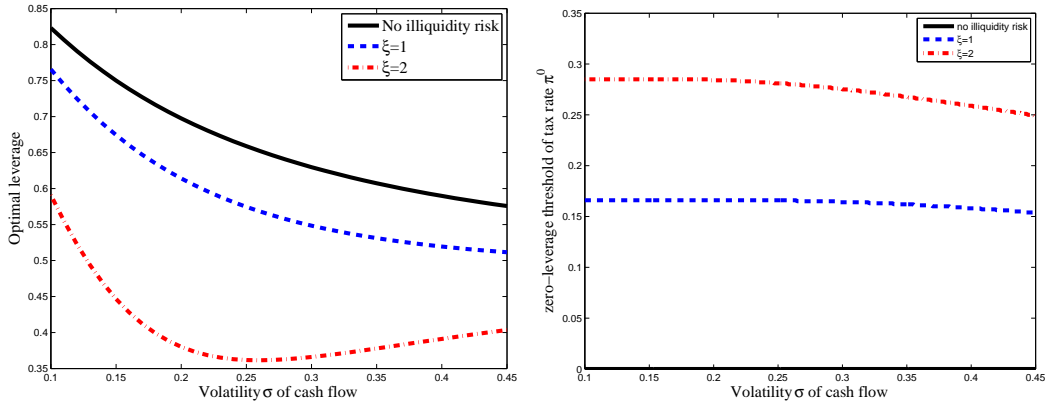
Fig. 2. The figure displays (a) optimal debt D^* ; and (b) optimal coupon C^* against the tax rate π for three levels of liquidity shock intensity ξ .

under a high liquidity risk. These large coupon payments correspond to the liquidity risk premium mentioned before. Indeed, though a high value of debt brings large tax benefits, it makes the firm much riskier if it bears a high liquidity risk. The debtholders in turn demand large risk premium.

3.3. Optimal leverage with debt conservatism puzzle and zero-leverage puzzle

Fig. 3(a) depicts that the firm generally decreases the optimal leverage with the project risk σ , which is consistent with Mauer and Sarkar (2005). Moreover, we find that it substantially reduces its leverage due to liquidity risk. For instance, the leverage with $\xi = 0$ is reduced by around 30 percent ($\sigma = 0.1$) up to 45 percent ($\sigma = 0.25$) under the high liquidity risk $\xi = 2$.

Our novel insight that a firm responds aggressively to liquidity risk by largely decreasing leverage provides a new explanation to the “debt conser-



(a) Optimal leverage versus volatility (b) Optimal leverage versus bankruptcy costs

Fig. 3. The figure plots the optimal leverage against (a) the diffusion volatility σ ; and (b) the bankruptcy cost rate α for three levels of intensity ξ .

vatism puzzle”. Intuitively, the firm incurs a higher liquidity-risk cost is more vulnerable to default. Hence, it chooses a lower leverage to reduce its default risk (see Section 3.4.1) and to survive when a liquidity shock arrives.¹⁰

In addition, our results capture the “zero leverage puzzle”. Strebulaev and Yang (2013) state that zero leverage firms are more profitable and pay higher taxes. Future financial flexibility is an important factor in the determination of zero leverage policy. Indeed, most characteristics of zero leverage firms are equivalent to those with low liquidity risk, since the firms with a higher level of financial flexibility are generally less influenced by market liquidity deterioration. Similarly, the firms paying higher taxes choose a zero

¹⁰Bolton et al. (2014a) present another explanation to the puzzle. They show that financially constrained firms limit debt to preserve cash holdings and to avoid debt overhang situations that would reduce equityholders’ incentives to raise new funds ex post.

leverage and give up tax shield benefits.

Figure 3(b) shows the key role played by liquidity risk in determining the optimal zero leverage. The figure portrays the zero leverage tax rate threshold π^0 . Above the threshold the firm will issue debt. Figure 3(b) reflects the mixed effects: tax shield effect, liquidity risk effect, and option effect. There is only the optimal zero leverage when π^0 equals to zero for no liquidity risk case, while there is a negative relationship between non-zero π^0 and σ for liquidity risk cases. In these cases, the option effect and tax shield effect are complementary as an increase in diffusion volatility will make the firm liable to choosing a non-zero leverage with tax shield benefits.

3.4. Optimal default decisions and default risk

In this subsection, not only do we confirm the standard results on default risk under the new situation with liquidity risk, but we also find that liquidity risk and tax dramatically amplify default risk. Furthermore, compared with He and Xiong (2012), our results highlight the opposite effect of liquidity risk on the firm's default policy due to a different debt structure.

3.4.1. Credit spreads and new effect of tax

In addition to the standard result that the project risk σ and tax rate π increase the firm's credit spread, $(C^*/D^* - r) \times 10^4 bp$, as Mauer and Sarkar (2005), Fig. 4 shows that the liquidity risk ξ raises the credit spread as well. This result is consistent with He and Xiong (2012). Such high credit spreads indicate high risk premium demanded by debtholders.

Moreover, the noticeable gaps of credit spread among different values of ξ in Fig. 4(a) indicates that the credit spread is more sensitive to the liquidity

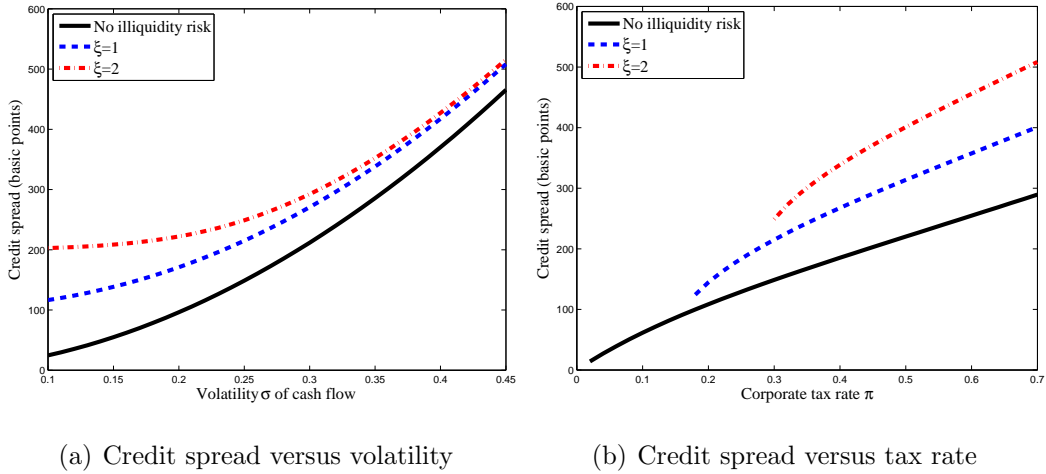


Fig. 4. The figure exhibits the credit spread against (a) the diffusion volatility σ ; and (b) the tax rate π for three levels of liquidity shock intensity ξ .

risk for small values of σ . However, these gaps gradually become narrow when the project risk σ ascends. This narrowing trend confirms our previous claim in Section 3.3 that the firm prefers substantially low leverage to reduce credit risk in anticipation of liquidity risk. Consequently, credit risk with high liquidity risk is close to that with low liquidity risk.

On the contrary, the tax rate significantly enlarges the gaps of credit spread as shown in Fig. 4(b). The credit spread increases by around 50 (resp. 100) percent for the liquidity risk $\xi = 1$ (resp. $\xi = 2$). To disentangle the liquidity risk from other risk factors, we depict three curves for three liquidity shock intensities when keeping other parameters and risk factors, e.g., the project risk σ , unchanged. Our new insight to risk management is that the high levels of liquidity risk under even moderate tax rates can almost double the credit spread, since under these circumstances debtholders

demand large coupon payments (see Section 3.2.1). This finding sheds light on the importance of liquidity risk management and provides a theoretical explanation to the empirical findings in Longstaff et al. (2005).¹¹

In addition, Mauer and Sarkar (2005) explain that an increase in tax rate has two effects: decreasing after-tax operating cash flows (which increases the credit spread) and increasing interest tax shields. Bolton et al. (2014a) show that the credit spread becomes more sensitive to cash liquidity when the firm's liquidity status is poor. We complement the literature by emphasizing the third effect of tax rate: It results in lower retained earnings under a higher liquidity risk due to the heavier burden of coupon payment.

3.4.2. *Endogenous default boundaries*

Table 1 implies two opposite effects of project risk σ on the firm's default boundary x_b determined endogenously. First, the boundaries in bold font show that the option effect of σ adds the values of waiting and therefore the firm is less vulnerable to default. This effect makes the default boundaries decrease with σ until $\sigma = 20\%$ (resp. $\sigma = 25\%$) for $\xi = 0$ (resp. $\xi = 1, 2$). Second, after these turning points, the default boundaries rise markedly with σ since the high levels of project risk considerably increase credit risk. In addition, it seems ambiguous that the default boundaries are sometimes higher than the initial value x_0 . This is due to the fact that the firm will

¹¹Fig. 4(b) displays that liquidity risk roughly contributes to 20% to 50% (i.e. double the credit spread without liquidity risk) of credit spreads. Longstaff et al. (2005, Table II) report that the nondefault component (related to illiquidity) in credit spreads explain 16% (BB), 32% (BBB), 47% (A), and 51% (AAA/AA) in average.

Table 1. The default boundary x_b under different levels of debt liquidity risk (ξ) and project risk (σ).

σ	0.15	0.2	0.25	0.3	0.35	0.4	0.6
$\xi = 0$	0.8325	0.8319	0.8548	0.8966	0.9544	1.0265	1.4407
$\xi = 1$	0.8611	0.8242	0.8218	0.8455	0.8905	0.9532	1.3485
$\xi = 2$	0.6330	0.5490	0.5421	0.5801	0.6440	0.7253	1.1719

wait until x is high enough to invest and after that the firm will consider the optimal default policy.

Contrary to He and Xiong (2012), we find that the liquidity risk generally decreases default boundaries and delays default. The difference originates from different model structures. The firm in our model issues a perpetual interest-only (consol) debt as Leland (1994) followed by Mauer and Sarkar (2005) and Wang et al. (2015) among others in a real option framework, while He and Xiong (2012) examine a rollover debt structure with a given amount of financing. They illustrate that a shorter debt maturity makes a firm default at a larger boundary due to higher rollover risk. Our finding is consistent with theirs if we close our real option channel by using given constant levels of coupon and current cash flow. In fact, our default strategy is feasible since the firm has reduced default risk by dramatically cutting its leverage as discussed before.

The interesting result reveals our novel insight to default decisions. When the firm considers the investment and financing decisions together, the firm should not default too early even under severe debt liquidity risk. It can delay default to maximize its equity value and wait longer for a lower leverage and

payment.

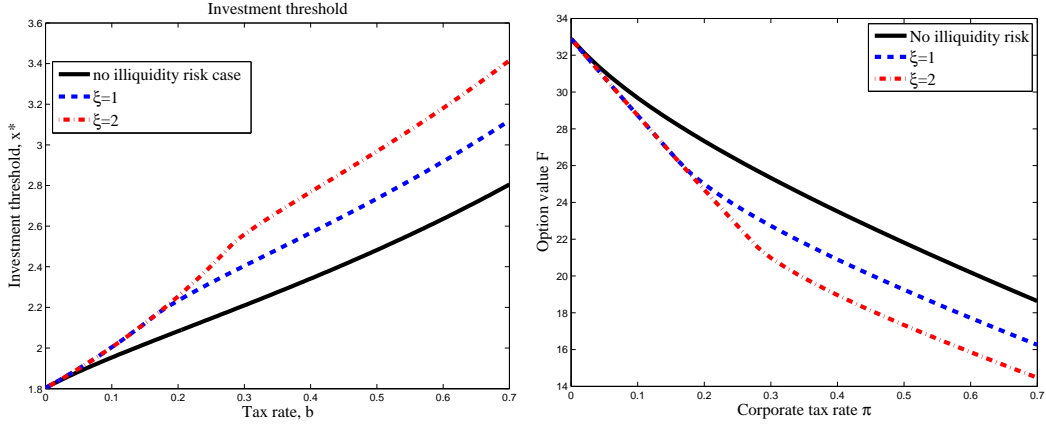
3.5. Optimal investment decisions

The real options literature discovers several factors delaying the exercise of option with a flow payoff (via raising investment thresholds), e.g. volatility and risk aversion (Miao and Wang, 2007) and estimation risk (Song et al., 2014). We complement the literature by emphasizing that liquidity risk and tax also play the role of postponing investment. Additionally, we identify the driving force: the liquidity-risk cost and tax that diminish operating cash flows and option values.

3.5.1. Investment timing and option values

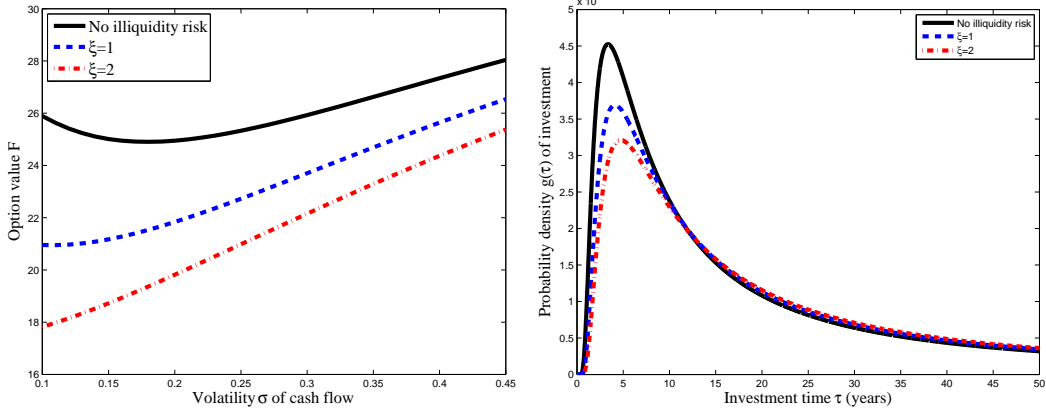
Figs. 5(a) and 5(b) demonstrates our new insight that the firm has the incentive to delay investment for a larger liquidity risk ξ or tax rate π , which *reduces* the option value F . The increasing trend of investment threshold x^* in Fig. 5(a) is similar to the standard option effect of volatility raising x^* .

However, the underlying driving forces for postponing investment are different. A higher volatility makes the option more valuable because the option holder can bound downside losses and acquire upside gains by waiting until the option is sufficiently in the money and exercising it. On the contrary, the option value declines with ξ or π in Fig. 5(b) since both of liquidity-risk cost and tax reduce the firm's value by decreasing debt value or increasing tax and coupon payments. As a result, the firm has to wait longer to invest until the option is sufficiently in the money via debt and equity financing. Similarly, Bolton et al. (2014b) find that a firm postpones investment and hoards cash until it has sufficient internal funds to invest in order to reduce



(a) Investment threshold versus volatility

(b) Option value versus tax rate



(c) Option value versus volatility

(d) Probability density of investment

Fig. 5. The figure demonstrates (a) the investment threshold x^* ; and (b) the option value F against the tax rate π ; (c) the option value F against the volatility σ ; and (d) the probability density $g(\tau)$ of investment for three ξ s.

dilution effects caused by external financing costs.

The interaction of the two opposing forces above is further illustrated in Fig. 5(c). For low volatilities and small option effects, the liquidity risk

ξ dramatically drags the option value down. When the volatility rises, the option effect gradually grows and it pulls up the two curves with $\xi = 1, 2$ close to the one with $\xi = 0$.

What's more, the option value of firm with no liquidity risk shows a slight decline when volatility is low. Generally, the standard option effect will increase the option value and investment threshold. However, in the no liquidity risk case, both investment threshold and default boundary experience only a relative small change when volatility is low, though the liquidity risk case leads to a much lower default boundary and a much higher investment threshold. Hence, an increase in volatility will lead to a less certainty-equivalent value for low values of volatility.

Fig. 5(d) furthermore displays that the likelihood of investment noticeably falls with the liquidity risk ξ , which is indicated by the First Passage Time Density (FPTD) for the investment time up to 20 years. The option is exercised when the underlying process x first time passes the optimal threshold x^* given the initial value x_0 . Applying results in stochastic process to real options, one can obtain the probability density $g(\tau)$ of investment below:

$$g(\tau) = \frac{\log(x^*/x_0)}{\sigma\tau\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left[\frac{\log(x^*/x_0) - (\mu - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right]^2},$$

where τ is the investment time and the π here is the mathematical constant.

In addition, Fig. 5(a) depicts that under liquidity risk the investment threshold increases faster with the tax rate before the turning points than after the points. Before these points, x^* grows faster since the firm has neither debt nor tax benefits due to the dominant liquidity-risk cost. After tax benefits are sufficiently large to issue debt, the growing speed of x^* is mitigated. A similar feature is displayed in Fig. 5(b) for the option value F .

Table 2. The option value $F_i(x_0)$ and liquidity loss $L_i(x_0) = (F_0(x_0) - F_i(x_0))/F_0(x_0)$ under three levels of debt liquidity risk ($\xi_i = 0, 1, 2$).

x_0	0.5	1	1.5	2.0	2.5	3.0	3.5	4.0
F_0	10.39	25.33	42.65	61.73	81.79	101.08	119.80	138.19
F_1	9.33	22.73	38.27	55.38	73.74	92.14	110.23	128.14
$F_0 - F_1$	1.07	2.60	4.38	6.34	8.05	8.94	9.58	10.06
L_1 (%)	10.28	10.28	10.28	10.28	9.84	8.84	7.99	7.28
F_2	8.61	20.98	35.33	51.13	68.12	85.70	103.25	120.78
$F_0 - F_2$	1.78	4.35	7.32	10.59	13.67	15.37	16.55	17.41
L_2 (%)	17.16	17.16	17.16	17.16	16.72	15.21	13.81	12.60

3.5.2. Liquidity losses with initial cash flows

Table 2 reports option values $F_i(x_0)$ and *relative liquidity loss* $L_i(x_0) = [F_0(x_0) - F_i(x_0)]/F_0(x_0)$ with $\xi_i = i = \{1, 2\}$, for different initial cash flow x_0 ranging from 0.5 to 4.0.

We observe that though an increase in x_0 reduces the relative liquidity loss L_i , from 17.16% ($\xi = 2$) with $x_0 = 0.5$ to 12.60% ($\xi = 2$) with $x_0 = 4.0$, the absolute loss $F_0 - F_i$ is in fact becoming larger, from 1.78 ($\xi = 2$) with $x_0 = 0.5$ to 17.41 ($\xi = 2$) with $x_0 = 4.0$. The reason is that the option value F_0 without liquidity risk grows much faster than liquidity risk. This finding on liquidity loss implies the new insight that the liquidity risk largely erodes projects' cash flows and values of investment options.

The implication of cash flows x_0 for option values allows us to make an ex post heuristic discussion of the joint effect of internal and external liquidity shocks on firm strategies. An increase in x_0 usually indicates high internal

liquidity and cash holdings. Given a level of external liquidity shock, e.g. $\xi = 1$, an increasing internal liquidity by keeping more cash holdings indicated by x_0 substantially raises the option value F_1 from 9.33 with $x_0 = 0.5$ to 128.14 with $x_0 = 4.0$. Large option values imply that the adverse effect of market liquidity shock is alleviated to some degrees by internal liquidity.

3.6. Issues on stockholder-bondholder conflicts

We have described that liquidity deterioration in the secondary debt market brings significant effects on firm's decisions. A natural question is whether these effects have implications for typical conflicts between stockholders and bondholders. To this end, we illustrate the conflicts through three respects: agency cost, risk shifting incentive, and debt overhang.

3.6.1. Agency cost due to the second-best exercise policy

Our model of investment option follows the widely applied setting that the firm makes an investment decision to maximize the *firm* value. This policy is called the “first-best” exercise policy. Considering the conflicts of interest, the firm might instead act in equityholders' interests and maximize the *equity* value. This policy is referred to as the “second-best” exercise policy. Mauer and Sarkar (2005) find that the latter leads to early investment or overinvestment relative to the former. They define the *agency cost* of overinvestment as $(F^1 - F^2)/F^2$, where F^1 and F^2 denote the firm (option) values given by the first- and second-best policies respectively.

One of the main findings in Mauer and Sarkar (2005) is that a smaller volatility σ results in a larger agency cost. In other words, the small volatility magnifies the disparity of firm value between the first- and second-best ex-

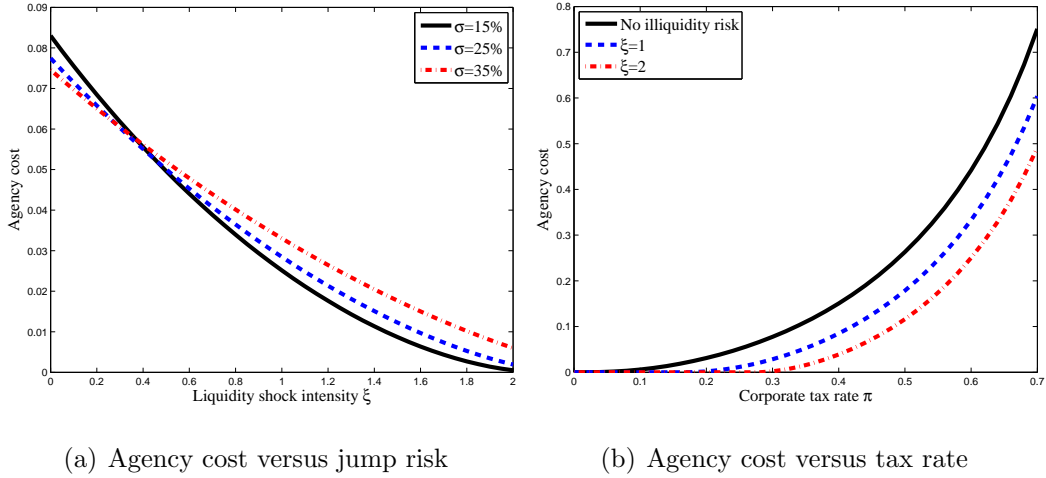


Fig. 6. The figure displays the agency cost against (a) the liquidity shock intensity ξ for three levels of σ ; and (b) the tax rate π for three levels of ξ .

exercise policies. They point out that the underlying reason is that the option is exercised sooner and closer to time zero by the decision makers with both policies due to the option effect.

However, our Fig. 6(a) shows that their conclusion only holds when the liquidity risk ξ is negligible ($\xi < 0.3$ here). We provide new insight to the agency cost theory by revealing the reverse effect of volatility on the agency cost for different ξ . When it is strong enough ($\xi > 0.3$), a fall in the volatility leads to a decline in the agency cost rather than a rise. In fact, when the liquidity risk is strong, the second-best firm value F^2 has already been fairly low. In this case a decrease in volatility reduces F^2 only to a lower margin than the first-best firm value F^1 . Indeed, we find that the agency cost is surprisingly low for a small project risk or a large liquidity risk.

Moreover, Figs. 6(a) and 6(b) display that the agency cost decreases

with ξ as well due to the same reason above. Intuitively, the liquidity risk makes equityholders be more concerned about default risk, which reduces the leverage and causes a smaller disparity between the first- and second-best investment policies.

In addition, the result that the agency cost ascends with the tax rate π for all of three curves in Fig. 6(b) is similar to Mauer and Sarkar (2005). According to their explanation, the disparity between the first- and second-best exercise policies steeply widens because of the net result of two effects of increasing tax rate: reducing cash flows and adding tax benefits.

3.6.2. Risk shifting incentives

Since the equity of a levered firm is comparable to a call option, Jensen and Meckling (1976) find that equityholders have a moral hazard incentive to raise the value of equity by increasing the volatility (risk) at the expense of debtholders' debt value. This incentive rises with the firm's leverage.

We investigate the implications of debt liquidity shock for equityholders' *risk shifting incentives*. Following previous studies (e.g. Pennacchi et al., 2014) we quantify the incentive using the sensitivity $\partial E/\partial\sigma$. Note that the firm's asset value dose not vary with this comparative static exercise.

Fig. 7(a) shows that for the moderate tax rate π around 0.25 to 0.4, the risk shifting incentive increases with π and declines with the liquidity risk ξ . Within such domain of tax rate, the effect of tax benefits dominate and the firm issues more debt with a higher tax rate or a lower liquidity risk. Thus, equityholders have more incentives to raise the volatility and equity value. Similarly, Fig. 7(b) depicts that the incentive decreases with bankruptcy costs because of costly financial distress. The incentive is also

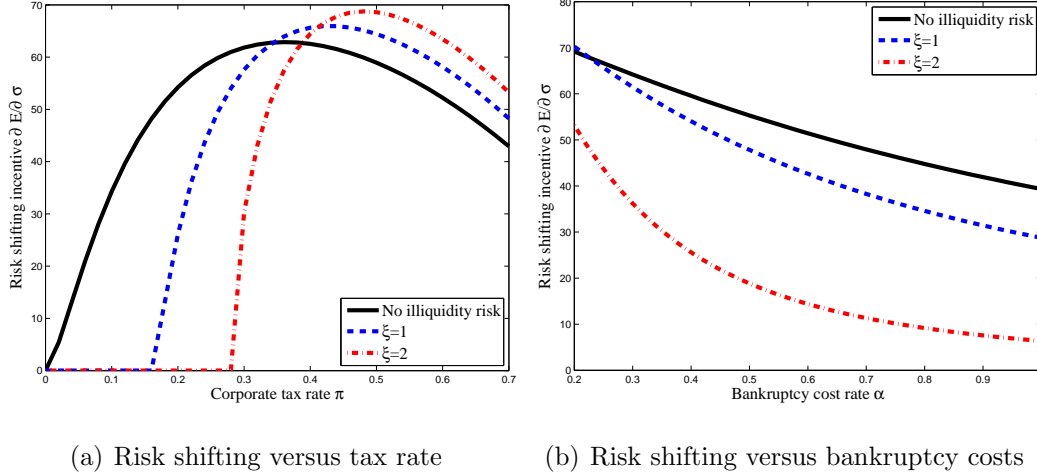


Fig. 7. The figure displays the risk shifting incentive $\frac{\partial E}{\partial \sigma}$ against (a) the tax rate π ; and (b) the bankruptcy cost rate α for three levels of intensity ξ .

much less with a higher liquidity risk. Our result implies that considering debt liquidity risk, the firm’s decision maker can substantially reduce risk shifting incentives compared with the case of ignoring debt liquidity shock.

The curves of sensitivity $\frac{\partial E}{\partial \sigma}$ reverse for high tax rates over 0.4. Equityholders have less risk shifting incentives with a higher tax rate since the tax effect on reducing cash flows dominates. This situation adds default risk and diminishes the incentive since equityholders would lose everything at liquidation. By contrast, a larger liquidity risk enhances the incentive during the domain of high tax rate. In this scenario debtholders demand particularly large coupon payments as compensation for a high liquidity risk (Fig. 2(b)), which aggravates equityholders’ moral hazard incentives.

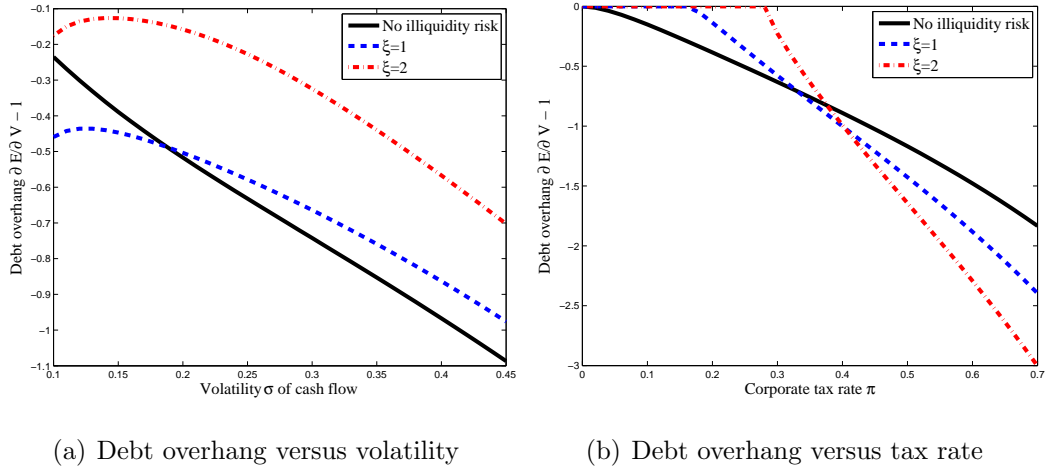


Fig. 8. The figure depicts the debt overhang $\frac{\partial E}{\partial V} - 1$ against (a) the volatility σ ; and (b) the tax rate π for three levels of liquidity shock intensity ξ .

3.6.3. Debt overhang problems

A firm can issue new equity to make default less likely while this issuance increases the debt value at the expense of initial shareholders' equity, given that the new equity is issued at a fair price. Considering such a loss, shareholders are reluctant to replenish equity when there is a drop in the firm's capital. Myers (1977) refers to this issue as "debt overhang" problem.

To examine shareholders' disincentives to issue new equity under debt liquidity risk, we observe the quantity $\partial E/\partial V - 1$. It represents the change in the initial shareholders' equity following a new equity issue (e.g. Pennacchi et al., 2014). A negative value for $\partial E/\partial V - 1$ indicates debt overhang and its absolute value implies the extent of debt overhang.

Fig. 8 demonstrates that the debt overhang problem increases with the project risk σ and tax rate π . It declines with the liquidity risk ξ within

the moderate tax rate $\pi \in (0.25, 0.4)$ while it reverses in the region with π higher than 0.4. Note that the shapes of $\partial E/\partial V - 1$ in Figs. 8(a) and 8(b) are opposite to those of coupon payments in Fig. 1(b) and Fig. 2(b).

Our finding shows that the optimal coupon plays an important role in determining the debt overhang problem due to the perpetual debt structure. An increase in coupon causes less retained earnings, a heavier debt burden, and higher default risk, which discourages equityholders from issuing new equity and worsens the debt overhang and underinvestment problems. Myers (1977) reports that an increase in leverage makes a firm be more likely to have debt obligations larger than its asset value, which leads to sub-optimal investment policies. Our finding further specifies that the optimal coupon of perpetual debt is a direct factor determining the debt overhang problem.

3.7. Finite maturity debt

In this section, we extend perpetual debt to a finite maturity debt structure. Then we examine whether the effects of liquidity shocks on firms' strategies are robust under different debt structures.

The setting and solution of finite maturity debt follow Leland and Toft (1996) and He and Xiong (2012). The firm issues a constant amount of new debt with a T years maturity and a principal at a rate $p = P/T$ per annum, where P is the principal of all outstanding bonds. The debt pays an annual coupon rate $c = C/T$, where the total coupon of all bonds is C . Denotes the time-to-maturity of debt by τ . The value of one unit bond $d(V, \tau)$ depending on the fundamental V and time-to-maturity τ follows the standard PDE:

$$rd(V, \tau) = c - \xi kd(V, \tau) - \frac{\partial d(V, \tau)}{\partial \tau} + \mu V \frac{\partial d(V, \tau)}{\partial V} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 d(V, \tau)}{\partial V^2}$$

with two boundary conditions: $d(V_B, \tau) = (1 - \alpha)V_B/T$ and $d(V, 0) = p$. Combining the boundary conditions with the PDE gives the value of $d(V, \tau)$:

$$d(V, \tau) = \frac{c}{r+\xi k} + e^{-(r+\xi k)\tau} \left(p - \frac{c}{r+\xi k} \right) (1 - M(\tau)) + \left(\frac{(1-\alpha)V_B}{T} - \frac{c}{r+\xi k} \right) G(\tau). \quad (13)$$

where

$$\begin{aligned} M(T) &= N(h_1(T)) + \left(\frac{V}{V_B} \right)^{-2a} N(h_2(T)), \\ G(T) &= \left(\frac{V}{V_B} \right)^{-a+\hat{z}} N(q_1(T)) + \left(\frac{V}{V_B} \right)^{-a-\hat{z}} N(q_2(T)), \\ h_1(\tau) &= \frac{-v-a\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad h_2(\tau) = \frac{-v+a\sigma^2\tau}{\sigma\sqrt{\tau}}, \\ q_1(\tau) &= \frac{-v-\hat{z}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad q_2(\tau) = \frac{-v+\hat{z}\sigma^2\tau}{\sigma\sqrt{\tau}}. \\ v &\equiv \ln \left(\frac{V}{V_B} \right), \quad a \equiv \frac{\mu-\sigma^2/2}{\sigma^2}, \quad \hat{z} \equiv \frac{[a^2\sigma^4+2(r+\xi k)\sigma^2]^{1/2}}{\sigma^2}, \end{aligned}$$

and $N(x)$ is the cumulative standard normal distribution function.

Integrating the value of one unit bond $d(V, \tau)$ in (13) from 0 to T gives the value of all outstanding bonds:

$$\begin{aligned} D(V; C, V_B, T) &= \frac{C}{r+\xi k} + \left(P - \frac{C}{r+\xi k} \right) \left[\frac{1-e^{-(r+\xi k)T}}{(r+\xi k)T} - I(T) \right] \\ &\quad + \left[(1-\alpha)V_B - \frac{C}{r+\xi k} \right] J(T), \end{aligned} \quad (14)$$

where

$$\begin{aligned} I(T) &= \frac{1}{(r+\xi k)T} [G(T) - e^{-(r+\xi k)T} M(T)], \\ J(T) &= \frac{1}{\hat{z}\sigma\sqrt{T}} \left\{ - \left(\frac{V}{V_B} \right)^{-a+\hat{z}} N[q_1(T)]q_1(T) + \left(\frac{V}{V_B} \right)^{-a-\hat{z}} N[q_2(T)]q_2(T) \right\}. \end{aligned}$$

Based on finite maturity debt, the firm in our model maximizes the real option value $F(V)$ by choosing the optimal investment threshold V^* and

optimal coupon. The debt is issued when the firm exercises the option at V^* . We omit the expressions of equity value $E(V; C, V_B, T)$ and optimal default boundary V_B since they are the same as the ones in He and Xiong (2012). The option value satisfies the following partial differential equation:

$$rF = \mu V F_V + \frac{1}{2} \sigma^2 V^2 F_{VV}$$

with three boundary conditions. Firstly, the option value is zero when $V = 0$:

$$F(0) = 0.$$

Secondly, the option value must satisfy the value-match condition:

$$F(V^*; C, V_B, T) = E(V^*; C, V_B, T) + D(V^*; C, V_B, T) - I.$$

Thirdly, the smooth pasting condition is

$$\frac{dF(V; C, V_B, T)}{dV} \Big|_{V=V^*} = \frac{dE(V; C, V_B, T)}{dV} \Big|_{V=V^*} + \frac{dD(V; C, V_B, T)}{dV} \Big|_{V=V^*}.$$

Then the real option value is given by

$$F(V; C, V_B, T) = [E(V^*; C, V_B, T) + D(V^*; C, V_B, T) - I] \left(\frac{V}{V^*}\right)^{w_2}. \quad (15)$$

Note that the option value in (15) depends on the value of total finite maturity debt D in (14), which implies that the option value depends on the time-to-maturity τ since all outstanding bonds rely on the unit bond $d(V, \tau)$.

3.7.1. Robust effects of liquidity risk on firms' optimal strategies

We discuss the robustness of our results under finite maturity debt. The parameters are the same as the perpetual debt structure for comparison. Table 3 displays the optimal coupon, leverage, default boundary, and option

value for the case of finite maturity debt. Comparing Table 3(a) with 3(b) shows that the effects of liquidity risk on firms' optimal strategies remain robust. Different to He and Xiong (2012), liquidity risk delays the firm's default decisions and meanwhile reduces the optimal coupon, leverage, and real option value. The difference is explained by the same underlying force as that under perpetual debt. In the model of He and Xiong (2012), the coupon C and aggregate principal P are given constants because neither optimal investment nor optimal coupon for investment is considered. Indeed, when investment is considered, the optimal strategy in our model is feasible since firms can reduce leverage when they choose to invest and delay default.

In short, our findings on firm strategies under debt liquidity risk are robust to the finite maturity debt structure and our model provides an extension of He and Xiong (2012) for firms' optimal decisions.

3.8. Extension to cyclical liquidity risk

In this subsection, we extend our baseline mode to a case in which the magnitude of the effect of liquidity risk varies depending on the firm's fundamental. We show that our previous results are robust to this model extension.

Liquidity deterioration in the secondary debt market affects the debt value $D(x)$ through the channel of transaction cost rate k , see Equation (7) of debt value following He and Xiong (2012). The transaction cost k measures the magnitude of the effect of liquidity risk. Intuitively, the transaction cost of corporate debt varies in accordance with the cyclical variation of the firm's cash flow. The firm with a high level of cash can alleviate its exposure to market liquidity.

To formulate the cyclical effect of liquidity risk, we let the trading cost

Table 3. Robust effects of liquidity risk on the optimal coupon C^* , leverage L^* , default boundary x_b , and option value F under finite maturity debt.

(a) Liquidity Shock Intensity $\xi = 1$

Maturity	T=1	T=5	T=10	T=20	T=50	T=100	T= ∞
C^*	0.93	1.47	1.63	1.86	2.65	3.36	3.70
L^*	0.1720	0.2622	0.2724	0.2902	0.3851	0.4893	0.5746
x_b	0.4919	0.5812	0.5483	0.5353	0.6549	0.7818	0.8218
F	21.213	21.461	21.613	21.766	22.160	22.513	22.726

(b) Liquidity Shock Intensity $\xi = 2$

Maturity	T=1	T=5	T=10	T=20	T=50	T=100	T= ∞
C^*	0.25	0.42	0.58	1.02	1.90	2.25	2.44
L^*	0.0396	0.1511	0.1828	0.2829	0.4636	0.5233	0.5420
x_b	0.1164	0.5812	0.5483	0.5353	0.6549	0.7818	0.8218
F	20.809	20.816	20.824	20.843	20.908	20.947	20.983

for the firm with large cash flow be lower than that for the firm with small cash flow. We denote the threshold of the cash flow that distinguishes the two types firms by \underline{x} . When the firm's cash flow is larger (resp. smaller) than \underline{x} , the transaction cost rate for its debt is denoted by k_h (resp. k_l), where $k_h < k_l$. Without loss of generality, we set the threshold $\underline{x} = 1$, $k_h = 1\%$, and $k_l = 2\%$, which implies that the liquidity risk is 1% (resp. 2%) when the firm has cash flow $x_t \geq 1$ (resp. $x_t < 1$). The other parameters are the same as the baseline model.

Table 4 summarizes the results under the case of cyclical liquidity risk, which are similar to the results under the baseline mode without the cyclical

variation of liquidity. It shows that an increasing liquidity risk decreases the optimal leverage and delays bankruptcy. The investment threshold x^* increases with the liquidity risk for all values of diffusion volatility. The firm generally issues less debts and coupons in a market with higher liquidity risk. Similar to Fig. 1(b), the optimal coupons with the liquidity risk $\xi = 1$ are slightly higher than those without liquidity risk when the volatility is small because the liquidity-risk-premium effect dominates under this situation.

Table 4. Robust effects of cyclical liquidity risk on the optimal coupon C^* , optimal debt D^* , investment threshold x^* , optimal leverage L^* , default boundary x_b , and option value F .

(a) Liquidity Shock Intensity $\xi = 0$

σ	0.15	0.2	0.25	0.3	0.35	0.4	0.45
C^*	2.83	3.22	3.86	4.85	6.20	7.74	9.65
D^*	50.861	53.987	59.495	67.782	78.028	87.967	99.095
x^*	1.54	1.82	2.16	2.60	3.10	3.62	4.20
L^*	0.7623	0.7077	0.6702	0.6470	0.6310	0.6144	0.6009
x_b	0.8557	0.8360	0.8560	0.9190	1.007	1.0828	1.1700
F	25.156	25.121	25.518	26.133	26.840	27.571	28.296

(b) Liquidity Shock Intensity $\xi = 1$

σ	0.15	0.20	0.25	0.3	0.35	0.4	0.45
C^*	2.92	3.37	3.95	4.85	5.88	7.49	8.91
D^*	45.656	49.846	54.702	61.994	69.185	80.061	87.543
x^*	1.72	2.05	2.41	2.84	3.30	3.86	4.36
L^*	0.6894	0.6405	0.6029	0.5821	0.5616	0.5556	0.5379
x_b	0.8829	0.8749	0.8760	0.9190	0.9549	1.0478	1.0800
F	21.261	21.943	22.851	23.843	24.841	25.801	26.704

(c) Liquidity Shock Intensity $\xi = 2$

σ	0.15	0.20	0.25	0.3	0.35	0.4	0.45
C^*	2.45	2.45	2.85	3.71	4.52	6.27	7.70
D^*	34.346	33.599	37.502	45.684	52.001	65.368	73.976
x^*	1.91	2.21	2.58	3.02	3.47	4.06	4.61
L^*	0.5107	0.4307	0.4116	0.4276	0.4221	0.4516	0.4485
x_b	0.7408	0.6360	0.6319	0.7029	0.7339	0.8769	0.9330
F	18.729	19.825	21.004	22.199	23.358	24.452	25.466

4. Conclusion

In conclusion, deterioration in debt market liquidity brings shocks to debt. Considering such risk, a firm makes various decisions to maximize its value. Our contributions to this optimization problem are threefold. First, we derive analytic solutions. Second, we provide a comprehensive description of firm decisions and agency conflicts. Third, we reveal six new insights: trade-off theory, effect of tax, debt conservatism resulting in late bankruptcy and investment, doubled credit spread, small agency cost, and strengthened risk shifting incentive and debt overhang under moderate or high tax rates.

Appendix A Derivation of analytical solutions with leverage

The firm issues debt when the liquidity shock intensity ξ is not too large or the tax rate τ is not too small, as revealed in Section 3. In this case, we have positive debt value $D > 0$ and coupon $C > 0$.

Option. We immediately obtain the solution to the real option value $F(x; C)$ in (9) of Proposition 2.1 applying the result of one unit contingent claim in (4). Alternatively, one can write down the partial differential equation (PDE) satisfied by $F(x)$. Then one needs to specify the general solution to the PDE using three conditions: $F(0) = 0$, the smooth pasting condition (8), and the value-matching condition (e.g. Dixit and Pindyck, 1994):

$$F(x^*) = E(x^*) + D(x^*) - I. \tag{A.1}$$

Threshold. We obtain (10) determining x^* by expanding (8) as follows.

$$\frac{C}{r} - \frac{x_b}{r-\mu} = \frac{C}{r} - \frac{w_1(r-\mu)C}{(r-\mu)r(w_1-1)} = \frac{C}{r(1-w_1)}.$$

$$\left. \frac{\partial F(x)}{\partial x} \right|_{x=x^*} = w_2 [E(x^*) + D(x^*) - I] \frac{1}{x^*}. \quad (\text{A.2})$$

$$\left. \frac{\partial E(x)}{\partial x} \right|_{x=x^*} = (1 - \pi) \left[\frac{1}{r-\mu} + w_1 \left(\frac{C}{r(1-w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \frac{1}{x^*} \right]. \quad (\text{A.3})$$

$$\left. \frac{\partial D(x)}{\partial x} \right|_{x=x^*} = \gamma \left((1 - \alpha)V^U(x_b) - \frac{C}{r+\xi k} \right) \left(\frac{x^*}{x_b} \right)^\gamma \frac{1}{x^*}. \quad (\text{A.4})$$

Substituting (A.2) – (A.4) into (8) and expanding $E(x^*)$ and $D(x^*)$ lead to

$$\begin{aligned} & w_2 \left[\left(\frac{x^*}{r-\mu} - \frac{C}{r} \right) + \left(\frac{C}{r(1-w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} + \frac{C}{(r+\xi k)(1-\pi)} + \frac{1}{1-\pi} H \left(\frac{x^*}{x_b} \right)^\gamma - \frac{1}{1-\pi} I \right] \\ & = \frac{x^*}{r-\mu} + w_1 \left(\frac{C}{r(1-w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} + \frac{\gamma}{1-\pi} H \left(\frac{x^*}{x_b} \right)^\gamma, \quad H := (1 - \alpha)V(x_b) - \frac{C}{r+\xi k}. \end{aligned}$$

Gathering items of $\frac{x^*}{r-\mu}$, $\left(\frac{x^*}{x_b} \right)^{w_1}$, and $\left(\frac{x^*}{x_b} \right)^\gamma$ gives (10) in Proposition 2.1.

Coupon. The optimal coupon C^* is given by $\partial F/\partial C = 0$ using (9). Using the value-matching condition (A.1), we simplify it as (11) that depends on dx^*/dC . We further obtain the expression (12) for dx^*/dC by doing the derivative of x^* with respect to C in (10) below.

$$\begin{aligned} 0 &= \frac{(w_2-1)}{r-\mu} \frac{dx^*}{dC} + (w_2 - w_1) \left(\frac{1}{r(1-w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \\ &+ w_1(w_2 - w_1) \left(\frac{C}{r(1-w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1-1} d \left(\frac{x^*}{x_b} \right) / dC \\ &+ \frac{w_2-\gamma}{1-\pi} \left((1 - \alpha) \frac{\partial V(x_b)}{\partial C} - \frac{1}{r+\xi k} \right) \left(\frac{x^*}{x_b} \right)^\gamma \\ &+ \frac{(w_2-\gamma)\gamma}{1-\pi} H \left(\frac{x^*}{x_b} \right)^{\gamma-1} d \left(\frac{x^*}{x_b} \right) / dC + \frac{w_2}{1-\pi} \left(\frac{1}{r+\xi k} - \frac{1-\pi}{r} \right). \end{aligned} \quad (\text{A.5})$$

To simplify equation (A.5), we first derive some details as follows.

$$\frac{dx_B}{dC} = \frac{w_1(r-\mu)}{r(w_1-1)} = \frac{x_B}{C}.$$

$$d \left(\frac{x^*}{x_b} \right) / dC = \frac{1}{x_b} \frac{dx^*}{dC} - \frac{x^*}{(x_b)^2} \frac{dx_B}{dC} = \frac{x^*}{x_b} \left(\frac{1}{x^*} \frac{dx^*}{dC} - \frac{1}{C} \right).$$

$$(1 - \alpha) \frac{\partial V(x_b)}{\partial C} - \frac{1}{r+\xi k} = (1 - \alpha) \frac{V(x_b)}{C} - \frac{1}{r+\xi k} = \left((1 - \alpha)V(x_b) - \frac{C}{r+\xi k} \right) \frac{1}{C} = \frac{H}{C}.$$

$$\begin{aligned}
\frac{\partial E(x^*)}{\partial C} &= (1 - \pi) \left[\left(\frac{1}{r - \mu} \frac{dx^*}{dC} - \frac{1}{r} \right) + \left(\frac{1}{r} - \frac{1}{r - \mu} \frac{dx_b}{dC} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \right. \\
&\quad \left. + w_1 \left(\frac{C}{r} - \frac{x_b}{r - \mu} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \left(\frac{1}{x^*} \frac{dx^*}{dC} - \frac{1}{x_b} \frac{dx_b}{dC} \right) \right] \\
&= (1 - \pi) \left[\left(\frac{1}{r - \mu} \frac{dx^*}{dC} - \frac{1}{r} \right) + \frac{w_1 C}{r(1 - w_1)} \left(\frac{1}{x^*} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \frac{dx^*}{dC} + \frac{1}{r} \left(\frac{x^*}{x_b} \right)^{w_1} \right]. \\
\frac{\partial D(x^*)}{\partial C} &= \frac{1}{r + \xi k} + \left((1 - \alpha) \frac{\partial V(x_b)}{\partial C} - \frac{1}{r + \xi k} \right) \left(\frac{x^*}{x_b} \right)^\gamma + \gamma H \left(\frac{x^*}{x_b} \right)^\gamma \left(\frac{1}{x^*} \frac{dx^*}{dC} - \frac{1}{x_b} \frac{dx_b}{dC} \right) \\
&= \frac{1}{r + \xi k} + \left(\frac{1 - \gamma}{C} + \frac{\gamma}{x^*} \frac{dx^*}{dC} \right) H \left(\frac{x^*}{x_b} \right)^\gamma.
\end{aligned}$$

Then, substituting all of the above details into (A.5) we obtain

$$\begin{aligned}
0 &= \frac{w_2}{1 - \pi} \left(\frac{1}{r + \xi k} - \frac{1 - \pi}{r} \right) + \frac{(w_2 - 1)}{r - \mu} \frac{dx^*}{dC} + (w_2 - w_1) \left(\frac{1}{r(1 - w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \\
&\quad + w_1 (w_2 - w_1) \left(\frac{C}{r(1 - w_1)} \right) \left(\frac{x^*}{x_b} \right)^{w_1} \left(\frac{1}{x^*} \frac{dx^*}{dC} - \frac{1}{C} \right) \\
&\quad + \frac{w_2 - \gamma}{1 - \pi} \left((1 - \alpha) \frac{V(x_b)}{C} - \frac{1}{r + \xi k} \right) \left(\frac{x^*}{x_b} \right)^\gamma + \frac{(w_2 - \gamma)\gamma}{1 - \pi} H \left(\frac{x^*}{x_b} \right)^\gamma \left(\frac{1}{x^*} \frac{dx^*}{dC} - \frac{1}{C} \right).
\end{aligned}$$

Rearranging items gives the next equation and (12) in Proposition 2.1.

$$\begin{aligned}
&\left[\frac{(w_2 - 1)}{r - \mu} + \frac{(w_2 - \gamma)\gamma}{1 - \pi} H \left(\frac{x^*}{x_b} \right)^\gamma \left(\frac{1}{x^*} \right) + \frac{w_1(w_2 - w_1)}{r(1 - w_1)} \left(\frac{x^*}{x_b} \right)^{w_1} \frac{C}{x^*} \right] \frac{dx^*}{dC} \\
&= \frac{w_1 - w_2}{r} \left(\frac{x^*}{x_b} \right)^{w_1} + \frac{(\gamma - w_2)(1 - \gamma)}{(1 - \pi)C} H \left(\frac{x^*}{x_b} \right)^\gamma + \frac{w_2}{1 - \pi} \left(\frac{1 - \pi}{r} - \frac{1}{r + \xi k} \right).
\end{aligned}$$

Appendix B Option value and threshold with zero leverage

The firm chooses zero leverage ($D = C = 0$) facing large liquidity risk or low tax rate (see Section 3). In this case, our model recovers the standard case of real options (Dixit and Pindyck, 1994).

The option value $F(x)$ satisfies the PDE as follows:

$$rF = \mu x F_x + \frac{1}{2} \sigma^2 x_t^2 F_{xx}, \quad x < x^*,$$

where x^* is the investment threshold determined later. The option value also satisfies the following three boundary conditions: $F(0) = 0$, $F(x^*) = x^*(1 - \pi)/(r - \mu) - I$, and $F'(x^*) = (1 - \pi)/(r - \mu)$. Using these conditions

one can specify the general solution to the PDE and obtain the solutions to $F(x)$ and x^* below:

$$F(x) = \frac{I}{w_2 - 1} \left(\frac{x}{x^*} \right)^{w_2}, \quad x^* = \frac{w_2(r - \mu)}{(1 - \pi)(w_2 - 1)} I.$$

References

- Almeida, H., Campello, M., Laranjeira, B., Weisbenner, S., 2009. Corporate debt maturity and the real effects of the 2007 credit crisis. National Bureau of Economic Research Working Paper No. 14990.
- Bao, J., Pan, J., Wang, J., 2011. The illiquidity of corporate bonds. *Journal of Finance* 66 (3), 911–946.
- Bolton, P., Chen, H., Wang, N., 2014a. Debt, taxes, and liquidity. National Bureau of Economic Research Working Paper No. 20009.
- Bolton, P., Wang, N., Yang, J., 2014b. Investment, liquidity, and financing under uncertainty. Working Paper, Columbia University and Shanghai University of Finance and Economics.
- Boyle, G. W., Guthrie, G. A., 2003. Investment, uncertainty, and liquidity. *The Journal of finance* 58 (5), 2143–2166.
- Dixit, A. K., Pindyck, R. S., 1994. *Investment under uncertainty*. Princeton University Press.
- Glosten, L. R., Milgrom, P. R., 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics* 14 (1), 71–100.

- Gryglewicz, S., 2011. A theory of corporate financial decisions with liquidity and solvency concerns. *Journal of Financial Economics* 99 (2), 365–384.
- Hackbarth, D., Johnson, T., 2015. Real options and risk dynamics. *Review of Economic Studies* 82 (4), 1449–1482.
- He, Z., Xiong, W., 2012. Rollover risk and credit risk. *Journal of Finance* 67 (2), 391–430.
- Jensen, M. C., Meckling, W. H., 1976. Theory of the firm: Managerial behavior, agency costs and ownership structure. *Journal of Financial Economics* 3 (4), 305–360.
- Kyle, A. S., 1985. Continuous auctions and insider trading. *Econometrica* 53 (6), 1315–1335.
- Leland, H. E., 1994. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* 49 (4), 1213–1252.
- Leland, H. E., Toft, K. B., 1996. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *Journal of Finance* 51 (3), 987–1019.
- Longstaff, F. A., Mithal, S., Neis, E., 2005. Corporate yield spreads: Default risk or liquidity? new evidence from the credit default swap market. *Journal of Finance* 60 (5), 2213–2253.
- Mauer, D. C., Sarkar, S., 2005. Real options, agency conflicts, and optimal capital structure. *Journal of Banking & Finance* 29 (6), 1405–1428.

- Miao, J., Wang, N., 2007. Investment, consumption, and hedging under incomplete markets. *Journal of Financial Economics* 86 (3), 608–642.
- Myers, S. C., 1977. Determinants of corporate borrowing. *Journal of Financial Economics* 5 (2), 147–175.
- Pennacchi, G., Vermaelen, T., Wolff, C. C., 2014. Contingent capital: The case of coerced. *Journal of Financial and Quantitative Analysis* 49 (03), 541–574.
- Song, D., Wang, H., Yang, Z., 2014. Learning, pricing, timing and hedging of the option to invest for perpetual cash flows with idiosyncratic risk. *Journal of Mathematical Economics* 51, 1–11.
- Strebulaev, I. A., Yang, B., 2013. The mystery of zero-leverage firms. *Journal of Financial Economics* 109 (1), 1–23.
- Wang, H., Yang, Z., Zhang, H., 2015. Entrepreneurial finance with equity-for-guarantee swap and idiosyncratic risk. *European Journal of Operational Research* 241 (3), 863–871.