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An improvement in the two-packing bound related to Vizing's conjecture

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1. Introduction and Definitions

One of the central problem in domination theory and product graphs is Vizing's conjecture [**5**] which states that for any graphs G and H,

$$\gamma(G \,\Box\, H) \geq \gamma(G)\gamma(H). \tag{1.1}$$

In this formulation, $\gamma(G)$ is the domination number of G and $G \,\Box\, H$ is the Cartesian product of G and H.

The truth of this statement is known for various classes of graphs. For more on this as well as the history of this problem, we refer the reader to the excellent survey [**2**].

Results that approximate the conjectured bound take the form of

$$\gamma(G \,\Box\, H) \geq c\gamma(G)\gamma(H)$$

for largest possible c. The first published result of this kind was that of Clark and Suen in 2000 [**3**], showing

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H). \tag{1.2}$$

Twelve years later, Suen and Tarr [**4**] improved this to

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

In 2017, Zerbib [**6**] further improved this bound to

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\max\{\gamma(G), \gamma(H)\}.$$

These results all used the method of Clark and Suen which is called "double projection" in [**2**]. Again in 2017, Brešar [**1**] generalized this method to a lower bound in terms of the two-packing number, $\rho(H)$, showing

$$\gamma(G \,\Box\, H) \geq \gamma(G)\frac{2\gamma(H) - \rho(H)}{3}.$$

Depending on the relationship between $\gamma(G)$ and $\rho(G)$, Brešar's bound, together with Zerbib's, and the standard result that $\gamma(G \,\Box\, H) \geq \rho(G)\gamma(H)$, produce the largest known lower bound for Vizing's conjecture.

In this note, we combine the technique of Zerbib [**6**] with that of Brešar [**1**], to improve the best current bound for Vizing's conjecture. We show that for any two graphs G and H,

$$\gamma(G \,\Box\, H) \geq \gamma(G)\frac{2\gamma(H) - \rho(H) + 1}{3}. \tag{1.3}$$

Next, we provide some necessary definitions and notation. The graphs $G(V, E)$ in this paper are finite, simple, connected, and undirected with vertex set V and edge set E. We may refer to the vertex set and edge set of G as $V(G)$ and $E(G)$, respectively.

For any graph $G(V, E)$, and any $v \in V$, we write $N(v)$ to mean the *open neighborhood* of v, which is the set of vertices adjacent to v. By $N[v]$ we mean the *closed neighborhood* of v which is $N(v) \cup \{v\}$. We write $N^2(v)$ to mean the set of vertices of distance at most two to v and $N^2[v]$ to mean $N^2(v) \cup v$. A subset $S \subseteq V$ *dominates* G if $N[S] = V(G)$. The minimum cardinality of $S \subseteq V$, so that S dominates G is called the *domination number* of G and is denoted $\gamma(G)$.

A *two-packing* of a graph G is a set of vertices such that every pair is of distance at least 3 apart. The maximum size of a two-packing in G is called the *two-packing number* of G, denoted by $\rho(G)$.

The *Cartesian product* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1 \square G_2) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\}$.

For a vertex $h \in V(H)$, the *G-fiber*, G^h, is the subgraph of $G \square H$ induced by $\{(g, h) : g \in V(G)\}$.

Let D be any minimum dominating set of $G \square H$ and let $C_1, \dots, C_k$ be a partition of $V(G)$. For every $h \in V(H)$ and i, $1 \leq i \leq k$, we call $C_i^h = C_i \times \{h\}$ a *G-cell* for h and refer to i as the index of the G-cell. We write $D_i = D \cap (C_i \times V(H))$ and denote the projection of D_i onto H by P_i.

To vizualize, we represent G horizontally and H vertically when discussing the Cartesian product of G and H.

Let D be a minimum dominating set of $G \square H$ and C_i^h be a cell which is dominated by vertices of $D \cap G^h$. We say that C_i^h is a *flat* cell. In other words, flat cells are dominated horizontally.

2. Dominating Cartesian Products

THEOREM 2.1. *For any graphs G and H,*

$$\gamma(G \square H) \geq \gamma(G) \frac{2\gamma(H) - \rho(H) + 1}{3}.$$

PROOF. Let D be a minimum dominating set of $G \square H$. Notice that the projection of D onto G is a dominating set of G. Choose a minimum subset of vertices $\Gamma = \{v_1, \dots, v_k\}$ from the projection of D onto G which dominate G. By defintion, $k \geq \gamma(G)$. We now partition $V(G)$ into parts C_i, $i \in [k]$ where $v_i \in C_i$ and $N[v_i] \supseteq C_i$. Any vertex not in Γ which is a common neighbor of more than one vertex of Γ may be placed in any allowed part arbitrarily.

First, we claim that for any fixed $h \in V(H)$, the number of flat cells in G^h, r^h, is no more than $|D \cap G^h|$. Otherwise, if we let R^h be the set of flat cells in G^h and $S^h = \bigcup_{i=1}^k (C_i^h) - R^h$, we may construct the dominating set of G^h composed of $(D \cap G^h) \cup \{(v_i, h) : (v_i, h) \in S^h, i \in [k]\}$. Since we assumed $|D \cap G^h| < r^h$, such a set contains fewer than k vertices. This leads to the contradiction that there exists a subset of vertices of D of size less than k which can be projected onto G to form a dominating set.

Next, for any fixed $i \in [k]$, if we let R_i be the flat cells of $\bigcup_{h \in V(H)} C_i^h$ and $D_i = D \cap \bigcup_{h \in V(H)} C_i^h$, we notice that projecting the vertices $D_i \cup R_i$ onto H, produces a dominating set of H, since every vertex which is not a neighbor of $proj_H(D_i)$ must be contained in $proj_H(R_i)$. Furthermore, we note that for every $i \in [k]$, by definition of Γ, at least one vertex of D_i can be projected onto $v_i \in \Gamma$, and such a vertex dominates its cell. Thus, $|D_i \cap R_i| \geq 1$. Let $r_i = |R_i|$ and $n_i = |D_i \cap R_i|$.

CLAIM 2.2. *For any $i \in [k]$, $|D_i| + \frac{r_i + \rho(H) - 1}{2} \geq \gamma(H)$.*

PROOF. If $r_i - n_i \leq \rho(H)$, then $r_i - n_i \leq \frac{r_i - n_i + \rho(H)}{2}$ and we see that $\gamma(H) \leq |D_i \cup R_i| \leq |D_i| + r_i - n_i \leq |D_i| + \frac{r_i - n_i + \rho(H)}{2} \leq |D_i| + \frac{r_i + \rho(H) - 1}{2}$.

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Suppose next that $r_i - n_i > \rho(H)$. If we let R be the set of vertices obtained from projecting $R_i - (D \cap R_i)$ onto H, then notice that there exist at least two vertices in R which are of distance less than 3. Call two such vertices x and y and note that if they are at distance 1, then any one of them dominates the other, if chosen in a dominating set. If x and y are at distance 2, then notice that there exists a vertex in H which, if chosen in a dominating set, dominates both x and y. This means that there exists a vertex in H which can be chosen to dominate at least 2 vertices of R. Pick such a vertex and remove its closed neighborhood from R. Repeat this procedure for the remaining vertices in R until there are at most $\rho(H)$ vertices in R. Finally choose the remaining vertices of R. Place all chosen vertices in a set S and notice that we chose at most $\frac{r_i - n_i - \rho(H)}{2} + \rho(H)$ vertices, which is $\frac{r_i - n_i + \rho(H)}{2}$. By construction, the set S dominates R. Let P_i be the projection of D_i onto H. We now have that $P_i \cup S$ dominates H, which gives

$$\gamma(H) \leq |P_i \cup S| \leq |P_i| + |S| \leq |P_i| + \frac{r_i - n_i + \rho(H)}{2} \leq |D_i| + \frac{r_i + \rho(H) - 1}{2}.$$

□

Note that

$$\sum_{i=1}^{k} r_i = \sum_{h \in V(H)} r^h \leq \sum_{h \in V(H)} |D \cap G^h|. \tag{2.1}$$

We now sum the inequality in Claim 2.2 over all values of i and obtain,

$$|D| + \frac{1}{2}\sum_{i=1}^{k} r_i + \frac{1}{2}k\rho(H) - \frac{1}{2}k \geq k\gamma(H).$$

Combining this with (2.1) we obtain

$$\frac{3}{2}|D| \geq k\gamma(H) - \frac{1}{2}(k\rho(H) - k).$$

Since $k \geq \gamma(G)$ this produces the bound

$$\gamma(G \,\square\, H) \geq \gamma(G)\frac{2\gamma(H) - \rho(H) + 1}{3}.$$

□

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