The Function Is Unsaturated*

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1 Opening

That there is a fundamental difference between objects and functions (among which are concepts) is among the most famous of Frege's mature views; it is the view encapsulated in the slogan that entitles this paper. It is also among the most puzzling of Frege's views. Commentators, we think it is fair to say, have by and large had very little idea what to make of it, perhaps with good reason. But we are going to suggest here that this doctrine is not only easy enough to understand, it is also in some sense so deeply embedded in contemporary logic and semantics that it is hard to imagine life without it. That is the reason for our perplexity: We do not understand the view Frege was opposing. To understand the doctrine of *unsaturatedness*, we must thus uncover its origin.

As we shall see, the notion of function with which Frege operates in 1879, in *Begriffsschrift*, is rather different from what we find in *Grundgestze* in 1893. The evolution of Frege's mature conception begins soon after the publication of the former volume, and is largely in place by 1882. What drives this development is Frege's confrontation with the work of George Boole. Ernst Schröder had argued in a scathing review of *Begriffsschrift* that Frege had simply replicated the work of the Boolean school—of which Schröder just happened to be the most prominent German member—in a new and excessively cumbersome notation:

> With the exception of what is said... about 'function' and 'generality' and up to [Part III], the book is devoted to the establishment of a formula language that essentially coincides with Boole's mode of presenting judgements and Boole's cal-

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> culus of judgements, and which certainly in no way achieves more. (Schröder, 1972, p. 221, emphasis removed)

John Venn—he of the Venn diagram—does not even mention Frege's notation for generality and simply dismisses Frege's system as clearly inferior to Boole's (Venn, 1972, p. 234). Both Schröder (1972, p. 220) and Venn (1972, p. 234) speculate that Frege was simply unfamiliar with Boole's writings, and they were probably right. As Terrell Bynum points out in the introduction to his translation of *Begriffsschrift*, Frege took no courses in logic as a student, and some of his claims about the originality of his own system reveal ignorance of Boole's work (Bynum, 1972, pp. 77–8).

But Frege's ignorance did not last for long: He wrote several papers over the next few years in which he compared his logic to those of Boole and his followers (Frege, 1979a,b, 1972b). As one would expect, Frege argues that Schröder has failed to appreciate the significance of his views about functions and generality and that his notation for generality is far more powerful than anything available to his opponents. But Frege's criticisms of the Booleans were not limited to this familiar point. It is from these other criticisms that the notion of unsaturatedness emerges.

2 Function and Argument in *Begriffsschrift*

In his mature period, Frege speaks of the distinction between function and object in broadly metaphysical terms. But Frege also regards the distinction between function and object as one that is central to logic, so much so that the very first section of *Grundgesetze*[1] is devoted to elucidating what it means for a function to be unsaturated. The distinction between function and object makes itself most clearly felt, however, in that functions, but not objects, are stratified into levels, a topic to which Frege devotes several sections of *Grundgesetze* (Frege, 1962, §§19, 21–24). In this discussion, Frege holds that functions are so different from objects that a function could not take both functions and objects as arguments.[2] A function's level is thus determined by the sort of argument it takes: A function is *first-level* if it takes objects as arguments; a

[1] There is an introductory section preceding this one. It is given the number zero in Furth's translation (Frege, 1964), but it has no number in the German original.

[2] We are speaking here only of monadic functions. Similar remarks of course apply to polyadic functions.

function is *second-level* if it takes first-level functions as arguments; and so forth.

An apparently similar distinction—between function and *argument*—is equally central to the logical theory of *Begriffsschrift*. It is in terms of it that Frege introduces the notion of quantification:

> In the expression of a judgement we can always regard the combination of signs to the right of $\vdash$ as a function of one of the signs occurring in it. If we replace this argument by a German letter and if in the content stroke we introduce a concavity with this German letter in it, as in
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> $$\vdash\!\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$$
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> this stands for the judgement that, whatever we may take for its argument, the function is a fact. (Frege, 1967, §11, emphasis removed)

The two sections of *Begriffsschrift* that immediately precede these remarks are devoted to the explanation of the very general notion of function that Frege is using here.

Although the distinction between function and argument is put to similar use in *Begriffsschrift* and *Grundgesetze*, Frege understood that distinction very differently in 1893 from how he had understood it in 1879. Frege mentions this fact himself in the introduction to *Grundgesetze*:

> . . . [T]he nature of the function, as distinguished from the object, is characterized more sharply here than in *Begriffsschrift*. From this results further the distinction between first- and second-level functions. (Frege, 1962, p. x)

Frege implies here that there was no distinction between levels in *Begriffsschrift*, and we shall see shortly that, indeed, there was not. Perhaps more interesting is Frege's remark concerning why there is no distinction between levels in *Begriffsschrift*: The distinction between function and object was not characterized sufficiently "sharply" there. But why would that have obscured the distinction of levels? Since the difference between first- and second-level functions is parasitic on the difference between their arguments, we will distinguish first- from second-level functions only if we have sharply distinguished functions from objects—only, that is, if functions differ so fundamentally from objects that it is impossible for a single (monadic) function to take both functions and objects as arguments. Frege's point is thus not that the distinction between function and object is drawn with more precision in *Grundgesetze* than in *Begriffsschrift*, though it certainly is. His point is that the distinction between function and object is *enforced* in his later work in a way that it was not in his early work. Or more strongly: Frege is telling us that there isn't really a distinction between function and object in *Begriffsschrift*, and accordingly that there is no distinction between levels, although there is a distinction between function and *argument*.

Familiarly, both what we would now call 'first-order' and what we would now call 'second-order' quantification appear to be available in Frege's formal language, both in *Grundgesetze* and in *Begriffsschrift*. In *Grundgesetze*, these two sorts of quantification are clearly distinguished, both by notation (miniscule versus majuscule gothic letters) and by the axioms that govern them: The axiom of universal instantiation comes in both a first-order form (Basic Law IIa) and a second-order form (Basic Law IIb).[3] The two sorts of quantification are separately introduced, as well: First-order quantification is introduced in §8; second-order quantification is not introduced until §20, and Frege's official statement of the meaning of his second-order quantifier does not appear until §24. It reads as follows:

> If after a concavity with a Gothic function-letter, there follows a combination of signs composed of the name of a second-level function of one argument and this [Gothic] function-letter, which fills the argument-places, then the whole denotes the True if the value of that second-level function is the True for every fitting argument [that is, for every function of appropriate type]; in all other cases, it denotes the False. (Frege, 1962, §24)

It is thus clear that Frege's explanation of second-order quantification depends upon the distinction between levels, and that is why the explanation has to wait until §24. The preceding sections contain a detailed explanation of the distinction between first- and second-level functions (Frege, 1962, §§21–23). At the very least, then, Frege cannot have understood the distinction between first- and second-order quantification in *Begriffsschrift,* where there is no distinction of levels, the same way he understood it in *Grundgesetze*, where there is.

[3] The rule of universal generalization, rule (5) in the list given in §48, does not need separate formulations, since Frege can speak quite generally of 'Gothic letters' and 'Roman letters', without specifying which sort of letter is at issue.

In fact, there is no distinction at all between first- and second-order quantification in *Begriffsschrift*. Frege's initial explanation of the quantifier in *Begriffsschrift*, partially quoted above, continues as follows:

> Since a letter used as a sign for a function, such as Φ in $\Phi(A)$, can itself be regarded as the argument of a function, its place can be taken, in the manner just specified, by a German letter. (Frege, 1967, §11)

That is to say, we can also write:

$$\vdash\!\overset{\mathfrak{F}}{\smile}\!-\, \mathfrak{F}(a)$$

In this passage, Frege is not suggesting that 'function quantification' is significantly different from 'argument quantification'. To the contrary, there is but one axiom of universal instantiation in the formal theory of *Begriffsschrift*, proposition 58:

$$\begin{array}{l} \vdash\!\!\!\!\!\top\!\!\!-\!\!\!-\!\!\!-\, f(c) \\ \quad\llcorner\!\overset{\mathfrak{a}}{\smile}\!-\, f(\mathfrak{a}) \end{array}$$

To a modern reader, this formula may appear to involve a first-order quantifier, but it does not. Frege is as happy to cite proposition 58 to justify inferences involving what we would regard as second-order quantifiers as he is to cite it to justify inferences involving what we would regard as first-order quantifiers.[4] Thus from proposition 58, we may infer:

$$\begin{array}{l} \vdash\!\!\!\!\!\top\!\!\!-\!\!\!-\!\!\!-\, f(a) \\ \quad\llcorner\!\overset{\mathfrak{F}}{\smile}\!-\, \mathfrak{F}(a) \end{array}$$

Frege regards the changes that have been made here as *substitutions*, and he would have indicated them as follows: We have replaced '$\mathfrak{a}$' with '$\mathfrak{F}$'; we have replaced '$f(\Gamma)$' with '$\Gamma(a)$'; and we have replaced 'c' with 'f'. Apparently, argument-symbols are being freely substituted for function-symbols and *vice versa*.

How can Frege enjoy such freedom in *Begriffsschrift*? Of his mature distinction between function and *object*, Frege wrote that it "is not made arbitrarily, but founded deep in the nature of things" (Frege, 1984c, op. 31). But concerning the distinction between function and *argument*, Frege insists that it "has nothing to do with the conceptual content [but]

[4] The actual examples in *Begriffsschrift* are needlessly complex for our purposes; see for instance Frege's instantiation of (60) just after (92).

comes about only because we view the expression [of that conceptual content] in a particular way" (Frege, 1967, §9). That is, what, on one way of viewing such an expression, we regard as a function, on another way of viewing it, we may regard as an argument (Frege, 1967, §10). Consider, for example, the expression "John swims". If we imagine "John" replaced by other expressions, then we are regarding "John" as the argument. But we may also imagine "swims" replaced by other expressions. Then we would be regarding it as the argument.[5]

Something similar is also true on Frege's mature view. The sentence "John swims", he would later hold, is most fundamentally composed of a name, "John", and a concept-expression, "ξ swims", where "ξ" indicates the 'incompleteness' that Frege then understood such expressions to have. But one can also regard the sentence as saying something like: Swimming is something John does. So to regard the sentence is—allowing ourselves an un-Fregean idiom for a moment—to take the original sentence's subject to be "ξ swims" and its predicate to be a 'second-level' concept-expression we might write "$\text{John}_x(\Phi x)$".[6] Here, the capital phi and bound variable 'x' together indicate what sort of incompeteness this expression has: Its argument-place must be filled by a first-level concept-expression. But this way of understanding what it means to treat "swims" as the argument cannot be how Frege understood it in *Begriffsschrift*: Doing so requries us to distinguish levels of functions in a way he simply doesn't then distinguish them. How, then, did Frege understand what it means to treat "swims" as the argument in 1879?

The most general statement of the distinction between function and argument in *Begriffsschrift* reads as follows:

> If in an expression. . . a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else. . . , then we call that part that remains invariant in the expression a function, and the replaceable part the argument of the function. (Frege, 1967, §9)

This explanation says quite plainly that functions are *expressions*, and similar passages can be found throughout *Begriffsschrift*. Nonetheless,

[5] Those who are bothered by the sloppiness about use and mention are congratulated and asked to be patient.

[6] Our notation here borrows from Frege's, which he introduces in *Grundgeseze* §25; here Frege is clearly anticipating λ-abstraction. The analysis gestures to that of Montague (1974).

we think it would be uncharitable to insist that Frege positively regarded functions as being expressions. What we can say is that Frege simply does not distinguish use from mention in *Begriffsschrift* at all clearly, and so tends to conflate functions with the expressions that name them. Perhaps the most charitable reading would note that, since the conceptual notation is intended transparently to represent functions and arguments, Frege's usage may be regarded as a transposition to the formal mode (Baker, 2001; May, 2011). On the other hand, however, in his exposition of Frege's work, Philip Jourdain mentions, in his list of "advances made by Frege from 1879 to 1893", that "the traces of formalism in the *Begriffsschrift* vanished: a function ceased to be called a name or expression" (Jourdain, 1980, p. 204). Frege himself commented extensively on Jourdain's piece, and many of his comments were included by Jourdain as (sometimes very long) footnotes. Given Frege's aversion to formalism, it seems unlikely that he would not have corrected Jourdain if he had regarded this remark as incorrect. Accordingly, in *Begriffsschrift*, Frege treats the distinction between function and argument as purely linguistic. In contrast, in his mature work, he regards the distinction between function and object as metaphysical.

Frege does not say explicitly what he regards as "that part that remains invariant" when "John" is imagined to vary in "John swims". In discussing examples, he tends to use gerunds and infinitives. Thus, he might have said that what remains fixed when "John" varies is "to swim" or "swimming". Frege does not use any notation in *Begriffsschrift* that would indicate any incompleteness in such an expression. On the contrary, gerunds and infinitives are *prima facie* complete in a way the finite form "swims" is not: Gerunds and infinitives occur as subjects in such sentences as "Swimming is exhausting" and "To swim is more difficult than to float"; the finite form cannot.[7]

What "remains invariant" when we vary "swims", then? The obvious thing to say is that, when "swims" is varied, what remains invariant is just "John". Nothing Frege says in *Begriffsschrift* contradicts this interpretation. The only relevant passage appears to be this one:

> Since the sign Φ occurs in the expression $\Phi(A)$ and since we can imagine that it is replaced by other signs, Ψ or X, which would then express other functions of the argument A, we

[7] Whether gerunds and infinitives are really incomplete is of course an empirical question. In linguistic theory, they are generally supposed to have lexically null pronominal subjects, as opposed to finite clauses.

> can also regard $\Phi(A)$ as a function of the argument Φ. (Frege, 1967, §10)

Frege simply does not say here what familiarity with his mature views would lead one to expect him to say: that, when we so regard $\Phi(A)$, the function is something *other* than A itself. But if he had held this view, surely he would have said so: It would have needed a great deal of explanation, the sort of explanation it gets in *Grundgesetze*. Frege's view in *Begriffsschrift* thus seems to have been that a sentence like "John swims" is composed of two parts, "John" and "to swim", each of which can be regarded either as argument or as function, with "to swim" being the function if "John" is the argument and *vice versa*. And so, indeed, we can see why Frege insisted that the distinction between function and argument "has nothing to do with the conceptual content [but] comes about only because we view the expression in a particular way" (Frege, 1967, §9).

The distinction between function and argument, as that distinction is used in mathematics, is every bit as fluid as the distinction we are attributing to the early Frege. Given any group G, for example, we may consider the set I_G of group isomorphisms on G: These are 1-1 functions on the underlying set that preserve the group operation; that is, if $+$ is the operation, we must have $\phi(a+b) = \phi(a)+\phi(b)$. Now taking composition as our operation, we may regard I_G as constituting a new group, a so-called permutation group. Permutation groups are of mathematical interest because the properties of a group's permutation group reflect properties of the original group in ways that can be systematically studied. On Frege's mature view, however, the permutation group is *not* a group in the same sense that the original group was a group: If the elements of the original group were objects, then the members of the permutation group are first-level functions, and so the group operations are first-level functions and second-level functions, respectively. That is not a natural view. The natural view is the one that reflects how mathematicians usually speak.[8]

There are, to be sure, differences between argument- and function-symbols in *Begriffsschrift*. When Frege is substituting something for a function-symbol—be it a name or a term—he always indicates the

[8] Frege would of course have said that, if we think of the isomorphism group as a group in the original sense, then we are taking its elements to be the value-ranges of the isomorphisms. And so this would be another example, he would have claimed, of mathematicians' tacit reliance upon his Basic Law V.

argument, thus: $f(\Gamma)$. One might compare this to his later convention of always writing "$f(\xi)$" rather than just f, so that the incompleteness of the function-symbol is indicated. The purpose of the capital gamma is, however, completely different. If we are going to replace a free variable "f" with a more complex expression, we need to indicate what the argument-places of that expression are to be: We cannot just say that "f" is to be replaced by "$g \to ha$", for it would not be clear, for example, whether "h" was a one- or two-place predicate. Hence Frege would have us say that we are replacing "$f(\Gamma)$" with "$f\Gamma \to ha\Gamma$", and now it is clear what is intended. Some such notational convention is obviously required if Frege is to indicate explicitly what substitutions he is making. What we are suggesting, however, is that Frege regarded it *merely* as a notational convention and so of no greater significance. One indication of this fact is that Frege only seems to think it necessary to indicate the argument-places of function-symbols when he is substituting something *for* a function-symbol: He does not indicates the argument-places of the function-symbol when he is substituting a function-symbol for a term. (That is why we said earlier, on page 5, that we were substituting 'f' for 'c', not '$f(\Gamma)$' for '$\Gamma(c)$'.)

The differences in how Frege understands quantification, early and late, run even deeper than has been indicated so far. In *Grundgesetze*, the quantifiers are themselves regarded as higher-level functions: The second-order quantifier is a third-level function; the first-order quantifier is a second-level function (Frege, 1962, §31). From the point of view of *Grundgesetze*, then, the first-order universal quantifier is but one among many second-level functions. The value-range operator is another, and so is the existential quantifier. Though Frege has no primitive symbol for it, it would be easy enough for him to define one, perhaps:

$$\frown^{\mathfrak{a}}\; F\mathfrak{a} \stackrel{df}{\equiv} \top\!\smile^{\mathfrak{a}}\!\top\; F\mathfrak{a}$$

Frege's mature view thus has much in common with how we understand quantifiers today, especially in light of the work on generalized quantifiers begun by Andrzej Mostowski (1957).

But this sort of view is wholly absent from *Begriffsschrift*, in which the purpose of the 'concavity' is conceived very differently.[9] Frege writes at the beginning of *Begriffsschrift*:

[9] We have a dim memory of having encountered this point elsewhere, perhaps in the work of Peter Geach.

> The signs customarily employed in the general theory of magnitudes are of two kinds. The first consists of letters, of which each represents either a number left indeterminate or a function left indeterminate. This indeterminacy makes it possible to use letters to express the universal validity of propositions, as in
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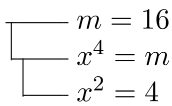
> $$(a + b)c = ac + bc$$
>
> The other kind consists of signs such as $+$, $-$, $\surd$, 0, 1, and 2, of which each has its particular meaning.
>
> *I adopt this basic idea of distinguishing two kinds of signs. . . in order to apply it in the more comprehensive domain of pure thought in general*. I therefore divide all signs that I use into *those by which we may understand different objects* and *those that have a completely determinate meaning*. The former are *letters* and they will serve chiefly to express *generality*. (1967, §1, emphasis in original)

It is important to read this afresh. What Frege is telling us is that '$(a + b)c = ac + bc$' is adequate, *on its own*, to express one form of the distributive law. What express generality here are the *letters* that occur in the formula. Generality is *not* expressed by the concavity. The concavity is necessary only because of cases like



of which Frege writes: ". . . the generality to be expressed by means of the x must not govern the whole. . . but must be restricted to" the antecedent of the outer conditional (Frege, 1979a, pp. 19–20). The sole purpose of the concavity is thus to "delimit[] the scope that the generality *indicated by the letter* covers" (Frege, 1967, §11, our emphasis). In that sense, then, the concavity itself has no independent meaning, and it is not a quantifier, but rather a syntactic scope indicator. Indeed, there are no quantifiers in *Begriffsschrift*. For the same reason, it would simply have been impossible, at that time, for Frege to introduce the upside-down concavity as a symbol for the existential quantifier. He could of course have introduced it as a kind of abbreviation, but there is a sense in which he could not have *defined* it. Generality is expressed by variables, and that generality is always universal.

3 From Function and Argument to Concept and Object

The familiar Fregean doctrine that functions differ fundamentally from objects is thus absent from *Begriffsschrift*. All we find there is the more basic logico-linguistic distinction between function and argument. The former distinction, however, is undoubtedly present in *Die Grundlagen*. One of the "three fundamental principles" Frege lists as shaping *Die Grundlagen* is "never to lose sight of the distinction between concept and object" (Frege, 1980a, p. x). Moreover, Frege explicitly distinguishes first- from second-order concepts in *Die Grundlagen*, including existence and 'oneness' among the second-order concepts (Frege, 1980a, §53).[10] And, as noted above, the distinction between first- and second-level functions is necessary only once we have sharply distinguished functions from objects, as Frege himself notes (Frege, 1962, p. x).

Frege does not use the language of 'unsaturatedness' or 'incompleteness' in *Die Grundlagen*, although it does figure prominently in his letter to Marty, written in August 1882: "A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own" (Frege, 1980b, p. 101). Frege remarks later in the letter that "... Kant's refutaton of the ontological argument becomes very obvious when presented in my way... " (Frege, 1980b, p. 102), foreshadowing his claim, in *Die Grundlagen*, that "[b]ecause existence is a property of concepts the ontological argument for the existence of God breaks down" (Frege, 1980a, §53). It would thus appear that both the doctrine that concepts are unsaturated and the distinction of levels were in place by 1882, just three years after the publication of *Begriffsschrift*. What happened?

The remark from the letter to Marty just cited, which contains Frege's earliest use of the term "unsaturated" (in the extant writings) occurs in the context of a lengthy explanation of his "distinction between individual and concept":

> ... [T]his distinction has not always been observed (for Boole only concepts exist). The relation of subordination of a concept under a concept is quite different from that of an individual's falling under a concept. It seems to me that logicians have clung too much to the linguistic schema of subject and predicate, which surely contains what are logically quite different relations. I regard it as essential for a concept that the question whether something falls under it have a sense. Thus I

[10] Frege's terminology changes over time: He uses "*Ordnung*" early and "*Stufe*" later.

> would call ‘Christianity’ a concept only in the sense in which it is used in the proposition ‘this (this way of acting) is Christianity’, but not in the proposition ‘Christianity continues to spread’. A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own. That an individual falls under the concept is a judgeable content, and here the concept appears as predicative and is always predicative. In this case, where the subject is an individual, the relation of subject to predicate is not a third thing added to the two, but it belongs to the content of the predicate, which is what makes the predicate unsatisfied. . . . In general, I represent the falling of an individual under a concept by $F(x)$, where x is the subject (argument) and $F(\ )$ is the predicate (function), and where the empty place in the parentheses after F indicates non-saturation. The subordination of a concept $\Psi(\ )$ under a concept $\Phi(\ )$ is expressed by
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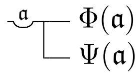
> 
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> which makes obvious the difference between subordination and an individual’s falling under a concept. Without the strict distinction between individual and concept, it is impossible to express particular and existential judgements accurately and in such a way as to make their close relationship obvious. For every particular judgement is an existential judgement.
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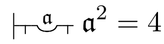
> 
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> means: ‘There is at least one square root of 4’.
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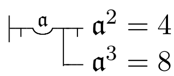
> 
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> means: ‘Some (at least one) cube roots of 8 are square roots of 4’. . . . Existential judgements thus take their place among other judgements. (Frege, 1980b, pp. 100–2)

We have quoted this passage at length to make it clear how wholly intertwined this early discussion of the distinction between concept and object—or, as Frege says here, concept and ‘individual’—is with fundamental questions in *logic*. Our task now is to understand what those questions are.

Given the manner in which Frege begins his remarks, it is tempting to read this passage in light of his earlier discussion in §3 of *Begriffsschrift*, where he famously insists that the distinction between subject and predicate is of no logical significance. But that discussion is limited to the contrast between active and passive voice: Frege tells us that logic need not represent the difference between “The Greeks defeated the Persians” and “The Persians were defeated by the Greeks”, the indifference he later labels ‘equipollence’. Frege does not suggest in *Begriffsschrift* that the subject–predicate form is actually ambiguous, that is, that “the linguistic schema of subject and predicate... contains what are logically quite different relations” (Frege, 1980b, p. 101). The topic here, then, is different, and that is because Frege has a new opponent: George Boole.

Boole[11] divides all judgements into two types. On the one hand, there are primary propositions, which express the sorts of relations between concepts studied in Aristotelian logic; on the other, there are secondary propositions, which concern the sorts of relations between judgements studied in sentential logic. The theory of the former is the ‘calculus of concepts’; the theory of the latter is the ‘calculus of judgements’. Given the dominance of this perspective in 1879, it is no surprise that Schröder, in his review of *Begriffsschrift,* should attempt to impose it on Frege’s system. Doing so, he concluded that “... Frege’s ‘conceptual notation’ actually has almost nothing in common with... the Boolean calculus of concepts; but it certainly does have something in common with... the Boolean calculus of judgements” (Schröder, 1972, p. 224).

In “Boole’s Logical Calculus and the Concept-script”, which Frege thrice submitted for publication, he argues in response that his notation for generality allows him to express everything that can be expressed in Boole’s calculus of concepts. But there is a more serious charge he wishes to bring against the Booleans:

> The real difference [between my system and Boole’s] is that I avoid such a division into two parts... and give a homogeneous presentation of the lot. In Boole, the two parts run alongside one another, so that one is like the mirror image of the other,

[11] Our rendition of Boole’s view is intended to represent Frege’s understanding of Boole, given that our present topic is the development of Frege’s views. How accurately Frege might have understood Boole is an interesting question, but not one for the present paper.

> but for that reason stands in no organic relation to it. (Frege, 1979a, p. 15)

The point here is partly aesthetic, but there is a logical point to be made, too.

Boole (and others) had tried to unify the treatment of primary and secondary propositions. Both the calculus of concepts and the calculus of judgements result from the imposition of an interpretation onto what is originally an uninterpreted formalism, a purely abstract algebra. For that reason, the two calculi are syntactically identical: Both contain expressions of the forms "$A \times B$" , "$A + B$", and "$\bar{A}$", for example.[12] In the calculus of concepts, the letters are taken to denote classes, or extensions of concepts, and the operations are then interpreted set-theoretically, in the now famliar way: Multiplication is intersection; addition is union;[13] the bar represents the relative complement. The formula "$A \times B = A$" then means: All A are B.

Precisely how the operations were to be interpreted in the calculus of judgements appears to have been a matter of some controversy, and Boole himself takes different views in *The Mathematical Analysis of Logic* (Boole, 1847) and *The Laws of Thought* (Boole, 1854). But, in both works, Boole takes the letters in this case, too, to denote classes. Schröder explains the view Boole held in the later work this way:

> ...[L]et 1 stand for the time segment during which the presuppositions of an investigation to be conducted are satisfied. Then let $a, b, c, \ldots$ be considered *judgements*... and at the same time, as soon as one constructs formulae or calculates (a small change of meaning taking place), the time segments during which these propositions are true. (Schröder, 1972, p. 224)

The virtue of this idea is that it allows for a reduction of the calculus of judgements to the calculus of classes, that is, of secondary propositions to primary propositions. To quote Boole:

> Let us take, as an instance for examination, the conditional proposition "If the proposition X is true, the proposition Y is

[12] The actual notation varies from logician to logician. We have used here something we hope will be familiar to modern readers.

[13] In some authors, it is something like a disjoint union, corresponding to exclusive disjunction.

> true". An undoubted meaning of this proposition is, that the *time* in which the proposition X is true, is *time* in which the proposition Y is true. (Boole, 1854, ch. XI, §5)

That is to say: All times at which X is true are times at which Y is true. The conditional proposition has thus become a universal affirmative proposition, and so "$A \times B = A$" now means: If A, then B.

It is clear enough both why and to what extent this idea works. The sentential operators are being treated as expressing set-theoretic operations on sets of times. The algebra so determined is of course a Boolean algebra, and so it satisfies the laws of classical logic. Now, it is surely safe to say that, just as Schröder had underestimated the importance of Frege's notion of generality, so Frege just as badly underestimated the importance of this parallel, that is, of the notion of a Boolean algebra: Frege has nothing positive to say about it. But we must surely also agree with Frege that the attempted reduction of sentential logic to quantification theory is a failure, and not only for the case of "eternal truths such as those of mathematics" (Frege, 1979a, p. 15).

Having rejected Boole's reduction, Frege then proceeds to turn the matter on its head, and "reduce [Boole's] *primary propositions* to the *secondary* ones" (Frege, 1979a, p. 17). The paradigmatic primary proposition is one expressing the subordination of one concept to another: Frege expresses such a judgement as a generalized conditional and thereby "set[s] up a simple and appropriate organic relation between Boole's two parts" (Frege, 1979a, p. 18).[14] Why the emphasis on the need to establish such an "organic relation"? Frege does not make his concern explicit, but it seems fairly obvious what is bothering him. Boole, he is implicitly claiming, cannot properly account for relationships between primary and secondary propositions: Boole's treatment does not, for example, reveal the relationship, clearly represented in Frege's system, between universal affirmative propositions and hypothetical judgements. As a consequence, Boole cannot account for the validity of inferences in which both primary and secondary propositions essentially occur. The simplest example of such an inference would, again, be that from a universal

[14] Frege remarks, in a similar spirit, that "[t]he precisely defined hypothetical relation between possible contents of judgement"—that is, the conditional—"has a similar significance for the foundation of my conceptual notation that identity of extensions has for Boolean logic" (Frege, 1979a, p. 16). (The reference to 'identity of extensions' reflects a feature of Boole's logic that is peculiar to his treatment: Frege might just as well have mentioned subordination.) One of the points Frege is making here is thus that sentential logic is more fundamental than predicate logic, a point to which we'll return.

affirmative proposition to a hypothetical judgement. In Frege's logic, the premise and conclusion of such an inference would be represented as

$$\vdash\!\smile^{\mathfrak{a}}\!\!\begin{array}{l}\Phi(\mathfrak{a})\\ \Psi(\mathfrak{a})\end{array}$$

and

$$\vdash\!\!\begin{array}{l}\Phi(a)\\ \Psi(a)\end{array}$$

respectively. In Boolean logic, one can represent them both as "$A \times B = A$". But the letters that occur in the two cases have nothing to do with one another: In one case, A is a concept; in the other, a set of times.

Having explained his reduction of primary propositions to secondary ones, Frege continues as follows:

> ...[O]n this view, we do justice to the distinction between concept and individual, which is completely obliterated in Boole. Taken strictly, his letters never mean individuals but always extensions of concepts. That is, we must distinguish between concept and thing, even when only one thing falls under a concept. In the case of a concept, it is always possible to ask whether something, and if so what, falls under it, questions which are senseless in the case of an individual. (Frege, 1979a, p. 18)

Frege does not make the connection between these remarks and the preceding ones terribly clear. In what way does Boole fail to respect the distinction between concept and object? How does Frege's view allow us to respect it? The connection is revealed by what Frege says next:[15]

> We must likewise distinguish the case of one concept's being subordinate to another from that of a thing falling under a concept, although the same form of words is used for both. The examples...
>
> $$\vdash\!\!\begin{array}{l}x^4 = 16\\ x^2 = 4\end{array}$$

[15] Between the previous quotation and this one, Frege gives a brief argument that we must distinguish concept from thing. The argument is of significant independent interest, but to consider it here would distract us from the point at issue. We have discussed it elsewhere (Heck and May, 2010, pp. 136–7).

> and
>
> $$\vdash 2^4 = 16$$
>
> show the distinction in the conceptual notation. (Frege, 1979a, p. 18)

Frege is alluding here to an aspect of Boole's logic that he does not explicitly mention but which would have been well-known to his contemporaries. Boole, as was then common, regards such propositions as "The sun shines" as expressing relations between concepts:

> To say, "The sun shines", is to say, "The sun is that which shines", and it expresses *a relation between two classes of things*, viz., "the sun" and "things which shine". (Boole, 1854, ch. IV, §1, our emphasis)

We can see, in retrospect, that Boole is attempting to reduce what we would now call 'atomic' propositions to universal affirmative propositions, that is, to one sort of primary proposition.[16] In a sense, he has no choice: Such propositions clearly are not secondary propositions—they express no relation between propositions—so there is nothing for them to be but primary ones.

From Frege's perspective, this treatment of atomic propositions is completely misconceived. His diagnosis of the problem is made most explicit in a passage from the letter to Marty, quoted earlier:

> The relation of subordination of a concept under a concept is quite different from that of an individual's falling under a concept. It seems that logicians have clung too long to the linguistic schema of subject and predicate, which surely contains what are logically quite different relations. I regard it as essential for a concept that the question whether something falls under it have a sense. (Frege, 1980b, pp. 100–01)

It should now be obvious what point Frege is making here, but it is worth spelling out explicitly. Frege is claiming that, whatever similarity of form there may be between "Dolphins are mammals" and "Flipper is a dolphin", it is a mistake to regard this similarity as logically significant: The relation between the subject and predicate in the first is very different from the relation between subject and predicate in the second. A

[16] Part of what lies behind Boole's failure, we suspect, at least in Frege's eyes, is a failure to distinguish classes from aggregates: Frege accuses Schröder of this conflation (Frege, 1984b).

proper treatment of the logic of these sentences will therefore require us to represent them differently, as Frege does in his conceptual notation.

The difference between subject and predicate is a topic Frege discusses in "Boole's Logical Calculus" on the pages just preceding the ones we have just been discussing ourselves. Boole, Frege says, takes concepts to be the basic building-blocks of logic and regards judgements as constructed from them. Frege, on the other hand, "start[s] out from judgements and their contents, not from concepts", and he explains how concepts are formed from judgements in a way reminiscent of his explanation of the distinction between function and argument in *Begriffsschrift*:

> If... you imagine the 2 in the content of possible judgement
>
> $$2^4 = 16$$
>
> to be replaceable by something else, by -2 or by 3 say, which may be indicated by putting an x in place of the 2:
>
> $$x^4 = 16,$$
>
> the content of possible judgement is thus split into a constant and a variable part. The former, regarded in its own right but holding a place open for the latter, gives the concept '4^{th} root of 16'. (Frege, 1979a, p. 16)

Frege goes on to explain that we may regard 4 as replaceable, rather than 2, or even in addition to 2, thus arriving at a different concept or at a relation. He then continues:

> And so instead of putting a judgement together out of an individual as subject and an already formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement. Of course, if the expression of the content of possible judgement is to be analysable in this way, it must already be itself articulated. We may infer from this that at least the properties and relations which are not further analysable must have their own simple designations. But it doesn't follow from this that the ideas of these properties and relations are formed apart from objects: on the contrary they arise with the first judgement in which they are ascribed to things. Hence, in the conceptual notation, their designations never occur on their own, but always in

> combinations which express contents of possible judgement. I could compare this with the behavior of the atom: we suppose an atom never to be found on its own, but only combined with others, moving out of one combination only in order to enter immediately into another. A sign for a property never appears without a thing to which it might belong being at least indicated, a desigation of a relation never without indication of the things which might stand in it. (Frege, 1979a, p. 17)

Frege's mature account of the distinction between concept and object is not quite present here. He does not use the term "unsaturated", for example, nor any equivalent, as he does in the letter to Marty. But the germ of that idea is present in the suggestion that a concept is what results when we vary an argument and regard what remains constant "in its own right but holding a place open for" the argument.

There are several other points to note about these remarks. One is that Frege is clearly moving away from his earlier view that the distinction between function and argument "has nothing to do with the conceptual content…" (Frege, 1967, §9). If it is to make any sense at all to speak of replacing the object "2 in the content of possible judgement $2^4 = 16$" with other objects, then the object 2 must itself occur in that content—it must somehow be a part of it—as must what remains constant when it is varied.[17] Another point is that Frege is no longer conflating functions with the expressions that denote them. On the contrary, he is carefully distinguishing the two and arguing that (what he would later call) the unsaturatedness of properties and relations has implications for the behavior of the expressions that denote them: It is *because* "the ideas of these properties and relations are [not] formed apart from objects" that "in the conceptual notation, their designations never occur on their own".

But the really crucial point is that this entire discussion, which constitutes the earliest appearance of something like the notion of unsaturatedness, occurs in a discussion of the differences between Frege's logic and the dominant logic of his day, which of course was Boole's. That is to say, the distinction between concept and object arises out of Frege's attempts to motivate and explain the crucial differences between these systems, as he understood them. If we want to understand the notion of

[17] It is a corollary of this point that Frege's insistence that we must begin with judgements rather than concepts does not express any view to the effect that judgements are intrinsically unstructured, as it is often taken to do.

unsaturatedness, then, what we need to understand is the *logical* point Frege is using it to make.

4 Unsaturatedness

What is that logical point? It has several aspects: Boole's secondary propositions are more fundamental than his primary ones; subsumption (an object's falling under a concept) is more fundamental than subordination (one concept's falling within another); judgements are more fundamental than concepts. These are the points to which Frege returns time and again in his discussions of Boole and out of which the distinction between concept and object arises. What underlies and unifies these various doctrines? The answer, we want to suggest, is something we now take largely for granted: From the standpoint of logical theory, the most basic sort of proposition is neither the primary proposition nor the secondary proposition but the *atomic* proposition; all other propositions are constructed from atomic propositions by means of certain syntactic operations.

Some of the operations by means of which propositions are constructed are common to Frege's and Boole's logics. Given some propositions, they may be related to one another in various ways: We may negate a proposition, form a conditional or disjunction from two propositions, or what have you. It is with respect to what Boole regarded as the primary propositions that disagreement arises. For Boole, such a proposition arises when we put concepts into relation with one another. In a sense, Frege does not disagree. But for Boole, concepts were logically primitive. Frege insists, by contrast, that to take concepts as primitive is to ignore one of the main questions an adequate logic must address, namely, how "true concept formation" is possible (Frege, 1979a, p. 35). Frege would have insisted, for example, that there is a straightforward sense in which the concept of a prime number is *not* primitive but derivative, or defined, and he would have expected Boole to agree. But the way this concept is constructed from other concepts is something Boole cannot explain. The concept of prime number is, in Sir Michael Dummett's apt phrase, *extracted from* such a judgement as

$$\forall x[\exists y(x \times y = 873) \rightarrow x = 1 \lor x = 873]$$

when we allow the argument 873 to 'become indeterminate'. To form the concept of a prime number thus involves perceiving a pattern in this judgement that it has in common with certain other judgements, such as:

$$\forall x[\exists y(x \times y = 26) \rightarrow x = 1 \vee x = 26].$$

And, according to Frege, this process of extraction is the key to an explanation of how scientifically fruitful concepts are formed (Frege, 1979a, p. 34).

But this non-Boolean mode of concept formation has a yet more basic role to play in Frege's logic: It is involved in almost every statement in which generality is expressed. For Frege, a universal affirmative proposition will take this sort of form:

$$\vdash\!\!\smile^{\mathfrak{a}}\!\!\begin{array}{l} \mathfrak{a}^3 = 8 \\ \mathfrak{a}^2 = 4 \end{array}$$

Such a formula, Frege tells us in *Begriffsschrift*, expresses "the judgement that, whatever we may take for its argument, the function is a fact" (Frege, 1967, §11, our emphasis). But what does Frege mean here by *the* function the formula contains? Isn't Frege's view in *Begriffsschrift* that the distinction between function and argument "has nothing to do with the conceptual content [but] comes about only because we view the expression in a particular way" (Frege, 1967, §9)? Well, yes, that is his view about some cases, but not about all:

> . . . [T]he different ways in which the same conceptual content can be considered as a function of this or that argument have no importance so long as function and argument are completely determinate. But if the argument becomes *indeterminate*, . . . then the distinction between function and argument acquires a *substantive* significance. . . . [T]hrough the opposition of the *determinate* and the *indeterminate*, the whole splits up into function and argument according to its own content, and not just according to our way of looking at it. (Frege, 1967, §9, emphasis in original)

Every general statement thus involves a particular function essentially. For example, the statement displayed above essentially involves the concept: *number whose cube is eight if its square is four*. Such functions cannot in general be primitive but must be formed by extraction. The just mentioned concept, for example, may be extracted from the sentence

$$\vdash\!\!\begin{array}{l} 5^2 = 16 \\ 5^2 = 4 \end{array}$$

by allowing the argument 5 to vary.

These remarks from *Begriffsschrift* once again conflate functions and the expressions that denote them. As we have seen, Frege quickly remedies that flaw. But, as we have also seen, Frege insists, from the moment he clearly distinguishes them, that *both* functions *and* the expressions that denote them[18] are in some sense incomplete. The obvious question is how these two sorts of incompleteness are supposed to be related.

Frege seems to answer this question three different ways. At the beginning, in 1881, his answer has a strikingly epistemological cast. The linguistic thesis that "[a] sign for a property never appears without a thing to which it might belong being at least indicated" is derived from the epistemological premise that ideas of properties "arise simultaneously with the first judgement in which they are ascribed to things" (Frege, 1979a, p. 17); if a metaphysical conception of unsaturatedness is present at all, it surfaces only in Frege's remarks about "the behavior of the atom", which are clearly intended as analogical. But things have changed already by 1882. In the letter to Marty, Frege's focus is on the metaphysical thesis that "[a] concept is unsaturated" and so "cannot exist on its own" (Frege, 1980b, p. 101). The epistemological doctrine that ". . . concept formation can[not] precede judgement. . . " (Frege, 1980b, p. 101) is present here, too, but it is not presented as fundamental. Rather, it is derived from the metaphysical thesis: "I do not believe that concept formation can precede judgement *because this would presuppose the independent existence of concepts*" (Frege, 1980b, p. 101, our emphasis).[19] By Frege's mature period, the epistemological thesis seems to have disappeared completely. We suggest, in fact, that Frege would then have regarded his earlier attempt to ground the distinction between concept and object in the priority of judgements over concept-formation as unacceptably psychologistic.[20]

[18] In his mature period, he will further insist that the senses of such expressions are also incomplete. What this might mean is a topic we have explored elsewhere (Heck and May, 2010).

[19] Even the linguistic thesis evolves between 1881 and 1882. Frege now indicates the fact that a predicate can occur only with an indication of its argument by using the notation: '$F(\ )$', "where the empty place in the parentheses after F indicates nonsaturation" (Frege, 1980b, p. 101). No such notation occurs in the (extant) papers on Boole.

[20] That Frege gives his distinction between concept and object an epistemological cast in 1881 may again be due to his reading of Boole, whose discussion of logic has, overall, a strongly psychologistic cast. Indeed, *The Laws of Thought* begins with the remark: "The design of the following treatise is to investigate the fundamental laws of those

Frege's mature view is different still. The direct way Frege tried to explain the incompleteness of concepts in 1881, by asking us to imagine replacing the number 2 in the content of the sentence '$2^4 = 16$', is no longer available to him once he has distinguished sense from reference: The content of the sentence is the thought it expresses, and objects simply do not occur in thoughts.[21] When Frege explains his view that functions are unsaturated in his mature period, then, what he explains first is always his view that functional *expressions* are unsaturated; he then explains the unsaturatedness of functions in terms of the unsaturatedness of the expressions that denote them.[22,23]

> ... [O]ne can always speak of the name of a function as having empty places, since what fills them does not, strictly speaking, belong to it. *Accordingly* I call the function itself unsaturated, or in need of supplementation, *because* its name has first to be completed with the sign of an argument if we are to obtain a meaning that is complete in itself. (Frege, 1979c, p. 119, our emphasis)

So, in the end, it is the unsaturatedness of the *expression* that is basic. The unsaturatedness of functions and concepts is to be explained in terms of the unsaturatedness of the expressions that denote them.

So we have two distinctions: There is the distinction between function and object, which is broadly metaphysical; and there is the distinction between a name and a predicate, which is essentially syntactic; the former is to be explained in terms of the latter. That can easily make it seem as if Frege is just conflating the incompleteness of predicates with the incompleteness of their denotations, but surely he is not: He draws this very distinction himself. Is the incompleteness of predicates

operations of the mind by which reasoning is performed... " (Boole, 1854, ch. I, §1). It is a nice question to what extent Boole is among Frege's targets in his anti-psychologistic rants.

[21] See the famous exchange about Mont Blanc and its snowfields in Frege's letter to Russell of 13 November 1904 (Frege, 1980b, p. 163) and Russell's reply of 12 December 1904 (Frege, 1980b, p. 169).

[22] One finds similar remarks in *Function and Concept* (Frege, 1984c, opp. 5ff), "On Concept and Object" (Frege, 1984d, opp. 194–95), and "What Is a Function?" (Frege, 1984e, opp. 665).

[23] The translation of the passage that follows has the last word of the first sentence being "them", as if it were anaphoric on "empty places". It is clear, however, that what Frege means is, as he puts it in *Function and Concept*, that "the argument does not belong with a function" (Frege, 1984c, op. 6).

simply being 'projected' onto their denotations, then? That would be unfortunate.

The answer, we want to suggest, is very simple. If the fact that predicates are unsaturated is to have any consequence whatsoever for the nature of what they denote—and, as we have said, in Frege's mature work, the unsaturatedness of concepts is explained in terms of the unsaturatedness of predicates—then surely such consequences must issue from the nature of the *connection* between predicates and what they denote, that is, from something about the *semantics* of predicates. The incompleteness of predicates manifests itself in the conceptual notation in the fact that predicates never appear without an argument's at least being indicated. If so, however, then any adequate account of the meaning of predicates must take note of this crucial fact about them: that they cannot occur without an appropriate argument.

One might therefore say that on Frege's view we do not need to answer the question what a predicate denotes, since a predicate can never occur on its own, anyway: We need only answer the question what the denotation is of the complete expression that is formed by inserting appropriate expressions into its argument-places. The semantic clause for "swims", then, should not take the form:

> "swims" denotes. . . ,

but rather:

> ⌜Δ swims⌝ denotes. . . ,

where Δ is a syntactic variable ranging over expressions that might occur as arguments.[24] This, indeed, is how Frege himself proceeds in *Grundgesetze*.[25] A Frege-inspired clause for "swims" might therefore

[24] More formally, Δ ranges over what may be called 'auxiliary names': We suppose that the language can always be expanded by the addition of a new name, whose reference may then be any object one wishes. Formally, a truth-definition using such a device requires us to quantify over languages that expand the original one (Heck, 1999). Frege uses some such device, and we have borrowed this use of Greek capitals from him. It is not clear, however, how Frege regarded these expressions, whose use he never explains. Sometimes, they seem to act like meta-linguistic variables ranging over objects; but then they also occur in quotation-marks, as in the semantic clause for identity in §7 of *Grundgesetze*, which suggests that they are substitutional variables. Auxiliary names let us have the best of both worlds.

[25] We will quote one of Frege's semantic clauses below, that for the horizontal. It is in no way exceptional. Regarding the other primitives, the clause for negation is in §6; identity, §7; the first-order universal quantifer, §8; the smooth breathing, §9; the definite article, §11; the conditional, §12; and the second-order universal quantifer, §24.

look like:

(1) ⌜Δ swims⌝ denotes the True iff,
for some x, denotes(Δ, x) and x swims.

But while clauses like (1) directly reflect the unsaturatedness of predicates, it is not clear what they imply about predicates' denotations, since they do not explicitly assign denotations to predicates at all. The most obvious way of doing so would be:

(2) The predicate "swims" denotes the concept *swimming*.

But that leads directly to the infamous problem of the concept *horse*.

One might want to deny that (1) assigns "swims" a denotation at all. But Frege would not agree: His semantics for quantification—especially for higher-order quantification—requires it to do so. And it is clear that Frege thought that a relation between a predicate and a concept was at least implied by (1). In §5 of *Grundgesetze*, for example, Frege explains the horizontal as follows:

> — Δ is the True if Δ is the True; on the other hand, it is the False if Δ is not the True,

and he takes this stipulation to be sufficient to assign the horizontal a function as its denotation, continuing: "Accordingly, — ξ is a function whose value is always a truth-value…".

The impression that (1) does not assign a denotation to "swims" is presumably a consequence of the fact that it does not take the form "This predicate denotes this concept". But, if one thinks it must take that form, then one is not thinking clearly about the logical structure of the relation of denotation itself. The relation that holds between a one-place predicate and its denotation is a relation of 'mixed level', taking as arguments an object—the predicate itself—and a concept: its denotation. So an expression denoting this relation must take as arguments a proper name denoting the predicate and a predicate denoting the concept. This predicate, being unsaturated, must occur with an argument, or at least the 'indication' of one, which is what we have in this case: the argument will be indicated by a bound variable. Thus, a 'denotation clause' for a predicate that is compatible with Frege's commitments must have the following form:

(3) denotes$_x$('ξ swims', x swims)

Now, suppose we formulate our semantic theory using clauses of this form rather than clauses like (1). To characterize the truth of an atomic sentence, we will then also need a compositional principle such as:

(4) $\ulcorner\Phi(\Delta)\urcorner$ denotes the True if, and only if,
for some ϕ and x, denotes$_x(\Phi(\xi), \phi x)$ and denotes(Δ, x) and ϕx.

We can now prove:[26]

(5) denotes$_x(\Phi(\xi), \phi x)$ iff, for every Δ, $\ulcorner\Phi(\Delta)\urcorner$ denotes the True iff, for some x, denotes(Δ, x) and ϕx.

It follows that (1) is indeed sufficient to determine the denotation of "swims", since (1) just is the right-hand side of the relevant instance of (5). It might therefore be thought that the question whether the semantics of predicates should be given by clauses like (1) or instead by clauses like (3) is of no real significance. We can take the latter as basic, in which case (3) and (4) obviously imply (1); or we can take (1) as basic, define denotation using (5), and then prove both (3) and (4). In that case, we could still regard (1) as assigning a denotation to "swims" as directly as it is possible to assign one, since, as already noted, (1) is the right-hand side of an instance of (5).

From Frege's perspective, however, the question whether (1) or (3) is more fundamental is critical. Recall the following remarks from the letter to Marty:

> A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own. That an individual falls under it is a judgeable content, and here the concept appears as predicative and is always predicative. In this case, where the subject is an individual, the relation of subject to predicate is not a third thing added to the two, but it belongs to the content of the predicate, which is what makes the predicate unsatisfied. (Frege, 1980b, p. 101)

Frege speaks here of "this case, where the subject is an individual". What is the other case, in which it is not? The contrast, as we have seen, is

[26] For the proof, we also need a principle stating that every predicate denotes at most one concept: denotes$_x(\Phi\xi, \phi x) \wedge$ denotes$_x(\Phi\xi, \psi x) \rightarrow \forall x(\phi x \equiv \psi x)$. But we need such a principle anyway, since we'd otherwise not be able to prove, say, that "$0 = 1$" is false: For that argument, we need to know that "$=$" denotes *only* the relation of identity. With this principle in place, we could then introduce an expression true-of(t, y), read 't is true of y', as equivalent to: $\exists F($denotes$_x(t, Fx) \wedge Fy)$.

between Frege's position and Boole's. The other case is thus the one traditional logic takes as fundamental, the case in which the subject is itself a concept. So this is a form of Frege's claim is that "the linguistic schema of subject and predicate... contains what are logically quite different relations" (Frege, 1980b, p. 101). The relation that is present when the subject is an individual is the one he calls "falling under"; the relation that is present when the subject is a concept is the one he calls "subordination". And in a proposition expressing subordination, Frege is insisting, the relation between subject and predicate *is* a "third thing added to the two", so it is something an adequate logical theory must make explicit. In the conceptual notation, the relation of subordination is of course represented as:

$$\begin{array}{l} -\!\overset{\mathfrak{a}}{\smile}\!\!\top\, \Phi a \\ \quad\;\;\llcorner\, \Psi a \end{array}$$

Part of what Frege is claiming here is thus, again, that 'atomic' sentences are what are fundamental for logic. As he writes about a decade later: "The fundamental logical relation is that of an object's falling under a concept: all relations between concepts can be reduced to this" (Frege, 1979c, p. 118).

If atomic sentences are *truly* fundamental, however, then they cannot assert the existence of a *relation* between the subject and the predicate. The correct analysis of "Bob swims" is not: falls-under(S, b): That is, in effect, simply a version of the traditional view. The correct analysis is just: $S(b)$. That is the sense in which a concept must contain the relation of predication within itself. But if we take the semantics of predicates to be given by clauses like (3), then we are *not* treating "the relation of subject to predicate" as something that "belongs to the content of the predicate". On the contrary, it is a "third thing" that must be "added to the two", and what must be added is made explicit by (4), which treats predication as a relation between the denotation of the predicate and the denotation of the subject. Frege's preference is thus for (1), which makes predication 'internal' to the concept.

In what sense is the denotation Frege assigns to "swims" incomplete, then? As we have seen, Frege uses the notion of incompleteness in an effort to explain certain crucial respects in which his logic differs from Boole's. One can try to press the notion into service here, too: What is assigned as the predicate's denotation is that part of the content expressed by (1) that is specific to the case of "swims" and that remains constant as the argument is varied, that is, that part expressed by "ξ

swims"; it is unsaturated because its argument is missing. But if the metaphor now seems to be doing no useful work, perhaps that is because its work is done, because it is no longer needed.

Frege says explicitly that these "figures of speech" are intended to play only a heuristic role (Frege, 1984d, p. 194, op. 205): He uses them when he is struggling to explain what he means by a 'concept' and, in particular, when he is trying to explain what was then a new understanding of concepts, different from the traditional one. But we hardly need such an explanation now. If there is something *we* need help understanding, it is not Frege's notion of a concept but Boole's. More seriously: We no longer need the metaphor of incompleteness because the claim that concepts are 'incomplete' is far more adequately expressed by the semantic thesis that the meaning of a predicate should be given by stating the meaning of an arbitrary atomic sentence in which it occurs, that is, that the proper form for a semantic clause governing a predicate is (1). The same goes for Frege's thesis that predicates are incomplete. We no longer need that metaphor because that thesis is more adequately expressed by the syntactic doctrine that a predicate must always occur with its argument.

5 Conclusion

In the Introduction to *Grundgesetze*, Frege describes the "progress" that he has made on the project he "had in view as early as my *Begriffsschrift* of 1879 and announced in my *Grundlagen der Arithmetik* of 1884" (Frege, 1962, p. viii): the reduction of arithmetic to logic. On his list of areas of significant progress is the understanding of "the nature of functions", which are now "characterized more precisely" by sharply distinguishing functions from objects. The result of this precision, Frege observes, is that functions can be stratified into levels, and that this result can be generalized to the case of primary logical interest by identifying concepts with functions whose values are truth-values. The notions of unsaturated and saturated, and of falling-under and falling-within, arise as aspects of Frege's nomenclature for elucidating these characteristics of functions.

As we have seen, Frege came by this elaboration through his engagement with Boolean logic. As Frege viewed the dispute, it was over a central point about logic: the primacy of atomic propositions. What blinded Booleans to this, on Frege's view, was their lack of appreciation of the predicative nature of such propositions. To Frege, it was essential that we understand the logical form of atomic propostions to be $P(a)$.

This logical form, composed of unsaturated and saturated parts, in itself represents predication. It would be a mistake, according to Frege, to think that some additional factor is needed to relate these parts predicationally. In this regard the Boolean analysis of atomic propositions is in error; so too would be analysis in Fregean terms as their having the logical form: falls-under(P, a). In neither way is predication inherently expressed, precisely because what is effaced is the distinction between concept and object: 'P' as it occurs here is just as much a saturated term as 'a'. We could, of course, give an analysis of the relation of falling under as predicational, but this is not Frege's concern. Rather, his concerns lie with the *semantics* of predication; this is what needs to be clearly articulated in order to understand the logical primacy of atomic propositions. What Frege is describing in *Grundgesetze* are the central advances in his thinking about functions that contribute to this goal.

For Frege, then, the function is unsaturated, but this notion only has significance in the context of predication; '$P(\xi)$' refers to a concept only when it occurs in the context '$P(a)$', with an argument at least indicated. Outside this context, by itself, it does not denote a function; it is an empty term, one that has no place in the conceptual notation. As early as his first explicit discussion of the matter in his letter to Marty, Frege insists that a term for a function has no denotation unless accompanied by a term for its argument. The generalization of this insistence is the Context Principle of *Die Grundlagen*, "never to ask for the meaning (*Beduetung)* of a word in isolation, but only in the context of a proposition", and Frege does not waver from this position, reiterating (and elaborating) it in *Grundgesetze*: "We can inquire about reference only if the signs are constituent parts of sentences expressing thoughts" (Frege, 1962, v. II, §97). Unsaturatedness is the notion through which Frege characterizes this composition: '$P(a)$' is composed of an unsaturated part '$P(\xi)$', whose reference in the context of '$P(a)$' is a concept, and a saturated part 'a', whose reference in that context is an object. And it is this notion of composition—predication—that sits at the core of Frege's conception of logic. It is only when we have recognized this structure, as it is represented in the conceptual notation, that we can appreciate the proper analysis of generality and so secure a notion of proof adequate to the needs of mathematics.

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