The Finite and the Infinite in Frege's *Grundgesetze der Arithmetik*

Richard G. Heck, Jnr.
Department of Philosophy
Harvard University

Published in M. Schirn, ed.,
Philosophy of Mathematics Today
Oxford and New York: Oxford University Press, 1998, pp. pp. 429-66

This is the manuscript on which the published version was based.
There may be, and almost certainly are, differences in the text.

0. Opening

Some of the best-known work on the foundations of mathematics done in the late nineteenth century was concerned with the concepts of the finite and the infinite, rigourous characterizations of which were famously given by both Cantor and Dedekind. It is little known, however, that, in his *Grundgesetze der Arithmetik*,[1] Frege also studied the notions of finitude and infinity. His work in *Grundgesetze* had little influence upon that of his contemporaries because, quite simply, almost no one appears to have read it. One obstacle was Frege's notation, which is notoriously difficult to learn to read (though easy enough to read, actually); following Russell's discovery that the formal system of *Grundgesetze* is inconsistent, mathematicians might justifiably have thought themselves to have had little reason to bother. Indeed, one might wonder why, given his system's inconsistency, the proofs Frege gives in it merit any serious study at all.

The answer, as I have argued elsewhere,[2] is that Frege's proofs depend far less upon the axiom responsible for the inconsistency than one might have supposed. For present purposes, the relevant axiom, Axiom V, may be taken to be:

$$\dot{\varepsilon}.F\epsilon = \dot{\varepsilon}.G\epsilon \text{ iff } \forall x(Fx \equiv Gx)$$

That is: The value-range (or extension) of the concept $F\xi$ is the same as that of the concept $G\xi$ just in case the Fs are exactly the Gs. Now, Frege uses value-ranges, and so Axiom V, throughout *Grundgesetze*, but he makes *essential* appeal to Axiom V only in the course of his proof of what may be called *Hume's Principle*, which states that the number of Fs is same as the number of Gs just in case the Fs can be correlated one-to-one with the Gs. Hume's Principle may be formulated, in second-order logic, thus:

---

[1]Gottlob Frege, *Grundgesetze der Arithmetik* (Hildesheim: Georg Olms Verlagsbuchhandlung, 1966). Further references are in the text, marked "*Gg*" with a roman numeral, for the volume number, and a section number. The first volume was published in 1893; the second, in 1903.

[2]See my "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*", *Journal of Symbolic Logic* 58 (1993), pp. 579-601, esp. §1, reprinted, with a postscript, in W. Demopoulos, ed., *Frege's Philosophy of Mathematics* (Cambridge MA: Harvard University Press, 1995).

$$\begin{aligned} Nx{:}Fx = Nx{:}Gx \text{ iff } (\exists R)[&(\forall x)(\forall y)(\forall z)(Rxy \mathbin{\&} Rxz \rightarrow y=z) \mathbin{\&} \\ &(\forall x)(\forall y)(\forall z)(Rxz \mathbin{\&} Ryz \rightarrow x=y) \mathbin{\&} \\ &(\forall x)(Fx \rightarrow (\exists y)(Gy \mathbin{\&} Rxy)) \mathbin{\&} \\ &(\forall y)(Gy \rightarrow (\exists x)(Fx \mathbin{\&} Rxy))] \end{aligned}$$

With the exception of Hume's Principle itself, the arithmetical theorems Frege proves in *Grundgesetze* can be, *and effectively are*, derived from Hume's Principle within second-order logic: That is to say, they are, in effect, proven in *Fregean Arithmetic*, that (standard, axiomatic) second-order theory whose sole non-logical axiom is Hume's Principle. Since Fregean Arithmetic is equi-consistent with second-order arithmetic,[3] all the important theorems concerning arithmetic which Frege proves in *Grundgesetze* are proven in a consistent sub-theory of his formal theory.

A large part of *Grundgesetze* is devoted to proofs of axioms for arithmetic, not the Dedekind-Peano Axioms—though each of them is indeed proven—but the axioms of Frege's own, somewhat different, axiomatization.[4] Frege's investigation of arithmetic does not, however, end with his proofs of these axioms. He goes on to prove a number of theorems concerning, for example, the infinite cardinal aleph-null, which he calls by the name "Endlos". Perhaps the most powerful of these results is that all structures satisfying Frege's axioms for arithmetic are isomorphic, a result analogous to one for which Dedekind's *Was sind und was sollen die Zahlen?* is celebrated.[5] Frege's efforts are also directed, however, towards the provision of characterizations of the finite and the infinite, characterizations which, as we shall see, are given in pure second-order logic, which do not rely even upon Hume's Principle.

---

[3] See George Boolos, "The Consistency of Frege's *Foundations of Arithmetic*", in *On Being and Saying: Essays in Honor of Richard Cartwright* (Cambridge MA: MIT Press, 1987), pp. 3-20, reprinted in Demopoulos, ed., *op. cit.*

[4] Frege's axioms are:

1. $\forall x \forall y \forall z(Sxy \mathbin{\&} Sxz \rightarrow y=z)$
2. $\neg\exists x[Nx \mathbin{\&} \mathscr{F}(S)(x,x)]$
3. $\forall x[Nx \rightarrow \exists y.Sxy]$
4. $Nx \equiv \mathscr{F}^{=}(S)(0,x)$

See below for the definitions of "$\mathscr{F}$" and "$\mathscr{F}^{=}$", which are the strong and weak ancestral, respectively. For an argument that these are Frege's axioms for arithmetic, see my "Development of Arithmetic", pp. 598-9, and "Definition by Induction in Frege's *Grundgesetze der Arithmetik*", *forthcoming* in M. Schirn, ed., *Frege: Importance and Legacy*, reprinted in Demopoulos, ed., *op. cit.*

[5] See Richard Dedekind, "The Nature and Meaning of Numbers", in his *Essays on the Theory of Numbers*, tr. by W.W. Beman (New York: Dover Publications, 1961), pp. 29-115, Theorem 132. For discussion of Frege's proofs, see my "Definition by Induction".

The main goal of this paper is to present Frege's proofs in a form in which they will be more accessible to potential readers. We shall also be trying to understand *why* he proves what he proves. The theorems proven in Part II are intended as proofs of what Frege calls, in its title, "The Basic Laws of Cardinal Number". The theorems we shall discuss do not, at first sight, look much like basic laws of cardinal number, and Frege does not spend much time, in *Grundgesetze*, explaining the importance of the results he proves. That does not, however, mean that his theorems do not each have some purpose, some significance, philosophical or mathematical, and it is implausible that the theorems included in what was intended to be the definitive statement and proof of Frege's logicism should be without such significance. Thus, though much of the present paper amounts to speculation about Frege's ulterior motives in *Grundgesetze*, it is well-motivated speculation.

We shall see that attention to what Frege proves, to how he proves it, and even to what he finds himself unable to prove can be fruitful. We shall see, for example, that Frege probably discovered, around 1892, that Dedekind's proof that every infinite set is Dedekind infinite depends upon an axiom of countable choice. This fact would, in itself, be of but passing interest were it not that Frege's interest in the axiom of choice vividly illuminates certain controversial aspects of his views, namely, his conception of logic and his views about the significance of formalization and axiomatization. For, *circa* 1897, Frege appears to have been thinking not only about the truth-value of the axiom he had discovered but also about its epistemological status. That Frege regarded it as an intelligible question whether the axiom of countable choice was a law of logic is inconsistent with the claim, advanced by some commentators, that he regarded all "meta-logical" discourse as non-sensical.

1. Definitions of Central Notions

In his proofs, Frege makes use of a number of defined notions. As most of the definitions are straightforward, and I have discussed them elsewhere, I shall present them with minimal commentary. Frege's definitions are not given in pure second-order logic, but instead make use of value-ranges. Here, however, they will silently be transformed into definitions in second-order logic (or in Fregean Arithmetic, in such cases as the operator "$Nx{:}\Phi x$" is needed). Later, when we discuss Frege's proofs, they too will be transformed into proofs in Fregean Arithmetic.

The first of Frege's definitions is that of the *converse of a relation* (*Gg* I §39):[6]

$\mathrm{Conv}_{\alpha\epsilon}(R\alpha\epsilon)(x,y) \equiv\mathrm{df}\ Ryx$

Thus, x stands in the converse of the relation $R\xi\eta$ to y if y stands in the relation $R\xi\eta$ to x. The second definition is that of a relation's being *functional*, i.e., function-like (*Gg* I §37):

$\mathrm{Func}_{\alpha\epsilon}(R\alpha\epsilon) \equiv\mathrm{df}\ \forall x\forall y\forall z(Rxy \,\&\, Rxz \rightarrow y=z)$

The third is that of a relation's *mapping* one concept into another (*Gg* I §38):

$\mathrm{Map}_{\alpha\epsilon xy}(R\alpha\epsilon)(Fx,Gy) \equiv\mathrm{df}\ \mathrm{Func}_{\alpha\epsilon}(R\alpha\epsilon) \,\&\, \forall x[Fx \rightarrow \exists y(Rxy \,\&\, Gy)]$

Thus, the relation $R\xi\eta$ maps the concept $F\xi$ into the concept $G\xi$ just in case $R\xi\eta$ is functional and every F stands in it to some (hence exactly one) G.[7] Additionally, we need Frege's famous definition of the *strong ancestral* of a relation (*Gg* I §45):

$\mathscr{F}_{\alpha\epsilon}(Q\alpha\epsilon)(a,b) \equiv\mathrm{df}\ \forall F[\forall x(Qax \rightarrow Fx) \,\&\, \forall x\forall y(Fx \,\&\, Qxy \rightarrow Fy) \rightarrow Fb]$

That is, *b follows after a in the Q-series* if, and only if, b falls under every concept (i) under which all objects to which a stands in the Q-relation fall and (ii) which is *hereditary in the Q-series,* i.e., under which every object to which an F stands in the Q-relation falls. Frege defines the *weak ancestral* as follows (*Gg* I §46):

$\mathscr{F}^{=}_{\alpha\epsilon}(Q\alpha\epsilon)(a,b) \equiv\mathrm{df}\ \mathscr{F}_{\alpha\epsilon}(Q\alpha\epsilon)(a,b) \vee a=b$

Thus, *b belongs to* (or *is a member of*) *the Q-series beginning with a* if, and only if, either b follows after a in the Q-series or b is identical with a.

Finally, we need Frege's definitions of specifically arithmetical notions. The number zero is defined as the number of objects which are not self-identical (*Gg* I §41; see *Gl* §74):[8]

$0 = Nx{:}x\neq x$

---

[6] I insert the bound variables in the definitions but will drop them when it causes no confusion to do so, for readability.

[7] Note that the definition does *not* say that $R\xi\eta$ is functional and is *onto* the Gs, but that $R\xi\eta$ is functional and that every F is mapped, by $R\xi\eta$, to some G. Indeed, one might well write "Map(R)(F,G)" as "R: F → G".

[8] References to Gottlob Frege, *The Foundations of Arithmetic*, tr. by J.L. Austin (Evanston IL: Northwestern University Press, 1980), are in the text, marked by "*Gl*" and a section number.

We say that *m (immediately) precedes n in the number-series* if, and only if, there is a concept Fξ, whose number is n, and an object y falling under Fξ, such that m is the number of Fs other than y (*Gg* I §43; see *Gl* §74):

$$\text{Pred}(m,n) \equiv\text{df } \exists F\exists y[n = Nx{:}Fx \;\&\; Fy \;\&\; m = Nx{:}(Fx \;\&\; x\neq y)]$$

The concept of a *finite* or *natural number* may then be defined as follows:

$$\mathbb{N}x \equiv\text{df } \mathscr{F}^{=}(\text{Pred})(0,x)$$

A number is finite, is a natural number, if and only if it belongs to the Pred-series (the number-series) beginning with zero. And the first transfinite cardinal may be defined thus (*Gg* I §122):

$$\infty =\text{df } Nx{:}\mathscr{F}^{=}(\text{Pred})(0,x)$$

'Endlos' is the number of natural numbers.

As was mentioned above, after completing the proofs of axioms for arithmetic, Frege turns to the proofs of theorems concerning the number Endlos. The most interesting of these constitute, together, a characterization of concepts whose number is Endlos. Frege's Theorem 207 is:

$$\exists Q[\text{Func}(Q) \;\&\; \neg\exists x.\mathscr{F}(Q)(x,x) \;\&\; \forall x(Gx \rightarrow \exists y.Qxy) \;\&\; \exists x\forall y(Gy \equiv \mathscr{F}^{=}(Q)(x,y))] \rightarrow Nx{:}Gx = \infty$$

Following Frege, we say that the Q-series is *simple* if (1) Q$\xi\eta$ is functional and (2) no object follows after itself in the Q-series. We say that the Q-series beginning with a is *endless* if, whenever x belongs to the Q-series beginning with a, there is an object onto which Q$\xi\eta$ maps x. Then Theorem 207 amounts to the claim that the number of Gs is Endlos if the Gs can be ordered as a series which is both simple and endless (or *simply endless*). Frege's Theorem 263 is the converse of Theorem 207. Putting these together, we thus have:

$$Nx{:}Gx = \infty \equiv \exists Q[\text{Func}(Q) \;\&\; \neg\exists x.\mathscr{F}(Q)(x,x) \;\&\; \forall x(Gx \rightarrow \exists y.Qxy) \;\&\; \exists x\forall y(Gy \equiv \mathscr{F}^{=}(Q)(x,y))]$$

That is: The number of Gs is Endlos *if and only if* the Gs can be ordered as a simply endless series. Note that the concept of a simply endless series, and so that of a concept the objects falling under which may be ordered as a simply endless series, is definable (because here defined) in second-order logic.

As we shall see, Frege's two main theorems concerning concepts whose number is finite are analogues of Theorems 207 and 263.

2. Frege's Characterization of Finitude

We turn now to Frege's characterization of finitude. To state it, we need an additional definition:

$\mathrm{Btw}_{\alpha\epsilon}(Q\alpha\epsilon;a,b)(x) \equiv_{df} \mathrm{Func}(Q) \,\&\, \neg\mathscr{F}(Q)(b,b) \,\&\, \mathscr{F}^{=}(Q)(a,x) \,\&\, \mathscr{F}^{=}(Q)(x,b)$

Frege reads "Btw(Q;a,b)(x)" as "x is a member of the Q-series running from a to b" (see *Gg* I §158). However, I prefer to read it as "x is between a and b in the Q-series" or, better, "x is Q-between a and b". So we say that x is Q-between a and b if, and only if: $Q\xi\eta$ is functional; b does not follow itself in the Q-series; and x belongs both to the Q-series beginning with a and to that ending with b.

The theorems of interest at present are Theorems 327 and 348. Theorem 327 is:

$\exists Q\exists x\exists y\forall z[Fz \equiv \mathrm{Btw}(Q;x,y)(z)] \rightarrow \mathscr{F}^{=}(\mathrm{Pred})(0,Nx{:}Fx)$

Theorem 348 is the converse of Theorem 327, so we have:

$\exists Q\exists x\exists y\forall z[Fz \equiv \mathrm{Btw}(Q;x,y)(z)] \equiv \mathscr{F}^{=}(\mathrm{Pred})(0,Nx{:}Fx)$

As we shall see below, the content of this theorem is, essentially, that the number of Fs is finite just in case the Fs can be ordered as a simple series which *ends*. Let us say that a concept is *Frege finite* under these circumstances. Note again that Frege finitude is definable (because so defined) in pure second-order logic.

To understand the point of Frege's so characterizing finitude, it is essential to look at his proofs of these two theorems. We begin our discussion, then, with his proof of Theorem 327, by far the more difficult of the two. Frege derives it from his Theorem 325:

$\mathscr{F}^{=}(\mathrm{Pred})[0, Nz{:}\mathrm{Btw}(Q;x,y)(z)]$

That is: The number of objects Q-between x and y is a natural number. Theorem 327 follows easily: If, for some $Q\xi\eta$, x, and y, the Fs just *are* the objects Q-between x and y, then certainly the number of Fs is a natural number.

Theorem 325 is a simple consequence of two lemmas, the first of which is Theorem 321:

$\mathscr{F}^{=}(Q)(x,y) \,\&\, \mathrm{Func}(Q) \,\&\, \neg\mathscr{F}(Q)(y,y) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})[0, Nz{:}\mathrm{Btw}(Q;x,y)(z)]$

The other lemma is Theorem 325,γ:

$\neg\{\mathscr{F}^{=}(Q)(x,y) \,\&\, \mathrm{Func}(Q) \,\&\, \neg\mathscr{F}(Q)(y,y)\} \rightarrow \mathscr{F}^{=}(\mathrm{Pred})[0, Nz{:}\mathrm{Btw}(Q;x,y)(z)]$

Theorem 325 follows by propositional logic. The proof of Theorem 325,γ is easy: If the antecedent holds, then *nothing* is Q-between x and y (by definition); so, by Frege's Theorem 97, the number of objects Q-between x and y is zero, which is a natural number.

The main work in the proof of Theorem 327 thus lies in the proof of Theorem 321. Frege's proof of this result is extremely interesting. One can perhaps best understand its significance by comparing it to a different proof, which is less complex, and rather shorter, than Frege's. This proof of Theorem 321 is, one should note, simple, utterly straightforward, and entirely obvious; indeed, since the antecedent of Theorem 321 includes a conjunct of the form "$\mathscr{F}^{=}(Q)(x,y)$", a proof by (logical) induction is what immediately suggests itself.[9] The induction is justified by Frege's Theorem 152 (of which mathematical induction is an instance):

$$\mathscr{F}^{=}(Q)(x,y) \ \&\ Fx \ \&\ \forall z \forall w[\mathscr{F}^{=}(Q)(x,z) \ \&\ Fz \ \&\ Qzw \rightarrow Fw] \rightarrow Fy$$

That is: If y belongs to the Q-series beginning with x, if x is itself F, and if Fξ is hereditary, as one might say, in the Q-series *beginning with x*, then y too is F. To prove Theorem 321, then, suppose that Qξη is functional. For Fξ in Theorem 152, we take: $\neg\mathscr{F}(Q)(\xi,\xi) \rightarrow \mathscr{F}^{=}(Pred)[0,Nz{:}Btw(Q;x,\xi)(z)]$. We must show that x falls under this concept, i.e., that

$$\neg\mathscr{F}(Q)(x,x) \rightarrow \mathscr{F}^{=}(Pred)[0,Nz{:}Btw(Q;x,x)(z)]$$

and that it is hereditary in the Q-series beginning with x, i.e., that

[9] To simplify the exposition of Frege's proofs, I shall make free use of what might be called *the basic facts* about the weak and strong ancestrals; these basic facts may be thought of as manifestations of the ancestrals' transitivity. (Special thanks here to George Boolos.) First, we have the following, which are easily derived from the definitions of the strong and weak ancestral:

$$\mathscr{F}^{=}(Q)(a,b) \ \&\ Qbc \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$
$$Qab \ \&\ \mathscr{F}^{=}(Q)(b,c) \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$

Note that the second conjunct of the consequent simply follows from the first. Moreover, for this same reason, we also have:

$$\mathscr{F}(Q)(a,b) \ \&\ Qbc \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$
$$Qab \ \&\ \mathscr{F}(Q)(b,c) \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$

All four of these also hold if 'Qξη' is weakened to '$\mathscr{F}(Q)(\xi,\eta)$, so we have:

$$[\mathscr{F}(Q)(a,b) \lor \mathscr{F}^{=}(Q)(a,b)] \ \&\ \mathscr{F}(Q)(b,c) \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$
$$\mathscr{F}(Q)(a,b) \ \&\ [\mathscr{F}(Q)(b,c) \lor \mathscr{F}^{=}(Q)(b,c)] \rightarrow \mathscr{F}(Q)(a,c) \ \&\ \mathscr{F}^{=}(Q)(a,c)$$

Note that the transitivity of the strong ancestral is a consequence of either of these. Finally, we have the transitivity of the weak ancestral:

$$\mathscr{F}^{=}(Q)(a,b) \ \&\ \mathscr{F}^{=}(Q)(b,c) \rightarrow \mathscr{F}^{=}(Q)(a,c)$$

For discussions of these results, and other results about the ancestral, see my "Development of Arithmetic" and "Definition by Induction".

$$\mathscr{F}^{=}(Q)(x,a) \;\&\; [\neg\mathscr{F}(Q)(a,a) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})[0,\mathrm{Nz}:\mathrm{Btw}(Q;x,a)(z)]] \;\&\; Qab \rightarrow \\ [\neg\mathscr{F}(Q)(b,b) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})[0,\mathrm{Nz}:\mathrm{Btw}(Q;x,b)(z)]]$$

Theorem 321 will then follow from Theorem 152.

For the former, we must show that the number of objects Q-between x and itself is a natural number. By Frege's Theorem 282, no object other than x is Q-between x and itself:[10]

$$\mathrm{Btw}(Q;x,x)(z) \rightarrow z=x$$

Hence, the number of objects Q-between x and itself is going to be a natural number: Namely, either zero or one, depending upon whether x is Q-between x and itself (which, actually, it is, given that $Q\xi\eta$ is functional and x does not follow itself in the Q-series).

We now show that the concept $\neg\mathscr{F}(Q)(\xi,\xi) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})[0,\mathrm{Nz}:\mathrm{Btw}(Q;x,\xi)(z)]$ is hereditary in the Q-series beginning with x. That is, we prove:

$$\mathscr{F}^{=}(Q)(x,a) \;\&\; [\neg\mathscr{F}(Q)(a,a) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})(0,\mathrm{Nz}:\mathrm{Btw}(Q;x,a)(z))] \;\&\; Qab \rightarrow \\ [\neg\mathscr{F}(Q)(b,b) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})(0,\mathrm{Nz}:\mathrm{Btw}(Q;x,b)(z))]$$

Suppose the antecedent. Suppose further that $\neg\mathscr{F}(Q)(b,b)$. Suppose, for reductio, that $\mathscr{F}(Q)(a,a)$. Then, we have Frege's Theorem 242, which is Theorem 124 of *Begriffsschrift*:[11]

$$\mathrm{Func}(Q) \;\&\; \mathscr{F}(Q)(a,x) \;\&\; Qab \rightarrow \mathscr{F}^{=}(Q)(b,x)$$

Since $\mathscr{F}(Q)(a,a)$ and Qab, Theorem 242 implies that $\mathscr{F}^{=}(Q)(b,a)$. But then $\mathscr{F}^{=}(Q)(b,a)$ and Qab, so $\mathscr{F}(Q)(b,b)$. Contradiction. Hence, $\neg\mathscr{F}(Q)(a,a)$.

By *modus ponens*, then, the number of objects between x and a in the Q-series is a natural number. To show that the number of objects Q-between x and b is a natural number, we show that the number of objects between x and b is the immediate successor of the number of objects between x and a,

[10]For the proof of Theorem 282, suppose that z is between x and itself in the Q-series and that $z \neq x$. By the definition of betweenness, $\mathscr{F}^{=}(Q)(x,z)$ and $\mathscr{F}^{=}(Q)(z,x)$; but $z \neq x$, so $\mathscr{F}(Q)(x,z)$, by the definition of the weak ancestral. But then $\mathscr{F}^{=}(Q)(x,z)$ and $\mathscr{F}(Q)(z,x)$, whence $\mathscr{F}(Q)(x,x)$. But that contradicts the supposition that z is Q-between x and itself, since, if $\mathrm{Btw}(Q;x,x)(z)$, then $\neg\mathscr{F}(Q)(x,x)$.

I shall confine the proofs of similarly unspectacular results to the footnotes.

[11]Gottlob Frege, *Begriffsschrift, a Formula Language, Modeled upon that of Arithmetic, for Pure Thought*, tr. by J. van Heijenoort, in J. van Heijenoort, ed., *Frege and Godel: Two Fundamental Texts in Mathematical Logic* (Cambridge MA: Harvard University Press, 1970).

whence the former is a natural number, since the latter is. That is, we show that, on the assumptions made so far:

Pred[Nz:Btw(Q;x,a)(z), Nz:Btw(Q;x,b)(z)]

To prove this, we employ Frege's Theorem 102

$Fb \ \& \ m = Nx{:}(Fx \ \& \ x \neq b) \rightarrow Pred(m, Nx{:}Fx)$

which follows from the definition of 'Pred(ξ,η)'. Instantiating the variable 'Fξ' with 'Btw(Q;x,b)(ξ)', and 'm' with 'Nz:Btw(Q;x,a)(z)', we have:

Btw(Q;x,b)(b) & Nz:Btw(Q;x,a)(z) = Nz:[Btw(Q;x,b)(z) & z≠b] →
Pred[Nz:Btw(Q;x,a)(z), Nz:Btw(Q;x,b)(z)]

We thus need only to establish the antecedent. The first conjunct is fairly obvious.[12] To prove the second conjunct, we show not only that the number of objects falling under Btw(Q;x,b)(ξ) & ξ≠b is the same as the number of objects falling under Btw(Q;x,a)(ξ), but that the objects falling under the former concept *just are* the objects falling under the latter, whence, by Frege's Theorem 96, the concepts have the same number.

For one direction, suppose that Btw(Q;x,a)(z); we must show that Btw(Q;x,b)(z) & z≠b. Since Btw(Q;x,a)(z), we have that $\mathscr{F}^{=}$(Q)(x,z) and $\mathscr{F}^{=}$(Q)(z,a), by definition. But Qab, so $\mathscr{F}^{=}$(Q)(z,b). Hence, z is between x and b in the Q-series (since ¬$\mathscr{F}$(Q)(b,b) and Func(Q), by hypothesis). So, suppose z=b. Then $\mathscr{F}^{=}$(Q)(b,a) and Qab; hence, $\mathscr{F}$(Q)(b,b), contradicting the assumption that ¬$\mathscr{F}$(Q)(b,b).

For the other direction, we show that, if Btw(Q;x,b)(z) and z≠b, then Btw(Q;x,a)(z). If Btw(Q;x,b)(z), then $\mathscr{F}^{=}$(Q)(z,b) and $\mathscr{F}^{=}$(Q)(x,z). Since, as argued above, ¬$\mathscr{F}$(Q)(a,a), we need only show that $\mathscr{F}^{=}$(Q)(z,a). We then have Frege's Theorem 243, which is closely related to Theorem 133 of *Begriffsschrift*:

Func(Q) & $\mathscr{F}^{=}$(Q)(x,a) & $\mathscr{F}^{=}$(Q)(x,z) → [$\mathscr{F}$(Q)(a,z) ∨ $\mathscr{F}^{=}$(Q)(z,a)]

Since $\mathscr{F}^{=}$(Q)(x,a), by hypothesis, and $\mathscr{F}^{=}$(Q)(x,z), either $\mathscr{F}$(Q)(z,a) or $\mathscr{F}^{=}$(Q)(a,z). Suppose that $\mathscr{F}$(Q)(a,z). Since Qab and Qξη is functional, $\mathscr{F}^{=}$(Q)(b,z), by Theorem 242. By hypothesis, b≠z, so $\mathscr{F}$(Q)(b,z). But then $\mathscr{F}$(Q)(b,z) and $\mathscr{F}^{=}$(Q)(z,b), so $\mathscr{F}$(Q)(b,b), contradicting the assumption that ¬$\mathscr{F}$(Q)(b,b). Hence, $\mathscr{F}^{=}$(Q)(z,a), and we are done.

[12]By definition: $\mathscr{F}^{=}$(Q)(x,b) & $\mathscr{F}^{=}$(Q)(b,b) & Func(Q) & ¬$\mathscr{F}$(Q)(b,b) → Btw(Q;x,b)(b).

That, then, completes the alternate proof of Theorem 321. We have made use of none of the theorems which Frege proves in the course of his proof of Theorem 321, other than certain very general facts about the ancestral and the notion of betweenness. Most importantly, however, we have not had to involve ourselves in the definition of any relations by induction, and we have not had to define any one-one correlations between objects falling under various concepts. As will shortly become clear, however, this proof, being a proof by brute force, has none of the interest of Frege's proof.

3. Frege's Method of Defining Functions by Induction

In order to discuss Frege's proof of Theorem 321, we need a few additional definitions, those required for the recursive definition of functions, *a la* Frege. Consider the definition of a function, by induction, on the natural numbers. Suppose given an object *a* and a function g(ξ); we wish to define a function φ(ξ), on the natural numbers, satisfying the recursion equations:

$$\varphi(0) = a$$
$$\varphi(Sn) = g(\varphi(n))$$

Consider, then, the following relation, which we may call Φξη:

$$\exists x \exists y \exists z \exists w[\xi=\langle x,z\rangle \ \& \ \eta=\langle y,w\rangle \ \& \ y=Sx \ \& \ w=g(y)]$$

An ordered pair stands in this relation to another just in case the first member of the latter is the successor of the first member of the former, and the second member of the latter is the value of g(ξ) for argument the second member of the former. Consider, then, the members of the Φ-series beginning with <0,*a*>: These are the ordered pairs <0,*a*>, <*S*0,g(*a*)>, <*SS*0,g(g(*a*))>, and so on. *Prima facie*, it would appear that the members of this series are exactly the ordered pairs we need to have in the extension of φ(ξ). And this is how Frege defines the (functional) relation η=φ(ξ), by means of the ancestral of the relation Φξη, viz.: $\mathscr{F}^{=}(\Phi)(\langle 0,a\rangle,\langle \xi,\eta\rangle)$.

More generally, Frege defines the *coupling* of relations Rξη and Qξη as follows:

$$(R\pi Q)(a,b) \equiv df \ \exists x \exists y \exists z \exists w[a=\langle x,y\rangle \ \& \ b=\langle z,w\rangle \ \& \ Rxz \ \& \ Qyw]$$

Thus, the coupling of Rξη and Qξη relates one ordered pair to another just in case the first member of the former bears Rξη to the first member of the latter, and the second member of the former bears Qξη to the second member of the latter. Obviously, in the definition of coupling, Frege makes use of ordered pairs, so these must be introduced into his system. For present purposes, we may take them as primitive, terms standing for ordered pairs being governed by the ordered pair axiom:[13]

$\langle x,y\rangle = \langle z,w\rangle$ iff $x=z$ & $y=w$

The central theorem on the recursive definition of functions on the natural numbers is Frege's Theorem 256, which is essentially:[14]

$$\begin{aligned}&\mathrm{Func}(Q) \,\&\, \forall x[\mathscr{F}^{=}(Q)(a,x) \rightarrow \exists y.Qxy] \rightarrow\\&\qquad \mathrm{Func}[\mathscr{F}^{=}(\mathrm{Pred}\ \pi\ Q)(\langle 0,a\rangle,\langle\xi,\eta\rangle)] \,\&\\&\qquad \forall x\{\mathscr{F}^{=}(\mathrm{Pred})(0,x) \rightarrow \exists y[\mathscr{F}^{=}(Q)(a,y) \,\&\, \mathscr{F}^{=}(\mathrm{Pred}\ \pi\ Q)(\langle m,a\rangle,\langle x,y\rangle)]\}\end{aligned}$$

That is: If $Q\xi\eta$ is functional, and if the Q-series beginning with a is endless, then the relation $\mathscr{F}^{=}(\mathrm{Pred}\ \pi\ Q)(\langle 0,a\rangle,\langle\xi,\eta\rangle)$ is functional and, strengthening the result slightly,[15] its *domain* is exactly the natural numbers and its *range* is wholly contained in the Q-series beginning with a. It is not difficult to see that, if $Q\xi\eta$ is taken to be $\eta=g(\xi)$, the relation so defined satisfies the recursion equations mentioned above.

4. Frege's Proof of Theorem 321

Because Frege's proof of Theorem 321 is both long and complex, I shall merely outline it here, noting those aspects of the proof which are of particular interest. A large portion of the proof will be confined to the appendix.

Recall that Theorem 321 is:

$\mathrm{Func}(Q) \,\&\, \neg\mathscr{F}(Q)(y,y) \,\&\, \mathscr{F}^{=}(Q)(x,y) \rightarrow \mathscr{F}^{=}(\mathrm{Pred})(0,\mathrm{N}z{:}\mathrm{Btw}(Q;x,y)(z))$

---

[13]Frege actually defines ordered pairs by means of value-ranges. It does not appear to be possible to define a *total* pairing function in Fregean Arithmetic: It is, of course, possible to define such a function on the natural numbers, but we have no assurance that the domain contains only natural numbers.

In any event, the uses Frege actually *makes* of ordered pairs may be eliminated, and he knew as much. For details, see "Definition by Induction". We need not concern ourselves with such complexities here, however.

[14]This theorem follows from a more general result, Frege's Theorem 254, on the recursive definition of functions on simple series in general. For discussion, see "Definition by Induction".

[15]The strengthening is justified by Frege's Theorem 232:

$\mathscr{F}^{=}(R\pi Q)(\langle m,a\rangle,\langle x,y\rangle) \rightarrow \mathscr{F}^{=}(R)(m,x) \,\&\, \mathscr{F}^{=}(Q)(a,y)$

That is: If $Q\xi\eta$ is functional, and if y is a member of the Q-series beginning with x which does not follow after itself, then the number of objects Q-between x and y is finite. The proof requires three lemmas, the first of which is Theorem 315:

$$\mathscr{F}^{=}(\text{Pred})(0,n) \ \& \ \mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x,1\rangle,\langle y,n\rangle) \ \& \ \text{Func}(Q) \ \& \ \neg\mathscr{F}(Q)(y,y) \rightarrow \\ \mathscr{F}^{=}(\text{Pred})[0, \text{N}z{:}\text{Btw}(Q;x,y)(z)]$$

That is: If n is finite, if <y,n> is a member of the (Q π Pred)-series beginning with <x,1>, if $Q\xi\eta$ is functional, and if y does not follow itself in the Q-series, then the number of objects Q-between x and y is finite. The second lemma is Theorem 317:[16]

$$\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x,1\rangle,\langle y,n\rangle) \rightarrow \mathscr{F}^{=}(\text{Pred})(0,n)$$

The third is Theorem 319:[17]

$$\mathscr{F}^{=}(Q)(x,y) \rightarrow \exists n.\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x,1\rangle,\langle y,n\rangle)$$

Theorem 321 is then an immediate consequence of these lemmas. Theorem 317 allows us to drop the assumption that n is finite from the antecedent of Theorem 315; Theorem 319 guarantees that an appropriate n exists, if y belongs to the Q-series beginning with x.

It is worth considering Theorem 319, whose point is not obvious, a little more carefully. Theorem 319 states that, if y belongs to the Q-series beginning with x, then, for some n, <y,n> belongs to the (Q π Pred)-series beginning with <x,1>. But what relation, exactly, is this relation $\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x,1\rangle,\langle \xi,\eta\rangle)$? Let us abbreviate it: $P(\xi,\eta)$. Consider, first, the case in which the Q-series beginning with x is simple. In this case, P(y,n) just in case y is the n^{th} member of the Q-series beginning with x. For let x_1=x. Then certainly, $\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x_1,1\rangle,\langle x_1,1\rangle)$, so $P(x_1,1)$, and x_1 is indeed the first member of the Q-series beginning with x_1. Suppose, then that Qx_1x_2. Since Qx_1x_2 and Pred(1,2), we have, by definition: $(Q \ \pi \ \text{Pred})(\langle x_1,1\rangle,\langle x_2,2\rangle)$. So, $\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x_1,1\rangle,\langle x_2,2\rangle)$; hence, $P(x_2,2)$. And so on. Now, if the Q-series is not simple, then, firstly, we can not say that P(y,n) just in case y is *the* n^{th} member of the Q-series beginning with x; at most, we can say that P(y,n) if y is *an* n^{th} member.[18] Moreover, in

[16] For the proof of this theorem, we need only Frege's Theorem 232, mentioned in note 17.

[17] The proof of Theorem 319 is by induction, the relevant concept being:
$$\exists n.\mathscr{F}^{=}(Q \ \pi \ \text{Pred})(\langle x,1\rangle,\langle \xi,n\rangle)$$

[18] That $P(\xi,\eta)$ is functional if the Q-series beginning with x is simple follows from Frege's Theorem 253: $\text{Func}(Q) \ \& \ \neg\exists y[\mathscr{F}^{=}(Q)(x,y) \ \& \ \mathscr{F}(Q)(y,y)] \ \& \ \text{Func}(R) \rightarrow \text{Func}[\mathscr{F}^{=}(Q \ \pi$

this case, members of the Q-series beginning with x need not be uniquely an n^{th} member, rather than an m^{th} member. Rather, we will have that P(y,n) just in case it is possible to get from x to y in n-1 "steps", that is, if, and only if, there is some sequence $x=x_1, x_2, ..., x_n=y$, where Qx_ix_{i+1}. This justifies our reading "$\mathscr{F}^{=}(Q\ \pi\ Pred)(<x,1>,<y,n>)$" as "y is an n^{th} member of the Q-series beginning with x" or, when justifiable, as "y is the n^{th} member of the Q-series beginning with x".[19]

Thus, what Theorems 319 and 317 together say is just this: If y belongs to the Q-series beginning with x, then, for some natural number n, y is an n^{th} member of the Q-series beginning with x. This theorem should reinforce one's confidence in Frege's definition of the ancestral, for y is, intuitively, an 'ancestor' of x just in case it is some finite number of 'steps' from x.[20]

Let us now return to the main line of argument. The main lemma in the proof of Theorem 321 is Theorem 315, which is, recall:

$$\mathscr{F}^{=}(Pred)(0,n)\ \&\ \mathscr{F}^{=}(Q\ \pi\ Pred)(<x,1>,<y,n>)\ \&\ Func(Q)\ \&\ \neg\mathscr{F}(Q)(y,y) \rightarrow \\ \mathscr{F}^{=}(Pred)[0, Nz{:}Btw(Q;x,y)(z)]$$

That is: If n is a natural number, if y is the n^{th} member of the Q-series beginning with x, if $Q\xi\eta$ is functional, and if y does not follow after itself in the Q-series, then the number of objects Q-between x and y is finite. Theorem 315 itself follows from two Lemmas, the first of which is Frege's Theorem 314:

$$\mathscr{F}^{=}(Pred)(0,n) \rightarrow n = Nz{:}Btw(Pred;1,n)(z)$$

$R)(<x,z>,<\xi,\eta>)]$.

[19]The foregoing can be formalized and proven, but these intuitive remarks should suffice for present purposes.

One of the differences between Frege's proof of the validity of definition by induction and Dedekind's is that Frege's proves the validity of recursive definitions of *relations*. The relation $\mathscr{F}^{=}(Q\ \pi\ Pred)(<x,1>,<\xi,\eta>)$ is a wonderful example of a useful and important *non-functional* relation naturally definable by Frege's methods. See "Definition by Induction", §4.

[20]Cf. Crispin Wright, *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press, 1983), pp. 159-60: "Does [Frege's definition] capture the intuitive meaning? ...[D]oes satisfaction of Frege's condition guarantee the accessibility of y from x by a series of [Q-]steps?" Yes, and provably so, though the reasoning required to show that it does is admittedly impredicative.

That is: Every natural number is the number of numbers Pred-between 1 and itself. The proof of Theorem 314 poses little difficulty.[21] The second lemma is Theorem 298:

$$\mathscr{F}^{=}(Q \,\pi\, \text{Pred})(\langle x,1\rangle,\langle y,n\rangle) \;\&\; \text{Func}(Q) \;\&\; \neg\mathscr{F}(Q)(y,y) \rightarrow \\ \text{Nz:Btw}(Q;x,y)(z) = \text{Nz:Btw}(\text{Pred})(1,n)(z)$$

That is: If y is the n^{th} member of the Q-series beginning with x, if $Q\xi\eta$ is functional, and if y does not follow itself in the Q-series, then the number of objects Q-between x and y is the same as the number of numbers Pred-between 1 and n.

The derivation of Theorem 315 is then straightforward. For suppose that n is finite, that y is the n^{th} member of the Q-series beginning with x, that $Q\xi\eta$ is functional, and that y does not follow itself in the Q-series. Then, by Theorem 298, the number of objects between x and y in the Q-series is the same as the number of numbers between 1 and n in the number-series. But, by Theorem 314, this number is n, which is, by hypothesis, finite.

The proof of Theorem 298, in turn, requires a number of lemmas, but the idea behind the proof is relatively straightforward. The main lemma is Theorem 288:

$$\mathscr{F}^{=}(R \,\pi\, \text{Pred})(\langle x,1\rangle,\langle y,n\rangle) \;\&\; \text{Func}(R) \;\&\; \neg\exists z.\mathscr{F}(R)(z,z) \rightarrow \\ \text{Nz:Btw}(R;x,y)(z) = \text{Nz:Btw}(\text{Pred})(1,n)(z)$$

Except for the change of variable, this result differs from Theorem 298 in only one respect: The condition, in Theorem 288, that no object follows itself is replaced, in Theorem 298, by the weaker condition that y not follow itself.

Theorem 298 is derived from 288 as follows. Suppose, as in the antecedent of Theorem 298, that y is an n^{th} member of the Q-series beginning with x, that $Q\xi\eta$ is functional, and that y does not follow itself in the Q-series. We now define, in terms of $Q\xi\eta$, a new relation, $R\xi\eta$, which will satisfy the antecedent of Theorem 288. The relation $R\xi\eta$ is to be $Q\xi\eta$ & $\mathscr{F}^{=}(Q)(\eta,y)$, the *restriction* of $Q\xi\eta$ to the

[21]The proof appeals to Frege's Theorem 160, which is:

$$\mathscr{F}^{=}(\text{Pred})(0,n) \rightarrow n = \text{Nx}:[\mathscr{F}^{=}(\text{Pred})(x,n) \;\&\; x\neq 0]$$

This in turn is a relatively easy consequence of the fact that n precedes $\text{Nx}:\mathscr{F}^{=}(\text{Pred})(x,n)$, if n is finite, which is Theorem 155. For $\text{Nx}:[\mathscr{F}^{=}(\text{Pred})(x,n) \;\&\; x\neq 0]$ precedes $\text{Nx}:\mathscr{F}^{=}(\text{Pred})(x,n)$, by Theorem 102, and $\text{Nx}:\mathscr{F}^{=}(\text{Pred})(x,n)$, like every number, has at most one predecessor.

Theorem 314 then follows from Theorem 313,ζ:

$$\mathscr{F}^{=}(\text{Pred})(0,n) \rightarrow \forall x\{\text{Btw}(\text{Pred})(1,n)(x) \equiv [\mathscr{F}^{=}(\text{Pred})(x,n) \;\&\; x\neq 0]\}$$

This, however, is an easy consequence of the definition of betweenness, the axioms of arithmetic, and Frege's Theorem 242, mentioned earlier.

Q-series beginning with y. To show that this relation satisfies the antecedent of Theorem 288, we thus have to prove:[22]

189*: $\mathrm{Func}(Q) \rightarrow \mathrm{Func}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)]$
290*: $\mathscr{F}^{=}(Q \;\pi\; \mathrm{Pred})(\langle x,1\rangle,\langle y,n\rangle) \rightarrow \mathscr{F}^{=}[(Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)) \;\pi\; \mathrm{Pred}](\langle x,1\rangle,\langle y,n\rangle)$
297: $\mathrm{Func}(Q) \;\&\; \neg\mathscr{F}(Q)(y,y) \rightarrow \neg\exists z.\mathscr{F}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)](z,z)$

From these results and Theorem 288, it then follows that the number of objects $[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)]$-between x and y is the same as the number of numbers Pred-between 1 and n. To complete the proof of Theorem 298, we thus need only show that the number of objects $[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)]$-between x and y is the same as the number of Q-between x and y:

295: $\mathrm{Func}(Q) \;\&\; \neg\mathscr{F}(Q)(y,y) \rightarrow$
$\mathrm{N}z{:}\mathrm{Btw}(Q;x,y)(z) = \mathrm{N}z{:}\mathrm{Btw}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y);x,y](z)$

As the proofs of these four lemmas are neither difficult nor terribly interesting, we leave them to the notes.[23]

---

[22] To simplify the exposition, I have here taken certain liberties. Theorem 189* actually follows from the more general result that, if $Q\xi\eta$ is functional, then, for any F, $Q\xi\eta \;\&\; F\eta$ is functional. Also, Frege's Theorem 290 is more general, '1' being replaced by 'm' and 'Pred' by 'R'.

[23] Theorem 189* is completely obvious. Any restriction of a functional relation is functional.

Theorem 290* is proven by induction, using Frege's Theorem 257, which is mentioned in the appendix, and the definition of coupling.

Theorem 297 follows from these two:

193: $\mathscr{F}[Q\xi\eta \;\&\; F\eta](x,y) \rightarrow Fy$
194: $\mathscr{F}[Q\xi\eta \;\&\; F\eta](x,y) \rightarrow \mathscr{F}(Q)(x,y)$

For suppose that z follows itself in the $[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)]$-series. Then, by (193), $\mathscr{F}^{=}(Q)(z,y)$; and, by (194), $\mathscr{F}(Q)(z,z)$. But following oneself is hereditary in any functional series (see Theorem 278), so y must also follow itself. Contradiction.

Finally, Theorem 295 follows from Theorem 295,ϵ (which is the theorem, marked 'ϵ', proved *during* the proof of Theorem 295):

295,ϵ: $\mathrm{Func}(Q) \;\&\; \neg\mathscr{F}(Q)(y,y) \rightarrow$
$\forall z\{\mathrm{Btw}(Q;x,y)(z) \equiv \mathrm{Btw}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y);x,y](z)\}$

The proof, from right-to-left, depends, for the most part, upon the following analogue of (194):

201: $\mathscr{F}^{=}[Q\xi\eta \;\&\; F\eta](x,y) \rightarrow \mathscr{F}^{=}(Q)(x,y)$

From left-to-right, it is a little more difficult. The crucial result is

292: $\mathscr{F}^{=}(Q)(x,d) \;\&\; \mathscr{F}^{=}(Q)(d,y) \rightarrow \mathscr{F}^{=}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)](x,d)$

Suppose that d is Q-between x and y. Then $Q\xi\eta$ is functional; hence, $Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)$ is. Moreover, $\neg\mathscr{F}(Q)(y,y)$, so $\neg\mathscr{F}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)](y,y)$, by (194). The antecedent of (292) holds, so we have that $\mathscr{F}^{=}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)](x,d)$; and, substituting into (292), we have

$\mathscr{F}^{=}(Q)(d,y) \;\&\; \mathscr{F}^{=}(Q)(y,y) \rightarrow \mathscr{F}^{=}[Q\xi\eta \;\&\; \mathscr{F}^{=}(Q)(\eta,y)](d,y)$

whose antecedent again holds. Done.

It remains to prove Theorem 288. Frege must show that, if the R-series is simple and if y is the n^{th} member of the R-series beginning with x, then some relation correlates the objects R-between x and y one-one with the numbers between 1 and n; one such relation is $\mathscr{F}^{=}(R\ \pi\ \mathrm{Pred})(\langle x,1\rangle,\langle\xi,\eta\rangle)$. What Frege proves is thus that, if the R-series is simple, and if y is the n^{th} member of the R-series beginning with x, then the objects R-between x and y are correlated one-one with the numbers from 1 to n by the relation: ξ is the η^{th} member of the R-series beginning with x. That this is so should seem overwhelmingly plausible. The proof is somewhat tortuous, however, and the details, while interesting, are not relevant to the central concerns of the present paper. I therefore leave Frege's proof of Theorem 288 to the appendix.

5. The Significance of Theorems 327 and 348

The differences between Frege's proof of Theorem 321 and that considered earlier are, in my judgement, not due to ignorance on Frege's part that there is a proof of Theorem 321 which is easier than his. Rather, Frege proves Theroem 321 as he does because he wants to establish a result somewhat stronger than Theorem 321 itself, a result which reveals the *intuitive* basis of his characterization of finitude.

Recall that the Q-series is *simple* if $Q\xi\eta$ is functional and if no object follows after itself in the Q-series. Now, the content of Theorems 207 and 263, as mentioned earlier, is that the number of a concept is Endlos just in case the objects falling under that concept can be ordered as a simple series which does not end. The content of Theorems 327 and 348, which Frege introduces as analogues, for the concept of finitude, is that the number of a concept is a natural number just in case "the objects falling under it can be ordered in a simple series which begins with a certain object and ends with a certain object", i.e., which contains a (unique) object after which no member of the series follows (*Gg* I §158).[24]

---

[24]This statement of the result is slightly stronger than what Theorems 327 and 348 themselves say, but the stronger claim follows easily from the weaker. To see this, recall that Theorems 327 and 348 together imply:

$$\exists Q\exists x\exists y\forall z[Fz \equiv \mathrm{Btw}(Q;x,y)(z)] \equiv \mathscr{F}^{=}(\mathrm{Pred})(0, Nz{:}Fz)$$

Plainly, if there is a concept $Q\xi\eta$ of the appropriate sort, there is such a concept whose domain and range consist only of Fs. (Consider $R\xi\eta \equiv_{df} Q\xi\eta \;\&\; F\xi \;\&\; F\eta$.) To simplify the exposition, we restrict attention to such concepts. If there is an F, then Func(Q) and $\neg\mathscr{F}(Q)(y,y)$, by the definition of 'Btw'. Moreover, that no object which precedes y in Q-series (so no F, so no object)

Given such an ordering of the Fs, we can show, essentially by brute force, that the number of Fs (the number of members of such an ordering) is a natural number, as we did in §2. That proof, however, does nothing to show, so to speak, *which* natural number is the number of objects so ordered. Frege's proof, on the other hand, rests upon the insight that, given such an ordering of the Fs, the Fs can then be mapped one-one onto an initial segment of the natural numbers: That is to say, it can be shown that the number of Fs is the same as the number of numbers between 0 and *n*, for some natural number *n*. It will then follow, from Frege's theorem on the infinity of the number series (Theorem 155), that the number of Fs is the successor of *n*.

As we have seen, however, Frege does not proceed in *quite* this way. His not so proceeding is our best indication of the role Theorems 327 and 348 play in *Grundgesetze* and were intended to play in his philosophy of mathematics. What Frege does, instead, is to produce a one-one mapping between the Fs and the numbers between *1* and a natural number *n*. On reflection, it is clear why Frege proceeds in this way. *One* way of producing a simple ordering of the Fs is to *enumerate* or *count* them, that is, to associate each of them, in turn, with a number, beginning with one and ending with some number *n*, which is then the number of Fs. Indeed, as was mentioned, the relation which Frege shows correlates the members of the relevant series with the numbers between 1 and n is the relation ξ is the η^{th} member of that series, so this correlation itself amounts to a 'counting' of the members of the Fs. The intuitive content of Theorems 327 and 348 is thus that the number of Fs is a natural number just in case the Fs can be counted and that, in that case, the number of Fs is the natural number one reaches *by* counting.

This is especially clear from the proof of Theorem 348. Recall that Theorem 348 is:

$$\mathscr{F}^{=}(\text{Pred})(0,\text{Nx:Gx}) \rightarrow \exists Q\exists x\exists y\forall z[\text{Gx} \equiv \text{Btw}(Q;x,y)(z)]$$

---

can follow itself in the Q-series, since following oneself in the Q-series is hereditary if $Q\xi\eta$ is functional: Hence, the Q-series is simple.

Finally, we show that no object follows after y in the Q-series. For *reductio*, suppose that $\mathscr{F}(Q)(y,z)$. By Frege's Theorem 242, mentioned earlier, z is in the range of $Q\xi\eta$ and so, by the simplifying assumption, is Q-between x and y; hence, $\mathscr{F}^{=}(Q)(z,y)$. But then $\mathscr{F}(Q)(y,z)$ and $\mathscr{F}^{=}(Q)(z,y)$, so $\mathscr{F}(Q)(y,y)$, contrary to hypothesis.

That is: If the number of Gs is a natural number, then the Gs can be ordered as a simple series which ends. This Theorem, like Theorem 321, admits of a direct and utterly uninteresting proof by induction.[25] Again, however, Frege's proof is more complicated than it needs to be and, therefore, more illuminating than it might have been. He derives Theorem 348 from two lemmas, the first of which is Theorem 347:

$$Nz{:}Gz = Nz{:}Btw(R;a,b)(z) \rightarrow \exists Q \exists x \exists y \forall z[Gx \equiv Btw(Q;x,y)(z)]$$

That is: If the number of Gs is the same as the number of objects R-between x and y, for some $R\xi\eta$, x, and y, then the Gs can be ordered as a simple series which ends. The other lemma is the previously mentioned Theorem 314:

$$\mathscr{F}^{=}(Pred)(0,n) \rightarrow n = Nz{:}Btw(Pred;1,n)(z)$$

Theorem 347 follows immediately. By Theorem 314, if the number of Gs is a natural number, then the number of Gs is the same as the number of numbers between 1 and the number of Gs; substitute into Theorem 347.

Now, Theorem 347 is a generalization of the following:

$$Nz{:}Gz = Nz{:}Btw(Pred;1,n)(z) \rightarrow \exists Q \exists x \exists y \forall z[Gx \equiv Btw(Q;x,y)(z)]$$

It is this which is really Frege's main goal in his proof of Theorem 348, not Theorem 347 (but Frege rarely passes up an oppotunity to prove the most general result possible). The proof proceeds by showing that, if the the numbers between 1 and n can be mapped one-one onto the Gs, then the Gs can be ordered as a simple series which ends; the relation which so orders the Gs will be the image of the predecessor relation under the relevant mapping. To correlate the Gs one-one with the numbers between 1 and n is, for all intents and purposes, to count them. So Theorem 348 amounts to this: If the number of Gs is finite, then they can be counted, that is, ordered as a simple series which ends.

Frege's proofs of Theorems 327 and 348 may thus be understood as falling into two parts. The hard work in the two proofs goes into establishing that the objects falling under a concept can be put in one-to-one correspondence with an initial segment of the counting numbers (the non-zero natural numbers) if, and only if, they can be ordered as a simple series which ends. (From left-to-right, this is

[25]For the base case, in which Nx:Gx = 0, take $Q\xi\eta$ to be any non-functional relation. For the induction step, suppose that, whenever the number of Fs is n, the Fs can be ordered as a simple series which ends; suppose, further, that the number of Gs is n+1. Let a be a G. The number of Gs other than a is n; so the Gs other than a can be ordered as a simple series which ends. Tack a onto the end. Done.

Theorem 347; from right-to-left, it is—roughly—Theorem 288.) From this intermediate result, Theorems 327 and 348 follow *via* Theorem 314, an immediate consequence of which is that the number of a given concept is finite if, and only if, the objects falling under it can be put in one-to-one correspondence with an initial segment of the counting numbers. Now, I do not want to deny that Theorems 327 and 348 have a purely *technical* point: They yield a nice characterization of finite sets. But the Theorems also have an epistemological point: They show that the concept of (Frege) finitude is a concept of *logic*; Frege would have regarded the characterization as *uncontroversially* logical, because purely second-order. And they have yet another point, for they reveal that the notion of finitude thus shown to be logical is compatible with, or is a rigorous version of, our *intuitive* conception of finitude: A finite set is one which can be counted. That is why Frege proves these theorems as he does, why both proofs go through Theorem 314.

Every philosophy of arithmetic must deal with the natural numbers' two aspects, their use as finite cardinals, and their use as finite ordinals. What is distinctive about Frege's development of arithmetic in *Grundgesetze* is that he takes the natural numbers fundamentally to be finite *cardinals*. The developments of arithmetic with which we are most familiar, on the other hand, take the natural numbers fundamentally to be finite ordinals. (It is for this reason that Dedekind's numbers begin, not with zero, but with one: There is no 'zeroth' position in an ordering and no ordinal 'zeroth', though of course there is a null order-type.) We are familiar with how the use of natural numbers as finite cardinals can be reconstructed from their use as finite ordinals. Indeed, Cantor, whose development of the theory of transfinite numbers was made possible by his taking the notion of an ordinal to be fundamental, showed how (at least part of) the theory of cardinal numbers could be reduced to the theory of ordinals: In the finite case, we say that the number of Fs is that finite ordinal *n* the ordinals less than which can be put into one-one correspondence with the Fs (here we are assuming that there is an ordinal number zero, but this is inessential). Theorems 327 and 348 are a kind of converse of this definition for finite cardinals: In these two theorems, Frege shows how the use of natural numbers as finite *ordinals* can be reconstructed from their use as finite *cardinals*.

The question of the relationship between finite cardinals and finite ordinals is, to my mind, of great importance, but it is not very often discussed. In part, this may be because it is thought, first, that Cantor showed that the cardinals could be reduced to the ordinals and, secondly, that the converse reduction can not be carried out. The reduction of finite cardinals to finite ordinals mentioned above is easily generalized: In set-theory, the cardinal number of a set x is typically taken to be the smallest ordinal number the ordinals less than which can be correlated one-one with the members of x. And there is no doubt that this provides for a reduction of (at least some) cardinals to ordinals. Frege's reduction of the finite ordinals to the finite cardinals, on the other hand, can not be so generalized. It is this which leads Dummett to remark that "if Frege had paid more attention to Cantor's work, he would have understood what it revealed, that the notion of an ordinal number is more fundamental than that of a cardinal number".[26]

Dummett is here expressing the view that the ordinals are *more* fundamental than the cardinals. And if the foregoing discussion of the significance of Theorems 327 and 348 is correct, Frege must have held the contrary view. But there appears to be no obvious reason to believe either the ordinals or the cardinals to be more fundamental. Indeed, Frege ought not to have held either of these views. According to him, a philosophical account of mathematical objects of a certain sort must not only be compatible with, but must from the outset *embody*, an account of the application of our theory of those objects. As Dummett puts the point, the foundations of the theory "must be so constructed as to display the most general form of [its] applications...".[27] It is on this sort of ground that Frege objects to Cantor's theory of the real numbers; presumably, Frege would have objected to the reduction of cardinals to ordinals on similar grounds, that it treats the use of natural numbers as cardinals not as "one of their distinguishing characteristics, which ought therefore to figure in their definition", but rather explains this use as one which could be made of *any* simply infinite system.[28] But the use of ordinals to answer questions of the form, "The whichth member of the ordering?" is as much one of *their* distinguishing characteristics as the use of cardinals to answer questions of the form "How many?" is one of theirs. One might therefore suppose that Frege was committed to developing the theory of ordinals on the basis of a principle, like

---

[26]Michael Dummett, *Frege: Philosophy of Mathematics* (Cambridge MA: Harvard University Press, 1992), p. 293. What Frege failed to understand may have been, more generally, the importance of the theory of ordinals.

[27]*Ibid*.

[28]See here *ibid*., pp. 47-51. The quotation is from p. 51.

Hume's Principle, which embodied this aspect of the ordinal numbers. Unfortunately, the most obvious way of doing so leads to the Burali-Forti Paradox, and it is a nice question whether a decent neo-Fregean theory of the ordinals can be constructed at all.

One should not, however, conclude from this that our only alternative is to reduce the cardinals to the ordinals (set-theoretically understood). For one might wonder whether the familiar reduction of cardinals to ordinals succeeds even on its own terms. I said earlier that one can reduce *at least some* cardinals to ordinals in the familiar way. What I had in mind was not that, without choice, not all sets can be shown to be equinumerous with the ordinals less than some ordinal (though one might, indeed, pursue that point): Rather, it seems to me that there is a cardinal number which is provably *not* the number of ordinals less than any ordinal, namely, the number of all the ordinals there are. Admittedly, the claim that there is such a number is widely rejected; indeed, Hume's Principle has been questioned on the ground that it entails that there is such a number.[29] But I know of no argument for the claim that there is no such number which does not beg the question—by implicitly appealing to the very reduction of ordinals to cardinals here under discussion (or to some similar claim).[30] There is nothing 'self-contradictory' about the claim that there is a number of all ordinals, or all sets, or all objects: That Fregean Arithmetic, and higher-order ZF+Hume's Principle, are consistent *implies* that there is nothing contradictory about it.

I conclude that, though Frege's reduction of ordinals to cardinals can not be generalized, his development of a theory of cardinals *independent* of the theory of ordinals is nevertheless worth our continued attention.

---

[29]George Boolos has frequently raised this worry in conversation; it is mentioned, though not endorsed, on p. 4 of his "Consistency".

[30]I should emphasize that the status of Zermelo-Fraenkel set-theory, or any other set-theory, as the proper response to the set-theoretic paradoxes is not presently in question; rather, the point is that *there are no set-theoretic paradoxes specifically concerning cardinal numbers.*

6. Finitude, the Least Number Principle, and Well-ordering

We have seen that Frege presents a purely second-order characterization of finitude in *Grundgesetze*. According to this characterization, a concept is Frege finite if, and only if, the objects falling under it can be ordered as a simple series which ends. Frege proves, in Fregean Arithmetic (*modulo* certain eliminable uses of ordered pairs) that a concept is Frege finite just in case the objects falling under it can be put in one-to-one correspondence with some initial segment of the natural numbers. His characterization is thus equivalent to Cantor's, not to Dedekind's. What I wish to discuss at this point is the importance, both to this characterization and to Frege's characterization of concepts whose number is Endlos, of the condition that the objects falling under the concept be ordered as a *simple* series of a certain form. This should help further to illuminate the relationship between Frege's characterization of finitude and more familiar ones.

Frege's most sustained discussion of the importance of the concept of simplicity occurs in a discussion of his Theorem 145, which states that no natural number follows after itself in the number-series. It is worth quoting this little-known discussion at length:[31]

> The importance of [Theorem 145] will be made more evident by the following considerations. If we determine the number belonging to a concept $\Phi(\xi)$, or, as one normally says, if we count the objects falling under the concept $\Phi(\xi)$, then we successively associate these objects with the number-words from "one" up to a number-word "*N*", which will be determined by the associating relation's mapping the concept $\Phi(\xi)$ into the concept "member of the series of number-words from 'one' to '*N*'" and the converse relation's mapping the latter concept into the former. "*N*" then denotes the sought number; i.e., *N* is this number. This process of counting may be carried out in various ways, since the associating relation is not completely determined.
>
> The question therefore arises whether, by another choice of this relation, one could reach another number-word '*M*'. Then, by our assumptions, *M* would be the same number as *N*, but, at the same time, one of these two number-words would follow after the other, e.g., '*N*' would follow '*M*'. Then *N* would follow in the number-series after *M*, which means that it would follow after itself. That is excluded by our proposition concerning finite numbers (*Gg* I §108).

Thus, it is, according to Frege, because the series of natural numbers is simple that the result of the process of counting is determinate.

Compare with this, then, Theorem 288:

$\mathrm{Func}(Q)\ \&\ \neg\exists z.\mathscr{F}(Q)(z,z)\ \&\ \mathscr{F}^{=}(Q\ \pi\ \mathrm{Pred})(\langle x,1\rangle,\langle y,n\rangle) \rightarrow$

[31]The translation of this passage is due to myself and Jason Stanley. I discuss this fascinating passage further in "Definition by Induction". It is, by the way, well worth reading this passage in conjunction with §56 of *Grundlagen*, on which it throws a great deal of light.

$$Nz{:}Btw(Q;x,y)(z) = Nz{:}Btw(Pred)(1,n)(z)$$

As said earlier, the first three conjuncts together imply that y is the n^{th} member of the Q-series beginning with x; it is essential to this that the Q-series be simple. Were the Q-series not simple, y might be not only an n^{th} but also an m^{th} member of the Q-series beginning with x: That is to say, there might be more than one way of correlating objects Q-"between" (though not in Frege's sense, of course) x and y with an initial segment of the counting numbers. Frege, in his discussion of Theorem 145, is concerned with the simplicity of the number-series *itself*. But the similarity between the importance of that fact and the importance of the simplicity of the Q-series in Theorem 288, is striking: It is because the Q-series is simple that its members have a well-defined ordinal position in it.

These reflections suggest a connection between the concept of a *simple* series and that of a *well-ordered* series, and such a connection there is. Eliminating a few definitions and contraposing, Frege's Theorem 359 is:

$$\begin{aligned}&Func(Q) \mathbin{\&} \neg\exists x.\mathscr{F}(Q)(x,x) \mathbin{\&} \exists x[\mathscr{F}^{=}(Q)(a,x) \mathbin{\&} Fx] \rightarrow \\ &\quad \exists x\{\mathscr{F}^{=}(Q)(a,x) \mathbin{\&} Fx \mathbin{\&} \neg\exists y[\mathscr{F}^{=}(Q)(a,y) \mathbin{\&} Fy \mathbin{\&} \mathscr{F}(Q)(y,x)]\}\end{aligned}$$

That is: If the Q-series is simple and if some F belongs to the Q-series beginning with *a*, then there is an F which belongs to the Q-series beginning with a which is ancestrally preceded in that series by no F; that is, there is some F in the Q-series beginning with a which is $\mathscr{F}$(Q)-*minimal*. Theorem 359 is thus a generalization, for all simple series, of the *least number principle*. Substituting 'Pred(ξ,η)' for 'Qξη' and '0' for 'a', we have:

$$\begin{aligned}&Func(Pred) \mathbin{\&} \neg\exists x.\mathscr{F}(Pred)(x,x) \mathbin{\&} \exists x[\mathscr{F}^{=}(Pred)(0,x) \mathbin{\&} Fx] \rightarrow \\ &\quad \exists x\{\mathscr{F}^{=}(Pred)(0,x) \mathbin{\&} Fx \mathbin{\&} \neg\exists y[\mathscr{F}^{=}(Pred)(a,y) \mathbin{\&} Fy \mathbin{\&} \mathscr{F}(Pred)(y,x)]\}\end{aligned}$$

But Pred(ξ,η) is functional (Theorem 71), and no natural number follows after itself in the Pred-series (Theorem 145). Thus, writing '$\mathbb{N}\xi$' for '$\mathscr{F}^{=}(Pred)(0,\xi)$' and '$\xi<\eta$' for '$\mathbb{N}\eta \mathbin{\&} \mathscr{F}(Pred)(\xi,\eta)$',[32] we have:

$$\exists x(\mathbb{N}x \mathbin{\&} Fx) \rightarrow \exists x[\mathbb{N}x \mathbin{\&} Fx \mathbin{\&} \forall y(y<x \rightarrow \neg Fy)]$$

And that is the least number principle.

---

[32]In general, '$\mathscr{F}(Pred)(\xi,\eta)$' does *not* mean that ξ is less than η, and it is highly confusing to read it that way, since $\mathscr{F}(Pred)(\infty,\infty)$ (and similarly for any Dedekind infinite cardinal—in fact, '$\mathscr{F}(Pred)(\xi,\xi)$' *is equivalent to* 'ξ is Dedekind infinite'.) However, '$\mathscr{F}(Pred)(\xi,\eta)$' is co-extensional with '$\xi<\eta$', if these relations are restricted to natural numbers.

Strengthening it slightly, Theorem 359, together with two other theorems, implies that, whatever $Q\xi\eta$ and a may be, if the Q-series beginning with a is simple, it is well-ordered by the ancestral of $Q\xi\eta$. The first of these other theorems is the previously mentioned Theorem 243, which states that the strong ancestral is connected on the Q-series beginning with a:

$$\mathrm{Func}(Q) \ \& \ \mathscr{F}^{=}(Q)(a,b) \ \& \ \mathscr{F}^{=}(Q)(a,c) \rightarrow [\mathscr{F}(Q)(b,c) \vee b=c \vee \mathscr{F}(Q)(c,b)]$$

The second auxiliary result is Theorem 275, which states the the strong ancestral is transitive:[33]

$$\mathscr{F}(Q)(x,y) \ \& \ \mathscr{F}(Q)(y,z) \rightarrow \mathscr{F}(Q)(x,z)$$

Now, the Fs are well-ordered by $\mathscr{F}(Q)(\xi,\eta)$ just in case (i) $\mathscr{F}(Q)(\xi,\eta)$ linearly orders the Fs[34] and, (ii) whenever some of the Fs are G, there is an $\mathscr{F}(Q)(\xi,\eta)$-minimal G which is F. But Theorems 243 and 275 together imply that, if the Q-series is simple, the members of the Q-series beginning with a are linearly ordered by $\mathscr{F}(Q)(\xi,\eta)$: For $\mathscr{F}(Q)(\xi,\eta)$ is transitive, by Theorem 275; irreflexive, since the Q-series is simple; and connected on the Q-series beginning with a, by Theorem 243. Moreover, by Theorem 359, every simple series satisfies the second condition. Thus, to repeat: Theorem 359, together with Theorems 243 and 275, implies that, if the Q-series beginning with a is simple, it is well-ordered by $\mathscr{F}(Q)(\xi,\eta)$.

There is thus, as was said, a close relationship between the concept of a simple series and that of a well-ordered series. For this reason, of the various characterizations of finitude given in the last hundred years, that most reminiscent of Frege's is due to Zermelo.[35] Say that the Fs are Zermelo finite if, and only if, the Fs can be well-ordered by a relation whose converse also well-orders them (if the Fs can, as is said, be *doubly well-ordered*). Frege's characterization is fairly easily seen to be equivalent to Zermelo's. The proof that, if the Fs are Zermelo finite, they are Frege finite is of no special interest at

---

[33]It is perhaps worth mentioning that these two theorems are the two main theorems of Part III of *Begriffsschrift*, Theorems 98 and 133, respectively. That these two results should become so important at this point is very suggestive.

[34]$Q\xi\eta$ linearly orders the Fs if, and only if, $Q\xi\eta$ is transitive and irreflexive and connected on the Fs (whenever Fx and Fy, either Qxy or Qyx or x=y).

[35]For a useful discussion of the history of definitions of finitude and infinity, see Charles Parsons, "Developing Arithmetic in Set Theory without Infinity: Some Historical Remarks", *History and Philosophy of Logic* 8 (1987), pp. 201-13. My understanding of Zermelo's work is largely due to this paper.

present.[36] For the other direction, it suffices to show that, if the Q-series beginning with a is simple and ends with b, then it is doubly well-ordered by $\mathscr{F}(Q)(\xi,\eta)$.

To simplify the exposition, let us assume the domain and range of $Q\xi\eta$ to contain only members of the Q-series beginning with a. By the foregoing, the Q-series beginning with a is well-ordered by $\mathscr{F}(Q)(\xi,\eta)$. Hence, we need only show that the members of the Q-series beginning with a are well-ordered by the converse of the ancestral of $Q\xi\eta$, i.e., by $\mathrm{Conv}[\mathscr{F}(Q)](\xi,\eta)$. Now, the converse of $\mathscr{F}(Q)(\xi,\eta)$ is just $\mathscr{F}(\mathrm{Conv}\ Q)(\xi,\eta)$, i.e., the ancestral of the converse of $Q\xi\eta$.[37] It will therefore suffice to show, first, that $\mathscr{F}(\mathrm{Conv}\ Q)(\xi,\eta)$ well-orders the Conv(Q)-series beginning with b and, secondly, that the members of the Conv(Q)-series beginning with b are the members of the Q-series beginning with a. To prove the latter claim, note that $\mathscr{F}^{=}(\mathrm{Conv}\ Q)(b,x)$ iff $\mathrm{Conv}[\mathscr{F}^{=}(Q)](b,x)$ iff $\mathscr{F}^{=}(Q)(x,b)$ iff $\mathscr{F}^{=}(Q)(a,x)$. To prove this last equivalence, note that, if $\mathscr{F}^{=}(Q)(x,b)$, then x is in the domain of $Q\xi\eta$, so $\mathscr{F}^{=}(Q)(a,x)$, by our simplifying assumption; and if $\mathscr{F}^{=}(Q)(a,x)$, then, since $\mathscr{F}^{=}(Q)(a,b)$, either $\mathscr{F}^{=}(Q)(x,b)$ or $\mathscr{F}(Q)(b,x)$, by Theorem 243; but, if $\mathscr{F}(Q)(b,x)$, then the assumption that the Q-series beginning with x can not end with b.

To prove the former claim, note first that $\mathscr{F}(\mathrm{Conv}\ Q)(\xi,\eta)$ is certainly a linear order. To show that it satisfies the well-ordering condition, we use Theorem 359 and need only show that the Conv(Q)-series is simple. Plainly, no object follows after itself in the Conv(Q)-series, since $\mathscr{F}(\mathrm{Conv}\ Q)(x,x)$ iff

[36]Suppose the Fs are doubly well-ordered by $\xi<\eta$. Consider the relation $Q\xi\eta$ which holds just in case η is a <-minimal object greater than ξ. $Q\xi\eta$ is functional, by linearity. No object follows itself in the Q-series, since, as can be established by induction, if $\mathscr{F}(Q)(x,y)$, then $x<y$, and $\xi<\eta$ is irreflexive. To complete the proof, one must show that all the Fs are Q-between the minimal element, say, a, of the <-ordering and the minimal element, say, b, of the >-ordering (the converse). Consider now the concept: $\forall x[x\leq\xi \rightarrow \mathrm{Btw}(Q;a,\xi)(x)]$; i.e., everything $\leq\xi$ is Q-between a and ξ. Call it $G\xi$. Plainly, a falls under $G\xi$. Since $\xi<\eta$ is a *double* well-ordering, there is a <-maximal object which is G, say, x. If $x\neq b$, then there is some maximal object <x, say, y. Hence, Qxy. But then y falls under $G\xi$, since everything <y is $\leq x$ and so Q-between a and x, and so Q-between a and y; and, of course, y itself is near-trivially Q-between a and y. So x=b. So b falls under $G\xi$. So everything $\leq b$ is Q-between a and b. So every F is Q-between a and b. Done.

[37]The two directions of this non-trivial result are Frege's Theorems 299 and 302, each of which is proven by (logical) induction.

Conv[$\mathscr{F}$(Q)](x,x) iff $\mathscr{F}$(Q)(x,x). That Conv(Q)(ξ,η) is functional follows from the following very general result about the ancestral, whose proof is confined to the notes:[38]

> Func(Q) & ¬∃x.$\mathscr{F}$(Q)(x,x) → ∀x∀y∀z[$\mathscr{F}^{=}$(Q)(a,x) & $\mathscr{F}^{=}$(Q)(a,y) & Qxz & Qyz → x=y]

That is: If the Q-series is simple, then the converse of Qξη, restricted to members of the Q-series beginning with any given object a, is functional. Since, by our simplifying assumption, the domain and range of Qξη contain only members of the Q-series beginning with *a*, the converse of Qξη is functional.

Frege's characterization of finitude is thus reasonably close to Zermelo's.

7. Frege and the Axiom of Choice

Frege offered no very complete account of the notions of finitude and infinity, though he did prove a number of important, preliminary results about them. Had Russell's Paradox not disrupted his work, Frege might well have continued his discussion of such matters later. I should like to speculate a bit about what, precisely, he might have discussed and argue that, probably no later than 1892, Frege had formulated at least the axiom of *countable* choice, if not the full axiom of choice, and that he was thinking both about its truth-value and about its epistemological status around 1897.[39]

About 1892, Frege became aware of Dedekind's proof that every concept which is not finite, in Frege's sense, is Dedekind infinite. He writes, in his review of Cantor's *Zum Lehre vom Transfiniten*:[40]

> ...Mr. Dedekind gives as the characteristic mark of the infinite that it is similar to a proper part of itself..., after which the finite is defined as the non-infinite, whereas Mr. Cantor tries to do what I have done: first to define the finite, after which the infinite

---

[38]Suppose that the Q-series is simple, that x and y belong to the Q-series beginning with a, and that both Qxz and Qyz. By Theorem 243, either $\mathscr{F}$(Q)(y,x), or x=y, or $\mathscr{F}$(Q)(x,y). Suppose that $\mathscr{F}$(Q)(y,x). Since Qyz, we have, by the previously mentioned Theorem 242, that $\mathscr{F}^{=}$(Q)(z,x). But then $\mathscr{F}^{=}$(Q)(z,x) and Qxz, so $\mathscr{F}$(Q)(z,z), contrary to the simplicity of the Q-series. Similarly, if $\mathscr{F}$(Q)(x,y), then $\mathscr{F}$(Q)(z,z). So x=y, and we are done.

[39]It is in the course of such speculations that one most laments the loss of Frege's *Nachlass* and, more specifically, Scholz's decision not to transcribe any of the formal material. This decision was, at the time, understandable, but still regrettable

[40]Gottlob Frege, "Review of Georg Cantor, *Zur Lehre vom Transfiniten*", in his *Collected Papers*, ed. by Brian McGuinness (Oxford: Blackwell, 1984), pp. 178-81, at p. 180, originally p. 271. The review was published in 1892, so that sets an upper bound on when Frege encountered *Was Sind?* And Frege writes, in the Introduction to *Grundgesetze*, composed in 1893, implies that the book had "lately come to [his] notice" (*Gg* I p. vii).

> appears as the non-finite. Either plan can be carried through correctly, and it can be proved that the infinite systems of Mr. Dedekind are not finite in my sense.[41] This proposition is convertible; but the proof of it is rather difficult....

This result would, indeed, have been of great interest to Frege. On the one hand, he had shown that a concept is of finite number if, and only if, the objects falling under it can be ordered as a simple series which *ends*; on the other, that a concept is of countably infinite number just in case the objects falling under it can be ordered as a simple series which does *not* end. It would be natural to wonder whether, and perhaps even more natural to *conjecture that*, every concept is either of finite number or has a countably infinite *subconcept*. (This is equivalent to the claim that every infinite concept is Dedekind infinite.) One might reason, informally, thus. Suppose we set out to build a simple series from objects falling under Fξ, successively choosing distinct Fs and adding them to the series so far constructed. Either we should at some point exhaust the Fs or we should not. If the former, the Fs would have been ordered as a simple series which ends. If the latter, would not *some* of the Fs *eventually* be ordered as a simple series which does not end? This proof would have suggested itself all the more readily to Frege, since it is by this *sort* of method that he establishes his Theorem 428, which states that, if every F is G, and if the Gs are countably infinite, then the Fs are either finite or countably infinite.[42]

The informal proof just given is, of course, hardly one with which Frege would have been prepared to settle. (Here we have exactly the sort of mix of reason and intuition which it was a large part of Frege's purpose to disentangle.) We know that he was not terribly happy with Dedekind's proof either, for what completes the above quotation from the review of Cantor are the words "and [the proof] is hardly executed with sufficient rigour in Mr. Dedekind's paper". Frege was right about this, for the theorem can not be proven without the axiom of (countable) choice, to which Dedekind tacitly appeals.

[41]This can be proven in Frege's system in much the same way as in *Was Sind?* In fact, it is the final theorem of Part II of *Grundgesetze*, Theorem 484. Indeed, since '$\mathscr{F}$(Pred)(ξ,ξ)' is equivalent to 'ξ is Dedekind infinite', Frege's Theorem 145 itself implies that no finite number is Dedekind infinite.

[42]Of course, the proof depends essentially upon the fact that countably infinite sets can be well-ordered. Theorem 359 is proved as a central lemma in the proof of Theorem 428.

Dedekind's proof is as follows.[43] Since the number of Fs is not a natural number, there is, for each natural number n, a one-one relation, call it $R_n(\xi,\eta)$, which correlates the numbers between 1 and n with some of the Fs. Now, say that $\varphi(x)$ just in case, for some n and y, $R_n(y,x)$. Every φ is obviously F, and the number of φs must surely be Endlos.

The problem with this proof certainly does not lie at the first step. It is not difficult to show, in Fregean Arithmetic, that[44]

$$\neg\mathscr{F}^{=}(\mathrm{Pred})(0, Nx{:}Fx) \rightarrow \forall n\{\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists G[\forall x(Gx \rightarrow Fx) \;\&\; n = Nx{:}Gx]\}$$

Given Frege's Theorem 314, mentioned earlier, it will then be easy to complete the first step by proving that

$$\begin{aligned}&\neg\mathscr{F}^{=}(\mathrm{Pred})(0, Nx{:}Fx) \rightarrow\\ &\quad \forall n\{\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists R[\mathrm{Func}(R) \;\&\; \mathrm{Func}(\mathrm{Conv}\,R) \;\&\\ &\qquad \forall x(\mathrm{Btw}(\mathrm{Pred})(1,n)(x) \rightarrow \exists y(Rxy \;\&\; Fy))]\}\end{aligned}$$

The difficulty lies in the next step, and it is Dedekind's terminology which misleads. We can, indeed, prove that, for each n, there is a relation which correlates the numbers between 1 and n with some of the Fs, and we may call this relation '$R\xi\eta$', '$R_n(\xi,\eta)$', 'ξ is greater than η', or whatever we wish. What we may not do is treat the subscript 'n' in '$R_n(\xi,\eta)$'—any more than the 'n' in 'ξ is greater than η'—as if it were a *variable*. More precisely, we may not suppose that 'n' here occupies an *argument-place*, and that is just what Dedekind tacitly supposes: In defining '$\varphi(x)$' to hold just in case, *for some n* and y, $R_n(y,x)$, he treats '$R_n(\xi,\eta)$' as if it were, not a two-place, but a three-place, predicate, 'n' filling the third argument-place; to put it a different way, he supposes that '$R_\zeta(\xi,\eta)$' is a functional expression, denoting a function from natural numbers to binary relations. A tacit, and crucial, inference is thus being made from

$$\begin{aligned}&\forall n\{\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists R[\mathrm{Func}(R\xi\eta) \;\&\; \mathrm{Func}(\mathrm{Conv}\,R\xi\eta) \;\&\\ &\qquad \forall x(\mathrm{Btw}(\mathrm{Pred})(1,n)(x) \rightarrow \exists y(Rxy \;\&\; Fy))]\}\end{aligned}$$

[43]See "The Nature and Meaning of Numbers", Theorem 159. Dedekind himself makes reference to a one-one *function*, not to a relation, but this is of no significance.

[44]The proof is, of course, by induction. For the case n=0, we may take $G\xi$ just to be $\xi \neq \xi$. Suppose then that every G is F, that the number of Gs is n, and that n is a natural number. Then some F is not G, since otherwise the number of Fs would be n, and so finite, contrary to hypothesis. Let a be such an F which is not G. Let $H\xi \equiv_{df} G\xi \vee \xi = a$. Every H is F, and the number of Hs is n+1. Done.

to

$$\forall n\{\mathscr{F}^{=}(\text{Pred})(0,n) \rightarrow \exists\Phi[\text{Func}(n\Phi\xi\eta) \,\&\, \text{Func}(\text{Conv } n\Phi\xi\eta) \,\&\, \forall x(\text{Btw}(\text{Pred})(1,n)(x) \rightarrow \exists y(n\Phi xy \,\&\, Fy))]\}$$

(Here, the variable 'Φ' ranges over ternary relations and is written as it is simply to emphasize that one should think of the ternary relation as if it were a function from objects to binary relations.) Given the validity of this inference, the remainder of Dedekind's proof can be carried out in Fregean Arithmetic.[45]

Frege's remark that Dedekind's proof "is hardly executed with sufficient rigour" strongly suggests that he had attempted to reproduce Dedekind's proof in his formal system and had failed. In formalizing the proof, Frege would quickly have discovered the gap, the inference upon whose validity Dedekind was tacitly relying: Its premise is what one can actually, and easily, prove in second-order arithmetic by following Dedekind's argument; its conclusion, what Dedekind, because of the way he speaks, takes himself to have proven. Now, Frege would probably have thought the inference plausible enough, but he would hardly have been content to let matters rest there. One would have expected him to investigate further, for example, to have attempted to establish the inference as a derived rule within his formal system. And so he did, for what appear to be fragments of this attempt survive in section O(micron) of Part II of *Grundgesetze*. This is a hodgepodge of results of the sort one might prove in attempting to make some progress towards a proof that every infinite set is Dedekind infinite. The very last theorem of Part II, Theorem 484, is the converse; Theorem 476 states that, if the number of Gs is Dedekind infinite, and every G is F, then the number of Fs is Dedekind infinite; and the remainder appear to form part of an attempt to prove Dedekind's theorem in something like the way sketched a few paragraphs back.

In the course of this investigation, Frege would naturally have considered the problematic inference in its general form. This form is obtained by generalizing the specific inference with respect to what is demanded of the relation whose existence is asserted, thus: If, for each natural number n, there is

[45]Showing that the number of φs is Endlos is not, as far as I can tell, as easy as one might have thought. The problem is that we need to define a one-one correlation between the φs and the natural numbers. One might just try mapping $\iota x.R_0(1,x)$ to 0, $\iota x.R_1(1,x)$ to 1, $\iota x.R_1(2,x)$ to 2, and so forth. But this may fail to be one-one. Still, the techniques for producing such a mapping were well-known to Frege and are employed in his proof of Theorem 428, mentioned previously, which states that if the number of Fs is Endlos and every G is F, then the number of Gs is either finite or Endlos. Indeed, that the number of φs is Endlos can be derived from Theorem 428.

a relation R$\xi\eta$ such that ...R$\xi\eta$..., then there is some ternary relation (so to speak, some function, from natural numbers to binary relations) which maps each number n to a relation n$\Phi\xi\eta$ such that ...n$\Phi\xi\eta$.... Formally:

$$\forall n[\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists R.M_{xy}(Rxy,n)] \rightarrow \exists\Phi\forall n[\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow M_{xy}(n\Phi xy,n)]$$

('M' is a third-order variable indicating relations between binary relations and objects; the subscripts indicate what (first-order) variables it binds.)[46] This is an axiom of countable choice, for binary relations, in Fregean Arithmetic.[47] The main reason to suppose Frege formulated (something like) it is, put as directly as possible, that one can not but do so if one investigates Dedekind's proof with the sorts of resources Frege had at his disposal. *That*, one might say, is the sort of thing formalization did for mathematics.

[46]Frege was perfectly familiar with such uses of higher-order variables. See *Gg* I §25.

[47]One can simplify the exposition here a little by considering an attempted proof simpler than Dedekind's. I have not done so in the text, because I have been attempting to reproduce a ttrain of thought which would have been available to anyone reading Dedekind who had access to Frege's formal theory and understood its workings. Still, it is worth sketching this alternative approach.

One argues that, for each natural number n, there is a *sub-concept* $\varphi_n\xi$ of Fξ, whose number is n, defines $\varphi\xi$ as $\exists n.\varphi_n\xi$, and then argues that the number of φs is Endlos. But, once again, while we can easily prove

$$\forall n\{\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists G[\forall x(Gx \rightarrow Fx) \;\&\; n = Nx{:}Gx]\}$$

we have no way to infer

$$\exists R\forall n\{\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow [\forall x(nRx \rightarrow Fx) \;\&\; n = Nx{:}nRx]\}$$

without an axiom of countable choice. What we need, this time, is to be able, quite generally, to infer

$$\forall n[\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow \exists G.M_{x}(Gx,n)] \rightarrow \exists R\forall n[\mathscr{F}^{=}(\mathrm{Pred})(0,n) \rightarrow M_{x}(nRx,n)]$$

And this is an axiom of countable choice, for concepts.

Interestingly enough, it would appear that showing that the number of φs is Endlos is *still* going to require an appeal to countable choice for binary relations. It may be for this reason that Dedekind proves the theorem as he does, given that the argument just mentioned seems in many ways more intuitive than Dedekind's own.

8. Closing: The Value of Formalization and Frege's Conception of Logical Truth

The supposition that Frege discovered the axiom of countable choice raises, in a very powerful way, a difficulty for those who would deny him the resources to discuss its truth or its epistemological status, that is, to discuss the question whether it is a law of logic.[48] To understand the sort of problem it raises, it is worth considering a more familiar argument for the claim that Frege regarded the question whether a given thought is a law of logic as intelligible.

Frege set out to derive the basic laws of number within a certain formal system, second-order logic plus Axiom V (it would not have mattered, for present purposes, had he set out to derive them within Fregean Arithmetic or within some other formal system). But he was not interested in formalization only for its mathematical benefits (which he rightly thought would be substantial). He, like his philosophical descendants, hoped that formalization would shed light upon epistemological questions. Indeed, Frege is admirably clear about the relation between formalization and epistemology:[49]

> I became aware of the need for a conceptual notation when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. Only *after* this question is answered can it be hoped to trace successfully the springs of knowledge on which this science thrives.

The epistemological status of arithmetic is not decided by formalization alone; it is only *after* its axioms have been isolated, that the question of arithmetic's epistemological status becomes tractable. But if logicism is not established by a "reduction" of arithmetic to the formal system of *Grundgesetze*, or Fregean Arithmetic, or any other formal system, it simply must be an intelligible question whether the axioms of that system are logical laws. What other question could remain at that point?[50]

This sort of argument concentrates upon Frege's attitude towards those thoughts he accepts as axioms of logic, those rules of inference he accepts as logically valid. I myself regard it as conclusive,

---

[48]Perhaps the best-known exponent of this view is Thomas Ricketts. See e.g. his "Frege, the *Tractatus*, and the Logocentric Predicament", *Noûs* 19 (1985), pp. 3-16; others are Burton Dreben, Warren Goldfarb, and Joan Weiner. For an important critical discussion of Ricketts's views in particular, see Jason Stanley, "Truth and Meta-theory in Frege", *forthcoming*.

[49]Gottlob Frege, "On Mr. Peano's Conceptual Notation and My Own", in *Collected Papers*, pp. 234-48, at p. 235, op. 362. My emphasis.

[50]Famously, the only principle to which Frege appealed about whose status as a law of logic Frege had any doubt was Axiom V: See *Gg* I p. vii. In my opinion, he probably had *no* real doubt that it was *true*.

but it nevertheless seems not to have carried conviction. A more powerful argument emerges, though, if we concentrate upon Frege's attitude toward a different sort of thought, a good example of which is the axiom of countable choice. In so far as one sets out to derive all the truths of some "branch of learning" from a fixed set of axioms, one is immediately confronted with the possibility that one's axioms might not be "complete",[51] might not suffice: There may be apparent truths, propositions of whose truth we had previously been convinced by informal argument, which can not be proven within the system as it stands. More interestingly, there may be a certain *kind* of inference which, though commonly made, can not be replicated within the system. Frege writes:[52]

> By [resolving inferences into their simple components] we shall arrive at just a few modes of inference, with which we must then attempt to make do at all times. And if at some point this attempt fails, then we shall have to ask whether we have hit upon a truth issuing from a non-logical source of cognition, whether a new mode of inference has to be acknowledged, or whether perhaps the intended step ought not to be taken at all.

Now, it is of course possible that Frege is here speaking completely hypothetically. But, I suggest, he may well be speaking from experience; he may have in mind something like the version of countable choice mentioned above. In any event, the interest of Frege's discussion does not depend upon the correctness of this speculation,[53] which merely serves to make vivid the problem that concerns him, which is one about the epistemological status of such principles. Frege is saying, quite reasonably, that the question whether e.g. the axiom of countable choice is true at all, and if so, whether it is a logical law, is not only intelligible but *important*.

---

[51]This is a term which Frege uses in this context (see "Peano", p. 235, op. 362), though not quite in its modern sense.

I am not, it is worth emphasizing, suggesting that Frege anticipated Gödel. There is no indication that Frege ever considered the possibility that there might be *no* complete axiomatization of arithmetic; it is this, of course, which makes Gödel's result so important. If the incompleteness of *Principia Mathematica* could have been remedied, its incompleteness would have been uninteresting.

[52]"Peano", p. 235, op. 363. There is nothing in what follows this passage to suggest that these remarks are not meant to be asserted. What follows is a discussion of how Frege's formal theory contributes to the attempt to formalize mathematical proof.

[53]It is worth noting, though, that it is demonstrable that Frege believed there were arithmetical truths which were unprovable within his system as it stood. See George Boolos and Richard G. Heck, Jnr., "*Die Grundlagen der Arithmetik* §§82-3", elsewhere in this volume.

Unable to formalize Dedekind's proof in the formal theory of *Grundgesetze*, or in Fregean Arithmetic, and having identified the proposition whose truth it assumes, we have, according to Frege, essentially three options: First, we may reject the proposition in question, as not being true, and reject the proof. If, instead, we accept the proposition's truth, we have two further options: To accept it as a truth peculiar to some special science, say, a part of mathematics not reducible to logic (a part similar, in that respect, to geometry); or to accept it as a truth of logic and to acknowledge some new logical axiom from which it follows or some rule of inference by means of which it can be proven. The presence of the first option shows that Frege did not think of himself merely as formalizing *accepted* mathematical practice: His project had, in his view, potentially revisionary consequences. The presence of the other two options shows that Frege regarded it as an intelligible question whether e.g. the axiom of countable choice was a law of logic. I thus regard it as demonstrable that Frege believed such questions to be intelligible.

Whatever the larger import of this point, however, I make it in the present context for purposes of illustration. It is sad enough that a work as great as *Grundgesetze der Arithmetik*, a work fully worthy of comparison with Dedekind's classic *Was sind und was sollen die Zahlen?*, should have been ignored for one hundred years. Our failure to study it is all the more regrettable, however, if an understanding of *Grundgesetze* promises a non-negligible improvement of our understanding of Frege's philosophy more generally. I admit, indeed, I insist that the offered interpretation of Frege's attitutde towards proposed new axioms does not depend upon the speculation inspired by our discussion of *Grundgesetze*. But that does not imply that our understanding of Frege will not be improved by an understanding of the kinds of mathematical problems with which he grappled, the sorts of solutions upon which he settled, and the philosophical significance he took them to have.[54]

[54]I should like to thank George Boolos, Burton Dreben, Warren Goldfarb, Mathieu Marion, Charles Parsons, William Tait, and Jason Stanley for discussions which have contributed greatly to this paper. I should also like to thank the members of the conference *Philosophy of Mathematics Today* whose proceedings this volume is. The paper I read there was not this one but a hybrid of "Development of Arithmetic" and "Definition by Induction"; nevertheless, the discussion which followed—indeed, all of our discussions at the conference—had an enormous effect on my thought about these and other issues. Thanks to Matthias for organizing the conference.

This paper is dedicated to the participants in a seminar, informally known as "Frege's Three Books", which George Boolos and I taught together in the Spring of 1993. Regular members, other than myself and George, were Emily Carson, Janet Folina, Michael Glanzberg, Delia Graff, David Hunter, Darryl Jung, Josep Macia-Fabrega, Ofra Rechter, Lisa Sereno, and

Appendix 1

Frege’s Proof of Theorem 288, and Reflections on the Least Number Principle

As was said above, the details of Frege’s proof of Theorem 288 are not relevant to the central concerns of the present paper. But, as the proof is somewhat difficult to follow, and as certain features of it are interesting for other reasons, it is worth explaining here. I shall discuss only those Theorems which Frege proves in the relevant sections of *Grundgesetze*, omitting the proofs of any and all theorems proven earlier in the book, i.e., all Theorems earlier than Theorem 264.[55]

Before we begin, we need one additional definition:

$$\mathscr{F}^{\wedge}(T;\langle a,x\rangle)(b,y) \equiv\mathrm{df}\ \mathscr{F}^{=}(T)(\langle a,x\rangle,\langle b,y\rangle)$$

Thus, $\mathscr{F}^{\wedge}(Q\pi R)(\langle a,x\rangle)(\xi,\eta)$ is a relation, defined by induction, according to Frege’s method: It is the relation which holds between ξ and η just in case $\langle\xi,\eta\rangle$ belongs to the $(Q\pi R)$-series beginning with $\langle a,x\rangle$.

Recall that Theorem 288 is:

$$\mathrm{Func}(Q)\ \&\ \neg\exists z.\mathscr{F}(Q)(z,z)\ \&\ \mathscr{F}^{=}(Q\ \pi\ \mathrm{Pred})(x,1;y,n) \rightarrow \\ \mathrm{N}z{:}\mathrm{Btw}(Q;x,y)(z) = \mathrm{N}z{:}\mathrm{Btw}(\mathrm{Pred})(1,n)(z)$$

That is: If the Q-series is simple and if $\langle y,n\rangle$ is a member of the $(Q\ \pi\ \mathrm{Pred})$-series beginning with $\langle x,1\rangle$, then the number of objects Q-between x and y is the same as the number of objects Pred-between 1 and n.

The proof of Theorem 288 requires two lemmas. The first of these is Theorem 287, whose proof is given in the notes:[56]

$$\neg\exists z[\mathscr{F}^{=}(\mathrm{Pred})(1,z)\ \&\ \mathscr{F}(\mathrm{Pred})(z,z)]$$

The second lemma is Theorem 284:

$$\mathrm{Func}(R)\ \&\ \neg\exists z[\mathscr{F}^{=}(R)(m,z)\ \&\ \mathscr{F}(R)(z,z)]\ \&\ \mathrm{Func}(Q)\ \&\ \neg\exists z[\mathscr{F}^{=}(Q)(x,z)\ \& \\ \mathscr{F}(Q)(z,z)]\ \&$$

Jason Stanley. What fun it was.

[55]Many of these are discussed in “Definition by Induction”.

[56]If $\mathscr{F}^{=}(\mathrm{Pred})(1,n)$, then, since $\mathrm{Pred}(0,1)$, we have $\mathscr{F}^{=}(\mathrm{Pred})(0,n)$, by Theorem 285. But then $\neg\mathscr{F}(\mathrm{Pred})(n,n)$, by Theorem 145.

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \rightarrow \mathrm{Nz{:}Btw}(Q;x,y)(z) = \mathrm{Nz{:}Btw}(R;m,n)(z)$$

That is: If both the R-series beginning with m and the Q-series beginning with x are simple, and if $\langle y,n\rangle$ is a member of the $(Q\pi R)$-series beginning with $\langle x,m\rangle$, then the number of objects Q-between x and y is the same as the number of objects R-between m and n.

Theorem 288 follows quickly from these two lemmas. For, taking $R\xi\eta$ as $\mathrm{Pred}(\xi,\eta)$ and m as 1, the following is a substitution instance of Theorem 284:

$$\begin{aligned}&\mathrm{Func}(\mathrm{Pred}) \;\&\; \neg\exists z[\mathscr{F}^{=}(\mathrm{Pred})(1,z) \;\&\; \mathscr{F}(\mathrm{Pred})(z,z)] \;\&\; \mathrm{Func}(Q) \;\&\\ &\quad \neg\exists z[\mathscr{F}^{=}(Q)(x,z) \;\&\; \mathscr{F}(Q)(z,z)] \;\&\; \mathscr{F}^{=}(Q\;\pi\;\mathrm{Pred})(\langle x,1\rangle,\langle y,n\rangle) \rightarrow\\ &\quad \mathrm{Nz{:}Btw}(Q;x,y)(z) = \mathrm{Nz{:}Btw}(\mathrm{Pred};1,n)(z)\end{aligned}$$

$\mathrm{Pred}(\xi,\eta)$ is functional, by Theorem 71, and the second conjunct is Theorem 287. Moreover, if $\neg\exists z.\mathscr{F}(Q)(z,z)$, then certainly $\neg\exists z[\mathscr{F}^{=}(Q)(x,z) \;\&\; \mathscr{F}(Q)(z,z)]$. So we are done.

We turn then to the proof of Theorem 284, which is derived from Theorem 283, which is:

$$\begin{aligned}&\mathrm{Func}(R) \;\&\; \neg\exists z[\mathscr{F}^{=}(R)(m,z) \;\&\; \mathscr{F}(R)(z,z)] \;\&\; \mathrm{Func}(Q) \;\&\; \neg\exists z[\mathscr{F}^{=}(Q)(x,z) \;\&\\ &\mathscr{F}(Q)(z,z)] \;\&\quad \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \rightarrow\\ &\qquad \mathrm{Map}[\mathscr{F}\hat{\ }(Q\pi R,\langle x;m\rangle)][\mathrm{Btw}(Q;x,y)(\xi),\mathrm{Btw}(R;m,n)(\xi)]\end{aligned}$$

That is: If the R-series beginning with m and the Q-series beginning with x are simple, and if $\langle y,n\rangle$ is a member of the $(Q\pi R)$-series beginning with $\langle x,m\rangle$, then $\mathscr{F}\hat{\ }(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ maps the objects Q-between x and y into the objects R-between m and n.

The method by which Theorem 284 is derived from Theorem 283 is very common in *Grundgesetze*. Theorem 283 asserts that, under certain conditions, the relation $\mathscr{F}\hat{\ }(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ will map the objects Q-between x and y into the objects R-between m and n. By exchanging 'R' and 'Q', 'x' and 'm', and 'y' and 'n' in Theorem 283, one can show that, under the same conditions,[57] $\mathscr{F}\hat{\ }(R\pi Q,\langle m,x\rangle)(\xi,\eta)$ will map the objects R-between m and n into those Q-between x and y. But by Frege's Theorem 259, $\mathscr{F}\hat{\ }(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ is the converse of $\mathscr{F}\hat{\ }(R\pi Q,\langle m,x\rangle)](\xi,\eta)$, so it follows that $\mathscr{F}\hat{\ }(R\pi Q,\langle m,x\rangle)](\xi,\eta)$ correlates the objects Q-between x and y one-to-one with those R-between m and n, whence the numbers of these concepts are the same, by Hume's Principle.

Most of the work in the proof of Theorem 288 thus lies in the proof of Theorem 283. To prove Theorem 283, suppose its antecedent. To show that $\mathscr{F}\hat{\ }(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ maps the objects Q-between

[57] This follows from Frege's Theorem 258: $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \rightarrow \mathscr{F}^{=}(R\pi Q)(\langle m,x\rangle,\langle n,y\rangle)$. In most such proofs, the conditions are not the same but merely corresponding.

x and y into the objects R-between m and n, we must show (i) that $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ is functional and (ii) that, if z is Q-between x and y, there is an object R-between m and n onto which $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ maps z. Now, (i) follows immediately from Frege's Theorem 253:

$$\begin{aligned}&\mathrm{Func}(R)\ \&\ \mathrm{Func}(Q)\ \&\ \neg\exists w[\mathscr{F}^{=}(Q)(m,w)\ \&\ \mathscr{F}(Q)(w,w)] \rightarrow\\&\mathrm{Func}[\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)]\end{aligned}$$

And (ii) is the consequent of Frege's Theorem 277, the first of two important lemmas:

$$\begin{aligned}&\mathrm{Func}(R)\ \&\ \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle)\ \&\ \neg\mathscr{F}(R)(n,n)\ \&\ \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w) \rightarrow\\&\qquad\{\mathrm{Btw}(Q;x,y)(z) \rightarrow \exists w[\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)\ \&\ \mathrm{Btw}(R;m,n)(w)]\}\end{aligned}$$

That is: If $R\xi\eta$ is functional, if $\langle y,n\rangle$ belongs to the $(Q\pi R)$-series beginning with $\langle x,m\rangle$, if n does not follow itself in the R-series, and if z is in the domain of $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$, then, if z is Q-between x and y, there is an object R-between m and n onto which $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ maps z. It will suffice to establish, on the assumption that the antecedent of Theorem 283 holds, the antecedent of

$$\begin{aligned}&\mathrm{Func}(R)\ \&\ \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle)\ \&\ \neg\mathscr{F}(R)(n,n)\ \&\\&\qquad\{\mathrm{Btw}(Q;x,y)(z) \rightarrow \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)\} \rightarrow\\&\qquad\{\mathrm{Btw}(Q;x,y)(z) \rightarrow \exists w[\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)\ \&\ \mathrm{Btw}(R;m,n)(w)]\}\end{aligned}$$

which is equivalent to Theorem 277. We have assumed the first two conjuncts. Moreover, since $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle)$, $\mathscr{F}^{=}(R)(m,n)$, by Frege's Theorem 232; so, since no member of the R-series beginning with m follows after itself, n does not. The fourth conjunct is, in turn, the consequent of the second important lemma, Frege's Theorem 283,ζ:

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \rightarrow \{\mathrm{Btw}(Q;x,y)(z) \rightarrow \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)\}$$

That is: If $\langle y,n\rangle$ belongs to the $(Q\pi R)$-series beginning with $\langle x,m\rangle$ and if z is Q-between x and y, then z is in the domain of $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$. (Since 'n' appears only in the antecedent, which is equivalent to '$\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(y,n)$', we may also say: If y is in the domain of $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$, then everything Q-between x and y is also in its domain.) But we have also supposed its antecedent, so Theorem 283 is proven.

We have now to prove Theorems 277 and 283,ζ. We begin with Theorem 277. Suppose that $R\xi\eta$ is functional, that $\langle y,n\rangle$ belongs to the $(Q\pi R)$-series beginning with $\langle x,m\rangle$, that z is in the domain of $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$—say, $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)$—that n does not follow itself in the R-series, and that z is Q-between x and y. We must show that $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(\xi,\eta)$ maps z to an object R-between m and n. Indeed, w itself is R-between m and n. For $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle z,w\rangle)$, whence, by Theorem 232,

$\mathscr{F}^{=}(R)(m,w)$. Thus, $\mathrm{Func}(R)$, $\neg\mathscr{F}(R)(n,n)$, and $\mathscr{F}^{=}(R)(m,w)$. So, if $\mathscr{F}^{=}(R)(w,n)$, then w is between m and n in the R-series and we are done. Therefore, by Theorem 232, it will suffice to show that $\mathscr{F}^{=}(Q\pi R)(\langle z,w\rangle,\langle y,n\rangle)$.

To show this, we employ Theorem 243, an instance of which is:

$$\begin{aligned}&\mathrm{Func}(Q\pi R) \,\&\, \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \,\&\, \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle z,w\rangle) \to\\ &\qquad \mathscr{F}^{=}(Q\pi R)(\langle z,w\rangle,\langle y,n\rangle) \vee \mathscr{F}(Q\pi R)(\langle y,n\rangle,\langle z,w\rangle)\end{aligned}$$

Now, we know that $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle)$ and $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle z,w\rangle)$. But we also know that $\mathrm{Func}(R)$; and, since $\mathrm{Btw}(Q;x,y)(d)$, $\mathrm{Func}(Q)$, by definition. But, then, Frege's Theorem 252 is:

$$\mathrm{Func}(Q) \,\&\, \mathrm{Func}(R) \to \mathrm{Func}(Q\pi R)$$

So, either $\mathscr{F}^{=}(Q\pi R)(\langle z,w\rangle,\langle y,n\rangle)$ or $\mathscr{F}(Q\pi R)(\langle y,n\rangle,\langle z,w\rangle)$. But if the latter, $\mathscr{F}(Q)(y,z)$, contradicting the assumption that z is Q-between x and y—since then $\mathscr{F}(Q)(y,z)$ and $\mathscr{F}^{=}(Q)(z,y)$, so $\mathscr{F}(Q)(y,y)$—and we are done.

To complete the proof of Theorem 283, and so of Theorem 288, we need now only prove Theorem 283,ζ, which is:

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \to \{\mathrm{Btw}(Q;x,y)(z) \to \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)\}$$

The proof is by induction, the induction justified by Frege's Theorem 257, which is:[58]

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle y,n\rangle) \,\&\, Fxm \,\&\, \forall t\forall s\{Fts \to \forall u\forall v[Qtu \,\&\, Rsv \to Fuv]\} \to Fyn$$

For the proof, we take $F\xi\eta$ to be:

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle \xi,\eta\rangle) \;\&\; \forall z[\mathrm{Btw}(Q;x,\xi)(z) \to \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)]$$

Let us call this 'the relevant relation'. To prove Theorem 283,ζ, we must therefore prove the base case, Theorem 283,ϵ:

$$\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle x,m\rangle) \,\&\, \forall z[\mathrm{Btw}(Q;x,x)(z) \to \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)]$$

and the induction step, Theorem 281,ρ:

$$\begin{aligned}&\forall t\forall s\{\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle t,s\rangle) \,\&\, \forall z[\mathrm{Btw}(Q;x,t)(z) \to \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)] \to\\ &\qquad \forall u\forall v[Qtu \,\&\, Rsv \to\\ &\qquad \mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle u,v\rangle) \,\&\, \forall z[\mathrm{Btw}(Q;x,u)(z) \to \exists w.\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(z,w)]]\}\end{aligned}$$

Since only x is Q-between x and x, Theorem 283,ϵ, follows from the trivial fact that $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle x,m\rangle)$ and so $\mathscr{F}^{\wedge}(Q\pi R,\langle x,m\rangle)(x,m)$.

[58] For a discussion of the significance of Theorem 257, see "Definition by Induction".

For the proof of Theorem 281,ρ, suppose that $\langle t,s\rangle$ belongs to the ($Q\pi R$)-series beginning with $\langle x,m\rangle$ and that everything Q-between x and t is in the domain of $\mathscr{F}$^($Q\pi R,\langle x,m\rangle$)(ξ,η); suppose further than Qtu and Rsv. We then have Theorem 281,η:

$$\mathscr{F}^{=}(Q)(x,t) \ \&\ Qtu \rightarrow [Btw(Q;x,u)(z) \rightarrow Btw(Q;x,t)(z) \vee z=u]$$

So, everything Q-between x and u is either Q-between x and t or is equal to u. But, by hypothesis, everything Q-between x and t is in the domain of $\mathscr{F}$^($Q\pi R,\langle x,m\rangle$)(ξ,η). Now, $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle t,s\rangle)$; but since Qtu and Rsv, we have $(Q\pi R)(\langle t,s\rangle,\langle u,v\rangle)$, by definition. So $\mathscr{F}^{=}(Q\pi R)(\langle x,m\rangle,\langle u,v\rangle)$, whence u is in the domain of $\mathscr{F}$^($Q\pi R,\langle x,m\rangle$)(ξ,η), and we are done.

The proof of Theorem 281,η, is not particularly difficult.[59] So that completes our discussion of Frege's proof of Theorem 288. I hope it is now clear that, when I said that the proof of Theorem 321 given in §2 is simpler, and shorter, than Frege's, I spoke truly.

It is worth discussing Theorem 281,η, a bit further, though, because is rather more important than it might at first appear. It is one of two results which Frege mentions, in the Forward to Volume II of *Grundgesetze*, as results he intends to use but which had not previously been marked for later use. In fact, he employs Theorem 281,η, frequently and at important points in the proofs in Volume II. The reason for this is that Theorem 281,η, has the force of an *induction* principle, one very closely related to (Frege's logicized version of) the least number principle. More precisely, suppose that the Q-series beginning with a is simple, and that, whenever everything Q-between a and some member of the Q-series beginning with a is F, that member is itself F. Then Theorem 281,η, very easily implies that every member of the Q-series beginning with a is F. That is, Theorem 281,η, implies what one might call *induction for betweenness*:

$$\begin{aligned}&(IB)\ \mathscr{F}^{=}(Q)(a,b) \ \&\ Func(Q) \ \&\ \neg\exists z[\mathscr{F}^{=}(Q)(a,z) \ \&\ \mathscr{F}(Q)(z,z)] \ \&\\ &\qquad \forall x\{\mathscr{F}^{=}(Q)(a,x) \ \&\ \forall z[Btw(Q;a,x)(z) \ \&\ x\neq z \rightarrow Fz] \rightarrow Fx\} \rightarrow Fb\end{aligned}$$

To prove (IB), suppose the antecedent. Now, the following is an instance of Frege's Theorem 144, a version of induction:

[59] Suppose that $\mathscr{F}^{=}(Q)(x,t)$, that Qtu, and that Btw(Q;x,u)(z). Then Func(Q), $\neg\mathscr{F}(Q)(u,u)$, $\mathscr{F}^{=}(Q)(x,z)$, and $\mathscr{F}^{=}(Q)(z,u)$. Suppose then that $z\neq u$. To show that Btw(Q;x,t)(z), we need only show that $\mathscr{F}^{=}(Q)(z,t)$. Since $\mathscr{F}^{=}(Q)(x,z)$ and $\mathscr{F}^{=}(Q)(x,t)$, either $\mathscr{F}^{=}(Q)(z,t)$ or $\mathscr{F}(Q)(t,z)$, by Theorem 243. And since Func(Q), $\mathscr{F}(Q)(t,z)$, and Qtu, $\mathscr{F}^{=}(Q)(u,z)$, by Theorem 242. Now, $u\neq z$, so $\mathscr{F}(Q)(u,z)$. But then $\mathscr{F}(Q)(u,u)$, since $\mathscr{F}^{=}(Q)(z,u)$. Contradiction.

$$\mathscr{F}^{=}(Q)(a,b) \ \& \ Fa \ \& \\ \forall x\{\mathscr{F}^{=}(Q)(a,x) \ \& \ \forall z[Btw(Q;a,x)(z) \to Fz] \to \forall y(Qxy \to \forall z[Btw(Q;a,y)(z) \to Fz])\} \to \\ \forall z[Btw(Q;a,b)(z) \to Fz]$$

Since b is a member of the Q-series beginning with a, it does not follow after itself in the Q-series. Thus, b is Q-between a and b, so it follows from the consequent that Fb. Hence, we need only establish the antecedent. We have assumed the first conjunct, so we need only show (i) that Fa and (ii) that

$$\forall x\{\mathscr{F}^{=}(Q)(a,x) \ \& \ \forall z[Btw(Q;a,x)(z) \to Fz] \to \forall y(Qxy \to \forall z[Btw(Q;a,y)(z) \to Fz])\}$$

For (i), note that

$$\forall y[Btw(Q;a,a)(z) \ \& \ a \neq z \to Fz] \to Fa$$

Since nothing other than a is Q-between a and itself, the antecedent is vacuously true; so Fa. For (ii), suppose that x belongs to the Q-series beginning with a, that everything Q-between a and x is F, and that Qxy. We must show that everything Q-between a and y is F. Now, the following is equivalent to an instance of Theorem 281,η:

$$\mathscr{F}^{=}(Q)(a,x) \ \& \ Qxy \ \& \ Btw(Q;x,y)(z) \ \& \ z \neq y \to Btw(Q;a,x)(z)$$

So everything Q-between x and y, other than y, is Q-between a and x. But since everything Q-between a and x is F, it follows that everything Q-between a and y, other than y, is F. But, by hypothesis:

$$\mathscr{F}^{=}(Q)(a,y) \ \& \ \forall z[Btw(Q;a,y)(z) \ \& \ y \neq z \to Fz] \to Fy$$

So, Fy. But then we have the following, which follows immediately from Theorem 281,η:

$$\mathscr{F}^{=}(Q)(a,x) \ \& \ \forall z[Btw(Q;a,x)(z) \to Fz] \ \& \ Fy \ \& \ Qxy \to \forall z[Btw(Q;a,y)(z) \to Fz]$$

But the antecedent holds, so everything Q-between a and y is F, and we are done.

Frege does not prove what I have called induction for betweenness. His logicized version of the least number principle is his Theorem 359, which (eliminating a couple definitions, again, and strengthening slightly) is:[60]

$$Func(Q) \ \& \ \neg\exists z[\mathscr{F}^{=}(Q)(a,z) \ \& \ \mathscr{F}(Q)(z,z)] \ \& \\ \forall x\{\mathscr{F}^{=}(Q)(a,x) \ \& \ \forall z[\mathscr{F}^{=}(Q)(a,z) \ \& \ \mathscr{F}(Q)(z,x) \to \neg Fz] \to \neg Fx\} \to \\ \forall x[\mathscr{F}^{=}(Q)(a,x) \to \neg Fx]$$

Now, if we substitute '¬Fξ' for 'Fξ', we have:

$$Func(Q) \ \& \ \neg\exists z[\mathscr{F}^{=}(Q)(a,z) \ \& \ \mathscr{F}(Q)(z,z)] \ \& \\ \forall x\{\mathscr{F}^{=}(Q)(a,x) \ \& \ \forall z[\mathscr{F}^{=}(Q)(a,z) \ \& \ \mathscr{F}(Q)(z,x) \to Fz] \to Fx\} \to$$

[60] Frege's proof of Theorem 359 is, of course, different from that given here of induction for betweenness, but it is strikingly similar.

$$\forall x[\mathscr{F}^{=}(Q)(a,x) \rightarrow Fx]$$

This is easily seen to be equivalent to induction for betweenness. It differs from induction for bewteenness only in that it has, as its third conjunct:

$$\forall x\{\mathscr{F}^{=}(Q)(a,x) \,\&\, \forall z[\mathscr{F}^{=}(Q)(a,z) \,\&\, \mathscr{F}(Q)(z,x) \rightarrow Fz] \rightarrow Fx\}$$

whereas, in induction for betweenness, we have:

$$\forall x\{\mathscr{F}^{=}(Q)(a,x) \,\&\, \forall z[Btw(Q;a,x)(z) \,\&\, x \neq z \rightarrow Fz] \rightarrow Fx\}$$

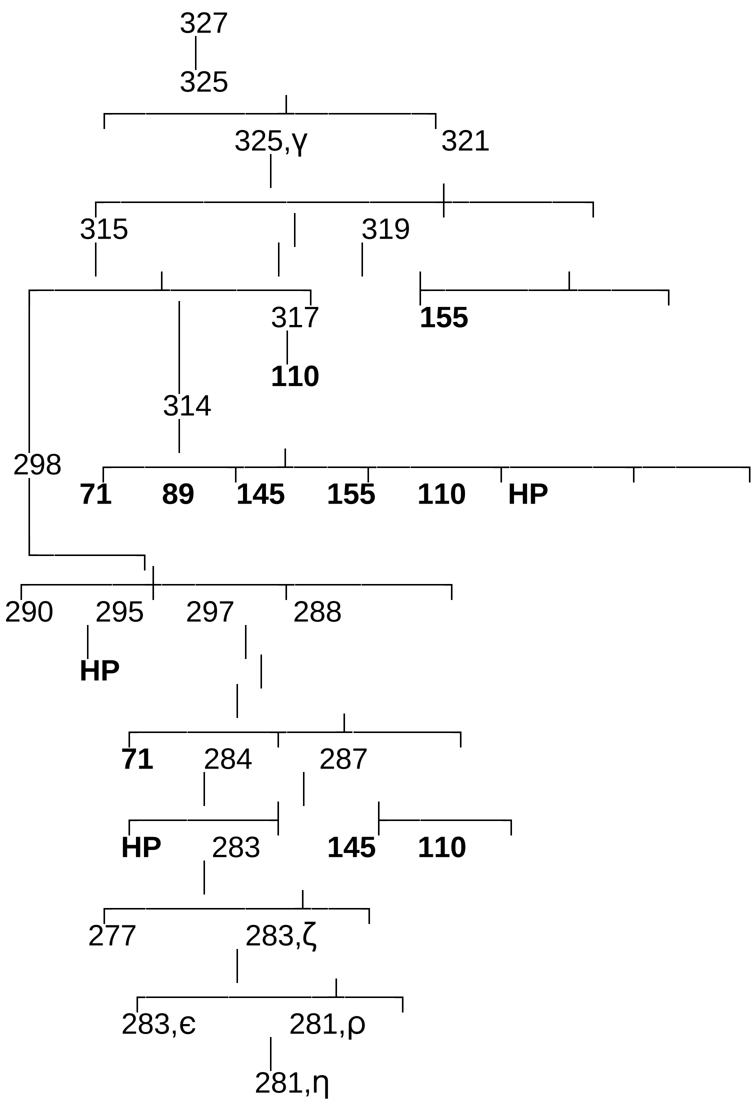
But, in the presence of the other conjuncts of the antecedent, these are equivalent. Indeed, the proof of Theorem 281,η, just is a demonstration that they are.

Appendix 2

Tree of Important Theorems in Frege's Proof of Theorem 327

The following is a graphical representation of the dependencies among various results used by Frege in his proof of Theorem 327. I have omitted references to general results about the ancestral, uses of definitions, and theorems which are immediate consequences thereof. I have also omitted reference to theorems proven in sections A-I of *Grundgesetze*, except for those to specificially arithmetical truths and to Hume's Principe, all of these being marked in boldface.[61]

[61]The arithmetical theorems in question are: (71) Func(Pred); (89) Func(Conv Pred); (110) Pred(0,1); (145) $\mathscr{F}^{=}(Pred)(0,x) \rightarrow \neg\mathscr{F}(Pred)(x,x)$; (155) $\mathscr{F}^{=}(Pred)(0,n) \rightarrow Pred[n,Nx:\mathscr{F}^{=}(Pred)(x,n)]$.


327
325
325,γ
321
315
319
317
155
110
314
298
71
89
145
155
110
HP
290
295
297
288
HP
71
284
287
HP
283
145
110
277
283,ζ
283,є
281,ρ
281,η