

On the complexity of evaluating highest weight vectors

Markus Bläser

Saarland University, Germany
mblaeser@cs.uni-saarland.de

Julian Dörfler 

Saarbrücken Graduate School of Computer Science, Germany
jdoerfler@cs.uni-saarland.de

Christian Ikenmeyer

University of Liverpool¹
christian.ikenmeyer@liverpool.ac.uk

Abstract

Geometric complexity theory (GCT) is an approach towards separating algebraic complexity classes through algebraic geometry and representation theory. Originally Mulmuley and Sohoni proposed (SIAM J Comput 2001, 2008) to use occurrence obstructions to prove Valiant’s determinant vs permanent conjecture, but recently Bürgisser, Ikenmeyer, and Panova (Journal of the AMS 2019) proved this impossible. However, fundamental theorems of algebraic geometry and representation theory grant that every lower bound in GCT can be proved by the use of so-called highest weight vectors (HWVs). In the setting of interest in GCT (namely in the setting of polynomials) we prove the NP-hardness of the evaluation of HWVs in general, and we give efficient algorithms if the treewidth of the corresponding Young-diagram is small, where the point of evaluation is concisely encoded as a noncommutative algebraic branching program! In particular, this gives a large new class of separating functions that can be efficiently evaluated at points with low (border) Waring rank.

2012 ACM Subject Classification Theory of computation → Algebraic complexity theory; Mathematics of computing → Mathematical software

Keywords and phrases Algebraic complexity theory, algebraic branching program, Waring rank, border Waring rank, representation theory, highest weight vector, treewidth

¹ part of this research was done when CI was at the Max Planck Institute for Software Systems, Germany, and the Simons Institute for the Theory of Computing, United States

1 Prelude: Border Waring rank and Algebraic Branching Programs

An algebraic branching program (ABP) is a layered directed acyclic graph (the vertex set is partitioned into numbered layers and edges only go from the i -th layer to the $(i + 1)$ -th layer) with two distinguished nodes, the *source* and the *sink*, and the edges are labeled with homogeneous linear polynomials. The weight $w(P)$ of a path P with edge labels ℓ_1, \dots, ℓ_d is defined as the product $w(P) := \ell_1 \cdots \ell_d$. We say that the ABP *computes* the sum $\sum_{\text{source-sink-path } P} w(P)$. We can view the same ABP both over commuting variables or noncommuting variables. If we interpret it over noncommuting variables, we call it an ncABP. If we want to stress that the variables commute, we call it a cABP. The size of an ABP is the number of its vertices. The width of an ABP is the largest number of vertices in any layer. For a homogeneous degree d polynomial let the *ABP width complexity* $w(f)$ be defined as the smallest width of a cABP computing f . A sequence (f_n) of polynomials is called a *p-family* if the number of variables and the degree of each f_n are polynomially bounded in n . p-families are the object of study in Valiant's algebraic complexity framework. Let VBP denote the set of all p-families (f_i) with polynomially bounded ABP width complexity $w(f_i)$.

The Waring rank $\text{WR}(f)$ of a homogeneous degree d polynomial is the smallest r such that f can be written as a sum of r powers of homogeneous linear polynomials. Let VWaring be the set of all p-families with polynomially bounded Waring rank.

Clearly, $w(f) \leq \text{WR}(f)$, because from a Waring rank r decomposition we can construct a width r cABP that computes f in the straightforward way: The cABP contains exactly r disjoint source-sink-paths (vertex-disjoint up to source and sink) so that on each path all edges have the same label. Therefore $\text{VWaring} \subseteq \text{VBP}$.

There is a natural way to associate to every algebraic complexity measure a corresponding border complexity measure: We define the *border Waring rank* $\overline{\text{WR}}(f)$ as the smallest r such that f can be approximated arbitrarily closely (coefficient-wise) by polynomials with $\text{WR}(f) \leq r$, or equivalently, the smallest r such that f lies in the closure (Zariski closure and Euclidean closure coincide) of the set $\{f \mid \text{WR}(f) \leq r\}$. Clearly $\overline{\text{WR}}(f) \leq \text{WR}(f)$. Let $\overline{\text{VWaring}}$ denote the set of sequences of polynomials with polynomially bounded border Waring rank. Clearly $\text{VWaring} \subseteq \overline{\text{VWaring}}$.

Analogously we can define the *border ABP width complexity* $\underline{w}(f)$ from w . Clearly $\underline{w}(f) \leq w(f)$. Let $\overline{\text{VBP}}$ be the set of polynomials with polynomially bounded border ABP width complexity. Clearly $\text{VBP} \subseteq \overline{\text{VBP}}$.

For noncommutative polynomials we define the analogous versions ncw and $\underline{\text{ncw}}$. It follows from Nisan's work [46] that $\text{ncw}(f) = \underline{\text{ncw}}(f)$.

In general, it is unknown by how much an algebraic complexity class grows when applying the closure. In particular, it is open whether $\text{VWaring} = \overline{\text{VWaring}}$ or whether $\text{VBP} = \overline{\text{VBP}}$. But the following result in this direction is known.

► **Theorem 1.** $\overline{\text{VWaring}} \subseteq \text{VBP}$.

We quickly sketch the standard proof. We will need the following concept only for this proof. A *read-once oblivious ABP* is a layered ABP whose edge labels have univariate polynomials in x_i on each edge in layer i . The first step in the proof is Saxena's duality trick [52, Lemma 1]:

If $f \in \mathbb{C}[x_1, \dots, x_m]_d$ has $\text{WR}(f) \leq s$, then there is a read-once oblivious ABP computing f with width at most $s \cdot (md + d + 1)$.

The proof uses a power series argument. The next crucial step is to use a variant of Nisan's result [46] to see that the *border* read-once oblivious ABP width equals the read-once oblivious

ABP width, so approximations can be removed [26, Sec. 4.5.2]:

If $f \in \mathbb{C}[x_1, \dots, x_m]_d$ has $\underline{\text{WR}}(f) \leq s$, then there is an read-once oblivious ABP computing f with width at most $s \cdot (md + d + 1)$.

We can unfold this read-once oblivious ABP, i.e., replace each edge (remember, each label is a univariate degree $\leq d$ polynomial) with a (non-layered) ABP computing it, where each edge has an affine linear label. If done properly, this requires $d - 1$ additional vertices per edge. Making the ABP layered and homogeneous blows up the ABP's width by a factor of $d + 1$. We conclude:

$$\text{For all } f \in \mathbb{C}[x_1, \dots, x_m]_d \text{ we have } \mathbf{w}(f) \leq \underline{\text{WR}}(f) \cdot (md + d + 1) \cdot (d + 1). \quad (1)$$

Eq. (1) proves Theorem 1 when we assume that m and d are polynomially bounded (which is usually assumed). We now strengthen eq. (1) with the following clean statement that is independent of m and d .

► **Theorem 2.** *For all $f \in \mathbb{C}[x_1, \dots, x_m]_d$ we have $\mathbf{w}(f) \leq \underline{\text{WR}}(f)$.*

The rest of Section 1 is devoted to the proof of Theorem 2. We start with introducing several main multilinear algebra concepts of this paper. The actual proof of Theorem 2 is then very short and natural.

When talking about homogeneous multivariate noncommutative polynomials, we use the standard language of multilinear algebra: An order d tensor in $\otimes^d \mathbb{C}^m$ is a d -dimensional $m \times m \times \dots \times m$ array of numbers. There is a canonical vector space isomorphism between the vector space of m -variate homogeneous degree d noncommutative polynomials $\mathbb{C}\langle x_1, \dots, x_m \rangle_d$ and $\otimes^d \mathbb{C}^m$, which is defined on monomials as

$$x_{i_1} x_{i_2} \cdots x_{i_d} \xrightarrow{\sim} E_{i_1, \dots, i_d},$$

where E_{i_1, \dots, i_d} is the tensor that is 0 everywhere, but has a single 1 at position (i_1, \dots, i_d) . Let (e_i) be the standard basis of \mathbb{C}^m . We use the notation $e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_d} := E_{i_1, \dots, i_d}$. More generally, for $v_1, \dots, v_d \in \mathbb{C}^m$, we write $v_1 \otimes v_2 \otimes \dots \otimes v_d$ to be the tensor whose entry at position (i_1, \dots, i_d) is the product $(v_1)_{i_1} \cdot (v_2)_{i_2} \cdots (v_d)_{i_d}$.

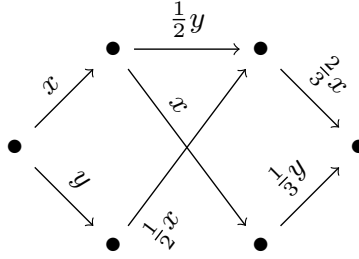
A tensor T is called *symmetric* if $T_{i_1, \dots, i_d} = T_{i_{\pi(1)}, \dots, i_{\pi(d)}}$ for all permutations $\pi \in \mathfrak{S}_d$. Let $\text{Sym}^d \mathbb{C}^m \subseteq \otimes^d \mathbb{C}^m$ denote the linear subspace of symmetric tensors. There is a canonical vector space isomorphism between the vector space of m -variate homogeneous degree d commutative polynomials $\mathbb{C}[x_1, \dots, x_m]_d$ and $\text{Sym}^d \mathbb{C}^m$, which is defined on monomials as

$$x_{i_1} x_{i_2} \cdots x_{i_d} \xrightarrow{\sim} \sum_{\pi \in \mathfrak{S}_d} \frac{1}{d!} E_{\pi(i_1), \dots, \pi(i_d)},$$

For example, the polynomial $x_1^2 x_2$ corresponds to the tensor $\frac{1}{3}(e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1)$.² We use e_i and x_i interchangeably.

It is crucial to note that *noncommutative ABPs can compute symmetric tensors*. An example is given in Figure 1, where we used $x := x_1$ and $y := x_2$. As before with cABPs, it is easy to see that every Waring rank r decomposition of f can be converted into a width r ncABP computing f in the straightforward way: The ncABP contains exactly r disjoint source-sink-paths (vertex-disjoint up to source and sink) so that on each path all edges

² This tensor is called the W-state in quantum information theory.



■ **Figure 1** An ncABP computing the symmetric tensor $\frac{1}{3}(x \otimes x \otimes y + x \otimes y \otimes x + y \otimes x \otimes x)$, which corresponds to the polynomial x^2y . If we reinterpret the ncABP as a cABP, it computes $\frac{1}{3}(xxy + xyx + yxx) = x^2y$. Such ncABPs can be efficiently constructed using Nisan’s construction technique [46]. Interestingly, in this example the width is only 2, while the Waring rank of x^2y is 3.

have the same label. Every ncABP can be reinterpreted as a cABP by letting the variables commute. If the ncABP computes a symmetric tensor, then clearly this cABP computes the corresponding polynomial. Now we can prove Theorem 2 in a very natural and short way as follows.

Given f with a border Waring rank s decomposition. We construct the corresponding border ncABP with s many edge-disjoint source-sink-paths, so $\underline{\text{ncw}}(f) \leq s$. Using Nisan’s result [46] that $\underline{\text{ncw}} = \text{ncw}$, it follows $\text{ncw}(f) \leq s$. This gives a width s ncABP that computes f . Reinterpreting this ncABP as a cABP finishes our proof of Theorem 2.

2 Introduction

Geometric complexity theory (GCT) is an approach towards the separation of algebraic complexity classes using algebraic geometry and representation theory [44, 45, 15]. Let $\text{per}_i := \sum_{\pi \in \mathfrak{S}_i} \prod_{j=1}^i x_{j,\pi(j)}$ be the permanent polynomial. Valiant’s famous $\text{VBP} \neq \text{VNP}$ conjecture (also known as the “determinant vs permanent conjecture”) can be phrased as: The sequence $w(\text{per}_i)$ is not polynomially bounded. Mulmuley and Sohoni strengthened the conjecture by conjecturing that even $\underline{w}(\text{per}_i)$ is not polynomially bounded, i.e., $\text{VNP} \not\subseteq \overline{\text{VBP}}$. Clearly, if $\text{VBP} = \overline{\text{VBP}}$, then both conjecture coincide, but this is an open question.

In the GCT approach, we set $m := d^2$ and let the group $\text{GL}_m := \text{GL}(\mathbb{C}^m)$ act on a the space of homogeneous degree d polynomials in m variables by linear transformation of the variables. The Mulmuley–Sohoni conjecture can be rephrased as $x_{11}^{d-i} \text{per}_i \notin \overline{\text{GL}_m \det_d}$ if d grows superpolynomially in i . Now we try to attack this problem by representation theoretic methods, so-called *obstructions*. A first crucial insight is that $x_{11}^{d-i} \text{per}_i \in \overline{\text{GL}_m \det_d}$ iff $\overline{\text{GL}_m(x_{11}^{d-i} \text{per}_i)} \subseteq \overline{\text{GL}_m \det_d}$. Thus, we compare two varieties.

An important object of study in this context are so-called highest weight vectors (HWVs) of weight $\lambda \in \mathbb{N}^m$, which are homogeneous degree d polynomials in the coefficients of homogeneous degree d polynomials in m variables, satisfying two properties (see Sec. 5). Their dimension is the called the plethysm coefficient. The dimension of their restriction to a GL_m -variety X is called the *multiplicity* $\text{mult}_\lambda \mathbb{C}[X]$ of λ in the coordinate ring $\mathbb{C}[X]$. They are important, because if $\text{mult}_\lambda \mathbb{C}[X] > \text{mult}_\lambda \mathbb{C}[Y]$, then Schur’s lemma implies that $X \not\subseteq Y$. In this case, λ is called a *multiplicity obstruction*. If additionally $\text{mult}_\lambda \mathbb{C}[X] > 0 = \text{mult}_\lambda \mathbb{C}[Y]$, then λ is called an *occurrence obstruction*.

Classically, the varieties X and Y are GL_m -orbit closures of the determinant or padded permanent polynomial. [14] proved that occurrence obstructions are not sufficient to prove

Valiant’s conjecture with this padded setting. Hence, multiplicity obstructions are a focus of recent research [23, 35] and other models of computation [28, 29]. Waring rank and border Waring rank are classical notions studied in algebraic geometry in the language of higher secant varieties [40].

3 Our contributions

To calculate a multiplicity $\text{mult}_\lambda \mathbb{C}[X]$, a common approach is to generate a basis of all HWVs of weight λ and evaluate them at enough points from X (points from all GL_m -varieties in GCT are efficiently samplable) and observe the dimension of their linear span, which equals $\text{mult}_\lambda \mathbb{C}[X]$. For this to work, one needs an algorithm to evaluate HWVs at points. An evaluation algorithm is even more important to make the following approach work: We know that if $X \not\subseteq Y$, then there exists a HWV f of some weight λ such that $f(Y) = \{0\}$ and $f(x) \neq 0$ for almost all points $x \in X$ [8, Cor. 11.4.2]. This evaluation is a challenging problem in algebraic geometry that is related to deep combinatorics, see [39, 18, 2].

To our best knowledge, we systematically study the complexity of evaluating highest weight vectors for the first time. In Section 5 we first present a known combinatorial method of exactly evaluating HWVs without expanding all the monomials explicitly which has been used to evaluate HWVs at points of small Waring rank as in [2, 13]. Additionally there have been attempts to improve the running time for evaluating at products of linear forms – the so called Chow variety – via dynamic programming [23]. We generalize both approaches in Section 6 to allow evaluation on all points with partial derivative spaces of small dimension, i.e., small ncABP width complexity. In particular, by Theorem 2, this includes for the first time all points of small border Waring rank. Note that our algorithms are particularly useful, because the ncABP width complexity can be determined in polynomial time, whereas determining the Waring rank of a polynomial is NP-hard, even when it is given explicitly as a list of coefficients, see [53].

A HWV can be encoded as a linear combination of Young tableaux, see e.g. [48, §3.9] or [33, Sec. 4.3]. All current evaluation algorithms have a running time exponentially dependent on the size of the Young tableaux. We improve this in Section 7 and establish an algorithm that only depends exponentially on the treewidth of the Young tableau.

Lastly we show in Section 8 that this dependency is basically optimal as we show two lower bounds under the exponential time hypothesis. A lower bound of $2^{\Omega(n)}$ for the decision problem when the HWV $f \in \text{Sym}^n \text{Sym}^d V$ is given by an arbitrary two row Young tableau and a lower bound of $2^{\Omega(\sqrt{n})}$ when it is given by a semistandard Young tableau. Additionally we show NP-hardness for both versions of the decision problem and even #P-hardness for exact evaluations.

We remark that if we restrict ourselves to two-row Young diagrams, then inheritance principles from representation theory [33, Sec. 5.3] let us replace V with \mathbb{C}^2 . Then $\text{Sym}^d \mathbb{C}^2$ is the Hilbert space corresponding to a system of d indistinguishable photons distributed among two modes, which is used in the study of 2-mode linear optical circuits on d indistinguishable particles.

4 Related work

Combinatorics on tableaux for describing highest weight vectors has a rich history dating back to the early invariant theory. This tableau calculus is equivalent to the classical *Feynman diagram calculus* explained in [1], also [48]). Highest weight vectors of a GL_m -representation

W are also called *covariants*, since they correspond to the invariants of $W \otimes (S_\lambda \mathbb{C}^m)^*$, see e.g. [47, Def. 3.9]. Recently, these methods have been applied in various areas, see [38, 3, 49, 22, 43, 2, 13, 19], to name a few.

Border complexity is classically studied in algebraic geometry, see [41]. Bini et al [7] (see also [6]) used it in their construction of fast matrix multiplication algorithms. Studying border complexity in algebraic circuit complexity started with [11, 44] and recently caught momentum [30, 9, 37].

Kronecker coefficients and plethysm coefficients are the dimensions of specific highest weight vector spaces. Algorithms for their computation or theorems about their positivity and value that depend heavily on the shape of the input Young tableau have a long history. For example, if the number of rows of all parameters is constant, then the Kronecker coefficient can be computed in polynomial time [20]. A similar statement is true for plethysm coefficients, see [24]. The software LiE [42] performs all representation theoretic computations with a fixed number of rows. In [34], positivity of Kronecker coefficients depends on comparing Young diagrams with respect to the dominance order, and in [4] the main parameter is the so-called *Durfee size* of the Young diagram, which is the side length of largest square that can be embedded into the Young diagram, see also the very recent [5]. The shape of the Young diagram also plays a crucial role in the recent breakthrough proof of Stembridge's stability conjecture [51]. For two-row Young diagrams much additional structure is known, for example Hermite's classical reciprocity law for plethysm coefficients [31], which makes our lower bound for two-row Young diagrams quite surprising.

Treewidth has been intensely studied by Robertson and Seymour and has been applied numerous times to construct faster graph algorithms for cases where the treewidth is bounded by a function $o(n)$, most notably some algorithms for NP-hard problems restricted to planar graphs, for example 3-coloring. See [21] for an introduction to treewidth algorithms.

Our paper is the first that formally connects the running time of algorithms in representation theory with a graph parameter.

5 Highest Weight Vectors and their combinatorial evaluation

Let $V = \mathbb{C}^m$ be a finite dimensional complex vector space with standard basis e_1, e_2, \dots, e_m . There is a canonical action of $g \in \mathrm{GL}(V)$ on the tensor power $\otimes^d V$ via $g(p_1 \otimes \dots \otimes p_d) := (gp_1) \otimes \dots \otimes (gp_d)$ and linear continuation. This action can be lifted to a linear action on $\mathrm{Sym}^n \otimes^d V$ via

$$(gf)(p) := f(g^t p) \text{ for } f \in \mathrm{Sym}^n \otimes^d V \text{ and } p \in \otimes^d V$$

Note that this makes $\mathrm{Sym}^n \otimes^d V$ a $\mathrm{GL}(V)$ -representation. We denote by $\mathrm{Sym}^d V \subseteq \otimes^d V$ the vector space of symmetric tensors over V of order d and by $p_1 \odot \dots \odot p_d := \sum_{\pi \in \mathfrak{S}_d} \frac{1}{d!} p_{\pi(1)} \otimes \dots \otimes p_{\pi(d)}$ the symmetric tensor product of $p_1, \dots, p_n \in V$. The linear subspace $\mathrm{Sym}^d V \subseteq \otimes^d V$ is closed under the action of $\mathrm{GL}(V)$. This action can be lifted to a linear action on $\mathrm{Sym}^n \mathrm{Sym}^d V$ via

$$(gf)(p) = f(g^t p) \text{ for } f \in \mathrm{Sym}^n \mathrm{Sym}^d V \text{ and } p \in \mathrm{Sym}^d V$$

Note that this makes $\mathrm{Sym}^n \mathrm{Sym}^d V$ a $\mathrm{GL}(V)$ -representation.

We call a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ a *partition of $N \in \mathbb{N}$* if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0$ and $\sum_{i \geq 1} \lambda_i = N$. In our case we will usually have $N = nd$. We denote the transpose partition λ^t by μ and define it as $\mu_i = |\{j \mid \lambda_j \geq i\}|$. Note that μ is also a partition of N . We will write partitions as finite sequences and omit all the trailing zeros.

For any GL_m representation W , a *highest weight vector* $f \in W$ of type λ is a vector that satisfies

1. f is invariant under the action of any $g \in \mathrm{GL}_m$ when g is upper triangular with 1s on the diagonal.
2. $\mathrm{diag}(\alpha_1, \dots, \alpha_m)f = \alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} f$ where $\mathrm{diag}(\alpha_1, \dots, \alpha_m)$ is the diagonal matrix with $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ on the diagonal.

The highest weight vectors of type λ form a vector space which we call $\mathrm{HWV}_\lambda(W)$. We denote by $\mathrm{HWV}(W)$ the vector space of all HWVs in W without any weight restriction.

The smallest example is the discriminant polynomial $b^2 - 4ac$ in $\mathrm{Sym}^2 \mathrm{Sym}^2 \mathbb{C}^2$, see [8, Exa. 9.1.4] for which we have $g(b^2 - 4ac) = \det(g)^2(b^2 - 4ac)$.

We first derive a combinatorial description of the evaluation of highest weight vectors. We follow [18, 13].

We can describe the highest weight vectors of $\mathrm{Sym}^n \mathrm{Sym}^d V$ in terms of so called Young tableaux (see also [48, §3.9]).

► **Definition 3.** A Young tableau T of shape $\lambda = (\lambda_1, \dots, \lambda_r)$ where λ is a partition is a left justified array of boxes where row i contains λ_i boxes and each box contains a positive integer. If the tableau contains the numbers 1 through n each d times it is said to have (rectangular) content $n \times d$, for example $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline \end{array}$ has content 3×2 . A Young tableau is said to be semistandard if the entries are strictly increasing in each column and non-decreasing in each row, for example $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & & \\ \hline \end{array}$ is semistandard, while $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 & 1 \\ \hline 2 & 3 & & \\ \hline \end{array}$ is not. A Young tableau is said to be standard if the entries are strictly increasing in each column and row and every entry occurs exactly once. For example, $\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 6 & 7 & 8 \\ \hline 2 & 5 & 9 & & & \\ \hline \end{array}$ is standard.

Fix a tableau T of shape λ with content $(nd) \times 1$ and fix a tensor $p = \sum_{i=1}^r \ell_{i,1} \otimes \cdots \otimes \ell_{i,d} \in \otimes^d \mathbb{C}^m$. We use arithmetic modulo d with the system of representatives $\{1, \dots, d\}$, so $a \bmod d \in \{1, \dots, d\}$. Each of the sets $\{1, \dots, d\}, \{d+1, \dots, 2d\}, \dots$ is called a *block*. We define $k(a) := \lceil a/d \rceil$. We define $j(a) := a \bmod d$, which gives the position of the element a in its block. A placement

$$\vartheta : \{1, \dots, nd\} \rightarrow \{\ell_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq d\}$$

is called *proper* if there is a map $\varphi : \{1, \dots, n\} \rightarrow \{1, \dots, r\}$ such that $\vartheta(a) = \ell_{\varphi(k(a)), j(a)}$. We define the determinant of a matrix that has more rows than columns as the determinant of its largest top square submatrix.

We define the polynomial f_T via its evaluation on p :

$$f_T(p) := \sum_{\text{proper } \vartheta} \prod_{c=1}^{\lambda_1} \det_{\vartheta, c} \quad \text{with} \quad \det_{\vartheta, c} := \det(\vartheta(T(1, c)) \dots \vartheta(T(\mu_c, c))) \quad (2)$$

Pictorially φ chooses one of the rank 1 tensors for each block of d numbers and places those onto T . Then we take the product of the columnwise determinants. The evaluation $f(p)$ is now the sum over all possible choices.

It is a classical result from multilinear algebra that this construction yields a well-defined polynomial of weight λ on $\otimes^d \mathbb{C}^m$. If T is the column-standard tableau, then $f_T \in \mathrm{HWV}_\lambda(\mathrm{Sym}^n \otimes^d \mathbb{C}^m)$ is not hard to verify. Schur-Weyl duality states that $\otimes^n \otimes^d \mathbb{C}^m = \bigoplus_\lambda S_\lambda(V) \otimes [\lambda]$, where the sum goes over all partitions λ of nd into at most m parts, and where $S_\lambda(V)$ is the irreducible GL_m -representation of type λ (called the Schur module) and $[\lambda]$ is the irreducible \mathfrak{S}_{dn} -representation of type λ (called the Specht module). Since a basis of $[\lambda]$ is given by the standard tableaux of shape λ , this immediately implies that

$$\mathrm{HWV}_\lambda(\mathrm{Sym}^n \otimes^d \mathbb{C}^m) \text{ is the linear span of the } f_T, \text{ where } T \text{ is standard of shape } \lambda. \quad (3)$$

See for example [48] or [8, Ch. 19] for a detailed exposition.

The following Lemmas 4 and 5 follow from eq. (2).

► **Lemma 4.** *Let T and T' be Young tableaux of the same shape with content $(nd) \times 1$ such that T' can be obtained from T by performing permutations within the blocks. The functions f_T and $f_{T'}$ coincide after restricting their domains of definition from $\otimes^d \mathbb{C}^m$ to $\text{Sym}^d \mathbb{C}^m$.*

Proof. If p is symmetric, then p has a Waring rank decomposition, i.e., there exists $r \in \mathbb{N}$ and homogeneous linear forms p_1, \dots, p_r such that $p = \sum_{i=1}^r p_i^{\otimes d}$. Using this decomposition for p , we see that the summands of $f_T(p)$ and $f_{T'}(p)$ in (2) coincide. ◀

Lemma 4 implies that in order to define the restriction of f_T to symmetric tensors we only need to define the blocks in T , but not the internal structure of the blocks. Thus for a tableau with content $(nd) \times 1$ we define the tableau \hat{T} by replacing all entries $a \in \{1, \dots, nd\}$ by $k(a)$. The resulting tableau \hat{T} has content $n \times d$. For example, if $n = 2, d = 4, T = \begin{bmatrix} 1 & 3 & 4 & 6 & 7 & 8 \\ 2 & 5 & & & & \end{bmatrix}$, then $\hat{T} = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & & & & \end{bmatrix}$. For a tableau \hat{T} with content $n \times d$ we define $f_{\hat{T}} \in \text{Sym}^d \text{Sym}^n \mathbb{C}^m$ as the restriction of f_T to $\text{Sym}^n \mathbb{C}^m$.

► **Lemma 5.** *Let T be a Young tableau that has a column in which there are two or more entries from the same block. Then $f_T = 0$.*

Proof. Let c be the column in T in which there are two or more entries from the same block. As in Lemma 4, consider the evaluation of f_T at a point p in its Waring rank decomposition. We observe that every summand in eq. (2) is zero, because the determinant corresponding to the column c has a repeated column. ◀

In other words, Lemma 5 says that $f_{\hat{T}} = 0$ if \hat{T} contains a column in which a number appears at least twice. Combining this insight with eq. (3), we conclude that

$$\text{HWV}_{\lambda}(\text{Sym}^n \text{Sym}^d \mathbb{C}^m) \text{ is the linear span of the } f_{\hat{T}},$$

$$\text{where } \hat{T} \text{ is semistandard of shape } \lambda \text{ with content } n \times d. \quad (4)$$

► **Remark 6.** From eq. 2 and writing p in its Waring rank decomposition, we immediately get an $O(\text{WR}(p)^n \cdot \text{poly}(n, d, m))$ algorithm to evaluate $f_{\hat{T}}(p)$.

6 Non-commutative algebraic branching programs

For an in-depth formal study of ncABPs we now introduce additional notation (cp. Section 1).

► **Definition 7.** *Let V be a vector space.*

- *A non-commutative algebraic branching program (ncABP) A is an acyclic directed graph with two distinguished nodes s and t and edges labeled with elements from V and every path from s to t having the same length. This makes A layered, with layer k containing all vertices of distance k from s .*
- *The weight $w(P)$ of a path P with edge labels $\ell_1, \dots, \ell_d \in V$ is defined as $w(P) := \ell_1 \otimes \dots \otimes \ell_d$.*
- *The tensor computed at a node v in A is $\hat{w}(v) = \sum_{s \rightarrow v \text{ path } P} w(P)$. By convention the tensor computed at s is 1.*
- *The tensor computed by A is the tensor computed at t .*
- *The size of an ncABP is the number of vertices.*
- *The width of an ncABP is the largest number of vertices in any layer.*

In particular we will be looking at ncABPs computing symmetric tensors p and the evaluation of highest weight vectors at p . An example is given in Figure 1.

Each node in layer k computes a tensor in $V^{\otimes k}$. We show in Proposition 12 that there is always a minimal ncABP where all these computed tensors are also symmetric and whose size is exactly the size of the partial derivative space of p . An example is given in Figure 1.

We can now use the “overlapping structure of the paths through ncABPs” to our advantage in evaluating HVWs by using dynamic programming.

► **Theorem 8.** *The evaluation $f_T(p)$ of a highest weight vector $f_T \in \text{Sym}^n \text{Sym}^d \mathbb{C}^m$ given by a Young tableau T with content $(nd) \times 1$ and r rows and a symmetric tensor $p \in \text{Sym}^d \mathbb{C}^m$ given by an ncABP of width w can be computed in time $O(w^{n+r} \text{poly}(n, d, m))$.*

Proof. Let A be an ncABP with source s , sink t , and width w computing a symmetric tensor $p \in \text{Sym}^d \mathbb{C}^m$. W.l.o.g. let the numbers $i \cdot d + 1, \dots, i \cdot d + d$ occur in order left to right in T for any $i \in \{0, \dots, n-1\}$, see Lemma 4. Note that left to right is a unique ordering since if one column contains multiple of these numbers we already know $f_T = 0$, see Lemma 5.

Combining eq. (2) with $p = \hat{w}(t) = \sum_{s-t \text{ path } P} w(P)$ (see Def. 7) we see that

$$f_T(p) := \sum_{\text{proper } \vartheta} \prod_{c=1}^{\lambda_1} \det_{\vartheta, c} \quad \text{with} \quad \det_{\vartheta, c} := \det(\vartheta(T(1, c)) \dots \vartheta(T(\mu_c, c))), \quad (5)$$

where here $\vartheta : \{1, \dots, nd\} \rightarrow V$ is called *proper* if there exists $\varphi : \{1, \dots, d\} \rightarrow \{s-t \text{ path } P\}$ such that $\vartheta(a) =$ the label of the $j(a)$ -th edge of $\varphi(k(a))$ (see the definitions of j and k in Section 5).

We now calculate partial evaluations in a column by column fashion from right to left. In order to do this we define a partial placement $\vartheta|_{\leq k}$ to be the restriction of ϑ to the boxes in the first k columns of T .

We now observe a common factor for a fixed partial placement $\vartheta|_{\leq k}$:

$$\sum_{\text{proper } \vartheta \text{ extending } \vartheta|_{\leq k}} \prod_{c=1}^{\lambda_1} \det_{\vartheta, c} = \left(\prod_{c=1}^k \det_{\vartheta|_{\leq k, c}} \right) \underbrace{\left(\sum_{\text{proper } \vartheta \text{ extending } \vartheta|_{\leq k}} \prod_{c=k+1}^{\lambda_1} \det_{\vartheta, c} \right)}_{=: \alpha(\vartheta|_{\leq k})}.$$

Each $\vartheta|_{\leq k}$ defines a set of n partial $s-t$ paths (of potentially different lengths, one path for each block), where $\alpha(\vartheta|_{\leq k})$ only depends on the endpoints of these paths. These paths are connected from s up to these endpoints due to the nature of T being ordered from left to right for each block of n numbers. This crucial observation allows us to store and reuse these values of α whenever two partial assignments correspond to lists of n paths ending in the same vertices of A .

We can now calculate the evaluation as $f_T(p) = \alpha(\vartheta|_{\leq 0}) = \alpha(\emptyset)$.

Since the length of each of the paths defined by any $\vartheta|_{\leq k}$ are fixed for fixed k , there are at most w^n possible different values for α that need to be computed. So in total this evaluation algorithm has running time $O(w^{n+r} \text{poly}(n, d, m))$. The w^r term comes from all the possibilities to extend a given $\vartheta|_{\leq k}$ by one column of T . ◀

► **Remark 9.** Note that Theorem 8 is a generalisation of the dynamic programming used in [23] to evaluate HVWs at the Chow variety Ch_m^d . The Chow variety Ch_m^d consists of products of d linear forms $\ell_1 \odot \dots \odot \ell_d \in \text{Sym}^d \mathbb{C}^m$. Here the minimal ncABP A of $\ell_1 \odot \dots \odot \ell_d$ corresponds to having subsets of $\{1, \dots, d\}$ as vertices where two vertices $U, V \subseteq \{1, \dots, n\}$ are connected by an edge labeled ℓ_i iff $U \setminus V = \{i\}$ and $U \supset V$. Then A has size exactly 2^d and width $\binom{d}{k}$ on layer k while layer k contains all the sets of size k .

We now give the connection between the width of ncABPs, and the dimension of the partial derivative spaces of the symmetric tensors computed by the ncAPB. We additionally show that ncABPs can efficiently compute partial derivatives.

First note that the following equivalence between partial derivatives and polynomial contractions is well known for fields of characteristic 0, see for example [32, Equation 1.1.2]. We reformulate this as an equivalence between partial derivatives and tensor contractions instead.

► **Lemma 10.** *Let φ be the canonical isomorphism between $\text{Sym}^d \mathbb{C}^m$ and $\mathbb{C}[x_1, \dots, x_m]_d$ defined via $\varphi(e_{i_1} \odot \dots \odot e_{i_d}) = x_{i_1} \dots x_{i_d}$. Then the partial derivative $\frac{\partial^k}{\partial \ell_1 \dots \partial \ell_k} t$ of a symmetric tensor $t \in \text{Sym}^d V$ is given by the tensor contraction $\frac{d!}{(d-k)!} \langle \ell_1 \otimes \dots \otimes \ell_k, t \rangle$.*

Since t is symmetric the partial derivative $\frac{\partial^k}{\partial \ell_1 \dots \partial \ell_k} t$ is also given by $\frac{d!}{(d-k)!} \langle \ell_1 \odot \dots \odot \ell_k, t \rangle$.

Proof. It suffices to prove this for the case $k = 1$ since repeated tensor contraction is the same as one big tensor contraction and the same holds for partial derivatives. Since both tensor contraction and taking derivatives are linear operations in both parameters we can restrict ourselves to the derivative $\frac{\partial}{\partial e_i} (e_{j_1} \odot \dots \odot e_{j_d})$.

In case e_i is not any of e_{j_1}, \dots, e_{j_d} clearly

$$\frac{\partial}{\partial e_i} (e_{j_1} \odot \dots \odot e_{j_d}) = 0 = \frac{d!}{(d-k)!} \langle e_i, e_{j_1} \odot \dots \odot e_{j_d} \rangle$$

so w.l.o.g. we can now assume due to symmetry $e_{j_1} = e_i$.

We can write $\varphi(e_i \odot e_{j_2} \odot e_{j_3} \odot \dots \odot e_{j_d}) = x_i^h \cdot q$ for some monomial $q \in \mathbb{C}[x_1, \dots, x_m]$ not containing x_i .

$$\begin{aligned} \varphi\left(\frac{\partial}{\partial e_i} (e_i \odot e_{j_2} \odot e_{j_3} \odot \dots \odot e_{j_d})\right) &= h \cdot x_i^{h-1} \cdot q \\ &= \varphi(h \cdot e_{j_2} \odot e_{j_3} \odot \dots \odot e_{j_d}) \\ &= \varphi(\langle e_i, h \cdot e_i \otimes (e_{j_2} \odot e_{j_3} \odot \dots \odot e_{j_d}) \rangle) \\ &= \varphi(\langle e_i, d \cdot e_i \odot e_{j_2} \odot e_{j_3} \odot \dots \odot e_{j_d} \rangle) \end{aligned}$$

The last equality follows from the fact that all terms of the symmetric tensor not containing e_i as the first component of the tensor vanish under the tensor contraction. ◀

► **Lemma 11.** *If A is an ncABP computing a symmetric tensor $p \in \text{Sym}^d V$, then the k -th derivatives are linear combinations of the tensors computed at the $(d-k)$ -th layer of A .*

Proof. As proven in Lemma 10 the derivatives are just tensor contractions. A tensor contraction on an ncABP replaces the last k edges on each s - t path by constants³, thus directly proving the claim. ◀

We will now characterize the minimal size of ncABPs via the dimension of the partial derivative spaces. For this we denote by $\partial^{=k}(t)$ the partial derivative space of k -th order for $t \in \text{Sym}^d V$:

$$\partial^{=k}(t) := \{ \langle q, t \rangle \mid q \in \text{Sym}^k V \}$$

³ due to the symmetry of p we could even choose any k layers and all outgoing edges out of these chosen layers would be replaced by constants for the derivative.

Analogously we define

$$\partial^{\leq k}(t) := \text{span} \bigcup_{i=0}^k \partial^{\leq i}(t).$$

Note that the usage of tensor contractions instead of derivatives is just for simplicity.

For a list $q \in \{1, \dots, m\}^k$ let $e_q := e_{q_1} \otimes \dots \otimes e_{q_m}$. For a tensor $p \in \otimes^d \mathbb{C}^m$ we define the $m^k \times m^{d-k}$ matrix $M_k(p)$ whose rows are indexed by elements $q \in \{1, \dots, m\}^k$ and whose columns are indexed by elements in $q' \in \{1, \dots, m\}^{d-k}$ via

$$M_k(p)[q, q'] := \text{the coefficient of } e_q \otimes e_{q'} \text{ in } p. \quad (6)$$

► **Proposition 12.** *If A is an ncABP computing a symmetric tensor $p \in \text{Sym}^d V$, then there is an ncABP B with the following properties:*

1. B also computes p .
2. Each layer of B has at most as many vertices as the same layer in A .
3. Each node of B computes a symmetric tensor.
4. The k -th layer of B has precisely $\dim \partial^{\leq k}(p)$ many vertices which is the optimal width.

Proof. We mainly follow Nisan [46] with this construction who constructed minimal ncABPs and extend this to also compute symmetric tensors at each node and establishing the connection to the dimensions of the partial derivative spaces.

Let v_1, \dots, v_t be the vertices in a fixed layer k . Let $M_k[q, q'] := M_k(p)[q, q']$ from eq. (6). Note that the row of M_k corresponding to q is given precisely by the tensor contraction $\langle q, p \rangle$ and it is thus by Lemma 10 a partial derivative of k -th order. Therefore $\text{rank } M_k = \dim \partial^{\leq k}(p)$.

Now we can construct two matrices L_k and R_k . Here $L_k[q, i]$ for indices $q \in \{e_1, \dots, e_{\dim V}\}^{\otimes k}$ is defined as the coefficient of q in $\hat{w}(v_i)$ and $R_k[i, q']$ for indices $q' \in \{e_1, \dots, e_{\dim V}\}^{\otimes (d-k)}$ is defined as the coefficient of q' in the tensor computed by the restricted ncABP with source v_i . It is easy to verify $M_k = L_k R_k$.

Hence if $t > \text{rank } L_k$ there must be some vertices v_i computing a linear combination of the other vertices in the same layer, thus all outgoing edges of v_i can be replaced by precisely this linear combination, allowing us to remove v_i . In this way we can remove some v_i as long as $t > \text{rank } R_k$.

After this process finishes we have $t = \text{rank } L_k = \text{rank } R_k = \text{rank } M_k = \dim \partial^{\leq k}(p)$ proving the claims on the width of the layers.

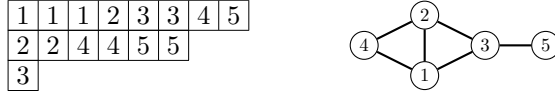
Since by Lemma 11 all the k -th partial derivatives are linear combinations of restrictions of the ncABP to the first k levels we can now replace all vertices on the k -th level by t vertices computing a symmetric tensor basis of the k -th partial derivatives thus proving the remaining claim. ◀

From this characterization of ncABP size as the rank of the partial derivative matrices we can also see that ncABP size is preserved under approximation. This was remarked by Michael Forbes [25], but we give a proof for the sake of completeness.

► **Corollary 13.** *Let $p \in \text{Sym}^d V$ and $(A_i)_{i \in \mathbb{N}}$ be ncABPs s.t. A_i computes $p_i \in \otimes^d V$ and has size $s_i \leq s$ and width $w_i \leq w$ with $\lim_{i \rightarrow \infty} p_i = p$. Then there is an ncABP A computing p with size s and width w .*

Proof. Let the matrices $M_{k, p_i} := M_k(p_i)$ from eq. (6). We have

$$M_{k, p} = \lim_{i \rightarrow \infty} M_{k, p_i}.$$



■ **Figure 2** An example of a Young tableau \hat{T} and the corresponding graph $G_{\hat{T}}$.

Since each A_i has width at most w , we know that $\text{rank } M_{k,p_i} \leq w_i \leq w$. This is characterized by all determinants of $(w+1) \times (w+1)$ minors of M_{k,p_i} vanishing. So by continuity of the determinant also all $(w+1) \times (w+1)$ minors of $M_{k,p}$ vanish and thus $\dim \partial^{\leq k}(p) = \text{rank } M_{k,p} \leq w$ and there is an ncABP A with width w by Proposition 12.

To get the upper bound on the size of A we note that the partial derivatives of different orders are linearly independent, so $\dim \partial^{\leq d}(p) = \sum_{j=0}^d \dim \partial^j(p) = s$. This is the same as looking at the rank of the direct sum $\bigoplus_{j=0}^d M_{j,p}$, so the bound on the size of A follows from the same argument. ◀

From this we can conclude an order of inclusion on the sets of symmetric tensors of small Waring rank, small border Waring rank and small non-commutative ncABP size.

► **Corollary 14.** *Let $k \in \mathbb{N}$ and*

$$W_{k,d} := \{p \in \text{Sym}^d V \mid \text{WR}(p) \leq k\},$$

$$\overline{W}_{k,d} := \{p \in \text{Sym}^d V \mid \underline{\text{WR}}(p) \leq k\},$$

$$B_{k,d} := \{p \in \text{Sym}^d V \mid \text{ncw}(p) \leq k\}.$$

$$\overline{B}_{k,d} := \{p \in \text{Sym}^d V \mid \underline{\text{ncw}}(p) \leq k\}.$$

Then

$$W_{k,d} \subsetneq \overline{W}_{k,d} \subseteq B_{k,d} = \overline{B}_{k,d}.$$

Proof. The inclusion $W_{k,d} \subseteq \overline{W}_{k,d}$ is trivial and $B_{k,d} = \overline{B}_{k,d}$ is proven in Corollary 13. To show $W_{k,d} \subsetneq \overline{W}_{k,d}$ is strict we refer to [17] showing that $x^{d-1}y$ has Waring rank d while it is known⁴ that $x^{d-1}y = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^d} ((x + \varepsilon y)^d - x^d)$ and thus $x^{d-1}y$ has border Waring rank at most 2. For the inclusion $W_{k,d} \subseteq B_{k,d}$ we can embed the k summands ℓ_i^d of the Waring rank decomposition as disjoint $s-t$ paths in an ncABP of width k and depth d . Here every edge on the path corresponding to ℓ_i^d has the label ℓ_i . Since $B_{k,d}$ is closed this immediately proves $\overline{W}_{k,d} \subseteq B_{k,d}$. ◀

Note that it is still unknown whether $B_{k,d} \subseteq W_{q(k),d}$ or $B_{k,d} \subseteq \overline{W}_{q(k),d}$ for any polynomial q .

7 Treewidth of Young tableaux

Let S be an arbitrary Young tableau containing the numbers $\{1, \dots, n\}$. We can associate with S the undirected graph $G_S = (V_S, E_S)$ where $V_S = \{1, \dots, n\}$ and $\{i, j\} \in E_S$ iff i and j are contained in some common column in S , see Figure 2.

⁴ Technically we need here that our base field is algebraically closed in order for this to be a border Waring rank decomposition, but \mathbb{C} satisfies this.

We are now going to study how we can use the graph parameter treewidth of G_S to speed up the evaluation of highest weight vectors. Treewidth has been intensely studied by Robertson and Seymour and has been applied numerous times to construct faster graph algorithms for cases where the treewidth is bounded by a function $o(n)$, most notably some algorithms for NP-hard problems restricted to planar graphs, for example 3-coloring. See [21] for an introduction to treewidth algorithms.

► **Definition 15.** A tree decomposition of a graph $G = (V, E)$ is a tree \mathcal{T} with nodes X_1, X_2, \dots, X_t called bags where $X_i \subseteq V$ and the following properties hold:

- $\cup_{i=1}^t X_i = V$
- For every edge $\{u, v\} \in E$ there is some bag X_i , s.t. $\{u, v\} \subseteq X_i$.
- For every vertex $v \in V$ the bags containing v form a subtree of \mathcal{T} .

The width of a tree decomposition is the size of the largest bag minus one. The treewidth of G is then the smallest possible width of a tree decomposition for G .

Often solving problems on graphs of bounded treewidth is easier than the general problem and indeed this is also the case for evaluating the highest weight vector corresponding to a graph if the graph $G_{\hat{T}}$ has bounded or low treewidth.

► **Theorem 16.** The evaluation $f_{\hat{T}}(p)$ for a highest weight vector $f_{\hat{T}} \in \text{Sym}^n \text{Sym}^d \mathbb{C}^m$ given by a Young tableau \hat{T} with content $n \times d$ and a symmetric tensor $p \in \text{Sym}^d \mathbb{C}^m$ given by an ncAPB of width w can be computed in time $w^{\omega(\tau+1)} \text{poly}(n, d, m)$ if a tree decomposition \mathcal{T} of $G_{\hat{T}}$ of width τ is given and given that we can multiply two matrices of size $\leq k \times k$ in time $O(k^\omega)$.

Proof. A tableau \hat{T} with its corresponding graph $G_{\hat{T}}$ is depicted in Figure 3(a) and (b). It is well known that every clique of a graph is fully contained in some bag of its tree decomposition. Every column c of \hat{T} corresponds to a clique in $G_{\hat{T}}$, so there is some bag X_i of \mathcal{T} which contains all the vertices corresponding to the entries of c . We modify \mathcal{T} by adding a new copy of X_i which is only adjacent to X_i . This copy is from now on associated with the column c . An example is given in Figure 3(c). From now on we only need the structure of the subtree \mathcal{T}' of \mathcal{T} whose leaves are the vertices associated with columns and every vertex removed that is not on the path between two of these vertices, see Figure 3(e). We interpret \mathcal{T}' as an ordered binary tree rooted at r . In case any vertex v of \mathcal{T}' has more than two children, we replace v by a binary tree. We now denote by $\sigma \in \mathfrak{S}_{\lambda_1}$ the order of the columns of \hat{T} in the in-order traversal of \mathcal{T}' . In the same way as for the evaluation in Theorem 8 we now need to choose a suitable tableau T of content $(nd) \times 1$ which is a preimage of \hat{T} under the $\hat{\cdot}$ -operation, see Section 5. So we choose T s.t. the entries with number i in \hat{T} are replaced by $(i-1) \cdot d + 1, (i-1) \cdot d + 2, \dots, i \cdot d$ in the order given by σ restricted to the columns containing i , see Figure 3(d).

We now calculate partial evaluations starting at the leaves of \mathcal{T}' and working our way up. In order to do this we define a partial placement $\vartheta|_{\geq j, \leq k}$ to be the restriction of ϑ to the boxes in the columns $\sigma(j), \sigma(j+1), \dots, \sigma(k)$.

Since we are working with ncABPs it is easier to interpret each $\vartheta|_{\geq j, \leq k}$ as a list of paths in the ncABP A . If we define $\kappa_{\leq j}(i)$ as the number of times the entry i occurs in the columns $\sigma(1), \sigma(2), \dots, \sigma(j-1)$ of \hat{T} , then the i -th path in $\vartheta|_{\geq j, \leq k}$ starts at some vertex in the $\kappa_{\leq j}(i)$ -th layer of A and ends in the $\kappa_{\leq k}(i)$ -th layer of A . Specifically we define the functions $\Phi_j : \{1, \dots, n\} \rightarrow V(A)$ with $\Phi_j(i) \in L_{\kappa_{\leq j}(i)}$ and say $\vartheta|_{\geq j, \leq k}$ respects (Φ_j, Φ_k) if the i -th path defined by $\vartheta|_{\geq j, \leq k}$ starts at $\Phi_j(i)$ and ends at $\Phi_k(i)$.

If the subtree of \mathcal{T}' rooted at v with bag X_v contains the columns $\sigma(j), \sigma(j+1), \dots, \sigma(k)$ we define

$$D[v, \Phi_j, \Phi_k] = \sum_{\substack{\vartheta|_{\geq j, \leq k} \\ \text{respects } (\Phi_j, \Phi_k)}} \prod_{c=j}^k \det_{\vartheta|_{\geq 1, \leq \lambda_1, \sigma(c)}}.$$

We claim that $D[v, \Phi_j, \Phi_k]$ only depends on the restrictions $\Phi_j|_{X_v}$ and $\Phi_k|_{X_v}$ which in turn implies that we only need to calculate at most $w^{2|X_v|} \leq w^{2(\tau+1)}$ entries of D for each vertex v . So let $i \notin X_v$, then either i does not appear in any column inside the subtree rooted at v , so $D[v, \Phi_j, \Phi_k]$ trivially doesn't depend on $\Phi_j(i)$ and $\Phi_k(i)$ or i does not appear in any column outside the subtree rooted at v . In that case $\Phi_j(i) \in L_0$ and $\Phi_k(i) \in L_d$, so both only have a single possibility and are the source and sink of A respectively.

Let s and t be the source and sink of A respectively. Then the evaluation $f_{\hat{T}}(p) = f_T(p)$ is given as

$$f_T(p) \stackrel{(2)}{=} \sum_{\text{proper } \vartheta} \prod_{c=1}^{\lambda_1} \det_{\vartheta, c} = \sum_{\substack{\vartheta|_{\geq 1, \leq \lambda_1} \\ \text{respects } (i \mapsto s, i \mapsto t)}} \prod_{c=1}^{\lambda_1} \det_{\vartheta|_{\geq 1, \leq \lambda_1, \sigma(c)}} = D[r, i \mapsto s, i \mapsto t].$$

Here $i \mapsto s$ and $i \mapsto t$ just denote the constant functions assigning s or t to every path.

Only remains to show how to calculate $D[v, \Phi_j, \Phi_k]$: For any leaf v of \mathcal{T}' with associated column $(c_1, c_2, \dots, c_{|c|})$ we have

$$D[v, \Phi_j, \Phi_k] = \det(\ell_{(\Phi_j(c_1), \Phi_k(c_1))}) \ell_{(\Phi_j(c_2), \Phi_k(c_2))} \cdots \ell_{(\Phi_j(c_{|c|}), \Phi_k(c_{|c|}))}$$

where $\ell_{(v_1, v_2)}$ denotes the label of the edge between the two vertices $v_1, v_2 \in A$ or 0 if this edge does not exist. Note that $\Phi_j(c_i)$ and $\Phi_k(c_i)$ are always two vertices of consecutive layers in A .

For any inner vertex v of \mathcal{T}' with only one child v' we have

$$D[v, \Phi_j, \Phi_k] = D[v', \Phi_j, \Phi_k].$$

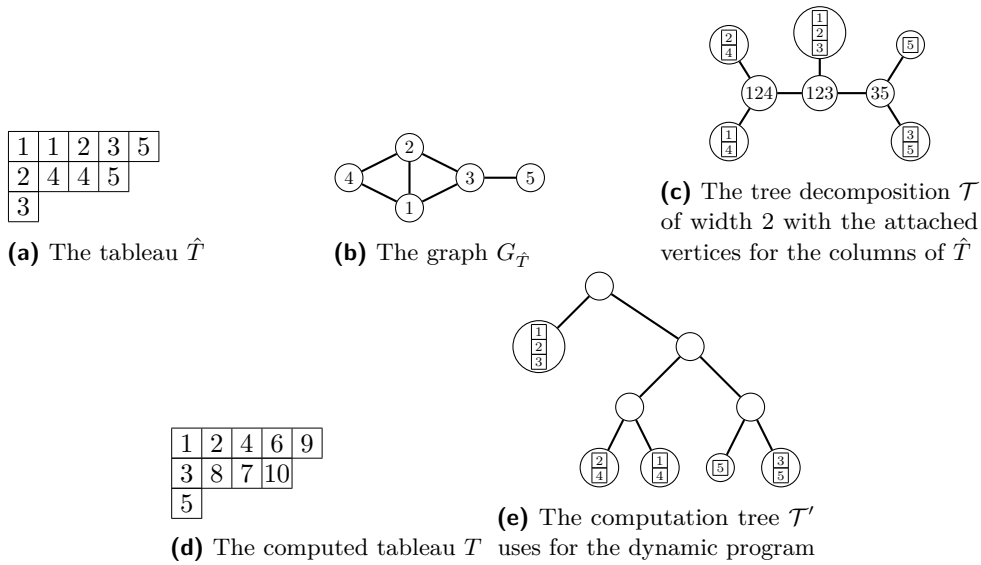
For any inner vertex v of \mathcal{T}' with two children v_1, v_2 where the subtree rooted at v_1 contains z columns we have

$$D[v, \Phi_j, \Phi_k] = \sum_{\Phi_{j+z}} D[v_1, \Phi_j, \Phi_{j+z}] \cdot D[v_2, \Phi_{j+z}, \Phi_k].$$

Here Φ_{j+z} runs over all functions $X_v \rightarrow V(A)$ with $\Phi_{j+z}(i) \in L_{\kappa \leq j+z}(i)$ for every $i \in X_v$. Note that we can calculate these entries simultaneously for all Φ_j, Φ_k as a matrix multiplication of matrices of size at most $w^{(\tau+1)} \times w^{(\tau+1)}$.

The total running time is thus $w^{\omega(\tau+1)} \text{poly}(n, d, m)$ given that we can multiply two matrices of size $\leq k \times k$ in time $O(k^\omega)$. \blacktriangleleft

This dependency on the treewidth instead of n is significant, since for example the graphs of semistandard Young tableaux with only two rows are planar and thus have a treewidth of $O(\sqrt{n})$. Additionally the dependency is tight: we can construct semistandard Young tableaux with rectangular content with two rows which induce multigraph versions of the $n \times n$ grid-graphs and thus have treewidth $\Omega(\sqrt{n})$. We prove both these observations in Proposition 18.



■ **Figure 3** An example execution of the preparation of the algorithm of Theorem 16 for the Young tableau \hat{T} .

Let p be given as a Waring rank decomposition of rank r . From this we can easily construct an ncABP of width w , in the same way we did to prove Theorem 2. Therefore the evaluation algorithm in Theorem 16 now takes time $O(w^{\omega(\tau+1)} \cdot \text{poly}(n, d, m)) = O(r^{\omega(\tau+1)} \cdot \text{poly}(n, d, m))$. Comparing this to the naive algorithm in Remark 6), we get a faster evaluation in the case $\tau \in o(n)$, which for example is achieved for all semistandard tableaux which we will now prove.

As a first step, we will prove

► **Proposition 17.** *Let S be a semistandard Young tableau with two rows. Then G_S is planar.*

Proof. Let S contain the numbers $\{1, \dots, n\}$. We first start by constructing a different graph $G'_S = (L_S \cup R_S, E'_S)$ which is a bipartite graph consisting of two copies of vertices $L_S = \{1_L, \dots, n_L\}$, $R_S = \{1_R, \dots, n_R\}$. Now $\{i_L, j_R\} \in E'_S$ iff $\begin{smallmatrix} i \\ j \end{smallmatrix}$ is a column in S . Here the vertical order in S matters, so due to S being semistandard we know $i < j$. Ordering the vertices, s.t. the vertices on the left and those on the right are each ordered in ascending order, we will now prove that G'_S is outerplanar and can be drawn with straight lines. So let $\{i, j\}, \{k, l\} \in E'_S$ be two different edges where the column $\begin{smallmatrix} i \\ j \end{smallmatrix}$ appears to the left of the column $\begin{smallmatrix} k \\ l \end{smallmatrix}$ in T . Due to T being semistandard this implies $i \leq k$ and $j \leq l$, which means those two edges do not cross. Since the edges were arbitrary no two edges intersect and G'_S is outerplanar.

Because both sets of vertices are ordered in ascending order we can now continuously rotate both vertex sets by 180 degrees and move them on top of each other, in this way unifying both copies of each vertex while still keeping the graph planar (the edges are not straight lines anymore, but they have the shape of a spiral). This resulting graph is precisely G_S , thus proving the claim. ◀

Now we can commence to prove the upper bound on the treewidth of Young tableaux with two rows. Additionally we prove that this bound is tight.

- **Proposition 18.** 1. Let S_n be a semistandard Young tableau with two rows containing the numbers $\{1, \dots, n\}$. Then G_{S_n} has treewidth at most $O(\sqrt{n})$.
2. Additionally there is a family (S'_n) of semistandard Young tableaux with two rows containing the numbers $\{1, \dots, n\}$ exactly 4 times each and $G_{S'_n}$ having treewidth $\Omega(\sqrt{n})$.

Proof. Let S_n be a semistandard Young tableau with 2 rows containing the numbers $\{1, \dots, n\}$. Then G_{S_n} is a planar graph with n vertices by Proposition 17. The fact that planar graphs on n vertices have treewidth bounded by $O(\sqrt{n})$ follows directly from the famous planar excluded grid theorem [50].

W.l.o.g. we can restrict n to be of the form $(2k)^2$ with $k \in \mathbb{N} \setminus \{0\}$, since we can always extend the tableau without increasing the treewidth by appending four columns containing only a single cell with the number $i+1$ to the end of S'_i to get S'_{i+1} . This change corresponds to adding a new isolated vertex to $G_{S'_i}$. We repeat this until $N = (2k)^2$, which scales n up by at most a factor of 8.

Every layered multigraph $G = (V, E)$ with the following properties is the graph G_S corresponding to some semistandard tableaux S where each number i appears exactly as often as the degree of i in G :

1. $V = \{1, \dots, n\}$
2. Edges in G only go from one layer to the next.
3. Edges between any two layers can be drawn with straight lines without crossing when the vertices in each layer are placed in ascending order.
4. All vertices in any layer j are labeled smaller than those in layer $j+1$ and each form a consecutive sequence of integers.

Some examples are provided in Figure 4. This can be shown constructively and separately for every pair of layers j and $j+1$. Since the edges between two layers are not crossing, there is a unique ordering on the set of edges from left to right. Adding columns corresponding to the edges in exactly this order to S forms exactly the wanted semistandard tableaux: For $\{u, v\} \in E$ we add the column $\begin{smallmatrix} u \\ v \end{smallmatrix}$ to S . Thus the entries corresponding to layer j are only in the first row while those corresponding to layer $j+1$ only appear in the second row. Because of property (4) the columns of edges from layer j to layer $j+1$ can directly be concatenated to the columns of edges from layer $j+1$ to layer $j+2$ without violating the property of being semistandard. Clearly S contains each number i exactly once for each incident edge of i in G .

We now take the $2k \times 2k$ grid $\boxplus_{2k} = (V_{2k}, E_{2k})$ where

$$V_{2k} = \{(x, y) \mid x, y \in \{1, \dots, 2k\}\}$$

$$E_{2k} = \{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| = 1\}$$

This graph is known to have treewidth exactly $2k$ [21]. We now create a multigraph by doubling all the edges $\{(1, 2i-1), (1, 2i)\}, \{(2k, 2i-1), (2k, 2i)\}, \{(2i-1, 1), (2i, 1)\}$ and $\{(2i-1, 2k), (2i, 2k)\}$ for every $i \in \{1, \dots, k\}$ which results in each vertex having degree exactly 4 while not changing the treewidth. To now apply the previous observations we now treat each diagonal $\{(x, y) \mid x + y = j + 1\}$ as layer j and label them by increasing x , thus proving the claim of the lower bound. The resulting graphs are also visualized in Figure 4. ◀

- **Question 19.** It is open whether the bound of $O(\sqrt{n})$ on the treewidth can be extended

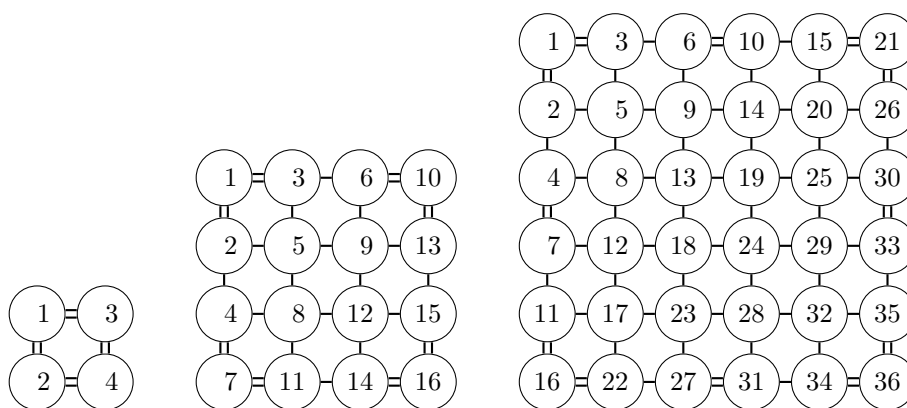


Figure 4 The grid graphs $\mathbb{E}_2, \mathbb{E}_4$ and \mathbb{E}_6 after doubling the correct edges around the border and relabeling the vertices. The layers in \mathbb{E}_2 are $\{1\}, \{2, 3\}, \{4\}$. The layers in \mathbb{E}_4 are $\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \{11, 12, 13\}, \{14, 15\}, \{16\}$. The layers in \mathbb{E}_6 are $\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \{11, 12, 13, 14, 15\}, \{16, 17, 18, 19, 20, 21\}, \{22, 23, 24, 25, 26\}, \{27, 28, 29, 30\}, \{31, 32, 33\}, \{34, 35\}, \{36\}$.

to any other constant number of rows, but starting at 3 rows G_S becomes non-planar⁵, so another approach to solving this problem would be needed. Additionally, if the number of rows is arbitrary G_S can contain an arbitrarily big clique, so it can have arbitrarily high treewidth. For example for any S with a first column with n distinct entries the graph G_S contains a clique on n vertices and thus has treewidth at least $n - 1$.

8 Hardness of evaluation

We will show that deciding whether a highest weight vector $f_{\hat{T}}$ of $\text{Sym}^n \text{Sym}^d \mathbb{C}^m$ vanishes at a point of Waring rank k for suiting parameters n, d, m, k is NP-hard. In particular we prove the NP-hardness of evaluating highest weight vectors given by Young tableaux with two rows in Theorem 20.

We can prove a similar – slightly weaker – result in Theorem 28, when the tableau \hat{T} is restricted to be semistandard. In this case we have to increase the number of rows, the inner degree of the symmetric tensors and the Waring rank of the points of evaluation. Furthermore we don't prove hardness for all constant d in this case, but only for those divisible by 16. This still rules out polynomial evaluation algorithms which allow d to be part of the input under $P \neq NP$. These reductions also yield more explicit lower bounds under the exponential time hypothesis (ETH) in Theorems 20 and 28. As a reminder, the exponential time hypothesis states, that 3SAT can not be solved in time $2^{o(n)}$. Finally we show in Theorem 21 that if we want to calculate the exact value of the evaluation we can even prove #P-hardness for evaluating highest weight vectors given as Young tableaux.

Most of these reductions start with the same base that deciding whether a graph admits a proper 3-coloring a graph is NP-hard even when restricted to planar graphs of maximum degree 4. This was originally proven by Gary, Johnson and Stockmeyer [27] and a modified version can be found in Lemmas 25 and 26.

⁵ For example, G_S is the complete graph on 5 vertices for $S = \begin{matrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 4 & 4 \\ 3 & 4 & 5 & 5 & 5 \end{matrix}$.

► **Theorem 20.** *Deciding whether a highest weight vector $f_{\hat{T}}$ of $\text{Sym}^n \text{Sym}^d \mathbb{C}^m$ given as a Young tableau \hat{T} evaluates to zero at a point $p \in \text{Sym}^d \mathbb{C}^m$ of Waring rank 3 is NP-hard for constant $d \geq 8, m \geq 2$.*

Assuming ETH no $2^{o(n)}$ algorithm for this evaluation can exist.

Proof. We use the NP-hardness of 3-coloring graphs of maximum degree at most 4, see [27] or Lemma 25.

Let $G = (V, E)$ be a graph of maximum degree at most 4. Assume w.l.o.g. that $V = \{1, \dots, n\}$. We now construct a Young tableau T with content $n \times d$ as follows: For every edge $(u, v) \in E$ we add two columns of the form $\begin{bmatrix} u \\ v \end{bmatrix}$ to \hat{T} . Now for every vertex $v \in V$ add $d - 2 \cdot \deg(v)$ single-box columns $\begin{bmatrix} v \end{bmatrix}$ to \hat{T} . It is easy to see that \hat{T} has content $n \times d$ and is not necessarily semistandard.

We now choose to evaluate the highest weight vector $f_{\hat{T}}$ at $p = \ell_1^d + \ell_2^d + \ell_3^d$ with $\ell_1 = (1, 0, 0, \dots), \ell_2 = (1, 1, 0, \dots), \ell_3 = (1, 2, 0, \dots) \in \mathbb{C}^m$. Note that the determinant of any two distinct linear forms of these is a real number, so its square is a positive real number.

Recall from (2) that

$$f_{\hat{T}}(p) = \sum_{\text{proper } \vartheta} \prod_{c=1}^{\lambda_1} \det_{\vartheta, c}.$$

We now show a 1-to-1 correspondence between summands of the evaluation and arbitrary – not necessarily proper – 3-colorings of G . A summand will be non-zero iff the corresponding 3-coloring is proper. Due to evaluating at p in its Waring decomposition, ϑ will be proper iff boxes with the same number j get assigned the same ℓ_i . We interpret this as vertex j receiving color i . Additionally every 3-coloring of G corresponds to some placement in this way.

We now take the product of determinants for each column. Since each column with two boxes is repeated twice, this product is a product of squares, and hence will always be positive iff none of the determinants is zero. This idea was first used in [12]. A determinant is non-zero iff different vectors ℓ_i and ℓ_j are chosen for both of the boxes, corresponding to coloring both vertices of this column with different colors. So a summand will be non-zero iff ϑ corresponds to a proper 3-coloring of G .

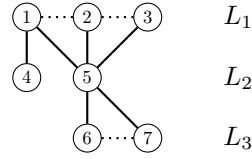
Note that any algorithm deciding whether $f_{\hat{T}}(p)$ is non-zero in time $2^{o(n)}$ can now be used to decide whether G allows for a proper 3-coloring in time $\text{poly}(|V|)2^{o(|V|)}$ which is a contradiction unless ETH fails as proven in [36]. ◀

Note that our algorithms for evaluation described in Theorems 8 and 16 both achieve a running time of $2^{O(n)}$ for evaluations at points of constant Waring rank with constant m and d . So Theorem 20 gives a matching lower bound under ETH.

The proof for #P-hardness is pretty similar and reduces from counting the number of 3-colorings of a graph with maximum vertex degree 3 which is known to be #P-complete [10]. The main idea is to use a more carefully chosen point of evaluation to ensure that every summand that corresponds to a proper 3-coloring will be exactly 1.

► **Theorem 21.** *Evaluating a highest weight vector $f_{\hat{T}}$ of $\text{Sym}^n \text{Sym}^d \mathbb{C}^m$ given as a Young tableau \hat{T} at a point $p \in \text{Sym}^d \mathbb{C}^m$ of Waring rank 3 is #P-hard for constant $d \geq 18, m \geq 2$.*

Proof. We reduce from counting the number of 3-colorings of a graph $G = (V, E)$ where every vertex has degree at most 3 which is known to be #P-complete [10]. We proceed in a similar manner as in the NP-hardness proof in Theorem 20. We construct \hat{T} by adding



■ **Figure 5** Example of a grid-like layered graph with 3 layers. Edges between layers are drawn as solid lines and edges inside layers as dotted lines.

the columns $\begin{bmatrix} u \\ v \end{bmatrix}$ for $(u, v) \in E$ 6-times each and for every vertex $v \in V$ add $d - 6 \cdot \deg(v)$ columns $\begin{bmatrix} v \end{bmatrix}$ to \hat{T} . This time we evaluate $f_{\hat{T}}$ at $p = \ell_1^d + \ell_2^d + \ell_3^d$ with $\ell_1 = (1, 0, 0, \dots)$, $\ell_2 = (1, e^{\frac{i\pi}{3}}, 0, \dots)$, $\ell_3 = (1, e^{\frac{2i\pi}{3}}, 0, \dots) \in \mathbb{C}^m$. Note that the determinant of any two distinct linear forms of these is a 6-th root of unity, so its 6-th power is always exactly 1. If we now analyse the summands of the evaluation again we see that each term contributes exactly 1 if it corresponds to a proper 3-coloring and 0 otherwise. Thus the evaluation $f_{\hat{T}}(p)$ counts exactly the number of 3-colorings of G . ◀

Extending this result to semistandard Young tableaux now proceeds in multiple steps, which we devote the rest of this section towards.

We first extend the NP-hardness of 3-coloring to a subclass of planar graphs which we call *grid-like layered* graphs. More specifically we prove NP-hardness for 8-regular, i.e. each vertex has degree exactly 8, grid-like layered graphs in Lemma 24, while we show a lower bound of $2^{o(\sqrt{|V|})}$ using Lemma 27.

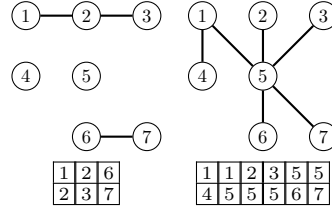
► **Definition 22.** We call a planar multigraph $G = (V, E)$ grid-like layered if there are disjoint layers $L_1, \dots, L_k \subseteq V$ of vertices and an embedding $e : V \rightarrow \mathbb{N} \times \{1, \dots, k\}$, s.t.

1. e is injective.
2. For every $i \in \{1, \dots, k\}$ we have $e^{-1}(\mathbb{N} \times \{i\}) = L_i$
3. Edges between layers only exist between layer L_i and L_{i+1} for all $i \in \{1, \dots, k-1\}$.
4. Edges inside layers only exist for vertices $v, u \in L_i$ where $e(v) = e(u) \pm (1, 0)$ for some $i \in \{1, \dots, k\}$.
5. All edges can be drawn as straight lines without crossing when vertices are placed according to e in \mathbb{R}^2 and the graph is treated as being simple.
6. Every vertex has a neighbour in a different layer.

Note that grid-like layered graphs are not necessarily subgraphs of a grid-graph, see Figure 5 for an example. The crucial property about grid-like layered graphs is, that they can be decomposed into two graphs over the same vertices each corresponding to a semistandard Young tableau with two rows. This decomposition is essential to encode the 3-coloring of such graphs into a single combined semistandard Young tableau.

► **Lemma 23.** Let $G = (V, E)$ be a grid-like layered graph. Then $G = (V, E(G_{\hat{T}_{\leftrightarrow}}) \cup E(G_{\hat{T}_{\downarrow}}))$ for two semistandard tableaux $\hat{T}_{\leftrightarrow}, \hat{T}_{\downarrow}$ for some relabeling of the vertices V . Additionally \hat{T}_{\downarrow} contains every number from 1 to $|V|$ at least once.

Proof. Let e be the embedding of G . We relabel the vertices in increasing order inside each layer according to e and then increasing order from layer L_i to layer L_{i+1} for every i . Let E_{\downarrow} now be the edges between different layers and E_{\leftrightarrow} those inside the layers. Clearly $E_{\leftrightarrow} \cup E_{\downarrow} = E$ and every vertex is incident to some edge in E_{\downarrow} by condition 22.6, so if we can construct semistandard tableaux $\hat{T}_{\leftrightarrow}, \hat{T}_{\downarrow}$ with $E_{\leftrightarrow} = E(G_{\hat{T}_{\leftrightarrow}})$ and $E_{\downarrow} = E(G_{\hat{T}_{\downarrow}})$ we are done.



■ **Figure 6** Example of $\hat{T}_{\leftrightarrow}$ and \hat{T}_{\downarrow} and the two graphs $G_{\hat{T}_{\leftrightarrow}}$ and $G_{\hat{T}_{\downarrow}}$ according to Lemma 23 for the grid-like layered graph given in Figure 5.

We start with \hat{T}_{\downarrow} . Since the labeling of the vertices is increasing from one layer to the next it suffices to show that we can create \hat{T}_{\downarrow} for a single pair of consecutive layers and afterwards concatenate them. Condition 22.5 gives a unique order of the edges between these layers from left to right. So for the edge $\{u, v\} \in E_{\downarrow}$ with $u < v$ we add the column $\begin{smallmatrix} u \\ v \end{smallmatrix}$ to \hat{T}_{\downarrow} . Assume two columns $\begin{smallmatrix} u \\ v \end{smallmatrix}$ and $\begin{smallmatrix} u' \\ v' \end{smallmatrix}$ would violate the semistandard property. Then either $u' < u$ in which case the edge $\{u', v'\}$ would be start left of $\{u, v\}$ or $v' < v$ in which case the edge $\{u', v'\}$ would end left of $\{u, v\}$, both a contradiction to our unique ordering from left to right. So \hat{T}_{\downarrow} is semistandard.

We continue with $\hat{T}_{\leftrightarrow}$. Again we only have to consider $\hat{T}_{\leftrightarrow}$ for a single layer as we can just concatenate the resulting tableaux afterwards. If we direct the edges in E_{\leftrightarrow} to only go from the smaller vertex to the larger one we see with condition 22.4 that each vertex can only be the first vertex of an edge once, and those edges have the form $\{v, v+1\}$. So the only columns in $\hat{T}_{\leftrightarrow}$ are of the form $\begin{smallmatrix} v \\ v+1 \end{smallmatrix}$. Those can clearly just be combined in order to make $\hat{T}_{\leftrightarrow}$ semistandard.

Note that since G is a multigraph we add every column to the tableaux k times if the edge appears with multiplicity k in G . ◀

An example for these of such a decomposition can be found in Figure 6.

We can now give an elegant proof of the NP-hardness of deciding whether a given 8-regular grid-like layered graph $G = (V, E)$ admits a proper 3-coloring. With this elegance comes the caveat, that this proof only yields a lower bound of $2^{o(\sqrt[4]{|V|})}$ under ETH, which we improve to $2^{o(\sqrt{|V|})}$ with a more technical proof in Lemma 27

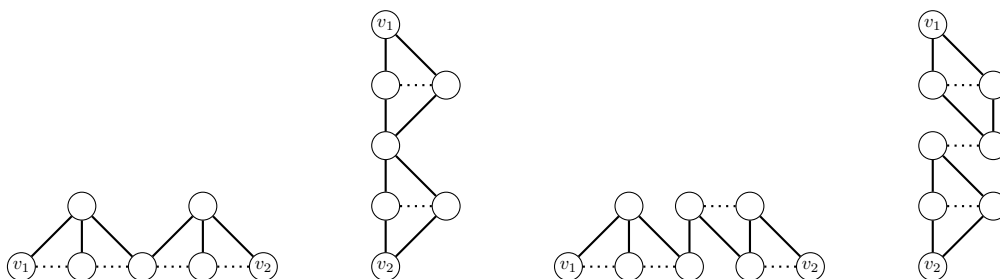
► **Lemma 24.** *Deciding whether a given graph $G = (V, E)$ admits a proper 3-coloring is NP-hard, even if the graph is restricted to be grid-like layered and 8-regular.*

Unless ETH fails, 3-coloring doesn't admit an $2^{o(\sqrt[4]{|V|})}$ time algorithm for grid-like layered graphs.

Proof. For this we reduce from the decision problem whether a planar graph G admits a proper 3-coloring.

In order to achieve this we find a graph minor model $(V_h)_{h \in V(G)}$ of embedding G into a grid \boxplus with $O(|V(G)|^2)$ vertices in linear time [54]. Let G_1 be the grid \boxplus after removing any vertices and edges which do not correspond to vertices or edges in G , i.e.

$$V(G_1) = \bigcup_{h \in V(G)} V_h$$



■ **Figure 7** The equality and inequality gadgets $H_1^=, H_2^=$ and H_1^\neq, H_2^\neq used in Lemma 24



■ **Figure 8** The nearly 8-regular versions of the gadgets $H_1^=$ and H_1^\neq used in Lemma 24. The edge labels denote the multiplicity of the edges in the multigraph and $2a = \deg(v_1)$ and $2b = \deg(v_2)$.

and

$$E(G_1) = \bigcup_{h \in V(G)} E(\boxplus[V_h]) \cup \bigcup_{uv \in E(G)} E(\boxplus[V_u \cup V_v])$$

Any proper 3-coloring of G can now be seen as a 3-coloring where each V_h is colored the same color as $h \in G$ and neighbouring components V_u, V_v for $\{u, v\} \in E(G)$ are colored with different colors.

In order to enforce these constraints we construct a new graph G_2 by replacing each edge inside any V_h by the equality gadgets $H_1^=$ or $H_2^=$ and replace each edge between neighbouring components V_u, V_v by the inequality gadgets H_1^\neq or H_2^\neq . These gadgets are shown in Figure 7. If an edge is horizontal in the canonical embedding of G_1 into the plane we choose variant 1 of the gadgets. If an edge is vertical we choose variant 2.

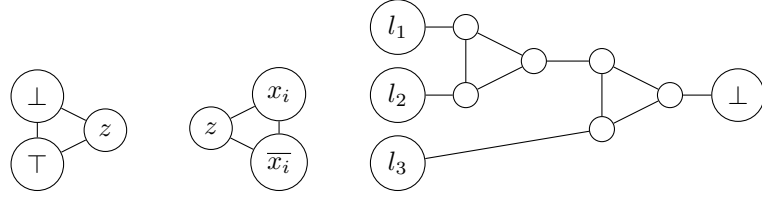
It can be easily checked that the only way to properly 3-color these gadgets is such that the colors of v_1 and v_2 are the same for the equality gadgets and different for the inequality gadgets.

Clearly G is now properly 3-colorable iff G_2 is properly 3-colorable.

Secondly all those gadgets are designed as grid-like layered graphs. It can be easily checked that replacing all edges in a subgraph of a grid yields a grid-like layered graph, so G_2 is grid-like layered.

So the only thing remaining to do is make the graph 8-regular by adding copies of existing edges to the graph. In order to achieve this it is sufficient to show that multigraph versions of $H_1^=, H_2^=, H_1^\neq$ and H_2^\neq exist which are 8-regular except for the vertices v_1 and v_2 , which can independently have a degree of 2, 4, 6 or 8 each. This is sufficient since every vertex of the grid graph has a degree between 1 and 4, so it has between 1 and 4 of these gadgets attached to it. The multigraph variations of the gadgets are shown for $H_1^=$ and H_1^\neq in Figure 8, for the other two gadgets these are constructed similarly.

Note that \boxplus has $O(|V(G)|^2)$ many vertices so we can conclude that G_2 also has $O(|V(G)|^2)$ many vertices. If we can decide whether the 8-regular grid-like layered graph G_2 allows for a proper 3-coloring in time $2^{o(\sqrt[4]{|V(G_2)|})}$ we can decide via this reduction whether the planar graph G allows for a proper 3-coloring in time $\text{poly}(|V(G)|) \cdot 2^{o(\sqrt{|V(G)|})}$. This contradicts



■ **Figure 9** The gadgets H_1 , H_2 and H_3 used in the proof of Lemma 25.

that planar 3-coloring is not solvable in time $2^{o(\sqrt{|V(G)|})}$ unless ETH fails. This was essentially observed by Cai and Juedes [16] and is also mentioned in [21, Theorem 14.9]. ◀

Looking at the reduction from 3-satisfiability to 3-coloring more closely we can improve the ETH lower bound of the previous proof. The fourth root was necessary because first embedding the 3SAT formula into a planar graph and then into a grid graph each resulted in quadratic blow-up. By abusing the structure of the intermediate graphs more closely we reduce the size of the grid graph to be only quadratic in the number of variables of the 3SAT formula and thus show a better lower bound in Lemma 27.

The proof uses similar gadgets to the standard textbook reduction of 3SAT to 3-coloring, which we show again for reference.

► **Lemma 25.** *Deciding whether a given graph $G = (V, E)$ admits a proper 3-coloring is NP-hard.*

Proof. We reduce from 3-satisfiability. So let $\phi = C_1 \wedge \dots \wedge C_m$ be a formula in 3-CNF on n variables x_1, \dots, x_n . We construct a graph G as follows. We start with the graph H_1 shown in Figure 9 (left) and call the three vertices \top , \perp , and z . Then for each $1 \leq i \leq n$ we add a vertex x_i and a vertex \bar{x}_i and add three edges: $\{x_i, \bar{x}_i\}$, $\{x_i, z\}$, $\{\bar{x}_i, z\}$. This is depicted in Figure 9 (middle). For each $1 \leq j \leq m$ we now add 6 vertices and connect them with the existing vertices as shown in Figure 9 (right): The vertices labeled l_1, l_2, l_3 in the figure stand for the vertices corresponding to the three literals (elements in $\{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\}$) in the clause C_j .

We now analyze potential proper 3-colorings of G . Our colors will conveniently be called \top , \perp , and z and we assume from now on w.l.o.g. that the three vertices in H_1 are colored according to their names. It follows from H_2 that in every proper 3-coloring the vertices corresponding to literals are colored with \top or \perp , but never with z . It is easy to see that H_3 has no proper 3-coloring if l_1, l_2 , and l_3 all are colored with \perp . Moreover, if at least one of l_1, l_2 , and l_3 is colored with \top and the others are colored with \perp , then a proper 3-coloring of H_3 exists.

Hence from a proper 3-coloring of G we can easily reconstruct a satisfying assignment of ϕ and vice versa. ◀

In Lemma 24 we then proceeded with a planar version of this theorem due to [27] and embedded these resulting graphs as minors of a grid. In essence we used a variant of 3-coloring where the graph is a subset of a grid graph and every edge can either be an equality or inequality edge, i.e. vertices connected by an equality edge have to be colored by the same color and vertices connected by an inequality edge have to be colored with different colors. We already implicitly showed NP-hardness of this variant which we call *relational 3-coloring on subgraphs of grids* in the proof of Lemma 24.

Note that equality edges are a necessity, since any subgraph of a grid graph is bipartite and thus can be 2-colored.

► **Lemma 26.** *Unless ETH fails, relational 3-coloring on subgraphs G of grids can not be solved in time $2^{o(\sqrt{|V(G)|})}$.*

Proof. We reduce from 3-satisfiability. So let $\phi = C_1 \wedge \dots \wedge C_m$ be a formula in 3-CNF on n variables x_1, \dots, x_n .

We again start with the color choosing gadget H'_1 assume that each vertex of H_1 is colored with its label to simplify the analysis. Note that vertices with the same labels will be connected by a path of equality edges, so they have the same color in each proper 3-coloring. H'_1 forms a border of width ≤ 2 around the rest of the graph.

Connected to the vertices labeled with z are the variable gadgets H'_2 . The vertices with labels x_i and \bar{x}_i corresponding to the literals of ϕ appear exactly as often as each of the literals appears in ϕ . Connected to the bottom vertices \perp are the clause gadgets H'_3 .

The only thing left is connecting the vertices corresponding to literals in the clause gadgets to those in the variable gadgets via an equality edge. Unfortunately this would make the graph not be a subgraph of a grid, so we need the crossing gadget H'_4 which is an embedding of the crossing gadget used in [27]. In H'_4 the vertices pairs labeled a, a' and b, b' each have the same color in every proper 3-coloring. Additionally there is a proper 3-coloring for every choice of colors of a and b .

We now need to “sort” the vertices corresponding to literals into the order $(l_{1,1}, l_{1,2}, l_{1,3}, l_{2,1}, l_{2,2}, l_{2,3}, \dots, l_{m,1}, l_{m,2}, l_{m,3})$ where $C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$. We do this via an iterative procedure. We add the crossing gadget H'_4 between every two consecutive vertices l_i, l_j which are in the wrong order in each step. In case some vertices could be part of multiple swaps choose the pairs in a way that maximizes the number of possible swaps per iteration. We connect l_i to the vertex a of H'_4 via an equality edge and similarly l_j to b . The vertices a' and b' now form the next step in the ordering process and have essentially swapped adjacent l_i and l_j . All the vertices l_i which do not change their position will be extended via paths made out of equality edges to be on the same layer as the outlets of the crossing gadgets. After at most $O(m)$ of these steps the vertices are sorted in our desired order and can be directly connected to the corresponding vertices of the clause gadgets.

We call this resulting graph G . Note that H'_4 enforces a finer subdivision of the grid than H'_1, H'_2 and H'_3 , but we can always split an equality edge into two equality edges connected by a vertex or split vertices into two vertices connected by an equality edge to stretch these gadgets, so G is a subgraph of a grid graph.

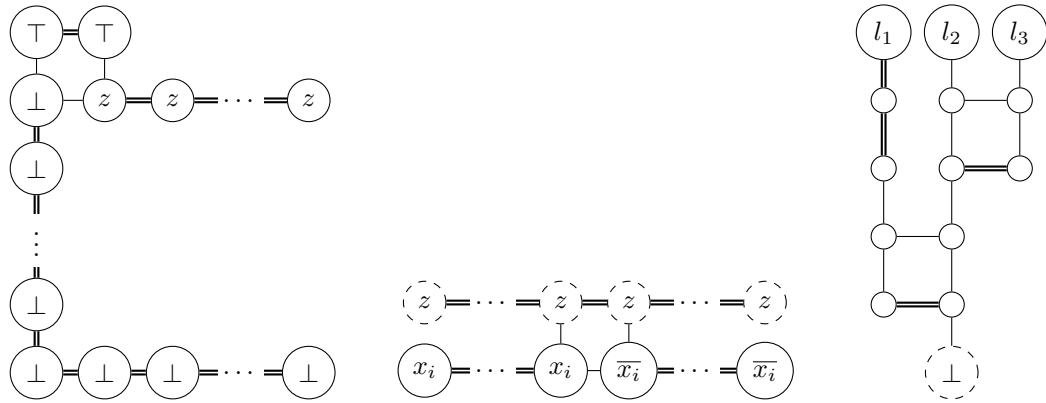
It can be easily checked that H'_1, H'_2 and H'_3 behave in exactly the same way as their counterparts in the proof of Lemma 25, so the correctness of this reduction can easily be seen with the same reasoning as there together with the properties of H'_4 .

G is a subgraph of an $O(m) \times O(m)$ grid, so $|V(G)| = O(m^2)$. If we could decide relational 3-colorability on subgraphs of grids in size $2^{o(\sqrt{|V(G)|})}$ we could thus decide 3-satisfiability in time $2^{o(m)}$ which is a contradiction unless ETH fails, see [36] for this lower bound for 3-satisfiability.

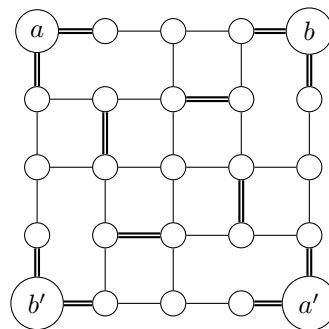
◀

► **Lemma 27.** *Unless ETH fails, 3-coloring doesn't admit an $2^{o(\sqrt{|V|})}$ time algorithm for 8-regular grid-like layered graphs $G = (V, E)$.*

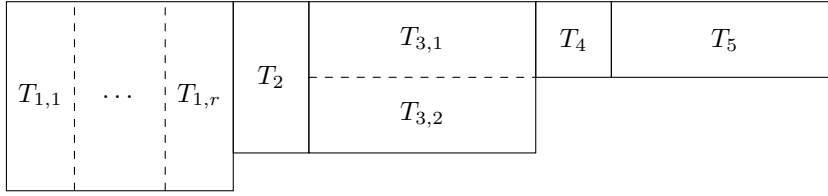
Proof. We reduce from relational 3-coloring on subgraphs of grids. Let $G = (V, E)$ be a subgraph of a grid. We proceed in the same way as Lemma 24 did except that \boxplus is replaced by G , each equality edge is replaced by the corresponding equality gadget and each inequality edge is replaced by the corresponding inequality gadget.



■ **Figure 10** The gadgets H'_1 , H'_2 and H'_3 used in the proof of Lemma 26. Double lines denote equality edges and single lines denote inequality edges. Vertices corresponding to other gadgets are visualized with dashed outline to show how to connect the gadgets.



■ **Figure 11** The gadget H'_4 used in the proof of Lemma 26. Double lines denote equality edges and single lines denote inequality edges.



■ **Figure 12** The general structure of the 5 row Young tableau \hat{T} constructed in Theorem 28

Note that the obtained graph G_2 now has $O(|V(G)|)$ many vertices. If we can decide whether the 8-regular grid-like layered graph G_2 allows for a proper 3-coloring in time $2^{o(\sqrt{|V(G_2)|})}$ we can decide via this reduction whether G allows for a relational 3-coloring in time $\text{poly}(|V(G)|) \cdot 2^{o(\sqrt{|V(G)|})}$, contradicting Lemma 26 unless ETH fails. ◀

We now have all the necessary intermediate results to prove that even evaluation of highest weight vectors given by semistandard tableaux is NP-hard. We use the same general idea of coloring the cells of the Young tableau s.t. all cells with the same number receive the same color. Additionally each column of the Young tableau has to be repeated often enough that any summands are guaranteed to be positive iff each column is colorful, i.e. does not contain any color multiple times and zero otherwise.

► **Theorem 28.** *The evaluation of highest weight vectors $f_{\hat{T}}$ of $\text{Sym}^n \text{Sym}^d \mathbb{C}^m$ is NP-hard for any constant $d \geq 16$ with $16 \mid d$ and $m \geq 5$, when $f_{\hat{T}}$ is given as a semistandard Young tableau \hat{T} . This even holds if evaluation is restricted to points of Waring rank 5 and the algorithm only has to decide whether the evaluation is non-zero.*

Additionally this evaluation can not be computed in time $2^{o(\sqrt{n})}$ unless ETH fails.

Proof. We reduce from checking whether an 8-regular grid-like layered graph allows for a proper 3-coloring which was proven in Lemma 24 to be NP-hard.

Let $G = (V, E_{\leftrightarrow} \cup E_{\updownarrow})$ be an 8-regular grid-like layered graph where E_{\leftrightarrow} denotes the edges inside layers and E_{\updownarrow} denotes edges between layers. W.l.o.g. the vertex set V are the numbers $1, \dots, |V|$ assigned in a layer by layer and left to right fashion, given by the embedding of G . To ease the description of the constructed semistandard tableaux \hat{T} we will describe it in 5 parts T_1, \dots, T_5 over the symbolic entries a_i, b_i, c_i, d_i, e_i . For better readability we will colorcode each of the symbolic entries in the constructions of this theorem.

The point of evaluation is now $p = \sum_{i=1}^5 \ell_i^d$ with $\ell_i = (1, i, i^2, i^3, \dots, i^m)$. Then any determinants arising in the evaluation are determinants of Vandermonde matrices and thus are well known to be non-zero.

p is a point of Waring rank 5, so analogously to Theorem 20 the summands of the evaluation will consist of assigning one of the 5 linear forms to each number and will be non-zero iff no column contains the same linear form twice. Since all vectors are real, any occuring determinants in the evaluation will also be real and hence every summand will be either 0 or positive due to every column being repeated an even number of times.

The general structure of \hat{T} can be seen in Figure 12 and we will first describe the main idea of each part. The parts T_3 and T_5 both encode the actual 3-coloring restrictions of the edge sets E_{\updownarrow} and E_{\leftrightarrow} respectively, in the entries e_i in the same way as in Theorem 20. To ensure that only 3 colors can be used for the graph coloring, the entries c_i are added into T_3 to use up the remaining colors. The consistency of the c_i , i.e. that exactly two colors are used by all of the c_i , is then ensured in T_1 with the help of the entries a_i and b_i . Everything

else, i.e. the d_i and the tableaux T_2 and T_4 are only used in order to make \hat{T} semistandard and with rectangular content.

\hat{T} is then the concatenation $T_1 T_2 \dots T_5$ where we assign increasing numbers starting from 1 first to all the a_i , then to all the b_i, c_i, d_i and e_i in order, increasing inside each group of symbolic entries with increasing index. This ensures that each of the T_i individually, but also the concatenation \hat{T} will be semistandard. The latter can be seen by looking at the symbolic entries at the left and right borders of the T_i in the following descriptions.

We first describe the construction of all the T_i : T_1 is built as a concatenation of smaller tableaux $T_{1,1}, \dots, T_{1,r}$ for $r = \lceil \frac{|E_\uparrow| - 1}{24} \rceil$. The construction of $T_{1,1}$ and of $T_{1,i}$ for $1 < i \leq r$ can be seen in Figure 13. T_2 and T_4 are always the same and are given in Figure 14. T_3 is given as a left aligned vertical concatenation of $T_{3,1}$ and $T_{3,2}$, where $T_{3,1}$ consists of the columns $\begin{array}{|c|} \hline c_1 \\ \hline c_2 \\ \hline \end{array}$, $\begin{array}{|c|} \hline c_3 \\ \hline c_4 \\ \hline \end{array}$, \dots , $\begin{array}{|c|} \hline c_{8r-3} \\ \hline c_{8r-2} \\ \hline \end{array}$ each repeated 12 times and $\begin{array}{|c|} \hline c_{8r-1} \\ \hline c_{8r} \\ \hline \end{array}$ repeated 14 times. $T_{3,2}$ is obtained from \hat{T}_\downarrow of Lemma 23 by doubling every column. Lastly T_5 is constructed in the exact same way as $T_{3,2}$, but is obtained from $\hat{T}_{\leftrightarrow}$ of Lemma 23.

We first prove that T is a semistandard Young tableau of rectangular content. T_1 fulfils the following properties which are easy to prove via induction:

- a_1, \dots, a_{3r} all appear exactly 16 times each in T_1 .
- b_1 and b_2 appear exactly twice in T_1 .
- c_1, \dots, c_{8r-2} all appear exactly 4 times in T_1 .
- c_{8r-1} and c_{8r} appear exactly twice in T_1 .
- If we replace the symbolic entries as previously described then T_1 is semistandard.

The only important properties of T_2 and T_4 are that b_1, b_2, d_1, d_2 all appear exactly 14 times in T_2 and d_1 and d_2 each appear twice in T_4 , while both are clearly semistandard.

The properties of T_3 are now:

- c_1, \dots, c_{8r-2} all appear 12 times in T_3 .
- c_{8r-1} and c_{8r} appear exactly 14 times in T_3 .
- If we replace the symbolic entries as previously described, T_3 is semistandard.
- $T_{3,1}$ has at least as many columns as $T_{3,2}$ by our choice of $r = \lceil \frac{|E_\uparrow| - 1}{24} \rceil$. $T_{3,1}$ has

$$(4r - 1) \cdot 12 + 14 \geq \left(\frac{|E_\uparrow| - 1}{6} - 1 \right) \cdot 12 + 14 = 2 \cdot |E_\uparrow|$$

columns while $T_{3,2}$ has exactly $2 \cdot |E_\uparrow|$ columns.

The last property of T_3 is important in order for T_3 and thus \hat{T} to be a of proper shape for a Young tableau, i.e. have non-decreasing row lengths.

Combining all the properties we see that \hat{T} contains every entry exactly 16 times each and is semistandard after replacing the symbolic entries. Additionally each column is repeated an even number of times, so no summands of the evaluation can be negative. In case $d > 16$ we repeat every column of T $\frac{d}{16}$ times in order to get the representation of a highest weight vector of $\text{Sym}^n \text{Sym}^d \mathbb{C}^m$ as a semistandard Young tableau.

Next we look at the effects of the gadgets on the possible non-zero summands of the evaluation.

Any further considerations will now assume w.l.o.g. that a_1, a_2, a_3 get assigned the first three linear forms of p , all other cases are symmetric. These three entries all occur together in the very first column of T_1 , so they have to be pairwise different in order to be part of a non-zero summand. $T_{1,1}$ then enforces c_1, \dots, c_8 to all be assigned the last two linear forms of p . Since $T_{1,i}$ and $T_{1,i+1}$ share the entries of c_{8i-1} and c_{8i} , inductively all of a_i, \dots, a_{3r}

$$T_{1,1} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 \\ \hline a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 \\ \hline a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \\ \hline b_1 & b_1 & c_1 & c_1 & c_1 & c_1 & c_3 & c_3 & c_3 & c_3 & c_5 & c_5 & c_5 & c_5 & c_7 & c_7 & c_7 \\ \hline b_2 & b_2 & c_2 & c_2 & c_2 & c_2 & c_4 & c_4 & c_4 & c_4 & c_6 & c_6 & c_6 & c_6 & c_8 & c_8 & c_8 \\ \hline \end{array}$$

$$T_{1,i} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} & a_{3i-2} \\ \hline a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} & a_{3i-1} \\ \hline a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} & a_{3i} \\ \hline c_{8i-9} & c_{8i-9} & c_{8i-7} & c_{8i-7} & c_{8i-7} & c_{8i-7} & c_{8i-5} & c_{8i-5} & c_{8i-5} & c_{8i-5} & c_{8i-3} & c_{8i-3} & c_{8i-3} & c_{8i-3} & c_{8i-1} & c_{8i-1} & c_{8i-1} \\ \hline c_{8i-8} & c_{8i-8} & c_{8i-6} & c_{8i-6} & c_{8i-6} & c_{8i-6} & c_{8i-4} & c_{8i-4} & c_{8i-4} & c_{8i-4} & c_{8i-2} & c_{8i-2} & c_{8i-2} & c_{8i-2} & c_{8i} & c_{8i} & c_{8i} \\ \hline \end{array}$$

■ **Figure 13** The Young tableaux $T_{1,1}$ and $T_{1,i}$ from the proof of Theorem 28

$$T_2 = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 \\ \hline b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 \\ \hline d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 & d_1 \\ \hline d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 & d_2 \\ \hline \end{array}$$

$$T_4 = \begin{array}{|c|c|} \hline d_1 & d_1 \\ \hline d_2 & d_2 \\ \hline \end{array}$$

■ **Figure 14** The Young tableaux T_2 and T_4 from the proof of Theorem 28

will be assigned the first three linear forms of p in some order and all of c_1, \dots, c_{8r} will be assigned the last two linear forms of p in some order.

The last important property is, that $e_1, \dots, e_{|V|}$ all appear at least once in T_3 since every vertex of a grid-like layered graph is incident to an edge going to another layer. This means that all the linear forms being chosen for any $e_1, \dots, e_{|V|}$ can only be the first three linear forms of p since the remaining two are already used for the c_i of which two appear in every column.

Now assume G admits a proper 3-coloring with the colors 1, 2, 3. We can now construct a placement of the linear forms onto the entries of \hat{T} as follows:

- The entries a_{3i+j} get assigned the linear form ℓ_j for every $i \in \{0, \dots, r-1\}$ and $j \in \{1, 2, 3\}$.
- The entries b_1 and b_2 get assigned the linear forms ℓ_4 and ℓ_5 respectively.
- The entries c_{2i+j} get assigned the linear form ℓ_{3+j} for every $i \in \{0, \dots, 4r-1\}$ and $j \in \{1, 2\}$.
- The entries d_1 and d_2 get assigned the linear forms ℓ_1 and ℓ_2 respectively.
- The entries e_i get assigned the linear form ℓ_j if vertex i was colored with color j in G .

It is now easy to check that in T_1, T_2 and T_4 no column contains any linear form twice. To see that the same holds for T_3 and T_5 note that the only way any column could contain the same linear form twice would be for two entries e_u and e_v to appear in the same column and be assigned the same linear form. That would mean that u and v got colored the same in G , but by our construction there is also an edge $\{u, v\} \in E_{\uparrow} \cup E_{\leftrightarrow}$, a contradiction to G being properly 3-colored. Since no column contains a repeated linear form this summand is strictly positive, making the whole evaluation $f_{\hat{T}}(p)$ non-zero.

Conversely assume that the evaluation of $f_{\hat{T}}(p)$ is non-zero. Thus there must be a non-zero summand, placing linear forms on each entry. As by the previous discussion there are only 3 different linear forms being placed on all of the e_i , directly inducing a 3-coloring of G . This 3-coloring is proper since every column can never contain the same linear form twice and every edge of G is represented by a column.

To now show that this evaluation is not possible in time $2^{o(\sqrt{n})}$ unless ETH fails, notice that if G has $|V|$ vertices, then \hat{T} has $n = O(|V|)$ many different entries. So any evaluation in time $2^{o(\sqrt{n})}$ would decide whether G admits a proper 3-coloring in time $2^{o(\sqrt{|V|})}$, which is a contradiction to Lemma 26⁶ unless ETH fails. ◀

► Remark 29. All these hardness results also hold if the highest weight vectors are given as a Young tableau T with content $(nd) \times 1$ opposed to \hat{T} with content $n \times d$ by replacing the entries containing 1 in \hat{T} by $1, \dots, d$ and 2 by $d+1, \dots, 2d$ and so on in a left-to-right, top-to-bottom fashion. This corresponds to undoing the projection of $\otimes^n \text{Sym}^d V$ onto $\text{Sym}^n \text{Sym}^d V$. In the cases when \hat{T} is semistandard T is standard.

References

- 1 Abdelmalek Abdesselam. Feynman diagrams in algebraic combinatorics. *Séminaire Lotharingien de Combinatoire [electronic only]*, 49:B49c, 45 p., electronic only–B49c, 45 p., electronic only, 2002. URL: <http://eudml.org/doc/123420>.
- 2 Abdelmalek Abdesselam, Christian Ikenmeyer, and Gordon Royle. 16,051 formulas for ottaviani’s invariant of cubic threefolds. *Journal of Algebra*, 447:649 – 663, 2016.
- 3 Daniel J. Bates and Luke Oeding. Toward a salmon conjecture. *Experimental Mathematics*, 20(3):358–370, 2011. doi:10.1080/10586458.2011.576539.
- 4 Christine Bessenrodt and Christiane Behns. On the durfee size of kronecker products of characters of the symmetric group and its double covers. *Journal of Algebra*, 280(1):132 – 144, 2004.
- 5 Christine Bessenrodt, Chris Bowman, and Rowena Paget. The classification of multiplicity-free plethysms of Schur functions. arXiv:2001.08763, 2020.
- 6 D. Bini. Relations between exact and approximate bilinear algorithms. applications. *CALCOLO*, 17(1):87–97, Jan 1980. doi:10.1007/BF02575865.
- 7 Dario Bini, Milvio Capovani, Francesco Romani, and Grazia Lotti. $O(n^{2.7799})$ complexity for $n \times n$ approximate matrix multiplication. *Inf. Process. Lett.*, 8(5):234–235, 1979.
- 8 Markus Bläser and Christian Ikenmeyer. Introduction to geometric complexity theory. lecture notes, summer 2017 at Saarland University, <http://people.mpi-inf.mpg.de/~cikenmey/teaching/summer17/introtogct/gct.pdf>, version from July 25, 2018, 2018.
- 9 Karl Bringmann, Christian Ikenmeyer, and Jeroen Zuiddam. On algebraic branching programs of small width. *J. ACM*, 65(5), August 2018. doi:10.1145/3209663.
- 10 Russ Bubley, Martin E. Dyer, Catherine S. Greenhill, and Mark Jerrum. On approximately counting colorings of small degree graphs. *SIAM J. Comput.*, 29(2):387–400, 1999. doi:10.1137/S0097539798338175.
- 11 Peter Bürgisser. The complexity of factors of multivariate polynomials. In *42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001)*, pages 378–385. IEEE Computer Soc., Los Alamitos, CA, 2001.
- 12 Peter Bürgisser, Matthias Christandl, and Christian Ikenmeyer. Even partitions in plethysms. *Journal of Algebra*, 328(1):322 – 329, 2011.
- 13 Peter Bürgisser and Christian Ikenmeyer. Fundamental invariants of orbit closures. *Journal of Algebra*, 477(Supplement C):390 – 434, 2017.
- 14 Peter Bürgisser, Christian Ikenmeyer, and Greta Panova. No occurrence obstructions in geometric complexity theory. *Journal of the American Mathematical Society*, 32:163–193, 2019. A conference version appeared in: Proceedings IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS 2016), 386–395.

⁶ or Lemma 24 for a weaker lower bound of $2^{o(\sqrt[4]{n})}$

- 15 Peter Bürgisser, J.M. Landsberg, Laurent Manivel, and Jerzy Weyman. An overview of mathematical issues arising in the Geometric complexity theory approach to VP v.s. VNP. *SIAM J. Comput.*, 40(4):1179–1209, 2011.
- 16 Liming Cai and David Juedes. Subexponential parameterized algorithms collapse the w-hierarchy. In *International Colloquium on Automata, Languages, and Programming*, pages 273–284. Springer, 2001.
- 17 Enrico Carlini, Maria Virginia Catalisano, and Anthony V Geramita. The solution to the Waring problem for monomials and the sum of coprime monomials. *Journal of algebra*, 370:5–14, 2012.
- 18 Man-Wai Cheung, Christian Ikenmeyer, and Sevak Mkrtychyan. Symmetrizing tableaux and the 5th case of the Foulkes conjecture. *Journal of Symbolic Computation*, 80:833–843, 2017.
- 19 Luca Chiantini, Jonathan D. Hauenstein, Christian Ikenmeyer, Joseph M. Landsberg, and Giorgio Ottaviani. Polynomials and the exponent of matrix multiplication. *Bulletin of the London Mathematical Society*, 50(3):369–389, 2018. URL: <https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/blms.12147>, arXiv:<https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/blms.12147>, doi:10.1112/blms.12147.
- 20 Matthias Christandl, Brent Doran, and Michael Walter. Computing multiplicities of lie group representations. In *Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science, FOCS '12*, page 639–648, USA, 2012. IEEE Computer Society. doi:10.1109/FOCS.2012.43.
- 21 Marek Cygan, Fedor V Fomin, Łukasz Kowalik, Daniel Lokshantov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized algorithms*, volume 4. Springer, 2015.
- 22 Noah Daleo, Jonathan Hauenstein, and Luke Oeding. Computations and equations for segre-grassmann hypersurfaces. *Portugaliae Mathematica*, 73, 08 2014. doi:10.4171/PM/1977.
- 23 Julian Dörfler, Christian Ikenmeyer, and Greta Panova. On geometric complexity theory: Multiplicity obstructions are stronger than occurrence obstructions. In *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece.*, pages 51:1–51:14, 2019. journal version accepted for publication in SIAM J Appl Alg Geom (SIAGA). doi:10.4230/LIPIcs.ICALP.2019.51.
- 24 Nick Fischer and Christian Ikenmeyer. The computational complexity of plethysm coefficients. arXiv:2002.00788, 2020.
- 25 Michael Forbes. Some concrete questions on the border complexity of polynomials. Talk presented at the Workshop on Algebraic Complexity Theory, WACT 2016, Tel Aviv, 2016. video available at <https://www.cs.tau.ac.il/~shpilka/wact2016/videos/index.php> accessed 10/17/2019. URL: <https://www.cs.tau.ac.il/~shpilka/wact2016/videos/index.php>.
- 26 Michael Andrew Forbes. *Polynomial Identity Testing of Read-Once Oblivious Algebraic Branching Programs*. PhD thesis, MIT, 2014. URL: <https://dspace.mit.edu/handle/1721.1/89843>.
- 27 M. R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified np-complete graph problems. *Theor. Comput. Sci.*, 1(3):237–267, 1976. doi:10.1016/0304-3975(76)90059-1.
- 28 Fulvio Gesmundo. Geometric aspects of iterated matrix multiplication. *Journal of Algebra*, 461:42 – 64, 2016. URL: <http://www.sciencedirect.com/science/article/pii/S0021869316300965>, doi:<https://doi.org/10.1016/j.jalgebra.2016.04.028>.
- 29 Fulvio Gesmundo, Christian Ikenmeyer, and Greta Panova. Geometric complexity theory and matrix powering. *Differential Geometry and its Applications*, 2017. in press, see <https://doi.org/10.1016/j.difgeo.2017.07.001>. URL: <http://www.sciencedirect.com/science/article/pii/S0926224517300876>, doi:<https://doi.org/10.1016/j.difgeo.2017.07.001>.
- 30 Joshua A. Grochow, Ketan D. Mulmuley, and Youming Qiao. Boundaries of VP and VNP. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016)*,

- volume 55 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 34:1–34:14, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: <http://drops.dagstuhl.de/opus/volltexte/2016/6314>, doi:10.4230/LIPIcs.ICALP.2016.34.
- 31 Charles Hermite. Sur la theorie des fonctions homogenes à deux indéterminées. *Cambridge and Dublin Mathematical Journal*, 9:172 – 217, 1854.
 - 32 Anthony Iarrobino and Vassil Kanev. *Power sums, Gorenstein algebras, and determinantal loci*. Springer Science & Business Media, 1999.
 - 33 Christian Ikenmeyer. *Geometric Complexity Theory, Tensor Rank, and Littlewood-Richardson Coefficients*. PhD thesis, Institute of Mathematics, University of Paderborn, 2012. URL: <http://nbn-resolving.de/urn:nbn:de:hbz:466:2-10472>.
 - 34 Christian Ikenmeyer. The Saxl conjecture and the dominance order. *Discrete Mathematics*, 338(11):1970 – 1975, 2015.
 - 35 Christian Ikenmeyer and Umangathan Kandasamy. Implementing geometric complexity theory: On the separation of orbit closures via symmetries. arXiv: 1911.03990, 2019.
 - 36 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*, 63(4):512–530, 2001. doi:10.1006/jcss.2001.1774.
 - 37 Mrinal Kumar. On top fan-in vs formal degree for depth-3 arithmetic circuits. <https://eccc.weizmann.ac.il/report/2018/068/revision/1/download>, 2018.
 - 38 Shrawan Kumar. A study of the representations supported by the orbit closure of the determinant. *Compositio Mathematica*, 151, 09 2011. doi:10.1112/S0010437X14007660.
 - 39 Shrawan Kumar and J.M. Landsberg. Connections between conjectures of alon-tarsi, hadamard-howe, and integrals over the special unitary group. *Discrete Math.*, 338(7):1232–1238, July 2015. doi:10.1016/j.disc.2015.01.027.
 - 40 J. M. Landsberg. Geometric complexity theory: an introduction for geometers. *ANNALI DELL’UNIVERSITA’ DI FERRARA*, 61(1):65–117, 2015. doi:10.1007/s11565-014-0202-7.
 - 41 Joseph Landsberg. *Tensors: Geometry and Applications*, volume 128 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2011.
 - 42 M.A.A. Leeuwen, van, A.M. Cohen, and B. Lissner. *Lie : a package for Lie group computations*. Centrum voor Wiskunde en Informatica, 1992.
 - 43 Laurent Manivel and Mateusz Michałek. Effective constructions in plethysms and weintraub’s conjecture. *Algebras and Representation Theory*, 17(2):433–443, Apr 2014. doi:10.1007/s10468-012-9402-y.
 - 44 K.D. Mulmuley and M. Sohoni. Geometric Complexity Theory. I. An approach to the P vs. NP and related problems. *SIAM J. Comput.*, 31(2):496–526 (electronic), 2001.
 - 45 K.D. Mulmuley and M. Sohoni. Geometric Complexity Theory. II. Towards explicit obstructions for embeddings among class varieties. *SIAM J. Comput.*, 38(3):1175–1206, 2008.
 - 46 Noam Nisan. Lower bounds for non-commutative computation. In *Proceedings of the 23rd ACM Symposium on Theory of Computing*, ACM Press. Citeseer, 1991.
 - 47 Luke Oeding and Claudiu Raicu. Tangential varieties of segre-veronese varieties. *Collectanea Mathematica*, 65, 11 2011. doi:10.1007/s13348-014-0111-1.
 - 48 Giorgio Ottaviani. Five lectures on projective invariants, lecture notes for trento school, september 2012. arXiv:1305.2749, to appear in *Rendiconti del Seminario Matematico*, Torino, 2013.
 - 49 Claudiu Raicu. 3×3 minors of catalecticants. *Mathematical Research Letters*, 20, 07 2013. doi:10.4310/MRL.2013.v20.n4.a10.
 - 50 Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Comb. Theory, Ser. B*, 62(2):323–348, 1994. doi:10.1006/jctb.1994.1073.
 - 51 Steven Sam and Andrew Snowden. Proof of stembridge’s conjecture on stability of Kronecker coefficients. *Journal of Algebraic Combinatorics*, 43:1–10, 2016.
 - 52 Nitin Saxena. Diagonal circuit identity testing and lower bounds. In *Automata, Languages and Programming*, pages 60–71, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.

- 53 Yaroslav Shitov. How hard is the tensor rank? arXiv:1611.01559, 2016.
- 54 Roberto Tamassia and Ioannis G Tollis. Planar grid embedding in linear time. *IEEE Transactions on circuits and systems*, 36(9):1230–1234, 1989.

