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A Mean-Risk MINLP for Transportation Network Protection

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Abstract

This research work focuses on pre-disaster transportation network protection in hedging against extreme events, e.g., earthquakes. Traditional two-stage stochastic programming method has been widely adopted to obtain solutions in presence of a set of uncertain scenarios, which however represents a risk-neutral preference due to the use of expectations in the recourse function. Decision makers in reality may hold different risk preferences. As a result, we develop a mean-risk two-stage stochastic programming model in this study, which allows for greater flexibility in handling risk preferences when allocating limited resources. In particular, the first stage minimizes the retrofitting cost by making strategic retrofit decisions whereas the second stage minimizes the travel cost. The conditional value-at-risk (CVaR) is included as the risk measure for the total system cost. The model is formulated as a mixed integer nonlinear programming (MINLP) problem, which is intrinsically difficult to solve using global solvers. A decomposition method based on Generalized Benders Decomposition (GBD) is developed, to overcome algorithmic challenges, particularly, embedded in non-convexity, nonlinearity, and non-separability of first- and second- stage variables. The model is used for developing retrofit strategies for networked highway bridges, which is one of the research fields that can significantly benefit from mean-risk models. We first justify the model and evaluate the proposed decomposition algorithm using a nine-node network. The model is then applied on the Sioux Falls network, which is a large-scale benchmark network in transportation research community. The effects of the risk measure and critical parameters on optimal solutions are empirically explored.

Keywords: Transportation, Stochastic programming, Mixed integer nonlinear optimization, Disjunctive programming, Benders decomposition
2010 MSC: 90C15, 90C26, 90C11, 90C90, 90C35

1. Introduction

Many highway bridges in the United States (U.S.), especially old bridges, can be seriously damaged or collapse even in relatively moderate natural disasters, e.g., earthquakes (Buckle et al., 2006). In the most recent infrastructure report card issued by the American Society of Civil Engineers (ASCE), one in nine of the bridges in U.S. are deemed structurally deficient (ASCE, 2013). Since 1960s, major structural damage has caused millions of dollars of economic losses in a number of states, including Alaska, California, Washington, and Oregon (Buckle et al., 2006). To improve this situation, at-risk

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bridges must be identified and evaluated and retrofitting programs should be in place to strengthen its resilience (Buckle et al., 2006).

Highway bridge retrofit is one of the most common approaches undertaken to mitigate negative effects of extreme events on highway transportation networks by federal and state departments of transportation. Bridge damages due to extreme events may result in direct social and economic losses as a result of post-disaster bridge repair and restoration as well as indirect impacts on transportation networks, due to short-term evacuations and emergency responses (Chang et al., 2012) and even long-term changes in travel activities (Fan and Liu, 2010; Liu et al., 2009). These adverse impacts can be avoided or alleviated if proactive bridge retrofit strategies are deployed.

The Federal Highway Administration (FHWA) estimates that to eliminate all deficient bridges backlog by 2028, an annual investment of $20.5 billion is needed while currently only $12.8 billion is being spent on (ASCE, 2013). Due to the limited retrofitting resources, it is neither practical nor economical to retrofit all bridges to their full health conditions and thus a prioritized retrofitting scheme is expected. In practice, resources are prioritized to bridges based on ranked structural deficiencies (Buckle et al., 2006), which neglects the effects of networked bridges and the resultant solution may not be optimal if indirect social losses (e.g., travel delay cost) are considered. This is because traffic flows may be redistributed over the transport network and affect other at-risk bridges. It justifies the need to consider bridge retrofitting strategies at a network level.

1.1. An example of a network based model

Let us consider the benchmark Sioux Falls network (LeBlanc et al., 1975, see Figure 1) to better understand the importance of a networked model. Assume that there are four bridges, labeled as A, B, C, and D, which are vulnerable to seismic hazards.

A failure of bridge C (i.e., functional obsolescence) would detour the traffic from node #20 to #18 that originally traverses via link 60 to a longer path that is consisted of links 61, 58, 52 and 50. It may result in higher travel cost, due to detours and resulted congestion. Additionally, the varied structural deficiencies of each bridge may require the use of different materials and labors for its rehabilitation. The main challenge is then formulating strategic allocations of limited resources to the bridges before they become structurally inadequate and cause undesirable consequences to the network. A strategy solely based on the ranked structural deficiency status would not guarantee system optimality. For instance, assuming that bridge D is in a worse condition than bridge C and that resources are insufficient to support retrofitting both, bridge D will outrank bridge C in retrofit priority, thus possibly exposing bridge C to a higher chance of reaching the state of functional obsolescence in extreme events. This solution could be counter-optimal, as the failure of bridge D only affects traffic on links among nodes #20, #21 and #22 while the failure of bridge C affects traffic on links among nodes #20, #19, #17, #16, and #18. From a network perspective, bridge C would be better positioned to be retrofitted .

1.2. Relevant literature

A network based bridge retrofitting is a general transportation network protection problem, which can be grouped into two broad categories, depending on if bridges are treated as links (Chang et al., 2012; Fan et al., 2010; Liu et al., 2009) or as paths (Mohaymany and Pirnazar, 2007; Viswanath and Peeta, 2003). The problems based on the links are formulated as maximum capacity or minimum cost flow network design problems with a focus on long-term economic effect of retrofit whereas the studies considering bridges as paths are formulated as maximal covering network design problems, which are more focused on short-term emergency response or maximal coverage of population centers. From a transportation system analysis viewpoint, the transportation network protection problem is essentially a

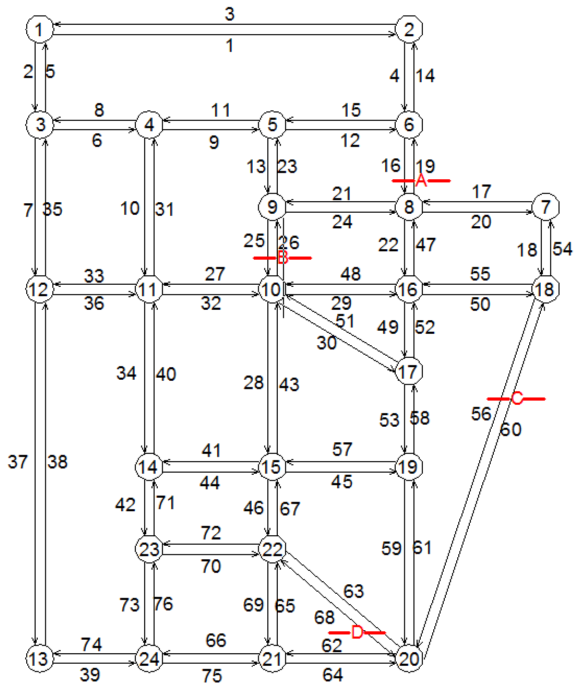


Figure 1: Sioux Falls network

network design problem (NDP), in which the upper-level problem involves optimal retrofit decisions for best social welfares (e.g., minimum retrofitting cost and travel delay) while lower-level problem accounts for the behaviors of network users which normally presents demand-performance equilibrium (Nagurney, 2006, 2007; Patriksson, 1994; Peeta and Ziliaskopoulos, 2001; Sheffi, 1985).

Uncertainty is naturally embedded in almost all transportation protection problems. Engineering methods based on the wait-and-see approach (Birge and Louveaux, 2011) seek optimal solutions upon the realizations of uncertainty (or scenario), which are deterministic. The resulting scenario dependent solutions are then aggregated in order to be implemented (Carturan et al., 2013; Chang et al., 2012; Rokneddin et al., 2013, 2011; Zhou et al., 2010). However, since future events are unknown at the time of making decisions, scenario-specific solutions (policies) may not even be feasible for other possible scenarios. Thus, a method that can account for a large number of possible scenarios needs to be developed. Previous studies use either stochastic programming (SP) (Barbarosoglu and Arda, 2004; Fan et al., 2010; Liu et al., 2009) or robust optimization (RO) method (Atamtürk and Zhang, 2007; Bertsimas and Sim, 2003; Lou et al., 2009; Sun et al., 2011; Yin et al., 2009) to take into account all scenarios. In general, the SP method takes the expectation of consequences of all scenarios and thus is risk-neutral and suitable for problems aiming to achieve long-term economic effects; however, it may have poor performance under extreme events. Though rare, these hazards exert more severe impacts on the system. RO approach, on the other hand, considers worst-case scenario with low-occurrence probability, which may lead to too conservative and in most cases costly solutions. Therefore, neither risk-neutral SP approach nor RO based method is best to capture the variability of risk, which motivates this study to seek a new method for economic yet robust solutions. As such, risk measures should be incorporated into decision making process of the stochastic modeling approach. In particular, we consider the conditional value-at-risk (CVaR) as the risk assessment in this study. Compared to other possible risk assessments (e.g., semi-deviation), CVaR is more adaptive to decision-makers' risk preference (i.e., users' preset confidence level).

In the field of transportation network protection problems, our study to the best of our knowledge, is the first study undertaken that specifies CVaR as the risk measure. CVaR was first introduced by (Andersson et al., 2001; Rockafellar and Uryasev, 2000, 2002) as a risk assessment technique in portfolio management to reduce the probability that a strategy incurs large losses (Krokhmal et al., 2002). Particularly, for a discrete distribution, CVaR at α-level is the conditional expected value exceeding value-at-risk (VaR) when there is no probability atom at VaR (Rockafellar and Uryasev, 2002). When the confidence level increases, the VaR increases, leading to a more risk-averse solution. When all scenarios are considered, the problem is equivalent to a RO problem. CVaR preserves convexity, which is a desirable mathematical property among risk measures (Ahmed, 2006). This risk measure has been broadened up in the past decade and applied to a number of engineering fields, including electricity operation decision (Yau et al., 2011), water resources allocation (Shao et al., 2011), facility location planning for reverse logistics (Toso and Alem, 2014), and hazard material routing (Kwon, 2011). On the other hand, the inherent computational challenges have motivated numerous algorithmic developments. Schultz and Tiedemann (2006a) developed a solution algorithm based on Lagrangian relaxation of nonanticipativity to solve a mixed-integer linear program with CVaR. Fábián (2008) developed decomposition methods for solving a two-stage SP linear program with CVaR and Noyan (2012) extended and solved a similar but two-stage SP mixed-integer linear program for disaster management. Comparitive computational studies with CVaR and other risk measures can be found in (Cotton and Ntaimo, 2015; Fábián et al., 2015).

1.3. Contributions of this study

In this study, we adopt CVaR as a risk measure in developing a mean-risk two-stage stochastic programming model, with the goal of minimizing the direct cost of retrofitting bridges in the first stage and indirect travel cost in the second stage. The first-stage decisions indicate the assignments of retrofit strategies to different bridges in an optimized manner, which are made simultaneously with second-stage traffic assignment decisions. CVaR is used to penalize the scenarios with large losses using a user-specified confidence level and the risk consequence is integrated with the two-stage stochastic program with a trade-off coefficient. The model is generic and generalizable for different kinds of natural and man-made disasters.

Our proposed model is closely related to the stochastic transportation protection model by Liu et al. (2009), in which a central semi-deviation is identified as the risk measure. However, our study is distinct from this prior study and advances the models in the following aspects. First, the semi-deviation can only capture the effects of scenarios that are worse than the expectation of second stage costs while the CVaR is flexible to incorporate a spectrum of scenarios, depending on the pre-defined confidence level and the weighting factors relative to cost terms in the objective. Second, the prior studies held the assumptions of the binary damage states (i.e., either no damage or collapse) and binary retrofit strategies (i.e., retrofit or no retrofit). Although these assumptions help reduce the problem size and consequently the computational challenges associated with solving large-scale problems, this simplification may result in less informative solutions and overestimate costs. In this study, we relax the assumption by defining multiple damage states and available retrofit strategies based on a recent study (Huang et al., 2014) where a set of binary decision variables are introduced to indicate whether a specific strategy is selected for a bridge. From the modeling perspective, it is not a trivial extension to the prior efforts, due to the inherent correlations between retrofit strategies, damage states, and resulting distributions of traffic flows on the network, which need to be explicitly modeled in the new problem. In addition, bridge retrofit strategies are subject to a budget limit, which makes the problem essentially a NP-hard knapsack problem (Kellerer et al., 2004).

The mean-risk two-stage stochastic programming model is formulated as a non-convex mixed integer nonlinear programming (MINLP) problem, wherein the travel cost for bridge links is a non-convex non-linear function of retrofit decisions. In general, it is known that non-convex MINLPs can be notoriously difficult to solve (Burer and Letchford, 2012). Thus another contribution of this study stems from the algorithmic development. In particular, we develop a novel decomposition that is based on the generalized Benders decomposition (GBD) method. Our decomposition resolves the issues of non-separability of first and second stage variables to enable efficient generations of Benders cuts. In this decomposition, we present a convex reformulation of the sub-problem. We justify our model and decomposition method on a hypothetical nine-node network and then apply the model and solution method to solve a stochastic transportation network protection problem based on a benchmark network - the Sioux Falls network (see Figure 1) and seismic events as a demonstration to explore the effects of risk measures and variations in critical parameters on the optimal solutions. The results provide managerial insights for state stakeholders on bridge retrofit schemes.

The remainder of the paper is organized as follows. The mean-risk two-stage SP model is presented in section 2, followed by the MINLP formulation presented in section 3. Decomposition method is described in section 4. The numerical results of the two networks are summarized in section 5. The paper is concluded in section 6 and the future research is outlined.

2. Mean-risk model

2.1. Parameters and variables

Let $G = (N, A)$ denote a transportation network, where N is the set of nodes and A is the set of directed arcs (or links) in the network. Denote by R and S, for some $\emptyset \neq R, S \subset N$, the set of origins and destinations in the network, respectively. The set of origin-destination (O-D) pairs is some subset $\mathcal{OD} \subseteq R \times S$. For every $(r, s) \in \mathcal{OD}$, $d^{rs} \in \mathbb{R}_+$ is the given travel demand on traffic originating at r and ending at s. Denote by $\bar{A}$, for some $\emptyset \neq \bar{A} \subset A$, as the set of links that are subject to hazards, which mainly comprises of the at-risk bridges. The nominal traffic capacity of each link $a \in A$ is equal to c_a. For $a \in A \setminus \bar{A}$, it is assumed that this link capacity remains unchanged after any disastrous event (e.g., natural or man-made disasters). However, the link capacities of links in $\bar{A}$ reduce due to the damage from the events and the extent of this change depends on how well the at-risk bridges were retrofitted before the events happened. The finite set H represents a list of applicable retrofit strategies for at-risk bridges in $\bar{A}$ in order to mitigate the adverse impacts caused by disastrous events in the future. The set H includes the do-nothing option and each at-risk bridge can be retrofitted with exactly only one strategy. The cost of retrofitting $a \in \bar{A}$ with strategy $h \in H$ is β_a^h. The total budget for retrofitting bridges is β_0.

In this study, two sets of probabilistic estimates – damage to a structure and the probabilities of various event occurrences, are combined to obtain a damage prediction. Let the finite set K denote the set of scenarios for possible damages to the network. Each scenario $k \in K$ is known to occur with a given probability $p_k \in (0, 1)$. For every $a \in \bar{A}, h \in H$, and scenario $k \in K$, we use the parameter $\theta_a^{h,k} \in (0, 1)$ to describe the ratio of post-event link capacity to the full link capacity, which can be determined externally by using bridge structural assessment, such as the study (Mackie and Stojadinovic, 2004) for seismic damages. When disaster happens, the post-event capacity of link $a \in \bar{A}$ that was retrofitted with strategy $h \in H$ is equal to $c_a \theta_a^{h,k}$.

We now describe the decision variables used to construct our mathematical formulation. For every $a \in \bar{A}, h \in H$, the binary variable u_a^h is equal to 1 if and only if link a is retrofitted by strategy $h \in H$. For $(r,s) \in \mathcal{OD}$, $a \in A$ and $k \in K$, $x_a^{rs,k}$ is the flow on link a corresponding to the traffic originating at r and terminating at s for scenario k. The total flow on link $a \in A$ due to all O-D pairs is v_a^k, and $v_a^k = \sum_{(r,s)\in\mathcal{OD}} x_a^{rs,k}, \forall a \in A$. In this model, we allow unsatisfied post-disaster travel demand for various reasons, such as shutdown of certain roadways, acute increased traffic congestion in the network, etc. The unsatisfied travel demand for any O-D pair (r,s) is captured by the decision variable $q^{rs,k}$ and we use a big M to impose a penalty cost for the unsatisfied demand in the objective function.

Remark 1. In the transportation network literature, traffic is often assumed to be in a user-equilibrium condition, where no traveler can further reduce their travel cost by simply changing their own routing decision (Yang and H. Bell, 1998). This assumption holds for a normal situation, where travelers have learned and adapted to daily traffic condition. However, modeling travelers' routing behavior in an environment following extreme events, such as earthquake, is still arguable (Fan and Liu, 2010). In this paper, it is assumed that traffic flow can be controlled to achieve system-optimization and the resulting estimated travel cost can be considered as a lower bound of actual travel cost.

2.2. Two-stage stochastic models

2.2.1. Risk-neutral model

We first present a basic two-stage stochastic programming model for our problem. The first stage considers the retrofit resource allocation problem and decides the retrofitting strategy for each of the links in $\bar{A}$. Define the set U as

$$U := \left\{ u \in \{0,1\}^{|\bar{A}|\times|H|} \mid \sum_{h\in H} u_a^h = 1 \;\; \forall a \in \bar{A},\; \beta^\top u \le \beta_0 \right\} \tag{1}$$

to include all first-stage decisions – each link in $\bar{A}$ can be retrofitted with exactly one strategy and the total budget is β_0. The problem is then to minimize $\beta^\top u + \mathbb{E}\, Q(u)$, which is the sum of total retrofitting cost and the expected travel cost incurred. Equivalently, the first-stage objective is to minimize $\mathbb{E}\, f(u)$, where $f(u) := \beta^\top u + Q(u)$ is the total cost function. The assumption of finitely many scenarios indexed by the set K allows us to discretize the expectation of expected travel cost function and state our two-stage stochastic program as

$$(\textsf{2-stage SP}): \quad \min_u \sum_{k\in K} p_k f^k(u) = \min_u \; \beta^\top u + \sum_{k\in K} p_k Q^k(u) \quad \text{s.t.} \quad u \in U, \tag{2}$$

where $f^k(u) = \beta^\top u + Q^k(u)$ is the total cost function for the k^{th} scenario, in which $Q^k(u)$ is the travel cost function based on an explicit traffic assignment model for the k^{th} scenario, defined as

$$Q^k(u) := \min_{x^k, q^k} \quad \gamma \sum_{a \in A} v_a^k t_a^k \; + \; M \sum_{(r,s) \in \mathcal{OD}} q^{rs,k} \tag{3a}$$

$$\text{s.t.} \quad v_a^k = \sum_{(r,s) \in \mathcal{OD}} x_a^{rs,k} \quad \forall a \in A \tag{3b}$$

$$t_a^k = t_{0a} \left[1 + \delta \left(\frac{v_a^k}{\hat{c}_a^k(u)} \right)^4 \right] \quad \forall a \in A \tag{3c}$$

$$(x^k, q^k) \in X^k \tag{3d}$$

In the second stage, v_a^k is the aggregation of link flow $x_a^{rs,k}$ over all O-D pairs (r, s), $q^{rs,k}$ is the unsatisfied demand between an O-D pairs (r, s), and t_a^k is the link travel time per unit flow. The objective function (3a) is to minimize the total cost of traffic flow on the network and consists of two terms. Each product $v_a^k t_a^k$ is equal to the travel time for the entire flow on link $a \in A$ and upon scaling this with the parameter γ that converts travel time to a monetary value[1]. The second term represents the penalty cost for unsatisfied demand. In our problem, all travel demand is satisfied (or penalized for economic concerns) in the second stage. The link travel time per unit flow is usually a non-decreasing link performance function of the aggregated link flow and a non-increasing function of the post-event link capacity in each scenario. Equation (3c) expresses the dependence of t_a^k on v_a^k using the Bureau of Public Records (BPR) function (of Public Roads, 1964), in which t_{0a} is a parameter for the free-flow-speed travel time of link a, δ is an empirical data (e.g., 0.15), and the denominator $\hat{c}_a^k(u)$, which is a function of the first stage decision u, denotes the remaining link capacity on link a in scenario k:

$$\hat{c}_a^k(u) = \begin{cases} c_a \sum_{h \in H} \theta_a^{h,k} u_a^h & a \in \bar{A} \\ c_a & a \in A \setminus \bar{A}. \end{cases} \tag{4}$$

The recourse function $Q^k(u)$ seeks to optimize flows over the set X^k, defined as:

$$X^k := \Big\{ (x, q) \geq (\mathbf{0}, \mathbf{0}) \mid \sum_{j:\,(r,j) \in A} x_{rj}^{rs} - \sum_{j:\,(j,r) \in A} x_{jr}^{rs} + q^{rs} = d^{rs} \quad \forall (r,s) \in \mathcal{OD} \tag{5a}$$

$$\sum_{j:\,(s,j) \in A} x_{sj}^{rs} - \sum_{j:\,(j,s) \in A} x_{js}^{rs} - q^{rs} = -d^{rs} \quad \forall (r,s) \in \mathcal{OD} \tag{5b}$$

$$\sum_{j:\,(i,j) \in A} x_{ij}^{rs} - \sum_{j:\,(j,i) \in A} x_{ji}^{rs} = 0 \quad \forall (r,s) \in \mathcal{OD}, i \in N \setminus \{r, s\} \Big\}. \tag{5c}$$

For each O-D pair (r, s), equations (5a) and (5b) allow a slack of q^{rs} in the flow balance at r and s, respectively, to account for unsatisfied travel demand, whereas equation (5c) balances flow exactly at all other nodes in the network.

As the post-earthquake link capacity (4) is a linear function of retrofit decisions for links in $\bar{A}$, the

[1] In practice, determining the value of γ requires approximation of value of travel time savings, which is assumed to be equal to a nationwide median gross compensation for business travel (U.S. Department of Transportation, 2014).

decision variable u appears on the denominator of the travel time cost function in (3c). This imparts non-convexity and nonlinearity to our two-stage stochastic problem and also leads to the following property.

Observation 1. *Problem (2-stage SP) has* complete recourse*, i.e., subproblem* (3) *is feasible for every* $u \in U$*.*

Proof. The only reliance of subproblem (3) on the first stage decision u is through the time variable t_a^k defined in equation (3c). Since t_a^k otherwise appears only in the objective function (3a), it follows that the subproblem is feasible regardless of the retrofit decisions made in the first stage. □

2.2.2. Mean-risk model

We now turn to introducing our mean-risk stochastic program, which combines the two-stage risk-neutral SP model and the CVaR function for risk assessment. Recall that the α-level CVaR is

$$\mathbb{CVaR}_\alpha Z := \inf_{\eta} \left[\eta + \frac{1}{1-\alpha} \mathbb{E} \max \{0, Z - \eta\} \right].$$

Since $f(u) = \beta^\top u + Q(u)$ is the total cost function, the mean-risk objective is $\mathbb{E} f(u) + \lambda \mathbb{CVaR}_\alpha f(u)$, where the coefficient $\lambda \in [0, \infty)$ represents a trade-off between the risk measure (CVaR in our case) and the expected first stage cost. The risk-neutral problem (2-stage SP) corresponds to $\lambda = 0$. Upon discretizing with finitely many scenarios as before, performing simple manipulations arising out of translation invariance of CVaR (cf. Schultz and Tiedemann, 2006b), and linearizing the $\max\{0, \cdot\}$ function in CVaR, the mean-risk SP program becomes

$$\text{(Mean-risk SP)}: \quad \min_{u,\eta,\xi} \quad (1+\lambda)\beta^\top u + \sum_{k\in K} p_k Q^k(u) + \lambda \left(\eta + \frac{1}{1-\alpha} \sum_{k \in K} p_k \xi^k \right) \tag{6a}$$

$$\text{s.t.} \quad u \in U \tag{6b}$$

$$\xi^k \geq Q^k(u) - \eta \quad \forall k \in K \tag{6c}$$

$$\xi^k \geq 0 \quad \forall k \in K, \tag{6d}$$

where $Q^k(u)$ is defined by equations (3)-(4). The objective is to minimize the total cost of retrofitting bridges, expected travel cost, unsatisfied demand penalty and the risk term (CVaR). Here λ is a pre-defined weighting factor. A larger λ value leans towards CVaR and thus results in a more conservative solution. On the other hand, a smaller λ value yields a solution that weighs more on the expected cost, and thus the solution is more risk-neutral. The variable η denotes the value-at-risk.

Formulation (6) may be thought of as a *disaggregated* stochastic program for mean-risk problems with CVaR because we introduce an auxiliary variable ξ^k for each $k \in K$. An alternative is to consider an aggregated formulation given as follows.

Proposition 1. *Let z^* be the optimal value of (Mean-risk SP). Then z^* is equal to*

$$\min_{u,\eta,\zeta} \quad (1+\lambda)\beta^\top u + \zeta_1 + \lambda \left(\eta + \frac{\zeta_2}{1-\alpha} \right) \tag{7a}$$

$$s.t. \quad u \in U, \quad \zeta_2 \geq 0 \tag{7b}$$

$$\zeta_1 \geq \sum_{k\in K} p_k Q^k(u), \qquad \zeta_2 \geq \sum_{k \in S} p_k (Q^k(u) - \eta) \quad \forall S \subseteq K. \tag{7c}$$

Proof. For $u \in U$ and $\eta \in \mathbb{R}$, define

$$T(u, \eta) := \{k \in K : Q^k(u) > \eta\}.$$

Since $p_k > 0\ \forall k \in K$, observe that for fixed $u \in U$ and $\eta \in \mathbb{R}$, we have

$$\sum_{k \in T(u,\eta)} p_k(Q^k(u) - \eta) \;=\; \max_{S \subseteq K} \sum_{k \in S} p_k(Q^k(u) - \eta),$$

with the maximum being zero if and only if $T(u, \eta) = \emptyset$. Denote $\tilde{z}$ to be the optimal value of (7). First let $(\bar{u}, \bar{\eta}, \bar{\xi})$ be a optimal solution to (6). Since $\lambda/(1-\alpha) \geq 0$, we have $\bar{\xi}^k = \max\{0, Q^k(\bar{u}) - \bar{\eta}\}$ for all $k \in K$. Setting $\zeta_1 = \sum_{k \in K} p_k Q^k(\bar{u})$ and $\zeta_2 = \sum_{k \in T(\bar{u},\bar{\eta})} p_k(Q^k(\bar{u}) - \bar{\eta})$ produces a feasible solution $(\bar{u}, \bar{\eta}, \zeta)$ to (7) with objective value z^*, implying that $\tilde{z} \leq z^*$. Now let $(\bar{u}, \bar{\eta}, \bar{\zeta})$ be optimal to (7). Clearly $\bar{\zeta}_1 = \sum_{k \in K} p_k Q^k(\bar{u})$. Since $\lambda/(1-\alpha) \geq 0$, we have

$$\bar{\zeta}_2 \;=\; \max\{0, \max_{S \subseteq K} \sum_{k \in S} p_k(Q^k(\bar{u}) - \bar{\eta})\} \;=\; \sum_{k \in T(\bar{u},\bar{\eta})} p_k(Q^k(\bar{u}) - \bar{\eta}).$$

Thus $\bar{\zeta}_2 = \sum_{k \in K} p_k \max\{0, Q^k(\bar{u}) - \bar{\eta}\}$. Set $\xi^k = \max\{0, Q^k(\bar{u}) - \bar{\eta}\}\ \forall k \in K$ to get a feasible solution $(\bar{u}, \bar{\eta}, \xi)$ to (6) with value $\tilde{z}$, thereby implying $z^* \leq \tilde{z}$. Combining this with the first part yields $z^* = \tilde{z}$. ☐

Formulation (7) uses ζ_1 and ζ_2 to replace $\sum_k p_k Q^k(u)$ and $\sum_k p_k \max\{0, Q^k(u) - \eta\}$, respectively. However, due to the presence of the $\max\{0, \cdot\}$ function in $\mathbb{CV}\text{a}\mathbb{R}_\alpha(\cdot)$, we need exponentially many constraints to reformulate $\zeta_2 \geq \sum_{k \in K} p_k \max\{0, Q^k(u) - \eta\}$ without adding any extra variables. The disaggregated stochastic program helps to restrict the formulation size and only adds a modestly many $|K|$ extra variables. Note that this tradeoff between a few extra variables and exponentially many constraints does not occur in a risk-neutral stochastic program. In terms of solving a two-stage stochastic program with a decomposition algorithm, disaggregated formulations are known to sometimes yield much stronger cuts than a aggregated formulation, thus accelerating convergence to the optimum. For these two reasons we henceforth work with formulation (6) and remark that it may be possible to handle the inequalities in (7c) via a cutting plane procedure.

3. Recourse function

For each scenario k, the recourse function $Q^k(u)$ is a nonlinear optimization problem in (3). This problem is *non-convex* due to presence of the bilinear terms $v_a^k t_a^k$ in the objective and nonlinear equality constraints defining t_a^k. More importantly, since the fractional function $\frac{v_a}{\hat{c}_a^k(u)}$ in (3c) has u appearing linearly in the denominator, the second stage variables are *non-separable* from the first stage variable in this formulation. Problem convexity and separability of the variables are both desirable properties of Benders-type decomposition methods for solving a two-stage stochastic program with a nonlinear second stage since they guarantee generation of valid supporting hyperplanes of the recourse function (see Floudas, 1995; Geoffrion, 1972). Our decomposition algorithm for solving (Mean-risk SP) is presented in §4. In this section, we derive a reformulation of $Q^k(u)$ that is not only a convex program for every $u \in U$ but also also achieves separability between first and second stage variables. This reformulation leads to a convex MINLP formulation for solving (Mean-risk SP) as a single optimization problem.

Substituting the variable t_a^k in formulation (3) with the nonlinear function in (3c) eliminates the bilinear terms $v_a^k t_a^k$ and leads to

$$Q^k(u) = \min_{v^k, q^k, x^k} \quad \gamma \sum_{a \in A} t_{0a} \left[v_a^k + \delta \frac{(v_a^k)^5}{\hat{c}_a^k(u)^4} \right] + M \sum_{(r,s) \in \mathcal{OD}} q^{rs,k} \tag{8a}$$

$$\text{s.t.} \quad v_a^k = \sum_{(r,s) \in \mathcal{OD}} x_a^{rs,k} \;\; \forall a \in A, \quad (x^k, q^k) \in X. \tag{8b}$$

Thus for a fixed $u \in U$, the recourse value $Q^k(u)$ can be obtained by solving the convex optimization problem (8). However this does not tell us anything about the convexity of $Q^k(\cdot)$. We exploit properties of the discrete set U to show that the recourse function is indeed convex. Our main approach is to eliminate u from the denominator in (8a) and make the subproblem separable in first and second stage variables. In particular, we obtain subproblem constraints that are linear in u, convex in v and do not contain product terms between u and any of v, q, x. There are different ways of achieving this and we present these next.

Let us introduce a auxiliary second stage non-negative continuous variable y_a^k for each $a \in \bar{A}$ and add the inequality

$$y_a^k \geq \frac{(v_a^k)^5}{(c_a \sum_{h \in H} u_a^h \theta_a^{h,k})^4} \quad \forall a \in \bar{A}. \tag{9}$$

The right hand side of the above inequality appears in the objective (8a) with a positive coefficient $\gamma \delta t_{0a}$. Hence we have

$$Q^k(u) = \min_{v^k, q^k, x^k, y^k} \quad \gamma \sum_{a \in A} t_{0a} \left[v_a^k + \delta y_a^k \right] + M \sum_{(r,s) \in \mathcal{OD}} q^{rs,k} \tag{10a}$$

$$\text{s.t.} \quad (8b), (9). \tag{10b}$$

The following lemma guides our convex reformulation for $Q^k(u)$.

Lemma 1. *For $a \in \bar{A}$ and $u \in U$, $\left(\sum_{h \in H} \theta_a^{hk} u_a^h\right)^p = \sum_{h \in H} \left(\theta_a^{hk}\right)^p u_a^h$ for all $p \in \mathbb{R}$.*

Proof. Since $\sum_{h \in H} u_a^h = 1 \; \forall a$ and $u_a^h \in \{0,1\} \; \forall a, h$, it must be that for every $a \in \bar{A}$ we have $u_a^h = 1$ for some $h \in H$ and $u_a^{h'} = 0$ for all $h' \in H \setminus h$. Hence both $(\sum_{h \in H} \theta_a^{hk} u_a^h)^p$ and $\sum_{h \in H} (\theta_a^{hk})^p u_a^h$ are equal to $(\theta_a^{hk})^p$. □

After clearing the denominator in (9) and applying Lemma 1 with $p = 4$, we obtain

$$(v_a^k)^5 \leq c_a^4 \left[\sum_{h \in H} \left(\theta_a^{hk}\right)^4 u_a^h \right] y_a^k \quad \forall a \in \bar{A}. \tag{11}$$

Remark 2. For $u \in U$, since $\sum_h u_a^h = 1$ and $u_a^h \in \{0,1\}$, the term $\sum_{h \in H} \left(\theta_a^{hk}\right)^4 u_a^h$ can be interpreted as the unary expansion of a discrete variable that takes values in $\cup_h \left(\theta_a^{hk}\right)^4$. The right hand side of inequality (11) is then the product of a discrete variable and a non-negative continuous variable and is therefore a bilinear term. Another formulation for this bilinear term can be obtained using the binary expansion of the discrete variable, where only $\log_2 |H|$ many $\{0,1\}$ variables (as opposed to $|H|$ $\{0,1\}$ u's in the unary case) are required. Gupte et al. (2013) theoretically compared unary and binary expansion reformulations for bilinear optimization problems, obtained new valid inequalities to strengthen the continuous relaxation of the binary reformulation and showed that these convexifications work well on hard test instances. This encourages the use of binary expansion formulations for general bilinear problems. However, in our case since the cardinality of H is quite small (e.g., 5), the discrete variable takes only up to 5 different values and there is no significant benefit of adopting the logarithmic formulation for $\cup_h \left(\theta_a^{hk}\right)^4$. Therefore we choose to not modify the term $\sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h$ in (11).

In mixed integer nonlinear optimization literature, it is common practice to replace each nonlinear constraint of $\leq$-type with the convex envelope of the corresponding nonlinear function; see Tawarmalani and Sahinidis (2004). Such a replacement usually relaxes the nonlinear constraint, although if some of the variables are discrete, one may sometimes also obtain an exact reformulation of the constraint. The $\leq$-inequality in (11) has different sets of variables on the left and right hand sides. Therefore, if we write the constraint as the difference of the left and right hand sides, taking the convex envelope of this difference is equivalent to separately taking the convex envelope of the left hand side and the concave envelope of the right hand side. The left hand side in (11) is a univariate convex function of v_a over $\mathbb{R}_+$ and hence does not require any convexification. For the right hand side, we have a bilinear term between a discrete variable $\sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h \in \cup_h \left(\theta_a^{hk}\right)^4$ (cf. Remark 2) and a continuous variable y_a^k. The concave envelope of this bilinear term is given by its McCormick inequalities (McCormick, 1976), which depend on lower and upper bounds on the variables appearing in the bilinear term. The bounds for $\sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h$ and y_a^k can be obtained as follows. It is clear that for $u \in U$,

$$\underline{\theta}_a^k \leq \sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h \leq \bar{\theta}_a^k, \quad \text{where} \quad \underline{\theta}_a^k := \left(\min_{h\in H} \theta_a^{hk}\right)^4, \; \bar{\theta}_a^k := \left(\max_{h\in H} \theta_a^{hk}\right)^4.$$

From equations (8b) and (9) we get the lower bound on y_a^k to be zero. For every $a \in \bar{A}$, let $\varsigma_a > 0$ be a large enough positive constant such that every optimal solution to (10) satisfies $v_a \leq c_a\varsigma_a \; \forall a \in \bar{A}$. Then by inequality (9) and Lemma 1, every optimal solution to (10) satisfies

$$y_a^k \leq \frac{c_a\varsigma_a^5}{\sum_h (\theta_a^{hk})^4 u_a^h}, \tag{12a}$$

which leads to

$$y_a^k \leq \frac{c_a\varsigma_a^5}{\underline{\theta}_a^k} \tag{12b}$$

since $u \in U$. Using these lower and upper bounds, the McCormick concave envelope of the bilinear term on the right hand side of (11) is

$$c_a^4 \min\left\{\bar{\theta}_a^k y_a^k,\; \underline{\theta}_a^k y_a^k \,+\, \frac{c_a\varsigma_a^5}{\underline{\theta}_a^k} \sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h \,-\, c_a\varsigma_a^5\right\}. \tag{13}$$

and hence we have two inequalities for (11):

$$(v_a^k)^5 \leq c_a^4 \bar{\theta}_a^k y_a^k, \quad (v_a^k)^5 \leq c_a^4 \underline{\theta}_a^k y_a^k \,+\, \frac{(c_a\varsigma_a)^5}{\underline{\theta}_a^k} \sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h \,-\, (c_a\varsigma_a)^5 \qquad \forall a \in \bar{A}. \tag{14}$$

Gupte et al. (2013, Proposition 2.1) tells us that modeling a bilinear term between a general integer and a continuous variable with the McCormick inequalities allows for more integer solutions. The same holds true when the bilinear term is between a general discrete and a continuous variable. Therefore (14) yields a relaxation but not a reformulation of (11). Although this relaxation can be used to under-estimate the recourse function $Q^k(u)$, doing so will only yield weaker cuts in the decomposition algorithm.

In order to derive an exact reformulation of (11), first note that $(v_a^k, (u_a^h)_h, y_a^k)$ is feasible to (11) if and only if there exists some w_a^k such that

$$(v_a^k)^5 \leq \; c_a^4 w_a^k, \tag{15a}$$

$$0 \leq w_a^k \leq \; c_a \varsigma_a^5, \quad w_a^k \leq \left[\sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h\right] y_a^k. \tag{15b}$$

This equivalence is correct because $0 \leq v_a \leq c_a\varsigma_a$ implies that the left hand side of (11) is upper-bounded by $(c_a\varsigma_a)^5$. Since $c_a\varsigma_a^5 < c_a\varsigma_a^5\bar{\theta}_a^k/\underline{\theta}_a^k$ and the right hand side is the product of the upper bounds on $\sum_{h\in H}\left(\theta_a^{hk}\right)^4 u_a^h$ and y_a^k, respectively, it is clear that the upper bound on w_a^k is nontrivial. We will convexify (15b) to obtain a strong reformulation of (11).

Remark 3. Inequalities (14) are precisely the projection of the McCormick relaxation of

$$v_a^5 \leq c_a^4 w_a^k, \quad 0 \leq w_a^k \leq \left[\sum_{h\in H} \left(\theta_a^{hk}\right)^4 u_a^h\right] y_a^k, \tag{16}$$

where the bilinear term is relaxed using (13). Observe that (15) and (16) are each equivalent to (11) but (16) is a relaxation of (15) in the $(v_a^k, (u_a^h)_h, y_a^k, w_a^k)$-space. Therefore convexifying (15b) produces a stronger reformulation in the $(v_a^k, (u_a^h)_h, y_a^k, w_a^k)$-space than convexifying $0 \leq w_a^k \leq \left[\sum_{h\in H}\left(\theta_a^{hk}\right)^4 u_a^h\right] y_a^k$ in (16) and hence could lead to stronger cuts in the decomposition algorithm.

Equation (15b) represents a bounded product term between a discrete and a continuous variable and for such bilinear terms, we note that the McCormick envelopes are neither a reformulation nor do they yield the convex hull.

Observation 2. *Denote* $\mathcal{I} := \{b_1, b_2, \ldots, b_m\} \times [0, d] \times [0, \eta]$ *with* $m \geq 3, 0 \leq b_1 < b_2 < \cdots < b_m$ *and* $\eta < b_m d$. *Let* $\mathcal{R} := \{(x, y, \nu) \in \mathcal{I} \mid \nu \leq xy\}$, $\mathcal{M} := \{(x, y, \nu) \in \mathcal{I} \mid \nu \leq b_m y,\ \nu \leq b_1 y + dx - db_1\}$ *be the McCormick relaxation of* $\mathcal{R}$, *and* $\mathcal{M}' := \{(x, y, \nu) \in [b_1, b_m] \times [0, d] \times [0, \eta] \mid \nu \leq b_m y,\ \nu \leq b_1 y + dx - db_1\}$ *be the continuous relaxation of* $\mathcal{M}$. *Then* $\mathcal{R} \subsetneq \mathcal{M}$ *and* $\operatorname{conv}\mathcal{R} \subsetneq \mathcal{M}'$.

Proof. $\mathcal{M}$ being the McCormick relaxation of $\mathcal{R}$ leads to $\mathcal{R} \subseteq \mathcal{M}$ and $\operatorname{conv}\mathcal{R} \subseteq \mathcal{M}'$. Denote

$$\chi_0 := \frac{d}{\eta(1 - \frac{b_1}{b_m})}, \quad \chi_1 := \eta \min\{1, (b_2 - b_1)\chi_0\}.$$

Then $(b_2, \frac{\chi_1}{b_m}, \chi_1) \in \mathcal{M} \setminus \mathcal{R}$. Note that $(b_1 + \frac{1}{\chi_0}, \frac{\eta}{b_m}, \eta)$ is an extreme point of $\mathcal{M}'$. Suppose this point can be written as a nontrivial convex combination of finitely many $(x^t, y^t, \nu^t) \in \mathcal{R}$. Then $\nu^t = \eta\ \forall t$. Since $\nu^t \leq x^t y^t$ and $x^t \leq b_m$, it follows that $y^t \geq \eta/b_m$ and hence $y^t = \eta/b_m$ and $x^t = b_m$ for all t. Consequently, $b_1 + \frac{1}{\chi_0} = b_m$, which is a contradiction because $\eta < b_m d$ by assumption. □

An extended formulation for the convex hull of $\mathcal{R}$ can be obtained using unary expansion of discrete variables. Let $\nu = \sum_i b_i z_i$ with $\sum_i z_i = 1, z_i \in \{0, 1\}\ \forall i$. Then disjunctive programming (Balas, 1979) implies that $\operatorname{conv}\mathcal{R}$ is equal to the projection onto the (χ, y, ν)-space of the polyhedron

$$\Big\{(\chi, y, \nu, z) \mid \ \chi = \sum_{i=1}^{m} \ell_i z_i,\ \nu = \sum_{i=1}^{m} \nu^i,\ y = \sum_{i=1}^{m} y^i \\ 0 \le y^i \le d z_i,\ 0 \le \nu^i \le \delta z_i,\ \nu^i \le \ell_i y^i \quad \forall i \\ \sum_{i=1}^{m} z_i = 1,\ z_i \ge 0\ \forall i\Big\}. \tag{17}$$

Since $\sum_h (\theta_a^{hk})^4 u_a$ is the unary expansion of a discrete variable as mentioned in Remark 2, we may apply (17) directly to (15b). However, we will first strengthen the bounds on the y_a^k variables in each disjunction. Of the two upper bounds on y_a^k in (12a) and (12b), the former is stronger but is a function of u whereas the latter is weaker but is a constant. The constant bound in (12b) is necessary for deriving McCormick relaxations of bilinear terms as in Observation 2. The bound in (12a) can be incorporated into the disjunctive programming approach of equation (17). Since $y_a^k \le \frac{c_a\varsigma_a^5}{\sum_h(\theta_a^{hk})^4 u_a^h}$ by equation (12a) and $\sum_h u_a^h = 1$ and $u_a^h \in \{0,1\}\ \forall h$ for every $u \in U$, it is clear that convexifying (15b) is equivalent to convexifying the following finite union of polytopes:

$$\bigcup_{h\in H} \Big\{(u_a, y_a^k, w_a^k) \mid u_a = \mathbf{e}_h,\ 0 \le y_a^k \le \frac{c_a\varsigma_a^5}{(\theta_a^{hk})^4},\ 0 \le w_a^k \le c_a\varsigma_a^5,\ w_a^k \le c_a^4(\theta_a^{hk})^4 y_a^k\Big\}. \tag{18}$$

where $\mathbf{e}_h$ is the h^{th} coordinate unit vector. A straightforward application of disjunctive programming (Balas, 1979) gives us the convex hull of (15b), which results in a strong reformulation of the recourse function.

Proposition 2. *The recourse function of the k^{th} scenario can be formulated as:*

$$Q^k(u) = \min_{\substack{v^k, q^k, x^k \\ y^k, w^k}} \quad \gamma \sum_{a\in A} t_{0a}\left[v_a^k + \delta y_a^k\right] + M \sum_{(r,s)\in\mathcal{OD}} q^{rs,k} \tag{19a}$$

$$\text{s.t.} \quad v_a^k = \sum_{(r,s)\in\mathcal{OD}} x_a^{rs,k} \quad \forall a \in A, \quad (x^k, q^k) \in X \tag{19b}$$

$$(v_a^k)^5 \le c_a^4 \sum_{h\in H} w_a^{hk},\ y_a^k = \sum_{h\in H} y_a^{hk} \quad \forall a \in \bar{A} \tag{19c}$$

$$w_a^{hk} \le c_a^4(\theta_a^{hk})^4 y_a^{hk} \quad \forall h \in H, a \in \bar{A} \tag{19d}$$

$$0 \le y_a^{hk} \le \frac{c_a\varsigma_a^5}{(\theta_a^{hk})^4} u_a^h,\ 0 \le w_a^{hk} \le c_a\varsigma_a^5 u_a^h \quad \forall h \in H, a \in \bar{A} \tag{19e}$$

where for every $a \in \bar{A}$, ς_a is a large enough positive constant such that $c_a\varsigma_a$ is the upper-bound on the traffic volume of link a.

Notice that Proposition 2 linearly separates the first stage variable u from the second stage variables, i.e., each subproblem constraint containing both first and second stage variables can be represented as $h_1(u) + g_1(w) \le 0$ or $h_2(u) + g_2(y) \le 0$ where h_1, h_2, g_1, g_2 are all linear functions. This structure fits in with the so-called P-property of Geoffrion (1972) and allows for an immediate application of the generalized Benders decomposition in the next section. It also implies that the recourse function $Q^k(u)$

can be approximated by supporting hyperplanes and hence is convex in u.

Proposition 2 also implies an exact convex MINLP formulation for the mean risk problem (6).

$$
\begin{aligned}
(\textsf{Convex MINLP}):\quad \min_{\substack{u,\eta,\xi\\ v,q,x,y,w}} \quad & (1+\lambda)\beta^\top u \;+\; \sum_{k\in K} p_k \left[\gamma \sum_{a\in A} t_{0a}\left[v_a^k + \delta y_a^k\right] \;+\; M \sum_{(r,s)\in\mathcal{OD}} q^{rs,k}\right] \\
& + \lambda\left(\eta + \frac{1}{1-\alpha}\sum_{k\in K} p_k \xi^k\right) \\
\text{s.t.} \quad & u\in U, \quad \xi^k \geq 0 \;\; \forall k\in K \\
& \xi^k \geq \gamma \sum_{a\in A} t_{0a}\left[v_a^k + \delta y_a^k\right] \;+\; M \sum_{(r,s)\in\mathcal{OD}} q^{rs,k} \;-\; \eta \;\; \forall k\in K \\
& \text{(19b)} - \text{(19e)} \;\; \forall k\in K
\end{aligned}
$$

This convex MINLP can be solved to ϵ-optimality using state-of-the-art MINLP solvers. However we demonstrate in §5 that even on a simple nine-node network, these generic global optimization solvers take a long time to converge, thereby making this approach intractable for larger-sized practical instances. This motivates our algorithmic development in the next section.

4. Decomposition method

Extensive algorithmic efforts have been made to improve the efficiency of solution algorithm for MINLPs, including the widely used branch and bound (Gupta and Ravindran, 1985) with its variants - LP/NLP based branch and bound method (Quesada and Grossmann, 1992) and spatial branch and bound (Smith and Pantelides, 1999), as well as Generalized Benders Decomposition (GBD) method (Geoffrion, 1972). The branch and bound method is essentially an implicit enumeration procedure, which can be computationally expensive when the number of integer variables is large. The GBD on the other hand is effective in handling large-scale problem by decomposing intractable MINLP to tractable sub-problems. In this study, we develop a decomposition method based on GBD. Also note that there are other plausible solution methods, including Extended Cutting Plane method (Westerlund and Pettersson, 1995), and outer approximation (Fletcher and Leyffer, 1994). Though beyond the scope of this study, comparisons between these different methods in terms of solution quality and performance are worth investigations in future works.

The mean-risk SP model (6) will be decomposed into a master problem and several subproblems. The master problem is a 0\1 mixed integer linear program (MILP) and contains first-stage integer variables u and the value-at-risk η. The sub-problems are evaluated for the second-stage cost, given the first-stage variable u. Combined with the first-stage cost, we can compute CVaR for the overall cost at the optimum of the master problem. We will discuss the details on decomposition method in this section.

The background on GBD can be found, for example, in Floudas (1995). In this method, when the first stage variables are temporally held fixed, the remaining optimization problem is considerably more tractable than the original one. As for this study, if bridge retrofit decision variable u and the value-at-risk η, are temporarily fixed, the remaining problem (3) becomes a traffic assignment problem based on system-optimization condition, which may be effectively solved by using commercial nonlinear program solvers. The CVaR value can be obtained once we have travel cost function values from the traffic assignment problems corresponding to different scenarios.

We decompose our mean-risk SP model in (6) into a master problem and one subproblem for each scenario k. In the objective function of the master problem, the recourse function travel cost and CVaR are not known explicitly in advance. Thus, two optimality cuts are added iteratively to approximate them. At iteration i, let $\pi^{ki}u \geq \pi_0^{ki}$ be a cut that lower approximates $Q^k(u)$. Then the master problem at any iteration l reads as

$$
\begin{aligned}
(\mathsf{Master}): \quad & \min_{u,\eta,\xi,\phi_1} && (1+\lambda)\beta^\top u + \phi_1 + \lambda\left(\eta + \frac{1}{1-\alpha}\sum_{k\in K} p^k \xi^k\right) \\
& \text{s.t.} && u \in U, \quad \xi^k \geq 0 \ \ \forall k \in K \\
& \text{Optimality cut 1} && \phi_1 \geq \sum_k p_k(\pi^{ki}u - \pi_0^{ki}) \quad i = 1,2,\ldots,l \\
& \text{Optimality cut 2} && \xi^k \geq \pi^{ki}u - \pi_0^{ki} - \eta \quad \forall k \in K,\, i = 1,2,\ldots,l
\end{aligned}
$$

The exact forms of these optimality cuts are presented in Proposition 3. According to Observation 1, the general problem has *relatively complete recourse* and the feasibility cut constraint can thus be omitted.

Let $(\bar{u}, \bar{\eta}, \bar{\xi}, \bar{\phi}_1)$ be a optimal solution to the master problem. If $\bar{\phi}_1 < \sum_{k\in K} p_k Q^k(\bar{u})$, then optimality cut 1 will be added to the master problem. Similarly, if $\sum_{k\in K} p_k\bar{\xi}^k < \sum_{k\in K} p_k \max\{0, Q^k(\bar{u}) - \bar{\eta}\}$, then optimality cut 2 will be added to the master problem. These optimality cuts are generated using Lagrange multipliers for each subproblem, which is a convex nonlinear problem, with u fixed to $\bar{u}$.

$$
\begin{aligned}
(\mathsf{Subproblem}\ k): Q^k(\bar{u}) = \quad & \min_{\substack{v^k,q^k,x^k\\ y^k,w^k}} && \text{(19a)} \\
& \text{s.t.} && \text{constraints (19b)} - \text{(19d)} \\
& && 0 \leq y_a^{hk} \leq \frac{c_a\varsigma_a^5}{(\theta_a^{hk})^4}\bar{u}_a^h,\ 0 \leq w_a^{hk} \leq c_a\varsigma_a^5\bar{u}_a^h \quad \forall h \in H, a \in \bar{A}
\end{aligned}
$$

The convex reformulation of $Q^k(u)$ in Proposition 2 and the arguments thereafter imply that it is straightforward to apply the GBD method for generating optimality cuts in the master problem.

Proposition 3. *Let $\bar{u}^l$ be a optimum solution of the master problem at l^{th} iteration. For each scenario k, let μ^{kl} and λ^{kl} be vectors of optimum Lagrange multipliers for the last two sets of constraints in subproblem $Q^k(\bar{u}^l)$. Denote*

$$\bar{y}_a^{hk} := \frac{c_a\varsigma_a^5}{(\theta_a^{hk})^4} \quad h \in H, a \in \bar{A}.$$

Then the optimality cuts for the l^{th} iteration are:

$$\phi_1 \geq \sum_{k\in K} p_k[Q^k(\bar{u}^l) - \mu^{kl}\bar{y}^k(u - \bar{u}^l) - \lambda^{kl}c_a\varsigma^5(u - \bar{u}^l)] \tag{20a}$$

$$\xi^k \geq Q^k(\bar{u}^l) - \mu^{kl}\bar{y}^k(u - \bar{u}^l) - \lambda^{kl}c_a\varsigma^5(u - \bar{u}^l) \quad \forall k \in K \tag{20b}$$

Multiple optimality cuts may help improve algorithm efficiency. Readers may refer to (Birge & Louveaux, 1988) for details. The multi-cut version of optimality cut for (20a) is

$$\phi_1^k \geq Q^k(\bar{u}^l) - \mu^{kl}\bar{y}^k(u - \bar{u}^l) - \lambda^{kl}c_a\varsigma^5(u - \bar{u}^l) \tag{21}$$

Accordingly, we should use the aggregation of cuts $\sum_{k\in K} p^k \phi_1^k$ to replace ϕ_1 in the objective function of (Master). Note that due to the CVaR function definition, optimality cut (20b) is already in multi-cut version. In each iteration, there are $|K|+1$ constraints added to the master problem, consisting of $|K|$ constraint (20b) and one constraint (20a).

The decomposition algorithm procedure:

1. Initialization $l = 0$.
2. Solve (Master). Let $(\bar{u}, \bar{\eta}, \bar{\xi}, \bar{\phi}_1, \bar{\phi}_2)$ be optimal solution and set $\bar{\phi} = \bar{\phi}_1 + \lambda(\bar{\eta} + \frac{1}{1-\alpha}\sum_k p_k \bar{\xi}^k)$.
3. Fix $u = \bar{u}$ and solve (Subproblem k) for all $k \in K$. Set $l = l + 1$ and calculate

$$\phi^* = \sum_{k\in K} p^k Q^k(\bar{u}) + \lambda\, \mathbb{CV}\mathrm{a}\mathbb{R}_\alpha\, Q(\bar{u}).$$

4. The procedure terminates if the optimality gap $|1 - \dfrac{\bar{\phi}}{\phi^*}| \le \epsilon$ (ϵ is a predefined small value) is met. Optimal solution is found. Otherwise, add optimality cuts (20a) (or the multi-cut version (21)) and (20b) to the master problem, and go back to step 2.

5. Numerical examples

The proposed mean-risk model and decomposition methods are first justified using a small nine-node hypothetical network. The Sioux-Falls network is then used to explore the impacts of uncertainty, network topology, and critical parameters on the strategic decisions on highway bridge retrofits.

5.1. Nine-node network

The nine-node network is shown in Figure 2, which consists of nine nodes, 24 directional links, and 72 (=8×9) O-D pairs. Assume that three bridges on both directions on the network, labeled as A, B, and C, are vulnerable to seismic disasters and their post-disaster capacities may be reduced while other road links are assumed intact. The nine-node instances were programmed in AMPL and conducted on a desktop with 8 GB RAM and Intel Core i5-2500@3.40GHz processor under Windows 7 environment.

The parameter $\theta_a^{h,k}$ is the ratio of bridge remaining capacity to the full capacity, which depends on the specific scenario, location of the bridge, and the retrofit strategies applied (Mackie and Stojadinovic,

Table 1: Sample values of $\theta_a^{h,k}$ for a fixed scenario k.

Link	Strategy				
	h_0	h_1	h_2	h_3	h_4
link5	0.05	0.5	0.5	0.5	1
link6	0.05	0.5	0.5	0.5	1
link13	0.5	0.5	0.5	0.75	0.75
link14	0.5	0.5	0.5	0.75	0.75
link21	0.17	0.33	0.33	0.67	0.67
link22	0.17	0.33	0.33	0.67	0.67

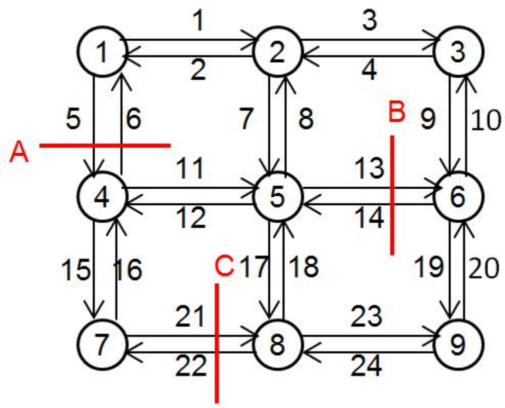


Figure 2: Nine-node network

Table 2: Comparisons between GBD and exact solutions

Number of scenarios	Obj. Value (10^6)			CPU seconds			GBD iterations
	BONMIN	FilMINT	GBD*	BONMIN	FilMINT	GBD*	
6	466.416	466.416	466.416	1280	466	7	15
12	466.144	466.144	466.144	3754	2487	21	29
18	462.063	462.063	462.063	6340	4673	21	21
24	460.483	460.483	460.484	4315	11377	22	18

2004). There are five strategies considered, as shown in Table 1, including "do nothing" strategy h_0 strategy (Huang et al., 2014). A higher numbered strategy indicates a more robust yet more costly strategy, and vice versa. In this numerical experiment, we randomly generate $\theta_a^{h,k}$ in (0,1] and assumed that a higher numbered strategy results in a higher $\theta_a^{h,k}$. As a demonstration, Table 1 only reports the ratios for one scenario. The initial points for all second-stage variables are set to be zero. The initial solution for the first-stage decision variable u is set as follows: $u_a^h = 1$, for $h = h_0$, and $u_a^h = 0$, for $h \neq h_0$, $\forall a \in A$. Other critical parameters are: $\alpha = 0.7, \beta = 0.15, \gamma = 1000, \lambda = 1$, and $\overline{y} = 1000$.

Two recently published papers (Klanšek, 2014; Lastusilta et al., 2009) demonstrate that commercial MINLP solvers, like BARON, AlphaECP, LindoGlobal and DICOPT, can successfully solve nonlinear, discrete transportation problems. We obtained benchmark solutions by using two commercial solvers - BONMIN (Bonami et al., 2008) and FilMINT (Abhishek et al., 2010), to justify our decomposition method by using the small-scale network. We tested the model using four different sizes of scenario sets. In each set, scenarios are randomly generated to create variations in uncertainty realizations in order to justify the effects of CVaR. The objectives and computational performances by the solvers are reported in Table 2, compared with the counterparts of the decomposition method (GBD).

We found that the optimal objective values obtained from BONMIN and FilMINT are most identical to the counterparts by GBD for all scenarios while the solution times by using GBD have been substantially reduced than using the BONMIN and FilMINT for all scenarios. This well justifies the use of our proposed solution method for larger scale of problems, such as Sioux Falls network. Also one may notice that the solution times rise quickly with the increase of number of scenarios, although it does not necessarily translate to a higher number of GBD iterations. This is because sub-problems become more difficult with larger number of scenarios, which takes longer time to finish each iteration. This explains why GBD results in almost identical solution time when solving 24 scenarios as solving 18 scenarios even if GBD involves with fewer iterations in the case of 24 scenarios. Based on the numerical results and solution performances, we apply our model and solution method to Sioux Falls network. The results are presented in the next subsection.

5.2. Sioux Falls network

The Sioux Falls network in Figure 1 is consisted of 24 nodes, 76 links, and 552 O-D pairs. The trip demands between all O-D demands are adopted from (LeBlanc et al., 1975). We adopted critical parameters from (Fan et al., 2010), including $\beta = 0.15$, and the peak 2-hour conversion value $\gamma = 2400$ to convert peak 2-hour delay to a monthly monetary value loss, which is set as $8 \times 30 \times 10 = 2400$, where 8 is daily adjust factor with 30 days duration and 10 is the value of travel time savings for drivers. The Sioux Falls instances were programmed in AMPL and conducted on a Linux cluster node with 16 Intel Cores and a total 64 GB RAM.

Traditional engineering method estimates earthquake damage of structures using discrete damage states (Choi et al., 2004); that is, the residual post-earthquake capacity ratio $\theta_a^{h,k}$ have discrete values. Note that there are possible noises in estimating the post-earthquake traffic capacity for each structure. We also need to keep in mind that the post-disaster traffic capacity can be highly varied with different location, retrofit strategies, and structural mechanisms. Thus, as accurate assessment of the capacity $\theta_a^{h,k}$ could be extremely complex and beyond the scope of this study. Without any existing data from the structural assessment, we randomly generated $\theta_a^{h,k}$ such that there are substantial variations among different scenarios to justify the use of stochastic programming method in our study.

That being said, we develop a simply mechanism to generate $\theta_a^{h,k}$ in two steps. First, we consider three levels of damages, which are low, median, and high damages and assume that the damages to the bridges at risk are independent. The $\theta_a^{h,k}$ with a low-damage scenario (k) and a retrofit strategy (h), is randomly selected from a set of finite numbers, which are generated as $\{n/N, n = 1, ..., N\}$, where the N is user defined (e.g., 6 is used in this study as an example). We then randomly pick five numbers out of the set for the five retrofit strategies (i.e., h_0 - h_4). Similarly, we pick five random numbers each for the $\theta_a^{h,k}$ with median- and high- damage scenarios, from the randomly generated sets $\{n/N, n = 1, ..., N-1\}$ and $\{n/N, n = 1, ..., N-2\}$, respectively. Note that the $\theta_a^{h,k}$ under a low-damage scenario has larger range of numbers to choose from than the $\theta_a^{h,k}$ under median- and high-damage scenarios. Statistically, bridge residual capacity under high-damage scenarios is lower than the counterpart under low-damage scenarios for the same bridge. However, due to the involving complexity in the estimation, such as locations and structures of different bridges, there are inevitable fluctuations in the residual capacities, which are captured in our developed mechanism. Second, for a given bridge a under scenario k, the $\theta_a^{h,k}$ value should be non-decreasing with an enhanced strategy h (higher numbered strategy), e.g., $\theta_a^{h4,k} \geq \theta_a^{h3,k}$. Based on this, we will assign the selected five numbers to $\theta_a^{h,k}$ according to the different retrofit strategies. We also assume that the occurrences of the three categories of scenarios follow a predefined ratio. For example, if a ratio of 5:3:2 is assumed for low-, median- and high-damage scenarios, respectively, for a total of 20 scenarios, the occurrences of each category of scenarios will be 10, 6, 4, respectively. The probabilities associated with scenarios are randomly generated following a uniform distribution.

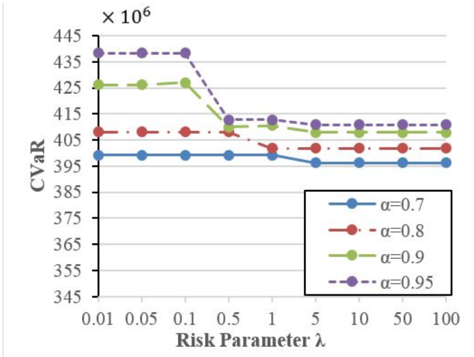
We adopted the same five-strategy scheme (i.e., h_0-h_4) and initial point settings from the nine-node network example. The results of the Sioux Falls network are thus obtained by using decomposition method. In this section, we aim to explore the impacts of uncertainty, network topology, and critical parameters on the retrofit strategies from numerical experiments.

The effects of risk parameters (i.e., α and λ) on model results and computational performances. We ran 36 combinations of four α (i.e., α= 0.7, 0.8, 0.9 and 0.95) and nine λ (i.e., λ = 0.01, 0.05, 0.1, 0.5, 1, 5, 10, 50, and 100) with $\overline{y}$ = 1500 for 10, 20, 50, and 100 scenarios. The confidence level parameter α controls the set of scenarios to be considered while the coefficient λ weighs the CVaR in the integrated mean-risk stochastic model. A higher α results in higher VaR value and a higher λ will increase the weight of the CVaR. Therefore, increased parameters α and λ both imply more risk-averse solutions. Through the numerical experiments, we intend to (i) understand the effects of risk parameters on systems costs; (ii) identify a best possible combination of the risk parameters; and (iii) highlight the modeling insights of a risk (i.e., CVaR) integrated stochastic program compared to a typical two-stage stochastic model.

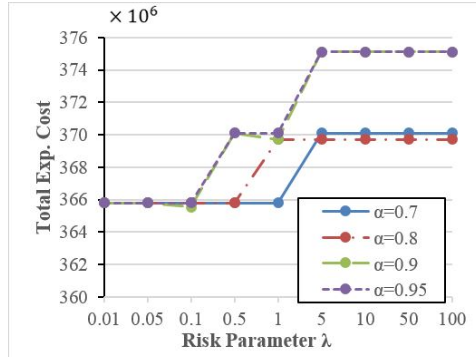
Let us first investigate the breakdown of the total cost plotted in Figure 3 based on the result of 20 scenarios. The total mean-risk cost or objective value is a combined total expected costs and weighted CVaR. The total expected cost can be further decomposed into the retrofit cost and the expected travel cost. The impacts of the risk parameters on the cost effectiveness and CVaR will be discussed separately. Note that the specified α level represents the risk preference, which quantifies the mean value of the worst $(1-\alpha)$% of the total costs. In Figure 3a, CVaR increases as α increases according to the definition, i.e., a larger value of α accounts for larger realizations of the total cost while decreasing as λ increases. The total expected cost shown in Figure 3b is comprised of the retrofit cost in Figure 3c and the expected travel cost in Figure 3d. The results show that increasing both λ and α generally increases retrofit cost, because it results in more risk-averse policy with enhanced yet more costly retrofit strategies. As a result, we expect a reduced expected travel cost, which implies a lower post-disaster capacity loss. However, the total expected cost, which is a combined retrofit and expected travel cost, is generally higher with a higher α. The retrofit cost roughly contributes 14.6%- 20% to the total expected cost.

Second, we plot the total expected cost against CVaR (see Figure 4) for both 20 and 100 scenarios to explore the trade-off between the two objectives as a result of different combinations of risk parameters. In both figures, each line represents the results of all nine λ values under a particular α. As some combinations of λ and α result in identical CVaR and the total expected cost, especially when fewer scenarios (e.g., 20 scenarios) are used, there are nodes overlapped. From the results, there is no single best combination of α and λ and the cost of retrofit and transport is comprised with CVaR. For a given confidence level α, a more risk-averse solution (a higher λ resulting in a higher CVaR) can generally help to reduce the total cost. We also note that a higher confidence level does not necessarily lead to a solution with lower total expected cost. For example, when K=20, the lowest total expected cost is occurred at $\alpha = 0.9$. In fact, in many cases, a lower confidence level (e.g., $\alpha = 0.7$) can just do as well as a higher confidence level (e.g., $\alpha = 0.95$). This could be attributed to many other involving factors, such as variations between scenarios and traffic flow distributions on the network.

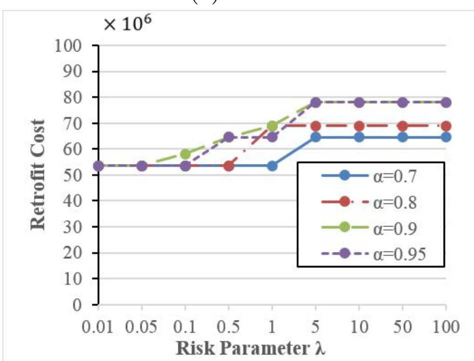
Third, we are also interested in understanding the managerial insights on the integration of risk assessment in a traditional risk-neutral two-stage SP method. We compare the results of our mean-risk model results with the counterparts of the two-stage SP model in Table 3, in which the first row contains the results of the two-stage SP model and other rows report the results of mean-risk model under different risk parameters. The results in the table provides some interesting insights. First, the lowest total expected cost is \$365.533M (when $\alpha = 0.9$ and $\lambda = 0.1$), which is only trivially better the result of two-stage SP model at \$365.798M. This triggered a worthy research question for the future on how to identify the bounds at early stage that better inform an appropriate formulation without having to completely solve the entire problem. Second, the lowest travel cost of \$291.106M, which is



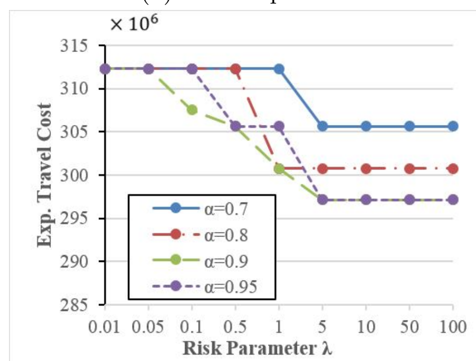
(a) CVaR



(b) Total exp. cost

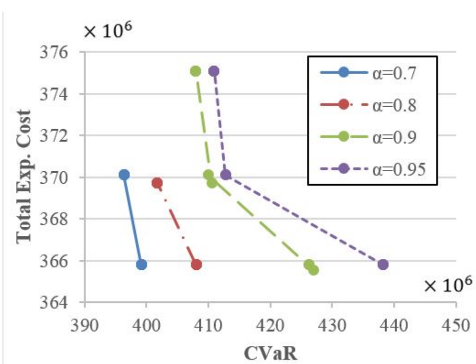


(c) Retrofit cost

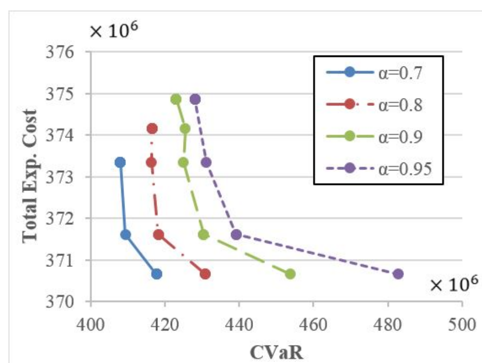


(d) Exp. travel cost

Figure 3: The breakdowns of the total cost under different combinations of risk parameters for 20 scenarios



(a) 20 scenarios



(b) 100 scenarios

Figure 4: The trade-off between the total expected cost and CVaR

Table 3: Comparisons between Two-Stage SP and Mean-Risk SP for 20 scenarios

		Total Exp. Cost (10^6)	Retrofit Cost (10^6)	Exp. Travel Cost (10^6)	Bridge A Strategy	Bridge B Strategy	Bridge C Strategy	Bridge D Strategy
Two Stage SP (=0)		365.798	53.5	312.298	3	3	2	1
$\lambda = 0.01$	α=0.7	365.798	53.5	312.298	3	3	2	1
	α=0.8	365.798	53.5	312.298	3	3	2	1
	α=0.9	365.798	53.5	312.298	3	3	2	1
	α=0.95	365.798	53.5	312.298	3	3	2	1
λ=0.05	α=0.7	365.798	53.5	312.298	3	3	2	1
	α=0.8	365.798	53.5	312.298	3	3	2	1
	α=0.9	365.798	53.5	312.298	3	3	2	1
	α=0.95	365.798	53.5	312.298	3	3	2	1
λ=0.1	α=0.7	365.798	53.5	312.298	3	3	2	1
	α=0.8	365.798	53.5	312.298	3	3	2	1
	α=0.9	365.533	58	307.533	3	3	2	2
	α=0.95	365.798	53.5	312.298	3	3	2	1
λ=0.5	α=0.7	365.798	53.5	312.298	3	3	2	1
	α=0.8	365.798	53.5	312.298	3	3	2	1
	α=0.9	370.116	64.5	305.616	3	3	3	1
	α=0.95	370.116	64.5	305.616	3	3	3	1
λ=1	α=0.7	365.798	53.5	312.298	3	3	2	1
	α=0.8	369.721	69	300.721	3	3	3	2
	α=0.9	369.721	69	300.721	3	3	3	2
	α=0.95	370.116	64.5	305.616	3	3	3	1
λ=5	α=0.7	370.116	64.5	305.616	3	3	3	1
	α=0.8	369.721	69	300.721	3	3	3	2
	α=0.9	375.106	78	297.106	3	3	3	3
	α=0.95	375.106	78	297.106	3	3	3	3
λ=10	α=0.7	370.116	64.5	305.616	3	3	3	1
	α=0.8	369.721	69	300.721	3	3	3	2
	α=0.9	375.106	78	297.106	3	3	3	3
	α=0.95	375.106	78	297.106	3	3	3	3
λ=50	α=0.7	370.116	64.5	305.616	3	3	3	1
	α=0.8	369.721	69	300.721	3	3	3	2
	α=0.9	375.106	78	297.106	3	3	3	3
	α=0.95	375.106	78	297.106	3	3	3	3
λ=100	α=0.7	370.116	64.5	305.616	3	3	3	1
	α=0.8	369.721	69	300.721	3	3	3	2
	α=0.9	375.106	78	297.106	3	3	3	3
	α=0.95	375.106	78	297.106	3	3	3	3

\$15.192M(=312.298-297.106) or 5% lower than the two-stage SP model, while it costs \$24.5 (=78-53.5) or 46% more in retrofit. It implies that reducing travel cost is costly. From a system-cost perspective, achieving the lowest network-wide travel cost it may not be the most economical solution.

We also investigate if increased availability of information would be helpful for a lower total system cost and also how significantly the increased problem size would impact solution performances. We conducted cross-comparisons between three sized scenario sets (i.e., $|K| = 20, 50, 100$) for the same four bridges. The model results and solution performances are represented in Tables 4 and 5. From results in Table 4, for a fixed pair of α and λ, the total expected cost does not vary much across different scenario sets, which implies that the increased availability of information may not necessarily help to improve the solution quality. In general, CVaR increases in a small extent as a result of the increasing number of high-damage scenarios in a larger scenario set. On the other hand, solution times experience more noticeable increase as a result of increased problem size. This is apparent that the running time for each iteration is generally proportional to the number of scenarios involved in the problem. With a larger scenario set, it thus takes longer finish the entire solution process.

6. Conclusions and future work

We develop a mean-risk MINLP for transportation network protection (e.g., retrofitting highway bridges) hedging against extreme disasters (e.g., earthquakes) on a system level, where CVaR is considered as the risk measurement and integrated into the single optimization framework. This is the first study that explicitly considers CVaR as the risk measure in the field of transportation network

Table 4: Costs under different sized scenario sets

		K=20		K=50		K=100	
		CVaR (10^6)	Total Exp. Cost (10^6)	CVaR (10^6)	Total Exp. Cost (10^6)	CVaR (10^6)	Total Exp. Cost (10^6)
Two Stage SP (λ=0)		399.143	365.798	410.502	368.437	417.665	370.661
λ=0.01	α=0.7	399.143	365.798	410.502	368.437	417.665	370.661
	α=0.8	408.018	365.798	419.683	368.437	430.958	370.661
	α=0.9	426.197	365.798	435.398	368.437	453.774	370.661
	α=0.95	438.264	365.798	447.975	368.437	482.858	370.661
λ=0.05	α=0.7	399.143	365.798	410.502	368.437	417.665	370.661
	α=0.8	408.018	365.798	419.683	368.437	430.958	370.661
	α=0.9	426.197	365.798	435.398	368.437	430.395	371.612
	α=0.95	438.264	365.798	447.975	368.437	439.092	371.612
λ=0.1	α=0.7	399.143	365.798	410.502	368.437	417.665	370.661
	α=0.8	408.018	365.798	419.683	368.437	418.16	371.612
	α=0.9	427.025	365.533	435.398	368.437	430.395	371.612
	α=0.95	438.264	365.798	447.975	368.437	439.092	371.612
λ=0.5	α=0.7	399.143	365.798	407.472	369.646	409.375	371.612
	α=0.8	408.018	365.798	413.893	369.646	418.16	371.612
	α=0.9	409.962	370.116	424.954	369.646	425.02	373.346
	α=0.95	412.771	370.116	426.484	372.338	431.136	373.346
λ=1	α=0.7	399.143	365.798	409.03	370.828	408.011	373.346
	α=0.8	401.749	369.721	413.893	369.646	416.344	373.346
	α=0.9	410.505	369.721	424.954	369.646	425.461	374.152
	α=0.95	412.771	370.116	426.484	372.338	428.039	374.875
λ=5	α=0.7	396.316	370.116	407.472	369.646	408.011	373.346
	α=0.8	401.749	369.721	413.893	369.646	416.55	374.152
	α=0.9	407.975	375.106	420.096	373.841	423.003	374.875
	α=0.95	410.876	375.106	424.233	373.494	428.039	374.875
λ=10	α=0.7	396.316	370.116	407.472	369.646	408.011	373.346
	α=0.8	401.749	369.721	413.893	369.646	416.55	374.152
	α=0.9	407.975	375.106	420.096	373.841	423.003	374.875
	α=0.95	410.876	375.106	424.233	373.494	428.039	374.875
λ=50	α=0.7	396.316	370.116	407.472	369.646	408.011	373.346
	α=0.8	401.749	369.721	413.893	369.646	416.55	374.152
	α=0.9	407.975	375.106	420.096	373.841	423.003	374.875
	α=0.95	410.876	375.106	424.233	373.494	428.039	374.875
λ=100	α=0.7	396.316	370.116	407.472	369.646	408.011	373.346
	α=0.8	401.749	369.721	413.893	369.646	416.55	374.152
	α=0.9	407.975	375.106	420.096	373.841	423.003	374.875
	α=0.95	410.876	375.106	424.233	373.494	428.039	374.875

Table 5: Solution performances under different sized scenario sets

		K=20			K=50			K=100		
		Optimality Gap	CPU Time (min)	# of Iterations	Optimality Gap	CPU Time (min)	# of Iterations	Optimality Gap	CPU Time (min)	# of Iterations
Two Stage SP (λ=0)		0.10%	147.68	31	0.00%	342.17	30	0.00%	687.05	28
λ=0.01	α=0.7	0.10%	167.57	31	0.00%	370.37	30	0.00%	648.33	28
	α=0.8	0.10%	166.23	31	0.00%	358.22	30	0.00%	691.42	28
	α=0.9	0.10%	167.9	32	0.00%	379.32	30	0.00%	689.07	28
	α=0.95	0.10%	159.62	31	0.00%	369.83	30	0.00%	685.17	28
λ=0.05	α=0.7	0.09%	120.4	30	0.00%	358.6	30	0.00%	689.87	28
	α=0.8	0.36%	155.72	28	0.00%	372.5	30	0.00%	649.68	28
	α=0.9	0.36%	159.23	29	0.26%	308.32	28	0.60%	1372.85	26
	α=0.95	0.28%	148.58	27	0.26%	308.4	29	0.00%	613.88	26
λ=0.1	α=0.7	0.34%	142.62	27	0.00%	291.35	31	0.00%	757.93	31
	α=0.8	0.34%	148.43	28	0.00%	377.1	31	0.00%	738.95	30
	α=0.9	0.00%	170.25	31	0.00%	284.38	29	0.17%	580.88	28
	α=0.95	0.00%	164.37	30	0.00%	350.08	30	0.00%	604.62	26
λ=0.5	α=0.7	0.09%	160.82	29	0.01%	309.52	26	0.38%	1336.77	26
	α=0.8	0.33%	152.93	29	0.33%	303.3	24	0.37%	579.1	26
	α=0.9	0.06%	141.15	25	0.01%	641.13	25	0.05%	632.5	27
	α=0.95	0.00%	110.53	22	0.43%	225.92	24	0.07%	553.78	23
λ=1	α=0.7	0.32%	126.02	29	0.47%	246.38	27	0.07%	494.85	27
	α=0.8	0.01%	153.35	28	0.16%	236.6	26	0.13%	489.22	26
	α=0.9	0.32%	84	21	0.44%	295.07	23	0.46%	530.88	23
	α=0.95	0.22%	102.82	20	0.24%	210.3	22	0.37%	464.2	19
λ=5	α=0.7	0.14%	131.42	24	0.08%	295.73	25	0.12%	536.78	27
	α=0.8	0.01%	81.6	20	0.05%	265.28	23	0.31%	453.12	23
	α=0.9	0.45%	104.13	20	0.03%	218.25	20	0.01%	459.77	19
	α=0.95	0.63%	235.45	21	0.03%	172.48	19	0.03%	374.17	18
λ=10	α=0.7	0.10%	126.15	24	0.31%	288.15	24	0.01%	675.3	27
	α=0.8	0.01%	112.2	22	0.14%	303.63	24	0.25%	427.93	22
	α=0.9	0.01%	114.38	21	0.03%	178.4	19	0.14%	367.65	16
	α=0.95	0.71%	92.55	20	0.05%	164.68	16	1.03%	379.42	17
λ=50	α=0.7	0.09%	96.72	23	0.02%	285.63	25	0.03%	596.13	26
	α=0.8	0.01%	119.47	22	0.00%	312.92	25	0.20%	445.3	22
	α=0.9	0.01%	102.7	20	0.03%	219.97	18	0.03%	409.72	18
	α=0.95	0.67%	107.27	20	0.20%	198.83	16	0.04%	349.57	17
λ=100	α=0.7	0.09%	128.43	23	0.02%	301.83	25	0.03%	488.6	26
	α=0.8	0.01%	114.7	21	0.01%	301.25	25	0.19%	494.52	22
	α=0.9	0.01%	81.02	20	0.03%	170.68	18	0.03%	412.75	18
	α=0.95	0.66%	92.45	20	0.19%	155.65	16	0.04%	401.73	17

protection. The mean-risk formulation is not obviously a convex optimization problem. By reformulation of the problem, we show that the recourse function is convex in the bridge retrofit variables. Thus another major contribution lies in the development of decomposition algorithm based on GBD to solve the large-scale MINLP.

We demonstrates the mean-risk model and decomposition method using two numerical examples, a small nine-node network and the benchmark Sioux Falls network. The nine-node network is used to justify the proposed decomposition method by comparing the solution quality and performances with the exact solutions that are obtained from using the global solvers BONMIN and FilMINT. We then use the Sioux Falls network to explore the correlations between risk parameters and retrofit decisions and their impacts on the system costs. We also investigated the capacity of the solution method in handling different sized scenario sets. We found that the increase of α and λ will generally be leading to a more risk-averse solution, with lower CVaRs. The resulting retrofit strategies are enhanced yet more costly and the expected travel cost is reduced. From the results, there is a clear trade-off between the total expected cost and CVaR and there is no single best combination of risk parameters. The choice of the α and λ depends on the risk preferences and budget. We also compared the results of our mean-risk model with the risk-neutral two-stage SP model to understand the cost and effects to be more risk-averse in modeling. From the results, there are several worthy notes. First, the best mean-risk model result (i.e., the minimum total expected cost (retrofit cost plus travel cost)) is only trivially better than the two-stage SP model, although the there is a discernible increase in retrofit cost and decrease in travel cost. It raise an interesting research question on how to possibly identify the bounds at early stage that could better inform an appropriate formulation without having to completely solve the entire problem. Second, there are quite few duplicated solutions with different combinations of risk parameters. Also, by running different sized scenario sets, note that the increased availability of information (e.g., larger scenario set) does not necessarily improve solution quality, but results in more computationally expensive problems.

Several future directions would be worth research efforts, which involve both the modeling and algorithmic development. From modeling perspective, the traffic equilibrium may be a more realistic assumption to model route choices of network users. The integration of equilibrium will make the model a Mathematical Program with Equilibrium Constraints (MPEC). One of the challenges would be converting this MPEC to a MINLP through regularization or penalization. Once it is in the form of MINLP, we may apply the developed decomposition method for obtaining solutions. In addition, more realistic assumptions on post-disaster traffic capacity (i.e., θ_a^k) may be included by integrating the network model with structural analysis. It concerns the nonlinear bridge traffic capacity-cost relationship of retrofitting each individual bridge in which the desired bridge performance varies with retrofit strategies that cost differently. Instead of assuming a constant or linear relationship in most optimization based transportation network protection problems, by using finite element analysis a structural performance-retrofit level relationship between the structural strength and allocated budget for each bridge can be constructed. From the algorithmic perspective, MINLP is difficult to solve. In addition to solving the model by using other solvers, such as AlphaECP, BARON, DICOPT, etc., more algorithmic development (including heuristics) may be worth further exploration to prepare the model for real-world scale networks. These method could be based on Outer Approximation, Branch-and-Bound, etc.

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