On L_p-Solvability of Stochastic Integro-Differential Equations

Citation for published version:
Gyongy, I & Wu, S 2020, 'On L_p-Solvability of Stochastic Integro-Differential Equations', *Stochastics and Partial Differential Equations: Analysis and Computations*. https://doi.org/10.1007/s40072-019-00160-8

Digital Object Identifier (DOI):
10.1007/s40072-019-00160-8

Published In:
Stochastics and Partial Differential Equations: Analysis and Computations

ON L_p-SOLVABILITY OF STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

ISTVÁN GYÖNGY AND SIZHOU WU

ABSTRACT. A class of (possibly) degenerate stochastic integro-differential equations of parabolic type is considered, which includes the Zakai equation in nonlinear filtering for jump diffusions. Existence and uniqueness of the solutions are established in Bessel potential spaces.

1. INTRODUCTION

We consider the equation

$$du_t(x) = (\mathcal{A}_t u_t(x) + f_t(x))\,dt + (\mathcal{M}_t^r u_t(x) + g_t^r(x))\;dw_t^r$$
$$+ \int_Z (u_{t-}(x+\eta_{t,z}(x)) - u_{t-}(x) + \gamma_{t,z}(x)u_{t-}(x+\eta_{t,z}(x)) + h_t(x,z))\;\tilde{\pi}(dz,dt) \qquad (1.1)$$

for $(t,x) \in [0,T] \times \mathbb{R}^d := H_T$ with initial condition

$$u_0(x) = \psi(x) \quad \text{for } x \in \mathbb{R}^d, \qquad (1.2)$$

on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$, carrying a sequence $w = (w_t^i)_{i=1}^\infty$ of independent $\mathcal{F}_t$-Wiener processes and an $\mathcal{F}_t$-Poisson martingale measure $\tilde{\pi}(dz,dt) = \pi(dz,dt) - \mu(dz) \otimes dt$, where $\pi(dz,dt)$ is an $\mathcal{F}_t$-Poisson random measure with a σ-finite characteristic measure $\mu(dz)$ on a measurable space $(Z, \mathcal{Z})$ with a countably generated σ-algebra $\mathcal{Z}$. We note that here, and later on, the summation convention is used with respect to repeated (integer-valued) indices and multi-numbers.

In the above equation $\mathcal{A}_t$ is an integro-differential operator of the form $\mathcal{A}_t = \mathcal{L}_t + \mathcal{N}_t^\xi + \mathcal{N}_t^\eta + \mathcal{R}_t$ with a "zero-order" linear operator $\mathcal{R}_t$ specified later, a second order differential operator

$$\mathcal{L}_t = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i + c_t(x),$$

and integral operators $\mathcal{N}_t^\xi$ and $\mathcal{N}_t^\eta$ defined by

$$\mathcal{N}_t^\xi\varphi(x) = \int_Z \varphi(x+\xi_{t,z}(x)) - \varphi(x) - \xi_{t,z}(x)\nabla\varphi(x) + \lambda_{t,z}^\xi(x)(\varphi(x+\xi_{t,z}(x)) - \varphi(x))\nu(dz),$$

$$\mathcal{N}_t^\eta\varphi(x) = \int_Z \varphi(x+\eta_{t,z}(x)) - \varphi(x) - \eta_{t,z}(x)\nabla\varphi(x) + \lambda_{t,z}^\eta(x)(\varphi(x+\eta_{t,z}(x)) - \varphi(x))\mu(dz) \quad (1.3)$$

for a suitable class of real-valued functions φ on $\mathbb{R}^d$ for each $t \in [0,T]$, where $\nu(dz)$ is a fixed σ-finite measure on $(Z,\mathcal{Z})$. The coefficients a^{ij}, b^i and c are real functions on $\Omega \times H_T$ for

2010 *Mathematics Subject Classification.* Primary 45K05, 60H15, 60H20, 60J75; Secondary 35B65.
Key words and phrases. Stochastic integro-differential equation, Bessel potential spaces, Interpolation couples.

$i,j=1,2,...,d$, and λ^{ξ}, λ^{η} and γ are real functions on $\Omega\times H_T\times Z$, and $\eta=(\eta^i_{t,z}(x))$ and $\xi=(\xi^i_{t,z}(x))$ are $\mathbb{R}^d$-valued functions of $(\omega,t,x,z)\in\Omega\times H_T\times Z$. Under "zero-order operators" we mean bounded linear operators $\mathcal{R}$ mapping the Sobolev spaces W^k_p into themselves for some $k\geq 0$. For each integer $r\geq 1$ the operator $\mathcal{M}^r_t$ is a first order differential operator of the form

$$\mathcal{M}^r_t=\sigma^{ir}_t(x)D_i+\beta^r_t(x).$$

The coefficients σ^{ir} and β^r are real functions on $\Omega\times H_T$ for $i,j=1,2,...,d$ and integers $r\geq 1$. The free terms f and g^r are real functions defined on $\Omega\times H_T$ for every $r\geq 1$, and h is a real function defined on $\Omega\times H_T\times Z$. The stochastic differentials in equation (1.1) are understood in Itô's sense, see the definition of a solution in the next section.

We are interested in the solvability of the above problem in L_p-spaces. We note that equation (1.1) may degenerate, i.e., the pair of linear operators $(\mathcal{L},\mathcal{M})$ satisfies only the *stochastic parabolicity* condition, Assumption 2.1 below, and the operators $\mathcal{N}^{\xi}$ and $\mathcal{N}^{\eta}$ may also degenerate. Our main result, Theorem 2.1 states that under the stochastic parabolicity condition on the operators $(\mathcal{L},\mathcal{M})$, $\mathcal{N}^{\xi}$, $\mathcal{N}^{\eta}$, and appropriate regularity conditions on their coefficients and on the initial and free data, the Cauchy problem (1.1)-(1.2) has a unique generalised solution $u=(u_t)_{t\in[0,T]}$ for any given T. Moreover, this theorem describes the temporal and spatial regularity of u in terms of Bessel potential spaces H^n_p, and presents also a supremum estimate in time. The uniqueness of the solution is proved by an application of a theorem on Itô's formula from [17], which generalises a theorem of Krylov in [24] to the case of jump processes. The existence of a generalised solution is proved in several steps. First we obtain a priori estimates in Sobolev spaces W^n_p for integers $n\in[0,m]$, where m is a parameter measuring the spatial smoothness of the coefficients and the data in (1.1)-(1.2). These estimates allow us to construct a generalised solution by approximating (1.1)-(1.2) with non-degenerate equations with smooth coefficients and compactly supported smooth data in $x\in\mathbb{R}^d$. Thus we see that a solution operator, mapping the initial and free data into the solution of (1.1)-(1.2), exists and it is a bounded linear operator in appropriate L_p-spaces. Hence by interpolation we get our a priori estimates in Bessel potential spaces H^n_p for any given $p\geq 2$ and real number $n\in[0,m]$. We obtain essential supremum estimates in time for the solution from integral estimates, by using the simple fact that the essential supremum of Lebesgue functions over an interval $[0,T]$ is the limit of their $L_r([0,T])$-norm as $r\to\infty$. Hence we get the temporal regularity of the solution formulated in our main theorem by using Theorem 2.2 on Itô's formula in [17], an extension of Lemma 5.3 in [9] and a well-known interpolation inequality, Theorem 4.1(v) below.

In the literature there are many results on stochastic integral equations with unbounded operators, driven by jump processes and martingale measures. A general existence and uniqueness theorem for stochastic evolution equations with nonlinear operators satisfying stochastic coercivity and monotonicity conditions is proved in [15], which generalises some results in [25] and [32] to stochastic evolution equations driven by semimartingales and random measures. Further generalisations are obtained and the asymptotic behaviour of the solutions are investigated in [28]. For a monograph on stochastic evolution equations driven by Lévy noise we refer to [33].

The main theorem in [15] implies the existence of a unique generalised solution to (1.1)-(1.2) in L_2-spaces when instead of the stochastic parabolicity condition (2.1) in Assumption

2.1 below, the strong stochastic parabolicity condition,

$$\sum_{i,j=1}^{d}\sum_{r=1}^{\infty}(a^{ij}-\frac{1}{2}b^{ir}b^{jr})z^iz^j \geq \lambda\sum_{i=1}^{d}|z^i|^2 \quad \text{for all } z=(z^1,z^2,...,z^d)\in\mathbb{R}^d$$

with a constant $\lambda > 0$ is assumed on $(\mathcal{L},\mathcal{M})$. Under the weaker condition of stochastic parabolicity the solvability of (1.1)-(1.2) in L_2-spaces is investigated and existence and uniqueness theorems are presented in [8] and [27]. The first result on solvability in L_p-spaces for the stochastic PDE problem (1.1)-(1.2) with $\xi=\eta=0$ and $h=0$ was obtained in [26], and was improved in [16]. However, there is a gap in the proof of the crucial a priori estimate in [26]. This gap is filled in and more general results on solvability in L_p-spaces for systems of stochastic PDEs driven by Wiener processes are presented in [12]. As far as we know Theorem 2.1 below is the first result on solvability in L_p-spaces of stochastic integro-differential equations (SIDEs) without any non-degeneracy conditions. It generalises the main result of [9] on deterministic integro-differential equations to SIDEs. Our motivation to study equation (1.1) comes from nonlinear filtering of jump-diffusion processes, and we want to apply Theorem 2.1 to filtering problems in a continuation of the present paper.

We note that under non-degeneracy conditions SIDEs have been investigated with various generalities in the literature, and very nice results on their solvability in L_p-spaces have recently been obtained. In particular, L_p-theories for such equations have been developed in [20], [21], [29], [30] and [31], which extend some results of the L_p-theory of Krylov [22] to certain classes of equations with nonlocal operators. See also [7] for an L_p-theory for stochastic PDEs driven by Lévy processes, [36] for an existence and uniqueness theorem for stochastic quasi-geostrophic equations driven by Poisson martingale measures, and [6], [10] and [35] for L_p theory of deterministic equations with nonlocal operators. Nonlinear filtering problems and the related equations describing the conditional distributions have been extensively studied in the literature. For results in the case of jump-diffusion models see, for example, [2], [4], [11] and [14].

In conclusion, we introduce some notions and notations used throughout this paper. All random elements are given on the filtered probability space $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\geq 0})$. We assume that $\mathcal{F}$ is P-complete, the filtration $(\mathcal{F}_t)_{t\geq0}$ is right-continuous, and $\mathcal{F}_0$ contains all P-zero sets of $\mathcal{F}$. The σ-algebra of the predictable subsets of $\Omega\times[0,\infty)$ is denoted by $\mathcal{P}$. For notations, notions and results concerning Lévy processes, Poisson random measures and stochastic integrals we refer to [1], [3] and [19].

For vectors $v=(v^i)$ and $w=(w^i)$ in $\mathbb{R}^d$ we use the notation $vw=\sum_{i=1}^m v^iw^i$ and $|v|^2=\sum_i|v^i|^2$. For real-valued Lebesgue measurable functions f and g defined on $\mathbb{R}^d$ the notation (f,g) means the integral of the product fg over $\mathbb{R}^d$ with respect to the Lebesgue measure on $\mathbb{R}^d$. A finite list $\alpha=\alpha_1\alpha_2,...,\alpha_n$ of numbers $\alpha_i\in\{1,2,...,d\}$ is called a multi-number of length $|\alpha|:=n$, and the notation

$$D_\alpha := D_{\alpha_1}D_{\alpha_2}...D_{\alpha_n}$$

is used for integers $n\geq 1$, where

$$D_iv=\frac{\partial}{\partial x^i}v$$

is the generalised derivative of a real-valued function v with respect to x_i, $i=1,2,...,d$. We use also the multi-number ϵ of length 0 such that D_ϵ means the identity operator. For an integer $n \geq 0$ and functions v on $\mathbb{R}^d$, whose generalised derivatives up to order n are functions, we use the notation $D^n v$ for the collection $\{D_\alpha v : |\alpha| = n\}$, and define

$$|D^n v|^2 = \sum_{|\alpha|=n} |D_\alpha v|^2.$$

For differentiable functions $v = (v^1, ..., v^d) : \mathbb{R}^d \to \mathbb{R}^d$ the notation Dv means the Jacobian matrix whose j-th entry in the i-th row is $D_j v^i$. When we talk about "derivatives up to order n" of a function for some nonnegative integer n, then we always include the zero-order derivative, i.e. the function itself. The space of smooth functions $\varphi = \varphi(x)$ with compact support on the d-dimensional Euclidean space $\mathbb{R}^d$ is denoted by C_0^∞.

For $p, q \geq 1$ we denote by $\mathcal{L}_q = \mathcal{L}_q(Z, \mathbb{R})$ the Banach spaces of $\mathbb{R}$-valued $\mathcal{Z}$-measurable functions $h = h(z)$ of $z \in Z$ such that

$$|h|^q_{\mathcal{L}_q} = \int_Z |h(z)|^q \, \mu(dz) < \infty.$$

The notation $\mathcal{L}_{p,q}$ means the space $\mathcal{L}_p \cap \mathcal{L}_q$ with the norm

$$|v|_{\mathcal{L}_{p,q}} = \max(|v|_{\mathcal{L}_p}, |v|_{\mathcal{L}_q}) \quad \text{for } v \in \mathcal{L}_p \cap \mathcal{L}_q.$$

The space of sequences $\nu = (\nu^1, \nu^2, ...)$ of real numbers ν^k with finite norm

$$|\nu|_{l_2} = \Big(\sum_{k=1}^{\infty} |\nu^k|^2\Big)^{1/2}$$

is denoted by l_2.

The Borel σ-algebra of a separable Banach space V is denoted by $\mathcal{B}(V)$, and for $p \geq 0$ the notations $L_p([0,T], V)$ and $L_p(\mathbb{R}^d, V)$ are used for the space of V-valued Borel-measurable functions f on $[0,T]$ and g on $\mathbb{R}^d$, respectively, such that $|f|^p_V$ and $|g|^p_V$ have finite Lebesgue integral over $[0,T]$ and $\mathbb{R}^d$, respectively. For $p \geq 1$ and $f \in L_p(\mathbb{R}^d, V)$ we use the notation $|f|_{L_p}$, defined by

$$|f|^p_{L_p} = \int_{\mathbb{R}^d} |f(x)|^p_V \, dx < \infty.$$

In the sequel, V will be $\mathbb{R}$, l_2 or $\mathcal{L}_{p,q}$. For integer $n \geq 0$ the space of functions from $L_p(\mathbb{R}^d, V)$, whose generalised derivatives up to order n are also in $L_p(\mathbb{R}^d, V)$, is denoted by $W^n_p = W^n_p(\mathbb{R}^d, V)$ with the norm

$$|f|_{W^n_p} := \sum_{|\alpha| \leq n} |D_\alpha f|_{L_p}.$$

By definition $W^0_p(\mathbb{R}^d, V) = L_p(\mathbb{R}^d, V)$.

For $m \in \mathbb{R}$ and $p \in (1, \infty)$ we use the notation $H^m_p = H^m_p(\mathbb{R}^d; V)$ for the Bessel potential space with exponent p and order m, defined as the space of V-valued generalised functions φ on $\mathbb{R}^d$ such that

$$(1-\Delta)^{m/2}\varphi \in L_p \quad \text{and} \quad |\varphi|_{H^m_p} := |(1-\Delta)^{m/2}\varphi|_{L_p} < \infty,$$

where $\Delta = \sum_{i=1}^{d} D_i^2$. Moreover, we use $\mathbb{H}_p^m$ to denote the space of $\mathcal{P}$-measurable functions from $\Omega \times [0,T]$ to H_p^m such that

$$|f|_{\mathbb{H}_p^m}^p := E\int_0^T |f_t|_{H_p^m}^p \, dt < \infty.$$

We will often omit the target space V in the notations $W_p^n(V)$, $H_p^m(V)$, and $\mathbb{H}_p^m(V)$ for convenience if $V = \mathbb{R}$, and we use $\mathbb{L}_p$ to denote $\mathbb{H}_p^0$. When $V = \mathcal{L}_{p,q}$ we use $W_{p,q}^n$, $H_{p,q}^m$ and $\mathbb{H}_{p,q}^m$ to denote $W_p^n(\mathcal{L}_{p,q})$, $H_p^m(\mathcal{L}_{p,q})$ and $\mathbb{H}_p^m(\mathcal{L}_{p,q})$ respectively, and we use $\mathbb{L}_{p,q}$ to denote $\mathbb{H}_{p,q}^0$.

Remark 1.1. If V is a UMD space, see for example [18] for the definition of UMD spaces, then by Theorem 5.6.11 in [18] for $p > 1$ and integers $n \geq 1$ we have $W_p^n(V) = H_p^n(V)$ with equivalent norms. Clearly, $\mathcal{L}_{p,q}$ is a UMD space for $p, q \in (1,\infty)$, which implies $W_{p,q}^n = H_{p,q}^n$ for non-negative integers n and $p, q \in (1,\infty)$.

2. Formulation of the results

To formulate our assumptions we fix a constant K, a non-negative real number m, an exponent $p \in [2,\infty)$, and non-negative $\mathcal{Z}$-measurable real-valued functions $\bar\eta$ and $\bar\xi$ on Z such that they are bounded by K and

$$K_{\bar\eta}^2 := \int_Z |\bar\eta(z)|^2 \, \mu(dz) < \infty \quad K_{\bar\xi}^2 := \int_Z |\bar\xi(z)|^2 \, \nu(dz) < \infty.$$

We denote by $\lceil m \rceil$ the smallest integer which is greater than or equal to m, and $\lfloor m \rfloor$ the largest integer which is less than or equal to m.

Assumption 2.1. The derivatives in $x \in \mathbb{R}^d$ of a^{ij}, b^i and c up to order $\max\{\lceil m \rceil, 2\}$, $\max\{\lceil m \rceil, 1\}$ and $\lceil m \rceil$, respectively, are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega \times H_T$, bounded by K for all $i, j = 1, 2, ...d$. The functions $\sigma^i = (\sigma^{ir})_{r=1}^{\infty}$ and $\beta = (\beta^r)_{r=1}^{\infty}$ and their derivatives up to order $\lceil m \rceil + 1$ are l_2-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions, bounded by K. Moreover $a^{ij} = a^{ji}$ for all $i, j = 1, ...d$, and for $P \otimes dt \otimes dx$-almost all $(\omega, t, x) \in \Omega \times H_T$

$$\tilde a_t^{ij}(x) z^i z^j \geq 0 \quad \text{for all} \quad z = (z^1, ..., z^d) \in \mathbb{R}^d, \tag{2.1}$$

where

$$\tilde a^{ij} = 2a^{ij} - \sigma^{ir}\sigma^{jr}.$$

Assumption 2.2. The mapping $\xi = (\xi^i)$ is an $\mathbb{R}^d$-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$-measurable function on $\Omega \times [0,T] \times \mathbb{R}^d \times Z$. Its derivatives in $x \in \mathbb{R}^d$ up to order $\max\{\lceil m \rceil, 3\}$ exist and are continuous in $x \in \mathbb{R}^d$ such that

$$|D^k \xi| \leq \bar\xi \quad k = 0, 1, 2, ..., \max\{\lceil m \rceil, 3\} := \bar m$$

for all $(\omega, t, x, z) \in \Omega \times H_T \times Z$. Moreover,

$$K^{-1} \leq |\det(\mathbf{I} + \theta D\xi_{t,z}(x))|$$

for all $(\omega, t, x, z, \theta) \in \Omega \times H_T \times Z \times [0,1]$, where $\mathbf{I}$ is the $d \times d$ identity matrix, and $D\xi$ denotes the Jacobian matrix of ξ in $x \in \mathbb{R}^d$.

Assumption 2.3. The function $\eta = (\eta^i)$ maps $\Omega \times [0,T] \times \mathbb{R}^d \times Z$ into $\mathbb{R}^d$ such that Assumption 2.2 holds with η and $\bar\eta$ in place of ξ and $\bar\xi$, respectively.

Remark 2.1. Assumptions 2.2 and 2.3 imply that for each $(\omega, t, z, \theta) \in \Omega \times [0,T] \times Z \times [0,1]$ the mappings

$$\tau_{\theta\xi}(x) := x + \theta\xi_{t,z}(x) \quad \text{and} \quad \tau_{\theta\eta}(x) := x + \theta\eta_{t,z}(x) \tag{2.2}$$

are C^k-diffeomorphisms on $\mathbb{R}^d$ with $k = \bar{m}$. Note that a C^k-diffeomorphism ρ for an integer $k \geq 1$ means a one-to-one mapping from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that ρ and its inverse ρ^{-1} are continuously differentiable up to order k.

Assumption 2.4. The functions λ^ξ, λ^η and γ are real-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$-measurable function on $\Omega \times [0,T] \times \mathbb{R}^d \times Z$. The derivatives in $x \in \mathbb{R}^d$ of λ^ξ, λ^η up to order $\max\{\lceil m \rceil, 1\}$ exist and are continuous in $x \in \mathbb{R}^d$ such that

$$|D^k\lambda^\xi| \leq K\bar{\xi}, \quad |D^k\lambda^\eta| \leq K\bar{\eta} \quad k = 0,1,2,...,\max\{\lceil m \rceil, 1\}.$$

The derivatives in $x \in \mathbb{R}^d$ of $\bar{\gamma} := \gamma - \lambda^\eta$ up to order $\lceil m \rceil$ exist and are continuous in $x \in \mathbb{R}^d$ such that

$$|D^l\bar{\gamma}| \leq K\bar{\eta}^2 \quad l = 0,1,2,...,\lceil m \rceil.$$

Assumption 2.5. The operator $\mathcal{R}_t$ is a linear mapping from $L_p(\mathbb{R}^d)$ into $L_p(\mathbb{R}^d)$ for every $t \in [0,T]$ and $\omega \in \Omega$, such that for every $\varphi \in C_0^\infty$ the function $\mathcal{R}\varphi$ is $\mathcal{P}$-measurable and

$$|\mathcal{R}_t\varphi|_{W_p^n} \leq K|\varphi|_{W_p^n} \quad \text{for all } \omega \in \Omega,\ t \in [0,T] \text{ and } n = 0, \lceil m \rceil.$$

Assumption 2.6. The free data $f = (f_t)_{t\in[0,T]}$, $g = (g_t^r)_{t\in[0,T]}$ and $h = (h_t)_{t\in[0,T]}$ are $\mathcal{P}$-measurable processes with values in H_p^m, $H_p^{m+1}(l^2)$ and $H_{p,2}^{m+1} = H_p^{m+1}(\mathcal{L}_{p,2})$, respectively, such that almost surely $\mathcal{K}_{p,m}^p(T) < \infty$, where

$$\mathcal{K}_{p,m}^p(t) := \int_0^t |f_s|_{H_p^m}^p + |g_s|_{H_p^{m+1}(l_2)}^p + |h_s|_{H_{p,2}^{m+1}}^p + \mathbf{1}_{p>2}|h_s|_{H_{p,2}^{m+2}}^p\,ds, \quad t \leq T.$$

The initial value ψ is an $\mathcal{F}_0$-measurable random variable with values in H_p^m.

Remark 2.2. By Taylor's formula we have

$$v(x+\eta(x)) - v(x) - \eta(x)\nabla v(x) = \int_0^1 \eta^k(x)(v_k(x+\theta\eta(x)) - v_k(x))\,d\theta$$

$$= \int_0^1 \eta^k(x)D_k(v(x+\theta\eta(x)) - v(x))\,d\theta - \int_0^1 \theta\eta^k(x)\eta_k^l(x)v_l(x+\theta\eta(x))\,d\theta,$$

where to ease notation we omit the arguments t and z, and write v_k instead of D_kv for functions v. Due to Assumption 2.3 these equations extend to $v \in W_p^1$ for $p \geq 2$ as well. Hence after changing the order of integrals, by integration by parts we obtain

$$\int_{\mathbb{R}^d}\int_Z (v(x+\eta(x)) - v(x) - \eta(x)\nabla v(x))\varphi(x)\,dz\,dx = -(\mathcal{J}_\eta^k v, D_k\varphi) + (\mathcal{J}_\eta^0 v, \varphi)$$

for $\varphi \in C_0^\infty$, with

$$\mathcal{J}_\eta^k(t)v(x) = \int_0^1\int_Z \eta^k(v(\tau_{\theta\eta}(x)) - v(x))\,\mu(dz)\,d\theta, \quad k = 1,2,...,d, \tag{2.3}$$

$$\mathcal{J}_\eta^0(t)v(x) = -\int_0^1\int_Z \{\sum_k \eta_k^k(v(\tau_{\theta\eta}(x)) - v(x)) + \theta\eta^k(x)\eta_k^l(x)v_l(\tau_{\theta\eta}(x))\}\,\mu(dz)\,d\theta, \tag{2.4}$$

where for the sake of short notation the arguments t, z of η and of $\eta_k = D_k\eta$ have been omitted, and $\tau_{\theta\eta}$ is defined in (2.2). Operators $\mathcal{J}^k_\xi$ and $\mathcal{J}^0_\xi$ are defined as $\mathcal{J}^k_\eta$ and $\mathcal{J}^0_\eta$ in (2.3) and (2.4) but with ξ everywhere in place of η.

Definition 2.1. An L_p-valued cadlag $\mathcal{F}_t$-adapted process $u = (u_t)_{t\in[0,T]}$ is a *generalised solution* to equation (1.1)-(1.2) with initial value $u_0 = \psi$, if $u_t \in W^1_p$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, such that almost surely $u \in L_p([0, T], W^1_p)$ and

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t \langle \mathcal{A}_s u_s, \varphi\rangle + (f_s, \varphi)\, ds + \int_0^t (\mathcal{M}^r_s u_s + g^r_s, \varphi)\, dw^r_s$$
$$+ \int_0^t \int_Z (u_{s-}(\tau_{\eta_{s,z}}) - u_{s-} + \gamma_{s,z} u_{s-}(\tau_{\eta_{s,z}}) + h_s(z), \varphi)\, \tilde{\pi}(dz, ds)$$

for all for each $\varphi \in C^\infty_0$ and $t \in [0, T]$, where $\tau_{\eta_{s,z}} = x + \eta_{s,z}(x)$, $x \in \mathbb{R}^d$,

$$\begin{aligned}\langle \mathcal{A}_s u_s, \varphi\rangle := &-(a^{ij}_s D_j u_s, D_i\varphi) + (\bar{b}^i_s D_i u_s + c_s u_s, \varphi) + (\mathcal{R}_s u_s, \varphi)\\ &-(\mathcal{J}^i_\xi u_s, D_i\varphi) + (\mathcal{J}^0_\xi u_s, \varphi) - (\mathcal{J}^i_\eta u_s, D_i\varphi) + (\mathcal{J}^0_\eta u_s, \varphi)\\ &+ \int_Z \int_{\mathbb{R}^d} \lambda^\xi_{s,z}(x)(u_s(x + \xi_{s,z}(x)) - u_s(x))\varphi(x)\, dx\, \nu(dz)\\ &+ \int_Z \int_{\mathbb{R}^d} \lambda^\eta_{s,z}(x)(u_s(x + \eta_{s,z}(x)) - u_s(x))\varphi(x)\, dx\, \mu(dz) \end{aligned} \tag{2.5}$$

with $\bar{b}^i_s = b^i_s - D_j a^{ij}_s$ for all $(s, \omega) \in [0, T] \times \Omega$, and the stochastic integrals are Itô integrals.

The next theorem is the main result of this paper.

Theorem 2.1. *If Assumptions 2.1 through 2.6 hold with $m \geq 0$, then there is at most one generalised solution to* (1.1)*. If Assumptions 2.1 through 2.6 hold with $m \geq 1$, then there is a unique generalised solution $u = (u_t)_{t\in[0,T]}$, which is a weakly cadlag H^m_p-valued adapted process, and it is a strongly cadlag H^s_p-valued process for any $s \in [0, m)$. Moreover,*

$$E \sup_{t\in[0,T]} |u_t|^q_{H^s_p}\, dt \leq N \left(E|\psi|^q_{H^s_p} + E\mathcal{K}^q_{p,s}(T)\right) \quad \textit{for } s \in [0, m],\ q \in (0, p] \tag{2.6}$$

with a constant $N = N(d, m, p, q, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$.

3. Preliminaries

For vectors $v = (v^1, ..., v^d) \in \mathbb{R}^d$ we define the operators T^v, I^v and J^v by

$$\begin{aligned} &T^v\varphi(x) = \varphi(x + v), \quad I^v\varphi(x) = \varphi(x + v) - \varphi(x),\\ &J^v\phi(x) = \phi(x + v) - \phi(x) - v^i D_i\phi(x) \quad \text{for } x \in \mathbb{R}^d, \end{aligned} \tag{3.1}$$

acting on functions φ and ϕ defined on $\mathbb{R}^d$ such that the generalised derivative $D_i\phi$ exist. If $v = v(x)$ is a function of $x \in \mathbb{R}^d$ then the notation T^v, I^v and J^v mean the operators defined by (3.1) with $v(x)$ in place v. For example, J^ξ and J^η mean for each $\omega \in \Omega$, $t \in [0, T]$ and $z \in Z$ the operators defined on differentiable functions φ on $\mathbb{R}^d$ by

$$\begin{aligned} &J^\xi\varphi(x) = \varphi(x + \xi(x)) - \varphi(x) - \eta^i(x) D_i v(x),\\ &J^\eta\varphi(x) = \varphi(x + \eta(x)) - \varphi(x) - \eta^i(x) D_i v(x), \quad x \in \mathbb{R}^d \end{aligned}$$

for each fixed variable (ω, t, z) suppressed in this notation. We will often use the Taylor formulas

$$I^v\varphi(x) = \int_0^1 \varphi_i(x+\theta v)v^i\,d\theta \tag{3.2}$$

and

$$J^v\phi(x) = \int_0^1 (1-\theta)\phi_{ij}(x+\theta v)v^iv^j\,d\theta \tag{3.3}$$

with $\varphi_i := D_i\varphi$ and $\phi_{ij} := D_iD_j\phi$, which hold for every $x \in \mathbb{R}^d$ when φ and ϕ are continuous functions on $\mathbb{R}^d$ with continuous derivatives up to first order and up to second order, respectively. These equations hold for dx-almost every $x \in \mathbb{R}^d$ when $\varphi \in W^1_p$ and $\phi \in W^2_p$.

We fix a non-negative smooth function $k = k(x)$ with compact support on $\mathbb{R}^d$ such that $k(x) = 0$ for $|x| \geq 1$, $k(-x) = k(x)$ for $x \in \mathbb{R}^d$, and $\int_{\mathbb{R}^d} k(x)\,dx = 1$. For $\varepsilon > 0$ and locally integrable functions v of $x \in \mathbb{R}^d$ we use the notation $v^{(\varepsilon)}$ for the mollification of v, defined by

$$v^{(\varepsilon)}(x) := S^\varepsilon v(x) := \varepsilon^{-d}\int_{\mathbb{R}^d} v(y)k((x-y)/\varepsilon)\,dy, \quad x \in \mathbb{R}^d. \tag{3.4}$$

Note that if $v = v(x)$ is a locally Bochner integrable function on $\mathbb{R}^d$, with respect to the Lebesgue measure, which takes values in a separable Banach space, then the mollification of v is defined as in (3.4) in the sense of Bochner integral.

The following lemmas are taken from [9] and for their proof we refer to [9].

Lemma 3.1. *Let Assumption 2.3 hold. Then for every $(\omega, t, z) \in \Omega \times [0,T] \times Z$ the operators T^η, I^η and J^η are bounded linear operators from W^k_p to W^k_p, from W^{k+1}_p to W^k_p and from W^{k+2}_p to W^k_p respectively, for $k = 0, 1, ..., \bar{m}$, such that $T^\eta\varphi$, $I^\eta f$ and $J^\eta g$ are $\mathcal{P} \otimes \mathcal{Z}$-measurable W^k_p-valued functions of (ω, t, z), and*

$$|T^\eta\varphi|_{W^k_p} \leq N|\varphi|_{W^k_p}, \quad |I^\eta f|_{W^k_p} \leq N\bar{\eta}(z)|f|_{W^{k+1}_p}, \quad |J^\eta g|_{W^k_p} \leq N\bar{\eta}^2(z)|g|_{W^{k+2}_p}$$

for all $\varphi \in W^k_p$, $f \in W^{k+1}_p$ and $g \in W^{k+2}_p$, where N is a constant only depending on K, m, d, p.

Lemma 3.2. *Let ρ be a $C^k(\mathbb{R}^d)$-diffeomorphism for some $k \geq 1$, such that*

$$M \leq |\det D\rho| \text{ and } |D^l\rho| \leq N \quad \text{for } l = 1, 2, ..., k \tag{3.5}$$

for some constants $M > 0$ and $N > 0$. Then there are positive constants $M' = M'(N,d)$ and $N' = N'(N, M, d, k)$ such that (3.5) holds with $g := \rho^{-1}$, the inverse of ρ, in place of ρ, with M' and N' in place of M and N, respectively.

The following lemma is a slight generalisation of Lemma 3.4 in [9].

Lemma 3.3. *Let ρ be a C^k-diffeomorphism for $k \geq 2$, such that (3.5) holds for some positive constants M and N. Then there is a positive constant $\varepsilon_0 = \varepsilon_0(M, N, d)$ such that $\rho_{\varepsilon,\vartheta} := \vartheta\rho + (1-\vartheta)\rho^{(\varepsilon)}$ is a C^k-diffeomorphism for every $\varepsilon \in (0, \varepsilon_0)$ and $\vartheta \in [0,1]$, and (3.5) remains valid for $\rho_{\varepsilon,\vartheta}$ in place of ρ, with $M'' = M/2$ in place of M. Moreover, $\rho^{(\varepsilon)}$ is a C^∞-diffeomorphism for $\varepsilon \in (0, \varepsilon_0)$.*

Proof. We show first that $|\det D\rho_{\varepsilon,\vartheta}|$ is separated away from zero for sufficiently small $\varepsilon > 0$. To this end observe that for bounded Lipschitz functions $v = (v^1, v^2, ..., v^d)$ on $\mathbb{R}^d$ and $v_{\varepsilon,\vartheta} := \vartheta v + (1-\vartheta)v^{(\varepsilon)}$ we have

$$|\Pi_{i=1}^d v^i - \Pi_{i=1}^d v^i_{\varepsilon,\vartheta}| \le \sum_{i=1}^d K^{d-1}|v^i - v^i_{\varepsilon,\vartheta}| \le K^{d-1}L\varepsilon \quad \text{for any } \varepsilon > 0 \text{ and } \vartheta \in [0,1],$$

where L is the Lipschitz constant of v and K is a bound for $|v|$. Using this observation and taking into account that $D_i\rho^l$ is bounded by N and it is Lipschitz continuous with a Lipschitz constant not larger than N, we get

$$|\det D\rho - \det D\rho_{\varepsilon,\vartheta}| \le d!\, N^d\varepsilon.$$

Thus setting $\varepsilon' = M/(2d!\, N^d)$, for $\varepsilon \in (0, \varepsilon')$ and $\vartheta \in [0,1]$ we have

$$\begin{aligned}|\det D\rho_{\varepsilon,\vartheta}| &\ge |\det D\rho| - |\det D\rho - \det D\rho_{\varepsilon,\vartheta}| \\ &\ge |\det D\rho|/2 \ge M/2.\end{aligned}$$

Clearly, $\rho_{\varepsilon,\vartheta}$ is a C^k function. Hence by the implicit function theorem $\rho_{\varepsilon,\vartheta}$ is a local C^k-diffeomorphism for $\varepsilon \in (0,\varepsilon')$ and $\vartheta \in [0,1]$. We prove now that $\rho_{\varepsilon,\vartheta}$ is a global C^k-diffeomorphism for sufficiently small ε. Since by the previous lemma $|D\rho^{-1}| \le N'$, we have

$$\begin{aligned}|x-y| \le& N'|\rho(x) - \rho(y)| \\ \le& N'|\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| + N'|\rho(x) - \rho_{\varepsilon,\vartheta}(x) + \rho_{\varepsilon,\vartheta}(y) - \rho(y)|\end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $\varepsilon > 0$ and $\vartheta \in [0,1]$. Observe that

$$|\rho(x) - \rho_{\varepsilon,\vartheta}(x) + \rho_{\varepsilon,\vartheta}(y) - \rho(y)| \le \int_{\mathbb{R}^d} |\rho(x) - \rho(x-\varepsilon u) + \rho(y-\varepsilon u) - \rho(y)|k(u)\,du$$

$$\le \int_{\mathbb{R}^d}\int_0^1 \varepsilon|u||\nabla\rho(x-\theta\varepsilon u) - \nabla\rho(y-\theta\varepsilon u)|k(u)\,d\theta\,du$$

$$\le \varepsilon N|x-y|\int_{|u|\le 1}|u|k(u)\,du \le \varepsilon N|x-y|.$$

Thus $|x-y| \le N'|\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| + \varepsilon N' N|x-y|$. Therefore setting $\varepsilon'' = 1/(2NN')$, for all $\varepsilon \in (0,\varepsilon'')$ and $\vartheta \in [0,1]$ we have

$$|x-y| \le 2N'|\rho_{\varepsilon,\vartheta}(x) - \rho_{\varepsilon,\vartheta}(y)| \quad \text{for all } x, y \in \mathbb{R}^d,$$

which implies $\lim_{|x|\to\infty}|\rho_{\varepsilon,\vartheta}(x)| = \infty$, i.e., under $\rho_{\varepsilon,\vartheta}$ the pre-image of any compact set is a compact set for each $\varepsilon \in (0,\varepsilon'')$ and $\vartheta \in [0,1]$. A continuous function with this property is called a *proper function*, and by Theorem 1 in [13] a local C^1- diffeomorphism from $\mathbb{R}^d$ into $\mathbb{R}^d$ is a global diffeomorphism if and only if it is a proper function. Thus we have proved that $\rho_{\varepsilon,\vartheta}$ is a global C^k-diffeomorphism for $\varepsilon \in (0,\varepsilon_0)$ and $\vartheta \in [0,1]$, where $\varepsilon_0 = \min(\varepsilon', \varepsilon'')$. Clearly, $\rho_{\varepsilon,0} = \rho^{(\varepsilon)}$ is a C^∞ function and hence it is a C^∞-diffeomorphism for every $\varepsilon \in (0,\varepsilon_0)$.

Now we can complete the proof of the lemma by noting that since $D_j\rho^{(\varepsilon)} = (D_j\rho)^{(\varepsilon)}$, the condition $|D^i\rho| \le N$ implies $|D^i\rho_{\varepsilon,\vartheta}| \le N$ for any $\varepsilon > 0$ and $\vartheta \in [0,1]$. □

For fixed $\varepsilon > 0$ and $\vartheta \in [0,1]$ let $\rho_{\varepsilon,\vartheta}$ denote any of the functions

$$\rho_{\varepsilon,\vartheta}(x) := x + \vartheta\eta_{t,z}(x) + (1-\vartheta)\eta^{(\varepsilon)}_{t,z}(x), \quad \rho_{\varepsilon,\vartheta}(x) := x + \vartheta\xi_{t,z}(x) + (1-\vartheta)\xi^{(\varepsilon)}_{t,z}(x) \quad x \in \mathbb{R}^d.$$

for each $(\omega, t, z) \in \Omega \times [0,T] \times Z$, and assume that Assumptions 2.2 and 2.3 hold. Then by the inverse function theorem ρ is a local C^1-diffeomorphism for each t, θ and z. Since

$$|\eta_{t,z}(x)| \leq \bar{\eta}(z) < \infty, \quad |\xi_{t,z}(x)| \leq \bar{\xi}(z) < \infty,$$

we have $\lim_{|x|\to\infty} |\rho_{\varepsilon,\vartheta}(x)| = \infty$. Hence $\rho_{\varepsilon,\vartheta}$ is a global C^1-diffeomorphism on $\mathbb{R}^d$, for $\varepsilon > 0$, $\vartheta \in [0,1]$, for each $t \in [0,T]$, $z \in Z$ and $\theta \in [0,1]$, by Theorem 1 in [13]. Note that by the formula on the derivative of inverse functions, a C^1-diffeomorphism and its inverse have continuous derivatives up to the same order. Thus Lemmas 3.2 and 3.3 imply the following statement, which is a slight generalisation of Corollary 3.6 in [9].

Corollary 3.4. *Let Assumptions 2.2 and 2.3 hold. Then there is a positive constant $\varepsilon_0 = \varepsilon_0(K,d)$ such that $\rho = \rho_{\varepsilon,\vartheta}$ is a C^k-diffeomorphism on $\mathbb{R}^d$ for $k = \bar{m}$, for any $\varepsilon \in (0,\varepsilon_0)$, $\vartheta \in [0,1]$ and $(\omega,t,z) \in \Omega \times [0,T] \times Z$. Moreover, for some constants $M = M(K,d,\bar{m})$ and $N = (K,d,\bar{m})$*

$$M \leq \min(|\mathrm{det}D\rho|, |\det(D\rho)^{-1}|), \quad \max(|D^k\rho|, |D^k\rho^{-1}|) \leq N \tag{3.6}$$

for any $\varepsilon \in (0,\varepsilon_0)$, $\vartheta \in [0,1]$, $(\omega,t,z) \in \Omega \times [0,T] \times Z$ and for $k = 1,2,...,\bar{m}$. Furthermore, if $\vartheta = 0$ then ρ is a C^∞-diffeomorphism for each $\varepsilon \in (0,\varepsilon_0)$, $(\omega,t,z) \in \Omega \times [0,T] \times Z$, and for each integer $m \geq 1$ there are constants $M = M(K,d,m)$ and $N = N(K,d,m)$ such the estimates in (3.6) *hold for all $\varepsilon \in (0,\varepsilon_0)$, $(\omega,t,z) \in \Omega \times [0,T] \times Z$ and $k = 1,2,...,m$.*

Lemma 3.5. *Let V be a separable Banach space, and let $f = f(x)$ be a V-valued function of $x \in \mathbb{R}^d$ such that $f \in L_p(V) = L_p(\mathbb{R}^d, V)$ for some $p \geq 1$. Then we have*

$$|f^{(\varepsilon)}|_{L_p(V)} \leq |f|_{L_p(V)} \quad \text{and} \quad \lim_{\varepsilon\to 0} |f^{(\varepsilon)} - f|_{L_p(V)} = 0.$$

Proof. This lemma is well-known. Its proof can be found, e.g., in [17], see Lemma 4.4 therein. For the convenience of the readers, we present the proof below. By the properties of Bochner integrals, Jensen's inequality and Fubini's theorem

$$|f^{(\varepsilon)}|^p_{L_p(V)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y)k_\varepsilon(x-y)dy \right|^p_V dx$$
$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)|^p_V k_\varepsilon(x-y)\,dy\,dx = |f|^p_{L_p(V)}.$$

Since V is separable, it has a countable dense subset V_0. Denote by $\mathcal{H} \subset L_p(V)$ the space of functions h of the form

$$h(x) = \sum_{i=1}^k v_i\varphi_i(x)$$

for some integer $k \geq 1$, $v_i \in V_0$ and continuous real functions φ_i on $\mathbb{R}^d$ with compact support. Then for such an h we have

$$|h^{(\varepsilon)} - h|_{L_p(V)} \leq \sum_{i=1}^k |\varphi^{(\varepsilon)}_i - \varphi_i|_{L_p}|v_i|_V \to 0 \quad \text{as } \varepsilon \to 0,$$

where $L_p = L_p(\mathbb{R}^d, \mathbb{R})$. For $f \in L_p(V)$ and $h \in \mathcal{H}$ we have

$$|f-f^{(\varepsilon)}|_{L_p(V)} \le |f-h|_{L_p(V)}+|h-h^{(\varepsilon)}|_{L_p(V)}+|(f-h)^{(\varepsilon)}|_{L_p(V)} \le 2|f-h|_{L_p(V)}+|h-h^{(\varepsilon)}|_{L_p(V)}.$$

Letting here $\varepsilon \to 0$ for each $f \in L_p(V)$ we obtain

$$\limsup_{\varepsilon\to 0} |f - f^{(\varepsilon)}|_{L_p(V)} \le 2|f-h|_{L_p(V)} \quad \text{for all } h \in \mathcal{H}.$$

Since $\mathcal{H}$ is dense in $L_p(V)$, we can choose $h \in \mathcal{H}$ to make $|f-h|_{L_p(V)}$ arbitrarily small, which proves $\lim_{\varepsilon\to 0} |f - f^{(\varepsilon)}|_{L_p(V)} = 0$. □

Recall that $L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$ denotes the L_p-space of $\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2}$-valued functions on $\mathbb{R}^d$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Since $(Z, \mathcal{Z}, \mu)$ is a σ-finite separable measure space, $\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2}$ is a separable Banach space for any $q_1, q_2 \in [1,\infty)$. Hence, by Lemma 3.6 in [17] for each $v \in L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$, $p > 1$ there is a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$-measurable function $\bar v = \bar v(x,z)$ such that for every $x \in \mathbb{R}^d$ we have $v(x,z) = \bar v(x,z)$ for μ-almost every $z \in Z$. Therefore if $v \in L_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$ for some $p > 1$, then we may assume that v is a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{Z}$-measurable real-valued function. Moreover, we will often use the following characterisation of $W^n_p(\mathcal{L}_{q_1} \cap \mathcal{L}_{q_2})$.

Lemma 3.6. *Let $v \in L_p(\mathcal{L}_p \cap \mathcal{L}_q)$ for some $p, q \in (1,\infty)$, and let α be a multi-index. Then the following statements hold.*

(i) If v_α, the $\mathcal{L}_p \cap \mathcal{L}_q$-valued generalised D_α-derivative, belongs to $L_p(\mathcal{L}_p \cap \mathcal{L}_q)$, then for μ-almost every $z \in Z$ the function $v_\alpha(\cdot, z)$ belongs to $L_p(\mathbb{R}^d, \mathbb{R})$ and it is the generalised D_α-derivative of $v(\cdot, z)$.

(ii) If $v_\alpha(\cdot, z)$, the generalised D_α-derivative of the function $v(\cdot, z)$, belongs to $L_p(\mathbb{R}^d, \mathbb{R})$ for μ-almost every $z \in Z$ such that

$$\int_{\mathbb{R}^d} \Big(\int_Z |v_\alpha(x,z)|^r \, \mu(dz) \Big)^{p/r} dx < \infty \quad \text{for } r = p, q, \tag{3.7}$$

then v_α belongs to $L_p(\mathcal{L}_p \cap \mathcal{L}_q)$, and it is the $\mathcal{L}_p \cap \mathcal{L}_q$-valued generalised D_α-derivative of v.

Proof. (i) Let $\bar v_\alpha$ denote the $\mathcal{L}_p \cap \mathcal{L}_q$-valued generalised D_α-derivative of v. Then

$$\int_{\mathbb{R}^d} \bar v_\alpha(x)\varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x) D_\alpha \varphi(x)\, dx$$

for every $\varphi \in C_0^\infty$, where the integrals are understood as Bochner integrals of $\mathcal{L}_p \cap \mathcal{L}_q$-valued functions. Hence

$$\int_{\mathbb{R}^d}\int_Z \bar v_\alpha(x,z)\psi(z)\varphi(x)\mu(dz)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d}\int_Z \bar v(x,z)\psi(z)\varphi_\alpha(x)\mu(dz)\, dx$$

for all $\varphi \in C_0^\infty$ and bounded $\mathcal{Z}$-measurable functions ψ supported on sets of finite μ-measure. We can use Fubini's theorem to get

$$\int_Z\int_{\mathbb{R}^d} \bar v_\alpha(x,z)\varphi(x)\, dx\, \psi(z)\mu(dz) = (-1)^{|\alpha|} \int_Z\int_{\mathbb{R}^d} v(x,z)\varphi_\alpha(x)\, dx\, \psi(z)\, \mu(dz).$$

Thus for each $\varphi \in C_0^\infty$ we have

$$\int_{\mathbb{R}^d} \bar v_\alpha(x,z)\varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x,z)\varphi_\alpha(x)\, dx \tag{3.8}$$

for μ-almost every $z \in Z$. Consequently, for μ-almost every $z \in Z$ equation (3.8) holds for all $\varphi \in \Phi$ for a separable dense set $\Phi \subset C_0^\infty$ in $L_{p/(p-1)}(\mathbb{R}^d, \mathbb{R})$. Notice that for μ-almost every $z \in Z$ the functions $\bar{v}_\alpha(\cdot, z)$ and $v(\cdot, z)$ belong to $L_p(\mathbb{R}^d, \mathbb{R})$. Hence there is a set $S \subset Z$ of full μ-measure such that for $z \in S$ equation (3.8) holds for all $\varphi \in C_0^\infty$, which proves that for $z \in S$ the function $\bar{v}_\alpha(\cdot, z)$ is the generalised D_α-derivative of $v(\cdot, z)$. To prove (ii) notice that if for μ-almost every $z \in Z$ the function $v_\alpha(\cdot, z)$ belongs to $L_p(\mathbb{R}^d, \mathbb{R})$ and it is the D_α generalised derivative of the function $v(\cdot, z)$, then for μ-almost every $z \in Z$ we have

$$\int_{\mathbb{R}^d} v_\alpha(x, z)\varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x, z)\varphi(x)\, dx$$

for every $\varphi \in C_0^\infty$. Using condition (3.7) and that $v \in L_p(\mathcal{L}_p \cap \mathcal{L}_q)$, it is easy to check that, as functions of z, both sides of the above equation are functions in $\mathcal{L}_p \cap \mathcal{L}_q$, and hence that these integrals define the same functions as the corresponding $\mathcal{L}_p \cap \mathcal{L}_q$-valued Bochner integrals. This proves that v_α is the $\mathcal{L}_p \cap \mathcal{L}_q$-valued generalised D_α-derivative of v. □

Lemma 3.7. *Let Assumptions 2.3 hold with $m = 0$. Then the following statements hold.*

(i) Let ζ be a $\mathcal{F} \otimes \mathcal{B}([0, T] \times \mathbb{R}^d) \otimes \mathcal{Z}$-measurable function on $\Omega \times H_T \times Z$ such that it is continuously differentiable in $x \in \mathbb{R}^d$ and

$$|\zeta| + |D\zeta| \le K\bar{\eta} \quad \text{for all } (\omega, t, x, z) \in \Omega \times H_T \times Z. \tag{3.9}$$

Then there is a constant $N = N(K, d)$ such that for $\varphi \in W_1^1$

$$A := \int_{\mathbb{R}^d} \zeta(t, x, z) I^\eta \varphi(x)\, dx \le N\bar{\eta}^2(z)\, |\varphi|_{L_1} \quad \text{for all } (\omega, t, z) \in \Omega \times [0, T] \times Z. \tag{3.10}$$

(ii) There is a constant $N = N(K, d)$ such that for all $\phi \in W_1^2$

$$B := \int_{\mathbb{R}^d} J^\eta \phi(x)\, dx \le N\bar{\eta}^2(z)|\phi|_{L_1}. \tag{3.11}$$

(iii) There is a constant $N = N(K, d)$ such that for all $\phi \in W_1^1$

$$C := \int_{\mathbb{R}^d} I^\eta \phi(x)\, dx \le N\bar{\eta}(z)|\phi|_{L_1}. \tag{3.12}$$

Proof. The proof of (3.11) and (3.12) is given in [8] and [9]. For the convenience of the reader we prove each of the above estimates here. We may assume that $\varphi, \phi \in C_0^\infty$. For each $(\omega, t, z, \theta) \in \Omega \times [0, T] \times Z \times [0, 1]$ let $\tau_{\theta\eta}^{-1}$ denote the inverse of the function $x \to x + \theta\eta_{t,z}(x)$. Using (3.2) and (3.3) by change of variables we have

$$A = \int_0^1 \int_{\mathbb{R}^d} \nabla\varphi(x)\chi_{t,z,\theta}(x)\, dx\, d\theta, \quad B = \int_0^1 \int_{\mathbb{R}^d} (1 - \theta) D_{ij}\phi(x)\varrho_{t,z,\theta}^{ij}(x)\, dx\, d\theta \tag{3.13}$$

$$C = \int_0^1 \int_{\mathbb{R}^d} \nabla\phi(x)\kappa_{t,z,\theta}(x)\, dx\, d\theta \tag{3.14}$$

with

$$\chi(x) := (\zeta\eta)(\tau_{\theta\eta}^{-1}(x))|\mathrm{det} D\tau_{\theta\eta}^{-1}(x)|, \quad \varrho^{ij}(x) := (\eta^i\eta^j)(\tau_{\theta\eta}^{-1}(x))|\mathrm{det} D\tau_{\theta\eta}^{-1}(x)|$$

and

$$\kappa(x) := \eta(\tau_{\theta\eta}^{-1}(x))|\mathrm{det} D\tau_{\theta\eta}^{-1}(x)|.$$

Due to (3.9) and Assumption 2.3 we have a constant $N = N(K, d)$ such that

$$|D\chi_{t,\theta,z}(x)| \leq N\bar{\eta}^2(z), \quad |D_{ij}\varrho^{ij}_{t,z,\theta}(x)| \leq N\bar{\eta}^2(z) \quad \text{and} \quad |D\kappa_{t,\theta,z}(x)| \leq N\bar{\eta}(z)$$

for all $(\omega, x, t, z, \theta) \in \Omega \times \mathbb{R}^d \times [0, T] \times Z \times [0, 1]$. Thus from (3.13) and (3.14) by integration by parts we get (3.10), (3.11) and (3.12). □

Lemma 3.8. *Let Assumption 2.3 hold with an integer $m \geq 0$. Then for all $\omega \in \Omega$, $t \in [0, T]$ and $r > 1$ we have*

$$\mathbf{H}(\varphi, h) := \int_Z |\int_{\mathbb{R}^d} I^\eta \varphi(x) h(x, z)\, dx| \mu(dz) \leq N|\varphi|_{L_{r/(r-1)}} |h|_{W^2_r(\mathcal{L}_{r,2})}, \tag{3.15}$$

$$\mathbf{K}(\phi, g) := \int_Z |\int_{\mathbb{R}^d} T^\eta \varphi(x) g(x, z)\, dx| \mu(dz) \leq N|\phi|_{L_{r/(r-1)}} |g|_{W^2_r(\mathcal{L}_{r,1})} \tag{3.16}$$

for $\varphi, \phi \in L_{r/(r-1)}$ and $h \in W^2_r(\mathcal{L}_{r,2})$ and $g \in W^2_r(\mathcal{L}_{r,1})$ with a constant $N = N(r, d, K, K_{\bar{\eta}})$.

Proof. First assume $\varphi, \phi \in C_0^\infty$, and notice that

$$\mathbf{H}(\varphi, h) \leq \sum_{j=0}^{1} \mathbf{H}^{(j)} \quad \text{and} \quad \mathbf{K}(\phi, h) \leq \sum_{j=0}^{2} \mathbf{K}^{(j)}$$

with

$$\mathbf{H}^{(0)} = \int_Z \Big| \int_{\mathbb{R}^d} J^\eta \varphi\, h\, dx \Big| \mu(dz), \quad \mathbf{H}^{(1)} = \int_Z \Big| \int_{\mathbb{R}^d} D_i \varphi\, \eta^i h\, dx \Big| \mu(dz)$$

and

$$\mathbf{K}^{(0)} = \int_Z \Big| \int_{\mathbb{R}^d} J^\eta \phi\, g\, dx \Big| \mu(dz), \quad \mathbf{K}^{(1)} = \int_Z \Big| \int_{\mathbb{R}^d} D_i \phi\, \eta^i g\, dx \Big| \mu(dz),$$

$$\mathbf{K}^{(2)} = \int_Z \Big| \int_{\mathbb{R}^d} \phi g\, dx \Big| \mu(dz).$$

By Fubini's theorem and Hölder's inequality

$$\mathbf{K}^{(2)} \leq \int_{\mathbb{R}^d} |\phi| |g(x, \cdot)|_{\mathcal{L}_1}\, dx \leq |\phi|_{L_{r/(r-1)}} |g|_{L_r(\mathcal{L}_1)}. \tag{3.17}$$

By Taylor's formula, Fubini's theorem, change of variables, integration by parts and using Assumption 2.3 we get

$$\mathbf{H}^{(0)} \leq \int_0^1 \int_Z \Big| \int_{\mathbb{R}^d} (1-\theta)(T^{\theta\eta}\varphi_{ij})\eta^i\eta^j h\, dx \Big| \mu(dz)\, d\theta$$

$$\leq N \int_0^1 \int_Z \int_{\mathbb{R}^d} \bar{\eta}^2(z) |\varphi| \sum_{|\beta| \leq 2} |h_\beta|(\tau^{-1}_{\theta\eta}(x), z)\, dx\, \mu(dz)\, d\theta$$

with $N = N(d, K)$, where $\tau_{\theta\eta}$ is defined by (2.2), and $\tau^{-1}_{\theta\eta}$ is its inverse. Hence by Hölder's inequality, change of variables, Fubini's theorem and using $|\bar{\eta}| \leq K$ we obtain

$$\mathbf{H}^{(0)} \leq N|\varphi|_{L_{r/(r-1)}} |h|_{W^2_r(\mathcal{L}_r)}, \tag{3.18}$$

and in the same way

$$\mathbf{K}^{(0)} \leq N|\phi|_{L_{r/(r-1)}} |g|_{W^2_r(\mathcal{L}_r)} \tag{3.19}$$

with a constant $N = N(K, d, r, K_{\bar\eta})$. By integration by parts, using Assumption 2.3, Cauchy-Schwarz and Hölder inequalities we get

$$\mathbf{H}^{(1)} = \int_Z \Big| \int_{\mathbb{R}^d} \varphi D_i(\eta^i h(x,z))\,dx \Big| \mu(dz) \leq \int_Z \int_{\mathbb{R}^d} |\varphi| \bar\eta(z) \sum_{|\beta|\leq 1} |h_\beta(x,z)|\,dx\,\mu(dz)$$
$$\leq K_{\bar\eta} \int_{\mathbb{R}^d} |\varphi| \sum_{|\beta|\leq 1} |h_\beta(x,\cdot)|_{\mathcal{L}_2}\,dx \leq N|\varphi|_{L_{r/(r-1)}} |h|_{W^2_r(\mathcal{L}_{r,2})} \tag{3.20}$$

and in the same way

$$\mathbf{K}^{(1)} \leq N|\phi|_{L_{r/(r-1)}} |g|_{W^1_r(\mathcal{L}_1)}$$

with a constant $N = N(K, d, r, K_{\bar\eta})$. Combining this with (3.17) through (3.20), we get (3.15) and (3.16) for $\varphi, \phi \in C_0^\infty$. Assume now that $\varphi, \phi \in L_{r/(r-1)}$ and $\mu(Z) < \infty$. It is easy to see that by Hölder inequality we have

$$\mathbf{H}(\varphi, h) \leq C|\varphi|_{L_{r/(r-1)}} |h|_{L_r(\mathcal{L}_r)} \quad \text{and} \quad \mathbf{K}(\phi, g) \leq C|\phi|_{L_{r/(r-1)}} |g|_{L_r(\mathcal{L}_r)}$$

for $\varphi, \phi \in L_{r/(r-1)}$ and $h, g \in L_r(\mathcal{L}_r)$ with a constant $C = C(K, K_{\bar\eta}, d, r, \mu(Z))$, which implies that estimates (3.15) and (3.16) for $\varphi, \phi \in C_0^\infty$ can be extended by continuity to $\varphi, \phi \in L_{r/(r-1)}$ for finite measures μ. In the general case of a σ-finite measure μ, there exist $Z_n \in \mathcal{Z}$, $n = 1, 2, ...$, such that $\mu(Z_n) < \infty$ and $\cup_{n=1}^\infty Z_n = Z$. We define measures μ_n for integers $n \geq 1$ such that $d\mu_n = \mathbf{1}_{Z_n} d\mu$, where $\mathbf{1}_{Z_n}$ is the indicator for Z_n. By the previous argument, for $\varphi, \phi \in L_{r/(r-1)}$ we have

$$\int_Z | \int_{\mathbb{R}^d} I^\eta \varphi(x) h(x,z)\,dx|\,\mu_n(dz) \leq N|\varphi|_{L_{r/(r-1)}} |h|^r_{W^2_r(\mathcal{L}_{r,2})}$$

and

$$\int_Z | \int_{\mathbb{R}^d} T^\eta \phi(x) g(x,z)\,dx|\,\mu_n(dz) \leq N|\phi|_{L_{r/(r-1)}} |g|_{W^2_r(\mathcal{L}_{r,1})}$$

with a constant $N = N(r, d, K, K_{\bar\eta})$. Then an application of Fatou's lemma finishes the proof of this lemma. □

Next we present two important Itô's formulas from [17] for the p-th power of L_p-norm of L_p-valued stochastic processes.

Lemma 3.9. *Let $(u^i_t)_{t\in 0,T}$ be a progressively measurable L_p-valued process such that there exist $f^i \in \mathbb{L}_p$, $g^i = (g^{ir})_{r=1}^\infty \in \mathbb{L}_p$, $h^i \in \mathbb{L}_{p,2}$, and an L_p-valued $\mathcal{F}_0$-measurable random variable ψ^i for each $i = 1, 2, ..., M$ for some integer M, such that for every $\varphi \in C_0^\infty$*

$$(u^i_t, \varphi) = (\psi, \varphi) + \int_0^t (f^i_s, \varphi)\,ds + \int_0^t (g^{ir}_s, \varphi)\,dw^r_s + \int_0^t \int_Z (h^i_s, \varphi)\,\tilde\pi(dz, ds) \tag{3.21}$$

for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$ and all $i = 1, 2, ..., M$. Then there are L_p-valued adapted cadlag processes $\bar u = (\bar u^1, \bar u^2, ..., \bar u^M)$ such that equation (3.21), with $\bar u^i$ in place of u^i, holds for every $i = 1, 2, ..., M$ and each $\varphi \in C_0^\infty$ almost surely for all $t \in [0, T]$. Moreover, $u^i = \bar u^i$ for $P \otimes dt$-almost every $(\omega, t) \in \Omega \times [0, T]$, and

$$|\bar u_t|^p_{L_p} = |\psi|^p_{L_p} + p \int_0^t \int_{\mathbb{R}^d} |\bar u_s|^{p-2} \bar u^i_s g^{ir}_s\,dx\,dw^r_s$$

$$+\tfrac{p}{2}\int_0^t\int_{\mathbb{R}^d}\big(2|\bar u_s|^{p-2}\bar u_s^i f_s^i+(p-2)|\bar u_s|^{p-4}|\bar u_s^i g_s^{i\cdot}|_{l_2}^2+|\bar u_s|^{p-2}\sum_{i=1}^M|g_s^{i\cdot}|_{l_2}^2\big)\,dx\,ds$$
$$+p\int_0^t\int_Z\int_{\mathbb{R}^d}|\bar u_{s-}|^{p-2}\bar u_{s-}^i h_s^i\,dx\,\tilde\pi(dz,ds)$$
$$+\int_0^t\int_Z\int_{\mathbb{R}^d}(|\bar u_{s-}+h_s|^p-|\bar u_{s-}|^p-p|\bar u_{s-}|^{p-2}\bar u_{s-}^i h_s^i)\,dx\,\pi(dz,ds)$$

holds almost surely for all $t\in[0,T]$, where the definition $0/0:=0$ is used where $0/0$ occurs.

Lemma 3.10. *Let $u=(u_t)_{t\in 0,T}$ be a progressively measurable W_p^1-valued process such that the following conditions hold:*
(i)

$$E\int_0^T|u_t|_{W_p^1}^p\,dt<\infty\,;$$

(ii) there exist $f\in\mathbb{L}_p$ for $\alpha\in\{0,1,...,d\}$, $g\in\mathbb{L}_p$, $h\in\mathbb{L}_{p,2}$, and an L_p-valued $\mathcal{F}_0$-measurable random variable ψ, such that for every $\varphi\in C_0^\infty$ we have

$$(u_t,\varphi)=(\psi,\varphi)+\int_0^t(f_s^\alpha,D_\alpha^*\varphi)\,ds+\int_0^t(g_s^r,\varphi)\,dw_s^r+\int_0^t\int_Z(h_s(z),\varphi)\,\tilde\pi(dz,ds) \tag{3.22}$$

for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$, where $D_\alpha^*=-D_\alpha$ for $\alpha=1,2,..,d$, and D_α^* is the identity operator for $\alpha=0$. Then there is an L_p-valued adapted cadlag process $\bar u=(\bar u_t)_{t\in[0,T]}$ such that for each $\varphi\in C_0^\infty$ equation (3.22) holds with $\bar u$ in place of u almost surely for all $t\in[0,T]$. Moreover, $u=\bar u$ for $P\otimes dt$-almost every $(\omega,t)\in\Omega\times[0,T]$, and almost surely

$$|\bar u_t|_{L_p}^p=|\psi|_{L_p}^p+p\int_0^t\int_{\mathbb{R}^d}|u_s|^{p-2}u_s g_s^r\,dx\,dw_s^r$$
$$+\tfrac{p}{2}\int_0^t\int_{\mathbb{R}^d}\big(2|u_s|^{p-2}u_sf_s^0-2(p-1)|u_s|^{p-2}f_s^iD_iu_s+(p-1)|u_s|^{p-2}|g_s|_{l_2}^2\big)\,dx\,ds$$
$$+\int_0^t\int_Z\int_{\mathbb{R}^d}p|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\,dx\,\tilde\pi(dz,ds)$$
$$+\int_0^t\int_Z\int_{\mathbb{R}^d}\big(|\bar u_{s-}+h_s|^p-|\bar u_{s-}|^p-p|\bar u_{s-}|^{p-2}\bar u_{s-}h_s\big)\,dx\,\pi(dz,ds)$$

for all $t\in[0,T]$, where $\bar u_{s-}$ denotes the left-hand limit in $L_p(\mathbb{R}^d)$ of $\bar u$ at $s\in(0,T]$.

The following slight generalisation of Lemma from [16] will play a role in obtaining supremum estimates.

Lemma 3.11. *Let $T\in[0,\infty]$ and let $f=(f_t)_{t\geq0}$ and $g=(g_t)_{t\geq0}$ be nonnegative $\mathcal{F}_t$-adapted processes such that f is a cadlag and g is a continuous process. Assume*

$$Ef_\tau\mathbf{1}_{g_0\leq c}\leq Eg_\tau\mathbf{1}_{g_0\leq c} \tag{3.23}$$

for any constant $c>0$ and bounded stopping time $\tau\leq T$. Then, for any bounded stopping time $\tau\leq T$, for $q\in(0,1)$

$$E\sup_{t\leq\tau}f_t^q\leq\tfrac{2-q}{1-q}E\sup_{t\leq\tau}g_t^q.$$

Proof. This lemma is proved in [16] when both processes f and g are continuous. A word by word repetition of the proof in [16] extends it to the case when f is cadlag. For the convenience of the reader we present the proof below. By replacing f_t and g_t with $f_{t\wedge T}$ and $g_{t\wedge T}$, respectively, we see that we may assume that $T=\infty$. Then we replace g_t with $\max_{s\leq t} g_s$ and see that without losing generality we may assume that g_t is nondecreasing. In that case fix a constant $c>0$ and let $\theta_f=\inf\{t\geq 0: f_t\geq c\}$, $\theta_g=\inf\{t\geq 0: g_t\geq c\}$. Then

$$P(\sup_{t\leq\tau} f_t>c)\leq P(\theta_f\leq\tau)\leq P(\theta_g\leq\tau)+P(\theta_f\leq\tau\wedge\theta_g,\theta_g>\tau)$$

$$\leq P(g_\tau\geq c)+P(g_0\leq c, f_{\tau\wedge\theta_g\wedge\theta_f}\geq c)\leq P(g_\tau\geq c)+\frac{1}{c}EI_{g_0\leq c}f_{\tau\wedge\theta_g\wedge\theta_f}.$$

In the light of (3.23) we replace the expectation with

$$EI_{g_0\leq c}g_{\tau\wedge\theta_g\wedge\theta_f}\leq EI_{g_0\leq c}g_{\tau\wedge\theta_g}=EI_{g_0\leq c}(g_\tau\wedge g_{\theta_g})$$
$$\leq EI_{g_0\leq c}(g_\tau\wedge c)\leq E(g_\tau\wedge c).$$

Hence

$$P(\sup_{t\leq\tau} f_t>c)\leq P(g_\tau\geq c)+\frac{1}{c}E(c\wedge g_\tau),$$

and it remains to substitute $c^{1/q}$ in place of c and integrate with respect to c over $(0,\infty)$. The lemma is proved. □

Finally we present a slight modification of Lemma 5.3 from [9] which we will use in proving regularity in time of solutions to (1.1)-(1.2).

Lemma 3.12. *Let V be a reflexive Banach space, embedded continuously and densely into a Banach space U. Let f be a U-valued weakly cadlag function on $[0,T]$ such that the weak limit in U at T from the left is $f(T)$. Assume there is a dense subset S of $[0,T]$ such that $f(s)\in V$ for $s\in S$ and $\sup_{s\in S}|f(s)|_V<\infty$. Then f is a V-valued function, which is cadlag in the weak topology of V, and hence $\sup_{s\in[0,T]}|f(s)|_V=\sup_{s\in S}|f(s)|_V$.*

Proof. Since S is dense in $[0,T]$, for a given $t\in[0,T)$ there is a sequence $\{t_n\}_{n=1}^\infty$ with elements in S such that $t_n\downarrow t$. Due to $\sup_{n\in\mathbb{N}}|f(t_n)|_V<\infty$ and the reflexivity of V there is a subsequence $\{t_{n_k}\}$ such that $f(t_{n_k})$ converges weakly in V to some element $v\in V$. Since f is weakly cadlag in U, for every continuous linear functional φ over U we have $\lim_{k\to\infty}\varphi(f(t_{n_k}))=\varphi(f(t))$. Since the restriction of φ in V is a continuous functional over V we have $\lim_{k\to\infty}\varphi(f(t_{n_k}))=\varphi(v)$. Hence $f(t)=v$, which proves that f is a V-valued function over $[0,T)$. Moreover, by taking into account that

$$|f(t)|_V=|v|_V\leq\liminf_{k\to\infty}|f(t_{n_k})|_V\leq\sup_{t\in S}|f(t)|_V<\infty,$$

we obtain $K:=\sup_{t\in[0,T)}|f(s)|_V<\infty$. Let ϕ be a continuous linear functional over V. Due to the reflexivity of V, the dual U^* of the space U is densely embedded into V^*, the dual of V. Thus for $\phi\in V^*$ and $\varepsilon>0$ there is $\phi_\varepsilon\in U^*$ such that $|\phi-\phi_\varepsilon|_{V^*}\leq\varepsilon$. Hence for arbitrary sequence $t_n\downarrow t$, $t_n\in[0,T]$ we have

$$|\phi(f(t))-\phi(f(t_n))|\leq|\phi_\varepsilon(f(t)-f(t_n))|+|(\phi-\phi_\varepsilon)(f(t)-f(t_n))|$$
$$\leq|\phi_\varepsilon(f(t)-f(t_n))|+\varepsilon|f(t)-f(t_n)|_V\leq|\phi_\varepsilon(f(t)-f(t_n))|+2\varepsilon K.$$

Letting here $n\to\infty$ and then $\varepsilon\to 0$, we get

$$\limsup_{n\to\infty}|\phi(f(t))-\phi(f(t_n))|\le 0,$$

which proves that f is right-continuous in the weak topology in V. We can prove in the same way that at each $t\in[0,T]$ the function f has weak limit in V from the left at each $t\in(0,T]$, which finishes the proof of the lemma. □

4. Some results on interpolation spaces

A pair of complex Banach spaces A_0 and A_1, which are continuously embedded into a Hausdorff topological vector space $\mathcal{H}$, is called an interpolation couple, and $A_\theta=[A_0,A_1]_\theta$ denotes the complex interpolation space between A_0 and A_1 with parameter $\theta\in(0,1)$. For an interpolation couple A_0 and A_1 the notations $A_0\cap A_1$ and A_0+A_1 is used for the subspaces

$$A_0\cap A_1=\{v\in\mathcal{H}:v\in A_0\,\text{and}\,v\in A_1\},\quad A_0+A_1=\{v\in\mathcal{H}:v=v_1+v_2,\ v_i\in A_i\}$$

equipped with the norms $|v|_{A_0\cap A_1}=\max(|v|_{A_0},|v|_{A_1})$ and

$$|v|_{A_0+A_1}:=\inf\{|v_0|_{A_0}+|v_1|_{A_1}:v=v_0+v_1,v_0\in A_0,\ v_1\in A_1\},$$

respectively. Then the following theorem lists some well-known facts about complex interpolation, see e.g., [5] or 1.9.3, 1.18.4 and 2.4.2 in [34] and 5.6.9 in [18].

Theorem 4.1. *(i) If A_0,A_1 and B_0,B_1 are two interpolation couples and $S:A_0+A_1\to B_0+B_1$ is a linear operator such that its restriction onto A_i is a continuous operator into B_i with operator norm C_i for $i=0,1$, then its restriction onto $A_\theta=[A_0,A_1]_\theta$ is a continuous operator into $B_\theta=[B_0,B_1]_\theta$ with operator norm $C_0^{1-\theta}C_1^\theta$ for every $\theta\in(0,1)$.*

(ii) For a σ-finite measure space $\mathfrak{M}$ and an interpolation couple of separable Banach spaces A_0, A_1 we have

$$[L_{p_0}(\mathfrak{M},A_0),L_{p_1}(\mathfrak{M},A_1)]_\theta=L_p(\mathfrak{M},[A_0,A_1]_\theta),$$

for every $p_0,p_1\in[1,\infty)$, $\theta\in(0,1)$, where $1/p=(1-\theta)/p_0+\theta/p_1$.

(iii) Let H^m_p denote the Bessel potential spaces of complex-valued functions. Then for $m_0,m_1\in\mathbb{R}$ and $1<p_0,p_1<\infty$

$$[H^{m_0}_{p_0},H^{m_1}_{p_1}]_\theta=H^m_p,$$

where $m=(1-\theta)m_0+\theta m_1$, and $1/p=(1-\theta)/p_0+\theta/p_1$. Moreover, for integers m one has $H^m_p=W^m_p$ with equivalent norms.

(iv) For a UMD Banach space V, denote by $H^m_p(V)$ the Bessel potential spaces of V-valued functions. Then for $1<p<\infty$ and $m_0,m_1\in\mathbb{R}$

$$[H^{m_0}_p(V),H^{m_1}_p(V)]_\theta=H^m_p(V)$$

for every $\theta\in(0,1)$, where $m=(1-\theta)m_0+\theta m_1$.

(v) For $\theta\in[0,1]$ there is a constant c_θ such that

$$|v|_{A_\theta}\le c_\theta|v|^{1-\theta}_{A_0}|v|^\theta_{A_1}$$

for all $v\in A_0\cap A_1$.

5. L_p ESTIMATES

Let Assumptions 2.1 through 2.6 hold with an integer $m \geq 0$, and assume in this section that $\mathcal{R} \equiv 0$. Let $u = (u_t)_{t\in[0,T]}$ be a W^{m+2}_p-valued solution to (1.1)-(1.2). Then for each $\varphi \in C_0^\infty$ and any multi-number α with $|\alpha| \leq m$, from the definition of the generalised solution and by integrating by parts we have

$$(D_\alpha u_t, \varphi) = (D_\alpha \psi, \varphi) + \int_0^t (D_\alpha(\mathcal{A}_s u_s + f_s), \varphi)\,ds + \int_0^t (D_\alpha(\mathcal{M}^r_s u_s + g^r_s), \varphi)\,dw^r_s$$
$$+ \int_0^t \int_Z (D_\alpha(I^\eta u_{s-} + \gamma_{s,z} T^\eta u_{s-} + h_s), \varphi)\,\tilde{\pi}(dz, ds)$$

for all $t \in [0, T]$ and almost all $\omega \in \Omega$. To shorten some expressions we introduce the operators $\mathbb{J}^\xi$, $\mathbb{J}^\eta$ and $\mathbb{I}^\eta$, defined by

$$\mathbb{J}^\xi \varphi = J^\xi \varphi + \lambda^\xi I^\xi \varphi, \quad \mathbb{J}^\eta \varphi = J^\eta \varphi + \lambda^\eta I^\eta \varphi, \quad \mathbb{I}^\eta \varphi = I^\eta \varphi + \gamma T^\eta \varphi$$

on functions φ. For functions $v \in W^m_p$ let $\mathbf{v} = \{v_\alpha : |\alpha| \leq m\}$ denote the vector whose coordinates are the derivatives $v_\alpha := D_\alpha v$ for $|\alpha| \leq m$. We use also the notation $I^\eta \mathbf{v}$, $J^\eta \mathbf{v}$ and $T^\eta \mathbf{v}$ for the vectors with coordinates $I^\eta v_\alpha$, $J^\eta v_\alpha$ and $T^\eta v_\alpha$, respectively for $|\alpha| \leq m$. Then applying Lemma 3.9 to $\mathbf{u}_t$ we have

$$\begin{aligned}
|\mathbf{u}_t|^p_{L_p} =& |\mathbf{u}_0|^p_{L_p} + p\int_0^t Q_p(s, u_s, f_s, g_s) + Q^\xi_p(s, u_s) + Q^\eta_p(s, u_s)\,ds \\
&+ p\int_0^t \int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-2} D_\alpha u_s D_\alpha(\mathcal{M}^r_s u_s + g^r_s)\,dx\,dw^r_s \\
&+ \int_0^t \int_Z \int_{\mathbb{R}^d} \{(\sum_{|\alpha|\leq m} |D_\alpha u_{s-} + D_\alpha(\mathbb{I}^\eta u_{s-} + h_s)|^2)^{p/2} - |\mathbf{u}_{s-}|^p \\
&- p\sum_{|\alpha|\leq m} |\mathbf{u}_{s-}|^{p-2} D_\alpha u_{s-} D_\alpha(\mathbb{I}^\eta u_{s-} + h_s)\}\,dx\,\pi(dz, ds) \\
&+ \int_0^t \int_Z \int_{\mathbb{R}^d} p|\mathbf{u}_{\mathbf{s}-}|^{p-2} \sum_{|\alpha|\leq m} D_\alpha u_{s-} D_\alpha(\mathbb{I}^\eta u_{s-} + h_s)\,dx\,\tilde{\pi}(dz, ds)
\end{aligned} \tag{5.1}$$

almost surely for all $t \in [0, T]$, where we use the notation

$$\begin{aligned}
Q_p(t, v, f, g) =& \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2}\{\sum_{|\alpha|\leq m} v_\alpha D_\alpha(\mathcal{L}v + f) + \frac{1}{2}\sum_{r=1}^\infty \sum_{|\alpha|\leq m} |D_\alpha(\mathcal{M}^r v + g^r)|^2\}\,dx \\
&+ \int_{\mathbb{R}^d} \tfrac{1}{2}p(p-2)|\mathbf{v}|^{p-4} \sum_{r=1}^\infty |\sum_{|\alpha|\leq m} v_\alpha D_\alpha(\mathcal{M}^r v + g^r)|^2\,dx,
\end{aligned} \tag{5.2}$$

$$Q^\xi_p(t, v) = \int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2} \sum_{|\alpha|\leq m} v_\alpha (\mathbb{J}^\xi v)_\alpha\,dx\,\nu(dz) \tag{5.3}$$

$$Q^\eta_p(t, v) = \int_Z \int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2} \sum_{|\alpha|\leq m} v_\alpha (\mathbb{J}^\eta v)_\alpha\,dx\,\mu(dz)$$

for $v\in W^{m+2}_p$, for each $f\in W^m_p$, $g\in W^{m+1}_p(l_2)$, $\omega\in\Omega$ and $t\in[0,T]$. In order to estimate $E|\mathbf{u}_t|^p_{L_p}$ we define also the "p-form"

$$\hat{Q}_p(t,v,h)=\int_Z\int_{\mathbb{R}^d}p|\mathbf{v}|^{p-2}\sum_{|\alpha|\le m}v_\alpha\big(\mathbb{J}^\eta v\big)_\alpha\,dx\,\mu(dz)$$

$$+\int_Z\int_{\mathbb{R}^d}|\sum_{|\alpha|\le m}(v+\mathbb{I}^\eta v+h)^2_\alpha|^{p/2}-|\mathbf{v}|^p-p|\mathbf{v}|^{p-2}\sum_{|\alpha|\le m}v_\alpha\big((\mathbb{I}^\eta v)_\alpha+h_\alpha\big)\,dx\,\mu(dz)\tag{5.4}$$

for $v\in W^{m+2}_p$, $h\in W^{m+2}_p(\mathcal{L}_{p,2})$, for each $\omega\in\Omega$ and $t\in[0,T]$.

Proposition 5.1. *Let Assumption 2.1 hold. Then for any $p\in[2,\infty)$ there is a constant $N=N(d,p,m,K)$ such that*

$$Q_p(v,t,f,g)\le N\big(|v|^p_{W^m_p}+|f|^p_{W^m_p}+|g|^p_{W^{m+1}_p(l_2)}\big)\tag{5.5}$$

for all $v\in W^{m+2}_p$, $f\in W^m_p$, $g\in W^{m+1}_p(l_2)$, $\omega\in\Omega$ and $t\in[0,T]$.

Proof. This estimate is proved in [12] in a more general setting. □

Proposition 5.2. *Let Assumption 2.3 hold with an integer $m\ge 0$. Then for $\omega\in\Omega$ and $t\in[0,T]$*

$$\hat{Q}_p(v,t,h)\le N(|v|^p_{W^m_p}+|h|^p_{W^{m+1}_p(\mathcal{L}_{p,2})}+\mathbf{1}_{p>2}|h|^p_{W^{m+2}_p(\mathcal{L}_{p,2})})\tag{5.6}$$

for $v\in W^{m+2}_p$ and $h\in W^{m+2}_p(\mathcal{L}_{p,2})$ with constants $N=N(d,m,p,K,K_{\bar\eta})$.

Proof. First we claim that for any $|\alpha|=n\le m$

$$(\mathbb{I}^\eta v)_\alpha=I^\eta v_\alpha+\sum_{|\beta|\le n}b^{\alpha\beta}T^\eta v_\beta+\sum_{|\beta|\le n}c^{\alpha\beta}T^\eta v_\beta\tag{5.7}$$

$$(\mathbb{J}^\eta v)_\alpha=J^\eta v_\alpha+\sum_{|\beta|\le n}b^{\alpha\beta}I^\eta v_\beta+\sum_{|\beta|\le n}\bar{c}^{\alpha\beta}T^\eta v_\beta\tag{5.8}$$

with some functions $b^{\alpha\beta}$, $c^{\alpha\beta}$ and $\bar{c}^{\alpha\beta}$ of (ω,t,z,x), such that for $|\beta|\le n$

$$|D^kb^{\alpha\beta}|\le N\bar\eta,\quad |D^kc^{\alpha\beta}|\le N\bar\eta^2,\quad |D^k\bar{c}^{\alpha\beta}|\le N\bar\eta^2\quad\text{for } k\le m-n\tag{5.9}$$

with a constant $N=N(d,m,K)$. The reader can prove this easily by induction on n, noticing that

$$(J^\eta v)_i=J^\eta v_i+\eta^k_iI^\eta v_k,\quad (I^\eta v)_i=I^\eta v_i+\eta^k_iT^\eta v_k,$$
$$(T^\eta v)_i=T^\eta v_i+\eta^k_iT^\eta v_k,$$

where, as before, the subscripts indicates derivatives in the corresponding coordinates. Using equations (5.7) and (5.8) we have

$$\hat{Q}_p(t,v,h)=\int_Z\int_{\mathbb{R}^d}|T^\eta\mathbf{v}+(\mathbf{b}+\mathbf{c})T^\eta\mathbf{v}+\mathbf{h})|^p-|\mathbf{v}|^p\,dx\mu(dz)$$

$$-\int_Z\int_{\mathbb{R}^d}p|\mathbf{v}|^{p-2}v_\alpha(I^\eta v_\alpha-J^\eta v_\alpha+b^{\alpha\beta}v_\beta+(c^{\alpha\beta}-\bar{c}^{\alpha\beta})T^\eta v_\beta+h_\alpha)\,dx\,\mu(dz),$$

where $\mathbf{b}$ and $\mathbf{c}$ denote the matrices with entries $b^{\alpha\beta}$ and $c^{\alpha\beta}$, respectively for $|\alpha| \le m$ and $|\beta| \le m$. Clearly,

$$-p|\mathbf{v}|^{p-2}v_\alpha\big(I^\eta v_\alpha - J^\eta v_\alpha\big) = p|\mathbf{v}|^{p-2}v_\alpha D_i v_\alpha \eta^i = J^\eta|\mathbf{v}|^p - I^\eta|\mathbf{v}|^p,$$

and

$$|T^\eta\mathbf{v} + (\mathbf{b}+\mathbf{c})T^\eta\mathbf{v} + \mathbf{h}|^p - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2}v_\alpha(b^{\alpha\beta}v_\beta + h_\alpha) = \sum_{i=1}^5 A_i$$

with

$$A_1 := |T\mathbf{v}|^p - |\mathbf{v}|^p = I|\mathbf{v}|^p, \quad A_2 := p|T\mathbf{v}|^{p-2}Tv_\alpha c^{\alpha\beta}Tv_\beta$$
$$A_3 := |T\mathbf{v} + (\mathbf{b}+\mathbf{c})T\mathbf{v} + \mathbf{h}|^p - |T\mathbf{v}|^p - p|T\mathbf{v}|^{p-2}Tv_\alpha((b^{\alpha\beta}+c^{\alpha\beta})Tv_\beta + h_\alpha)$$
$$A_4 := pI(|\mathbf{v}|^{p-2}v_\alpha v_\beta)b^{\alpha\beta}, \quad A_5 := pI(|\mathbf{v}|^{p-2}v_\alpha)h_\alpha,$$

where to ease notation we write I and T in place of I^η and T^η, respectively. Hence

$$\hat{Q}_p(v,t,h) = \int_Z\int_{\mathbb{R}^d} J^\eta|\mathbf{v}|^p\,dx\mu(dz) + \sum_{i=2}^5\int_Z\int_{\mathbb{R}^d} A_i\,dx\mu(dz)$$
$$+\int_Z\int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2}v_\alpha(\bar{c}^{\alpha\beta} - c^{\alpha\beta})T^\eta v_\beta\,dx\,\mu(dz).$$

By Lemma 3.7

$$\int_Z\int_{\mathbb{R}^d} J^\eta|\mathbf{v}|^p\,dx\mu(dz) \le N|v|^p_{W^m_p},$$

and due to (5.9)

$$\int_Z\int_{\mathbb{R}^d} A_2\,dx\mu(dz) \le N|v|^p_{W^m_p}, \quad \int_Z\int_{\mathbb{R}^d} A_4\,dx\mu(dz) \le N|v|^p_{W^m_p}$$

with constants $N = N(d,m,p,K,K_{\bar{\eta}})$. By Taylor's formula, (5.9), and Assumption 2.3

$$A_3 \le p(p-1)\int_0^1 |T\mathbf{v} + \theta(\mathbf{b}+\mathbf{c})T\mathbf{v} + \theta\mathbf{h})|^{p-2}|(\mathbf{b}+\mathbf{c})T\mathbf{v} + \mathbf{h}|^2\,d\theta$$
$$\le N(|\mathbf{h}|^p + T|\mathbf{v}|^{p-2}|\mathbf{h}|^2 + \bar{\eta}^2T|\mathbf{v}|^p). \tag{5.10}$$

It is easy to see that

$$\int_Z\int_{\mathbb{R}^d} A_2\,dx\mu(dz) \le N|v|^p_{W^m_p},$$

and

$$\int_Z\int_{\mathbb{R}^d} p|\mathbf{v}|^{p-2}v_\alpha(\bar{c}^{\alpha\beta} - c^{\alpha\beta})T^\eta v_\beta\,dx\,\mu(dz) \le N|v|^p_{W^m_p}.$$

Thus, from the above estimates we have

$$\hat{Q}_p(v,t,h) \le N(|v|^p_{W^m_p} + |h|^p_{W^m_p(\mathcal{L}_{p,2})}) + N\int_Z\int_{\mathbb{R}^d} T^\eta|\mathbf{v}|^{p-2}|\mathbf{h}|^2\,dx\,\mu(dz)$$
$$+p\int_Z\int_{\mathbb{R}^d} I^\eta(|\mathbf{v}|^{p-2}v_\alpha)h_\alpha\,dx\,\mu(dz).$$

It remains to estimate

$$H_p(v,h) := \int_Z\Big|\int_{\mathbb{R}^d} I^\eta(|\mathbf{v}|^{p-2}v_\alpha)h_\alpha\,dx\Big|\,\mu(dz),$$

$$K_p(v,h) := \int_Z \int_{\mathbb{R}^d} T^\eta |\mathbf{v}|^{p-2}|\mathbf{h}|^2 \, dx\mu(dz).$$

by the right-hand side of (5.6). If $p = 2$, then we have

$$K_2(v,h) = |\mathbf{h}|^2_{L_2(\mathcal{L}_2)}.$$

To estimate $H_2(v,h)$ notice that by Taylor's formula, the change of variable $y = \tau_{\theta\eta}(x) = x + \theta\eta(x)$ and by integration by parts we have

$$\int_{\mathbb{R}^d} I^\eta v_\alpha h_\alpha \, dx = \int_{\mathbb{R}^d} \int_0^1 v_{\alpha i}(\tau_{\theta\eta}(x))\eta^i(x) h_\alpha(x) \, d\theta \, dx$$

$$\int_{\mathbb{R}^d} \int_0^1 v_{\alpha i}(x)\eta^i(\tau^{-1}_{\theta\eta}(x)) h_\alpha(\tau^{-1}_{\theta\eta}(x))|\mathrm{det} D\tau^{-1}_{\theta\eta}(x)| \, d\theta \, dx$$

$$= \int_{\mathbb{R}^d} \int_0^1 v_\alpha(x)\zeta^{\alpha\beta}(x) h_{\alpha\beta}(\tau^{-1}_{\theta\eta}(x)) \, d\theta \, dx$$

where $\beta = 0, 1, 2, ..., d$ and $\zeta^{\alpha\beta}$ are some functions of $(\omega, t, z, x, \theta)$ such that

$$\sum_{|\alpha| \le m, |\beta| \le 1} |\zeta^{\alpha\beta}|^2 \le N\bar{\eta}^2$$

with a constant $N = N(d, m, K)$. Hence by the Cauchy-Schwarz and Young inequalities we get

$$\int_{\mathbb{R}^d} I^\eta v_\alpha h_\alpha \, dx \le N \int_{\mathbb{R}^d} \bar{\eta}^2 |\mathbf{v}(x)|^2 \, dx + N' \int_{\mathbb{R}^d} \sum_{|\beta| \le m+1} |h_\beta(x,z)|^2 \, dx$$

with $N' = N'(d, m, K)$, which gives the estimate for $H_2(v,h)$ and finishes the proof of (5.6) for $p = 2$. If $p > 2$ then by taking $r = p$, $\varphi = |\mathbf{v}|^{p-2} v_\alpha$ and $h = h_\alpha$ for each α in estimate (3.15) and using Young's inequality, we get

$$H_p(v,h) \le N(|v|^p_{W^m_p} + |h|^p_{W^{m+2}_p(\mathcal{L}_{p,2})}).$$

Similarly, by taking $r = p/2$, $\phi = |\mathbf{v}|^{p-2}$ and $g = |\mathbf{h}|^2$ in estimate (3.16) and using Young's inequality, we have

$$K_p(v,h) \le N(|v|^p_{W^m_p} + |h|^p_{W^{m+2}_p(\mathcal{L}_{p,2})}),$$

which completes the proof of the proposition. ☐

Corollary 5.3. *Let Assumption 2.3 hold with an integer $m \ge 0$. Then for any $p \ge 2$ and $v \in W^{m+2}_p$*

$$Q^\eta_p(t,v) \le N|v|^p_{W^m_p} \tag{5.11}$$

for all $\omega \in \Omega$ and $t \in [0,T]$ with a constant $N = N(d, p, m, K, K_{\bar{\eta}})$.

Proof. Notice that

$$Q^\eta_p(t,v) = \hat{Q}_p(t,v,0) - \mathfrak{Q}_p(t,v),$$

where

$$\mathfrak{Q}_p(t,v) := |\sum_{|\alpha| \le m} (v + \mathbb{I}^\eta v)^2_\alpha|^{p/2} - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2} \sum_{|\alpha| \le m} v_\alpha (\mathbb{I}^\eta v)_\alpha.$$

By Proposition 5.2, we have

$$\hat{Q}_p(t,v,0) \le N|v|^p_{W^m_p}$$

with a constant $N = N(d,p,m,K,K_{\bar\eta})$. Moreover by the convexity of the function $f(x) = |x|^p$, we have $\mathfrak{Q}_p(t,v) \ge 0$, which finishes the poof of this corollary. □

For integers $m \ge 0$ set

$$\mathfrak{P}^2_{m,p}(t,v,g) := \sum_{r=1}^{\infty} (p|\mathbf{v}|^{p-2}v_\alpha, (\mathcal{M}^r v + g^r)_\alpha)^2 \tag{5.12}$$

$$\begin{aligned}\mathfrak{Q}_{m,p}(t,v,h) := \int_Z \int_{\mathbb{R}^d} \{(\sum_{|\alpha|\le m} |(v + \mathbb{I}^\eta v + h)_\alpha|^2)^{p/2} - |\mathbf{v}|^p \\ - p|\mathbf{v}|^{p-2}v_\alpha(\mathbb{I}^\eta v + h)_\alpha\}\, dx\, \mu(dz)\end{aligned} \tag{5.13}$$

for $v \in W^{m+1}_p$, $g \in W^{m+1}_p(l_2)$, $h \in W^{m+1}_p(\mathcal{L}_{p,2})$, $\omega \in \Omega$ and $t \in [0,T]$, where repeated indices α mean summation over all multi-numbers of length m. (Recall that for functions $v \in W^m_p$ we use the notation $\mathbf{v}$ with coordinate $v_\alpha = D_\alpha v$ for $|\alpha| \le m$).

Proposition 5.4. *Let $m \ge 0$ be an integer and $p \in [2,\infty)$. Then the following estimates hold for all $(\omega,t) \in \Omega \times [0,T]$.*

(i) If Assumption 2.1 is satisfied with $m \ge 0$ then

$$\mathfrak{P}^2_{m,p}(t,v,g) \le N(|v|^{2p}_{W^m_p} + |v|^{2p-2}_{W^m_p}|g|^2_{W^m_p}) \tag{5.14}$$

for all $v \in W^{m+1}_p$ and $g \in W^{m+1}_p(l_2)$, with a constant $N = N(d,m,p,K)$.

(ii) If Assumption 2.3 is satisfied then

$$\mathfrak{Q}_{m,p}(t,v,h) \le N(|v|^p_{W^{m+1}_p} + |h|^p_{W^m_{p,2}}) \tag{5.15}$$

for all $v \in W^{m+1}_p$ and $h \in W^m_p(\mathcal{L}_{p,2})$ with $N = N(d,m,p,K,K_{\bar\eta})$.

Proof. Noticing that $p|\mathbf{v}|^{p-2}v_\alpha\sigma^{ir}D_i v_\alpha = \sigma^{ir}D_i|\mathbf{v}|^p$, by integration by parts and by Minkowski's and Hölder's inequalities we can see that

$$\bar{\mathfrak{P}}^2_{m,p}(t,v,g) := \sum_{r=1}^{\infty} (p|\mathbf{v}|^{p-2}v_\alpha, \mathcal{M}^r v_\alpha + g^r_\alpha))^2$$

can be estimated by the right-hand side of (5.14). By Minkowski and Hölder's inequalities it is easy to show that

$$\mathfrak{P}^2_{m,p}(t,v,g) - \bar{\mathfrak{P}}^2_{m,p}(t,v,g)$$

can also be estimated by the right-hand side of (5.14). To prove (ii) let $\mathbf{y}$ denote the vector with coordinates $y_\alpha = (\mathbb{I}^\eta v + h)_\alpha$ for $|\alpha| \le m$. Then the integrand in (5.13) can be written as

$$A := |\mathbf{v} + \mathbf{y}|^p - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2}v_\alpha y_\alpha.$$

By Taylor's formula

$$0 \le A \le N(|\mathbf{v}|^{p-2}|\mathbf{y}|^2 + |\mathbf{y}|^p) \le N'(|\mathbf{v}|^{p-2}|\mathbf{e}|^2 + |\mathbf{v}|^{p-2}|\mathbf{h}|^2 + |\mathbf{h}|^p + |\mathbf{e}|^p)$$

with constants $N = N(d,p,m)$ and $N' = N'(d,p,m)$, where $\mathbf{e}$ denotes the vector with coordinates $e_\alpha := (\mathbb{I}^\eta v)_\alpha$ for $|\alpha| \le m$. By Fubini's theorem and Hölder's inequality

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2}|\mathbf{h}|^2 \, dx\mu(dz) = \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2}|\mathbf{h}(x)|^2_{\mathcal{L}_2} \, dx \le |v|^{p-2}_{W^m_p}|h|^2_{W^m_p(\mathcal{L}_2)}.$$

Using Hölder's inequality, taking into account (5.7) and using Lemma 3.1 we obtain

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{v}|^{p-2}|\mathbf{e}|^2 \, dx\mu(dz) \le \int_Z |\mathbf{v}|^{p-2}_{L_p}|\mathbf{e}|^2_{L_p} \, \mu(dz) \le N^2 K^2_{\bar\eta}|v|^{p-2}_{W^m_p}|v|^2_{W^{m+1}_p}.$$

By Lemma 3.1 and (5.7) we have

$$\int_Z \int_{\mathbb{R}^d} |\mathbf{e}|^p \, dx \, \mu(dz) \le N^p K^{p-2} K^2_{\bar\eta}|v|^p_{W^{m+1}_p}.$$

Combining these inequalities and using Young's inequality we get (5.15). □

6. Proof of the main result

6.1. **Uniqueness of the generalised solution.** Let Assumptions 2.1 through 2.4 hold with $m = 0$. For a fixed $p \in [2,\infty)$ let $u^{(i)} = (u^{(i)}_t)_{t\in[0,T]}$ be generalised solutions to equation (1.1) with initial condition $u^{(i)}_0 = \psi \in L_p$ for $i = 1,2$. Then for $v = u^{(2)} - u^{(1)}$ by Lemma 3.10 on Itô formula we have that almost surely

$$\begin{aligned} y_t := |v_t|^p_{L_p} = &\int_0^t Q_s(v_s) + \bar Q^\xi_s(v_s) + \bar Q^\eta_s(v_s) + (p|v_s|^{p-2}v_s, \mathcal{R}_s v_s) \, ds \\ &+ \int_0^t \int_Z \int_{\mathbb{R}^d} P^\eta_s(z, v_{s-})(x) \, dx \, \pi(dz,ds) + \zeta_1(t) + \zeta_2(t) \end{aligned} \tag{6.1}$$

for all $t \in [0,T]$, where ζ_1 and ζ_2 are local martingales defined by

$$\zeta_1(t) := p\int_0^t \int_{\mathbb{R}^d} |v_s|^{p-2}v_s \mathcal{M}^r_s v_s \, dx \, dw^r_s,$$

$$\zeta_2(t) := p\int_0^t \int_Z \int_{\mathbb{R}^d} |v_{s-}|^{p-2}v_{s-}\mathbb{I}^\eta v_{s-} \, dx \, \tilde\pi(dz,ds),$$

$Q_s(\cdot)$, $\bar Q^\xi_s(\cdot)$, $\bar Q^\eta_s(\cdot)$ and $P^\eta_s(z,\cdot)$ are functionals on W^1_p, for each (ω,s) and z, defined by

$$Q_s(v) := p\int_{\mathbb{R}^d} -D_i(|v|^{p-2}v)a^{ij}_s D_j v + \bar b^i_s |v|^{p-2}vD_i v + c_s|v|^p + \tfrac{p-1}{2}|v|^{p-2}\sum_{r=1}^\infty |\mathcal{M}^r_s v|^2 \, dx,$$

$$\bar Q^\xi_s(v) = p\int_{\mathbb{R}^d} -D_i(|v|^{p-2}v)\mathcal{J}^i_\xi v + |v|^{p-2}v\mathcal{J}^0_\xi v \, dx + \int_Z\int_{\mathbb{R}^d} p|v|^{p-2}v\lambda^\xi_{s,z} I^\xi v \, dx \, dz,$$

$$\bar Q^\eta_s(v) = p\int_{\mathbb{R}^d} -D_i(|v|^{p-2}v)\mathcal{J}^i_\eta v + |v|^{p-2}v\mathcal{J}^0_\eta v \, dx + \int_Z\int_{\mathbb{R}^d} p|v|^{p-2}v\lambda^\eta_{s,z} I^\eta v \, dx \, dz,$$

$$P^\eta_s(z,v) := |v + \mathbb{I}^\eta v|^p - |v|^p - p|v|^{p-2}v\mathbb{I}^\eta v.$$

Recall that $\bar b^i = b - D_j a^{ij}$, $\mathcal{J}^i_\eta$ and $\mathcal{J}^0_\eta$ are defined by (2.3)-(2.4), and (v,w) denotes the Lebesgue integral over $\mathbb{R}^d$ of the product vw for real functions v and w on $\mathbb{R}^d$.

Note that due to the convexity of the function $|r|^p$, $r \in \mathbb{R}$, we have

$$P^{\eta}_s(z,v)(x) \geq 0 \quad \text{for all } (\omega,s,z,x) \tag{6.2}$$

for real-valued functions $v = v(x)$, $x \in \mathbb{R}^d$. Together with the above functionals we need also estimate the functionals $\mathfrak{Q}_s(\cdot)$ and $\hat{Q}^{\eta}_s(\cdot)$ defined for each $(\omega,s) \in \Omega \times [0,T]$ by

$$\mathfrak{Q}_s(v) := \int_Z \int_{\mathbb{R}^d} P^{\eta}_s(z,v)(x)\,dx\,\mu(dz), \quad \hat{Q}^{\eta}_s(v) := \mathfrak{Q}_s(v) + \bar{Q}^{\eta}_s(v)$$

for $v \in W^1_p$.

Proposition 6.1. *Let Assumptions 2.1, 2.2 and 2.3 hold with $m = 0$. Then for $p \geq 2$ there are constants $N = N(d,p,K)$, $N_1 = N_2(d,p,K,K_{\bar{\xi}})$ and $N_2 = N_2(d,p,K,K_{\bar{\eta}})$ such that*

$$Q_s(v) \leq N|v|^p_{L_p}, \quad \bar{Q}^{\xi}(v) \leq N_1|v|^p_{L_p}, \quad \bar{Q}^{\eta}(v) \leq N_2|v|^p_{L_p}, \quad \hat{Q}^{\eta}(v) \leq N_2|v|^p_{L_p}, \tag{6.3}$$

$$\mathfrak{Q}(v) \leq N_2|v|^p_{W^1_p} \tag{6.4}$$

for all $v \in W^1_p$ and $(\omega,s) \in \Omega \times [0,T]$.

Proof. Notice that the estimate (6.4) is the special case of Proposition 5.4 (ii), and for $v \in W^2_p$ the second and third estimates in (6.3) follow from the estimate (5.11) in Corollary 5.3. Notice also that for $v \in W^2_p$ the first estimate in (6.3) is a special case of (5.5) in Proposition 5.1, and the last estimate in (6.3) is a special case of the estimate in Proposition 5.2. It is an easy exercise to show that the functionals on the left-hand side of the inequalities in (6.3) are continuous in $v \in W^1_p$, that completes the proof of the proposition. ☐

Define now the stochastic process

$$X_t := |v_t|^p_{L_p} + \int_0^t |v_s|^p_{W^1_p}\,ds, \quad t \in [0,T]$$

and the stopping time

$$\tau_n := \inf\{t \in [0,T] : X_t \geq n\} \wedge \rho_n$$

for every integer $n \geq 1$, where $(\rho_n)_{n=1}^{\infty}$ is an increasing sequence of stopping times, converging to infinity such that $(\zeta_i(t \wedge \rho_n))_{t \in [0,T]}$ is a martingale for each $n \geq 1$ and $i = 1,2$. Then clearly, $E\zeta_i(t \wedge \tau_n) = 0$ for $t \in [0,T]$ and $i = 1,2$. Due to (6.2) and the estimate in (6.4) we have

$$E\int_0^{T\wedge\tau_n} \int_Z \int_{\mathbb{R}^d} |P^{\eta}_s(z,v_{s-})(x)|\,dx\,\mu(dz)\,ds \leq NE\int_0^{T\wedge\tau_n} |v_s|^p_{W^1_p}\,ds < \infty,$$

which implies

$$E\int_0^{t\wedge\tau_n} \int_Z \int_{\mathbb{R}^d} P^{\eta}_s(z,v_{s-})(x)\,dx\,\pi(dz,ds)$$
$$= E\int_0^{t\wedge\tau_n} \int_Z \int_{\mathbb{R}^d} P^{\eta}_s(z,v_{s-})(x)\,dx\,\mu(dz)\,ds = E\int_0^{t\wedge\tau_n} \mathfrak{Q}_s(v_{s-})\,ds.$$

Thus, substituting $t \wedge \tau_n$ in place of t in (6.1) and then taking expectation and using Proposition 6.1 and Assumption 2.5 we obtain

$$Ey_{t\wedge\tau_n} = E\int_0^{t\wedge\tau_n} Q_s(v_s) + \bar{Q}^{\xi}_s(v_s) + \hat{Q}^{\eta}_s(v_s) + (p|v_s|^{p-2}v_s, \mathcal{R}v_s)\,ds$$

$$\le NE\int_0^{t\wedge\tau_n}|v_s|^p_{L_p}\,ds \le N\int_0^t Ey(s\wedge\tau_n)\,ds \le NTn < \infty$$

for $t\in[0,T]$. Hence by Gronwall's lemma $Ey(t\wedge\tau_n)=0$ for each $t\in[0,T]$ and integer $n\ge 1$, which implies that almost surely $y_t=0$ for all $t\in[0,T]$, and completes the proof of the uniqueness.

6.2. **A priori estimates.**

Proposition 6.2. *Let Assumptions 2.1 through 2.4 and assume Assumption 2.6 hold with an integer $m\ge 0$ and $\mathcal{R}\equiv 0$. Let $u=(u_t)_{t\in[0,T]}$ be a W^{m+2}_p-valued generalised solution to* (1.1)-(1.2) *such that it is cadlag as a W^m_p-valued process and*

$$E\int_0^T |u_t|^p_{W^{m+2}_p}\,dt + E\sup_{t\le T}|u_t|^p_{W^m_p}<\infty.$$

Then

$$E\sup_{t\le T}|u_t|^p_{W^n_p}\le N(E|\psi|^p_{W^n_p}+E\mathcal{K}^p_{n,p}(T))\quad \text{for every integer } n\in[0,m] \tag{6.5}$$

with a constant $N=N(m,d,p,T,K,K_{\bar\xi},K_{\bar\eta})$.

Proof. We may assume that the right-hand side of the inequality (6.5) is finite. For a fixed integer $n\in[0,m]$, multi-numbers $|\alpha|\le n$ and $\varphi\in C_0^\infty$, we have

$$\begin{aligned} d(D_\alpha u_t,\varphi) =&(D_\alpha\mathcal{A}_t u_t + D_\alpha f_t,\varphi)\,dt + (D_\alpha\mathcal{M}^r_t u_t + D_\alpha g^r_t,\varphi)\,dw^r_t\\ &+\int_Z (D_\alpha(\mathbb{I}^\eta u_{t-}+h_t(z)),\varphi)\,\tilde\pi(dz,dt). \end{aligned}$$

For an integer $n\le m$ let $\mathbf{u}$ denote the vector with coordinates $u_\alpha := D_\alpha u$ for $|\alpha|\le n$. Recall, see (5.1), that by Lemma 3.9 on Itô's formula we have

$$d|\mathbf{u}_t|^p_{L_p} = (Q_p(t,u_t,f_t,g_t)+Q^\xi_p(t,u_t)+\hat Q_p(t,u_t,h_t))\,dt + \sum_{i=1}^3 d\zeta_i(t),$$

where the Q_p, Q^ξ_p and $\hat Q_{n,p}$ are defined in (5.2), (5.3) and (5.4), with n in place of m, and $\zeta_i=(\zeta_i(t))_{t\in[0,T]}$ is a cadlag local martingale starting from zero for each $i=1,2,3$, such that

$$d\zeta_1(t) = p(|\mathbf{u}_t|^{p-2}D_\alpha u_t, D_\alpha\mathcal{M}^r_t u_t + D_\alpha g^r_t)\,dw^r_t,$$

$$d\zeta_2(t) := p\int_Z (|\mathbf{u}_{t-}|^{p-2}D_\alpha u_{t-}, D_\alpha\mathbb{I}^\eta u_{t-}+D_\alpha h_{t,z})\,\tilde\pi(dz,dt) \tag{6.6}$$

and

$$d\zeta_3(t) := \int_Z P_p(t,u_{t-},h_t)\,\pi(dz,dt) - \int_Z P_p(t,u_{t-},h_t)\,\mu(dz)dt, \tag{6.7}$$

where

$$P_p(t,v,h) := \int_{\mathbb{R}^d} |\sum_{|\alpha|\le n}|(v+\mathbb{I}^\eta v+h)_\alpha|^2|^{p/2} - |\mathbf{v}|^p - p|\mathbf{v}|^{p-2}v_\alpha(\mathbb{I}^\eta v+h)_\alpha\,dx$$

for $v \in W_p^{m+2}$ and $h \in W_{p,2}^{m+2}$. By Proposition 5.1 and Corollary 5.3 we obtain

$$d|\mathbf{u}_t|^p \le N(|u_t|^p_{W^n_p}\,dt + d\mathcal{K}^p_{n,p}(t)) + \sum_{i=1}^{3} d\zeta_i(t).$$

Hence using the estimate (5.15) in Proposition 5.4 we have

$$E|\mathbf{u}_{t\wedge\tau_k}|^p \le E|\mathbf{u}_0|^p + N\int_0^t E|u_{s\wedge\tau_k}|^p_{W^n_p}\,ds + NE\mathcal{K}^p_{n,p}(T\wedge\tau_k)$$

for all $t \in [0,T]$, for a localising sequence $(\tau_k)_{k=1}^{\infty}$ of stopping times for ζ_i, $i = 1,2,3$. Hence by Gronwall's lemma

$$E|u_{t\wedge\tau_k}|^p_{W^n_p} \le N(E|\psi|^p_{W^n_p} + E\mathcal{K}^p_{n,p}(T))$$

for $t \in [0,T]$ and $k \ge 1$ with a constant $N = N(d,m,p,T,K,K_{\bar\xi},K_{\bar\eta})$, which implies

$$\sup_{t\le T} E|u_t|^p_{W^n_p} \le N(E|\psi|^p_{W^n_p} + E\mathcal{K}^p_{n,p}(T)) \tag{6.8}$$

by Fatou's lemma. To show that we can interchange the supremum and expectation it suffices to prove that for every $\varepsilon > 0$

$$E\sup_{t\le T}|\zeta_1(t)| \le \varepsilon E\sup_{t\le T}|u_t|^p_{W^n_p} + N(E|\psi|^p_{W^n_p} + E\mathcal{K}^p_{n,p}(T)) < \infty \tag{6.9}$$

and

$$E\sup_{t\le T}|\zeta_2(t)+\zeta_3(t)| \le \varepsilon E\sup_{t\le T}|u_t|^p_{W^n_p} + N(E|\psi|^p_{W^n_p} + E\mathcal{K}^p_{n,p}(T)) < \infty \tag{6.10}$$

with a constant $N = N(\varepsilon,d,m,p,T,K,K_{\bar\xi},K_{\bar\eta})$. The proof of (6.9) is well-known and it goes as follows. Recall the notation $\mathfrak{P}_p^2$ in (5.12) and notice that by the Davis inequality, using the estimate in (5.14) we have

$$\begin{aligned}
E\sup_{t\le T}|\zeta_1(t)| &\le 3E\left(\int_0^T \mathfrak{P}^2_p(t,u_t,g_t)\,dt\right)^{1/2} \\
&\le NE\left(\int_0^T |u_t|^{2p}_{W^n_p} + |u_t|^{2p-2}_{W^n_p}|g_t|^2_{W^n_p}\,dt\right)^{1/2} \\
&\le NE\left(\sup_{t\le T}|u_t|^p_{W^n_p}\int_0^T |u_t|^p_{W^n_p} + |u_t|^{p-2}_{W^n_p}|g_t|^2_{W^n_p}\,dt\right)^{1/2} \\
&\le \varepsilon E\sup_{t\le T}|u_t|^p_{W^n_p} + \varepsilon^{-1}N^2E\int_0^T |u_t|^p_{W^n_p} + |g_t|^p_{W^n_p}\,dt < \infty,
\end{aligned} \tag{6.11}$$

which gives (6.9) by virtue of (6.8). To prove (6.10) we first assume that μ is a finite measure. Notice that

$$\zeta(t) := \zeta_2(t) + \zeta_3(t) = \int_0^t\int_Z A(s,u_s,h_s)\,\tilde\pi(dz)ds,$$

where $A(s,v,h)$ is defined by

$$A(s,v,h) := |\sum_{|\alpha|\le n}(v + \mathbb{I}^\eta v + h)^2_\alpha|^{p/2} - |\mathbf{v}|^p$$

for $v \in W_p^m$, $\mathbf{v} = (v_\alpha)_{|\alpha| \leq n}$ and $h \in W_p^{m+2}(\mathcal{L}_{p,2})$. By similar calculations to those in the proof of Proposition 5.2 we can easily see that

$$A(s,v,h) = \sum_{i=1}^{6} B_i(s,v,h)$$

with

$$B_1 = I^\eta |\mathbf{v}|^p, \quad B_2 = p|\mathbf{v}|^{p-2} v_\alpha h_\alpha,$$
$$B_3 := |T^\eta \mathbf{v} + (\mathbf{b}+\mathbf{c})T^\eta \mathbf{v} + \mathbf{h}|^p - |T^\eta \mathbf{v}|^p - p|T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha ((b^{\alpha\beta} + c^{\alpha\beta})T^\eta v_\beta + h_\alpha),$$
$$B_4 = pI^\eta(|\mathbf{v}|^{p-2} v_\alpha) h_\alpha$$
$$B_5 = p|T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha b^{\alpha\beta} T v_\beta, \quad B_6 = p|T^\eta \mathbf{v}|^{p-2} T^\eta v_\alpha c^{\alpha\beta} T^\eta v_\beta,$$

where $\mathbf{b} = (b^{\alpha\beta})$ and $\mathbf{c} = (c^{\alpha\beta})$ are from (5.7). Hence

$$\zeta(t) = \sum_{i=1}^{6} (\rho_{i1}(t) - \rho_{i2}(t))$$

with

$$\rho_{i1}(t) = \int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h)\, \pi(dz, ds),$$
$$\rho_{i2}(t) = \int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h)\, dx \mu(dz) ds.$$

Note that for $\rho_i(t) := \rho_{i1}(t) - \rho_{i2}(t)$ one can always have the supremum estimate

$$E \sup_{t \leq T} |\rho_i(t)| \leq E \sup_{t \leq T} |\rho_{i1}(t)| + E \sup_{t \leq T} |\rho_{i2}(t)|$$
$$\leq 2E \int_0^T \int_Z |\int_{\mathbb{R}^d} B_i(t, u_s, h)\, dx| \mu(dz)\, dt. \tag{6.12}$$

This, however, is not always useful, and when almost surely

$$\langle \rho_i \rangle (T) = \int_0^T \int_Z \Big| \int_{\mathbb{R}^d} B_i(s, u_{s-}, h)\, dx \Big|^2 \mu(dz) ds < \infty,$$

then we can view $\rho_i(t)$ as the stochastic Itô integral

$$\int_0^t \int_Z \int_{\mathbb{R}^d} B_i(s, u_{s-}, h) \tilde{\pi}(dz, ds)$$

and apply the Davis inequality

$$E \sup_{t \leq T} |\rho_i(t)| \leq 3E \langle \rho_i \rangle^{1/2}(T).$$

By Minkowski's and Hölder's inequalities

$$\int_0^T \int_Z \Big| \int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-2} D_\alpha u_s D_\alpha h_s\, dx \Big|^2 \mu(dz)\, ds \leq \int_0^T \Big(\int_{\mathbb{R}^d} |\mathbf{u}_s|^{p-1} |\mathbf{h}_s|_{\mathcal{L}_2}\, dx \Big)^2 ds$$
$$\leq \int_0^T |u_s|^{2p-2}_{W_p^n} |h_s|^2_{W_{p,2}^n}\, ds < \infty.$$

Thus we can view $\zeta_2(t)$ as a stochastic Itô integral, and applying the Davis inequality we get

$$E\sup_{t\le T}|\zeta_i(t)|\le \varepsilon E\sup_{t\le T}|u_t|^p_{W^n_p}+\varepsilon^{-1}N(E|\psi|^p_{W^n_p}+E\mathcal{K}^p_{n,p}(T)) \tag{6.13}$$

for $i=2$ in the same way as estimate in (6.9) is proved. By Lemma 3.7 (iii) we have

$$\int_0^T\int_Z\Big|\int_{\mathbb{R}^d}I^\eta|\mathbf{u}_s|^p\,dx\Big|^2\mu(dz)\,ds\le N\int_0^T|u_s|^{2p}_{W^n_p}\,ds,$$

which, as before, allows us to get the estimate (6.13) for $i=1$. By estimate (3.15), Young's inequality and (6.8), we have

$$E\int_0^T\int_Z\Big|\int_{\mathbb{R}^d}I^\eta(|\mathbf{u}_{s-}|^{p-2}D_\alpha u_{s-})D_\alpha h_s\,dx\Big|\mu(dz)\,ds$$

$$\le NE\int_0^T|u_s|^p_{W^n_p}\,ds+NE\mathcal{K}^p_{n,p}(T)\le N'(E|\psi|^p_{W^n_p}+E\mathcal{K}^p_{n,p}(T))$$

with constants N and N' depending only on K, d, m, p, T, $K_{\bar\xi}$ and $K_{\bar\eta}$. Thus we can use estimate (6.12) to get

$$E\sup_{t\le T}|\zeta_i(t)|\le NE|\psi|^p_{W^n_p}+NE\mathcal{K}^p_{n,p}(T) \tag{6.14}$$

for $i=4$ with a constant $N=N(K,d,m,p,T,K_{\bar\xi},K_{\bar\eta})$. Similarly, using the estimate for A_3 in (5.10) and the estimate (3.16) we obtain (6.14) for $i=3$. Due to (5.9)

$$\Big|\int_{\mathbb{R}^d}B_5(t,v,h)\,dx\Big|\le N\bar\eta|v|^p_{W^n_p},\quad \Big|\int_{\mathbb{R}^d}B_6(t,v,h)\,dx\Big|\le N\bar\eta^2|v|^p_{W^n_p}$$

for $v\in W^m_p$. Consequently, viewing $\zeta_5(t)$ as Itô integral we get the estimate (6.13) for i=5, and applying estimate (6.12) to $\zeta_6(t)$ we obtain the estimate (6.14) for $i=6$. Combining the estimates (6.13) for $i=1,2,5$ and the estimate (6.13) for $i=3,4,6$ we obtain (6.11).

In the general case of σ-finite measure μ we have a nested sequence $(Z_k)_{k=1}^\infty$ of sets $Z_k\in\mathcal{Z}$ such that $\mu(Z_k)<\infty$ for every k and $\cup_{k=1}^\infty Z_k=Z$. For each integer $k\ge 1$ define the measures

$$\pi_k(F)=\pi((Z_k\times(0,T])\cap F),\quad \mu_k(G)=\mu(Z_k\cap G)$$

for $F\in\mathcal{Z}\otimes\mathcal{B}((0,T])$ and $G\in\mathcal{Z}$, and set $\tilde\pi_k(dz,dt)=\pi_k(dz,dt)-\mu_k\otimes dt$. Let $\zeta_2^{(k)}$ and $\zeta_3^{(k)}$ be defined as ζ_2 and ζ_3, respectively, but with $\tilde\pi_k$, π_k and μ_k in place of $\tilde\pi$, π and μ, respectively, in (6.6) and (6.7). By virtue of what we have proved above, for each k we have

$$E\sup_{t\le T}|\zeta_2^{(k)}(t)+\zeta_3^{(k)}(t)|\le\varepsilon E\sup_{t\le T}|u_t|^p_{W^n_p}+N(E|\psi|^p_{W^n_p}+E\mathcal{K}^p_{n,p}(T))<\infty \tag{6.15}$$

for $\varepsilon>0$ with a constant $N=N(\varepsilon,m,p,T,K,K_{\bar\xi},K_{\bar\eta})$. Note that for a subsequence $k'\to\infty$

$$\zeta_i^{(k')}(t)\to\zeta_i(t)\quad\text{almost surely, uniformly in } t\in[0,T]$$

for $i=2,3$. Hence letting $k=k'\to\infty$ in (6.15) by Fatou's lemma we obtain (6.10), which completes the proof of the proposition. □

6.3. **Existence of a generalised solution.** Before the construction of a generalised solution to (1.1)-(1.2), we introduce some notations. For integers $r > 1$, real numbers $n \geq 0$ and $p \geq 2$ let $\mathbb{U}^n_{r,p}$ denote the space of H^n_p-valued $\mathcal{F} \otimes \mathcal{B}([0,T])$-measurable functions v on $\Omega \times [0,T]$ such that

$$|v|^p_{\mathbb{U}^n_{r,p}} := E\left(\int_0^T |v_t|^r_{H^n_p}\,dt\right)^{p/r} < \infty.$$

The subspace of well-measurable functions $v : \Omega \times [0,T] \to H^n_p$ in $\mathbb{U}^n_{r,p}$ is denoted by $\mathbb{V}^n_{r,p}$, and we will use $\mathbb{V}^n_p$ to denote $\mathbb{V}^n_{p,p}$. Set $\Psi^n_p := L_p(\Omega, \mathcal{F}_0; H^n_p)$, and recall from the Introduction the definition of the spaces $\mathbb{H}^n_p = \mathbb{H}^n_p(\mathbb{R})$ and $\mathbb{H}^n_p(V)$ for separable Banach spaces V.

In the whole section we assume that Assumptions 2.1 through 2.6 with $m \geq 1$ are in force. By a standard stopping time argument we may assume that

$$E|\psi|^p_{H^m_p} + E\mathcal{K}^p_{p,m}(T) < \infty. \tag{6.16}$$

First we assume that m is an integer, $\mathcal{R} \equiv 0$, and make also the following additional assumption

Assumption 6.1. The initial condition ψ and the free data f, g and h vanish if $|x| \geq R$ for some $R > 0$.

Under the above conditions we approximate the Cauchy problem (1.1)-(1.2) by mollifying (in $x \in \mathbb{R}^d$) all data and coefficients involved in it. For $\varepsilon \in (0, \varepsilon_0)$ we consider the equation

$$\begin{aligned} dv_t(x) &= \left(\mathcal{A}^\varepsilon_t v_t(x) + f^{(\varepsilon)}_t(x)\right)\,dt + \left(\mathcal{M}^{\varepsilon r}_t v_t(x) + g^{(\varepsilon)r}_t(x)\right)\,dw^r_t \\ &+ \int_Z \left(I^{\eta^{(\varepsilon)}} v_{t-}(x) + \gamma^{(\varepsilon)}_{t,z}(x) T^{\eta^{(\varepsilon)}} v_{t-}(x) + h^{(\varepsilon)}_t(x,z)\right)\,\tilde{\pi}(dz,dt), \end{aligned} \tag{6.17}$$

with initial condition

$$v_0(x) = \psi^{(\varepsilon)}, \tag{6.18}$$

where ε_0 is given in Corollary 3.4,

$$\mathcal{M}^{\varepsilon r} = \sigma^{(\varepsilon)ir} D_i + \beta^{(\varepsilon)r}, \quad \mathcal{A}^\varepsilon = \mathcal{L}^\varepsilon + \mathcal{N}^{\xi^{(\varepsilon)}} + \mathcal{N}^{\eta^{(\varepsilon)}}$$

with operators

$$\mathcal{L}^\varepsilon = a^{\varepsilon ij} D_{ij} + b^{(\varepsilon)i} D_i + c^{(\varepsilon)}, \quad a^\varepsilon = a^{(\varepsilon)} + \varepsilon \mathbf{I},$$

and

$$\begin{aligned} \mathcal{N}^{\xi^{(\varepsilon)}}_t \varphi(x) &= \int_Z J^{\xi^{(\varepsilon)}} \varphi(x) + (\lambda^\xi_{t,z})^{(\varepsilon)}(x) I^{\xi^{(\varepsilon)}} \varphi(x)\,\nu(dz), \\ \mathcal{N}^{\eta^{(\varepsilon)}}_t \varphi(x) &= \int_Z J^{\eta^{(\varepsilon)}} \varphi(x) + (\lambda^\eta_{t,z})^{(\varepsilon)}(x) I^{\eta^{(\varepsilon)}} \varphi(x)\,\mu(dz) \end{aligned}$$

for real-valued differentiable functions φ. Recall that $\mathbf{I}$ denotes the identity matrix, and $v^{(\varepsilon)}$ denotes the mollification $v^{(\varepsilon)} = S^\varepsilon v$ of v in $x \in \mathbb{R}^d$ defined in (3.4).

Note that by virtue of standard properties of mollifications and by Corollary 3.4 and Lemma 3.5, we have that Assumptions 2.1 through 2.6 are satisfied for (6.17)-(6.18) with every integer $m \geq 0$ with non-negative functions $\bar{\xi} = \bar{\xi}_m(z)$, $\bar{\eta} = \bar{\eta}_m(z)$ of $z \in Z$ and constants $K = K_m$,

$$K^2_{\bar{\xi}} := K^2_{\bar{\xi},m} = \int_Z \bar{\xi}^2_m(z)\,\nu(dz) < \infty, \quad K^2_{\bar{\eta}} := K^2_{\bar{\eta},m} = \int_Z \bar{\eta}^2_m(z)\,\mu(dz) < \infty.$$

Moreover, there is a constant $\delta > 0$ such that $P \otimes dt \otimes dx$-almost all $(\omega, t, x) \in \Omega \times H_T$

$$(2a^{\varepsilon ij} - \sigma^{(\varepsilon)ir}\sigma^{(\varepsilon)jr})z^i z^j \geq \delta|z|^2 \quad \text{for all} \quad z = (z^1, ..., z^d) \in \mathbb{R}^d.$$

Due to (6.16) and Assumption 6.1

$$E|\psi|^2_{W^n_2} + E\mathcal{K}^2_{n,2}(T) < \infty$$

for each n. Hence by [15] the Cauchy problem (6.17)-(6.18) has a unique generalised solution u^ε, which is a W^n_2-valued cadlag process and for each integer $n \geq 0$, and there is a constant N such that

$$E\sup_{t\leq T}|u^\varepsilon_t|^2_{W^n_2} \leq N(E|\psi|^2_{W^n_2} + E\mathcal{K}^2_{n,2}(T)) < \infty.$$

Thus by Sobolev's embedding u^ε is a cadlag W^n_p-valued process for every n such that

$$E\sup_{t\leq T}|u^\varepsilon_t|^p_{W^n_p} < \infty.$$

Moreover, by Proposition 6.2 and Lemma 3.5 for $m \geq 1$ and $n = 0, 1, ..., m$ we have

$$|u^\varepsilon|_{\mathbb{V}^n_{r,p}} \leq N(|\psi|_{\Psi^n_p} + |f|_{\mathbb{H}^n_p} + |g|_{\mathbb{H}^{n+1}_p(l_2)} + |h|_{\mathbb{H}^{n+i}_p(\mathcal{L}_{p,2})}) \quad \text{for } n = 0, 1, 2, ..., m \tag{6.19}$$

for every integer $r > 1$ with a constant $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$, where $i = 1$ when $p = 2$ and $i = 2$ when $p > 2$. Recall that $\mathbb{V}^n_{r,p}$ denotes the subspace of well-measurable functions $v : \Omega \times [0, T] \to H^n_p$ in $\mathbb{U}^n_{r,p}$. Since $\mathbb{V}^n_{r,p}$ is reflexive, there exists a sequence $\{\varepsilon_k\}^\infty_{k=1}$ and a process $u \in \mathbb{V}^n_{r,p}$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and u^{ε_k} converges weakly to some u in $\mathbb{V}^n_{r,p}$. To show that a modification of u is a solution to (1.1)-(1.2) we pass to the limit in the equation

$$\begin{aligned}(u^\varepsilon_t, \varphi) = (\psi^{(\varepsilon)}, \varphi) + \int_0^t \langle \mathcal{A}^\varepsilon_s u^\varepsilon_s, \varphi\rangle + (f^{(\varepsilon)}_s, \varphi)\, ds + \int_0^t (\mathcal{M}^{\varepsilon r}_s u^\varepsilon_s + g^{(\varepsilon)r}_s, \varphi)\, dw^r_s \\ + \int_0^t \int_Z (I^{\eta^{(\varepsilon)}} u^\varepsilon_s + \gamma^{(\varepsilon)}_{s,z} T^{\eta^{(\varepsilon)}} u^\varepsilon_s + h^{(\varepsilon)}_s, \varphi)\, \tilde{\pi}(dz, ds)\end{aligned} \tag{6.20}$$

where $\varphi \in C^\infty_0$, and $\langle \mathcal{A}^\varepsilon_s u^\varepsilon_s, \varphi\rangle$ is defined as $\langle \mathcal{A}_s u_s, \varphi\rangle$ in (2.5) but with u^ε, a^ε, $b^{(\varepsilon)}$, $c^{(\varepsilon)}$, $\xi^{(\varepsilon)}$, $\eta^{(\varepsilon)}$, $(\lambda^\xi)^{(\varepsilon)}$ and $(\lambda^\eta)^{(\varepsilon)}$ in place of u, a, b, c, ξ, η, λ^ξ and λ^η respectively. To this end we take a bounded well-measurable real-valued process $\zeta = (\zeta_t)_{t\in[0,T]}$, multiply both sides of equation (6.20) with ζ_t and then integrate the expressions we get against $P \otimes dt$ over $\Omega \times [0, T]$. Thus we obtain

$$\begin{aligned}F(u^\varepsilon) =& E\int_0^T \zeta_t(\psi^{(\varepsilon)}, \varphi)\, dt + \sum_{i=1}^3 F^i_\varepsilon(u^\varepsilon) + E\int_0^T \int_0^t \zeta_t(f^{(\varepsilon)}_s, \varphi)\, ds\, dt \\ &+ E\int_0^T \zeta_t \int_0^t (g^{(\varepsilon)r}_s, \varphi)\, dw^r_s\, dt + E\int_0^T \zeta_t \int_0^t \int_Z (h^{(\varepsilon)}_s, \varphi)\, \tilde{\pi}(dz, ds)\, dt,\end{aligned} \tag{6.21}$$

where F and F^i_ε, $i = 1, 2, 3$, are linear functionals of $v \in \mathbb{V}^1_p$, defined by

$$F(v) = E\int_0^T \zeta_t(v_t, \varphi)\, dt, \quad F^1_\varepsilon = E\int_0^T \zeta_t \int_0^t \langle \mathcal{A}^\varepsilon_s v_s, \varphi\rangle\, ds\, dt$$

$$F^2_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t (\mathcal{M}^{\varepsilon r}_s v_s, \varphi)\, dw^r_s\, dt$$

and

$$F^3_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t \int_Z (I^{\eta^{(\varepsilon)}} v_s + \gamma^{(\varepsilon)}_{s,z} T^{\eta^{(\varepsilon)}} v_s, \varphi)\,\tilde{\pi}(dz, ds)\,dt.$$

For each $i = 1, 2, 3$ we also define the functional F^i in the same way as F^i_ε is defined above, but with $\mathcal{A}$, $\mathcal{M}$, η and γ in place of $\mathcal{A}^\varepsilon$, $\mathcal{M}^\varepsilon$, $\eta^{(\varepsilon)}$ and $\gamma^{(\varepsilon)}$ respectively. Obviously, by Hölder's inequality and the boundedness of ζ, for all $v \in \mathbb{V}^1_p$ we have

$$F(v) \le C|v|_{\mathbb{V}^1_p} |\varphi|_{L_q}$$

with $q = p/(p-1)$ and a constant C independent of v and ε, which means $F \in \mathbb{V}^{1*}_p$, the space of all bounded linear functionals on $\mathbb{V}^1_p$. Next we show that F^i_ε and F^i are also in $\mathbb{V}^{1*}_p$ for each $\varepsilon > 0$, and $F^i_\varepsilon \to F^i$ strongly in $\mathbb{V}^{1*}_p$ as $\varepsilon \to 0$ for $i = 1, 2, 3$.

Lemma 6.3. *For sufficiently small $\varepsilon > 0$ the functionals F^i and F^i_ε are in $\mathbb{V}^{1*}_p$ for $i = 1, 2, 3$.*

Proof. To show $F^1_\varepsilon \in \mathbb{V}^{1*}_p$ we notice that $F^1_\varepsilon = \sum_{k=1}^7 R^k_\varepsilon(v)$ with

$$R^1_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t -(a^{\varepsilon ij} D_j v_s, D_i\varphi) + (\bar{b}^{i(\varepsilon)} D_i v_s + c^{(\varepsilon)} v_s, \varphi)\,ds\,dt,$$

$$R^2_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t -(\mathcal{J}^i_{\xi^{(\varepsilon)}} v_s, D_i\varphi)\,ds\,dt, \quad R^3_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t (\mathcal{J}^0_{\xi^{(\varepsilon)}} v_s, \varphi)\,ds\,dt,$$

$$R^4_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t -(\mathcal{J}^i_{\eta^{(\varepsilon)}} v_s, D_i\varphi)\,ds\,dt, \quad R^5_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t (\mathcal{J}^0_{\eta^{(\varepsilon)}} v_s, \varphi)\,ds\,dt,$$

$$R^6_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t \int_Z ((\lambda^{\xi_{s,z}})^{(\varepsilon)} I^{\xi^{(\varepsilon)}} v_s, \varphi)\,\nu(dz)\,ds\,dt$$

$$R^7_\varepsilon(v) = E\int_0^T \zeta_t \int_0^t \int_Z ((\lambda^\eta_{s,z})^{(\varepsilon)} I^{\eta^{(\varepsilon)}} v_s, \varphi)\,\mu(dz)\,ds\,dt \tag{6.22}$$

where $\mathcal{J}^i_{\xi^{(\varepsilon)}}$, $\mathcal{J}^i_{\eta^{(\varepsilon)}}$, $\mathcal{J}^0_{\xi^{(\varepsilon)}}$, $\mathcal{J}^0_{\eta^{(\varepsilon)}}$ are defined by (2.3) and (2.4) with $\xi^{(\varepsilon)}$ and $\eta^{(\varepsilon)}$ in place of η respectively, for $i = 1, 2, ..., d$. Since the functions ζ, a^ε, $\bar{b}^{(\varepsilon)}$ and $c^{(\varepsilon)}$ are in magnitude bounded by a constant, by Hölder's inequality we have

$$R^i_\varepsilon(v) \le N|v|_{\mathbb{V}^1_p} |\varphi|_{H^1_q}, \tag{6.23}$$

for $i = 1$ with $q = p/(p-1)$ and a constant N independent of v and ε. which shows that $R^1_\varepsilon \in \mathbb{V}^{1*}_p$ for all ε. Using Taylor's formula

$$v(x + \theta\eta^{(\varepsilon)}) - v(x) = \int_0^1 D_i v(x + \vartheta\theta\eta^{(\varepsilon)})\theta\eta^{(\varepsilon)i}\,d\vartheta,$$

and taking into account that $|\zeta|$ is bounded by a constant, we have

$$R^2_\varepsilon(v) \le C\int_0^1\int_0^1\int_0^T\int_Z\int_{\mathbb{R}^d} |Dv(s, x + \vartheta\theta\eta^{(\varepsilon)}_{s,z}(x))|\bar{\eta}^2(z)|D\varphi(x)|\,dx\,\mu(dz)\,ds\,d\theta\,d\vartheta.$$

Hence by Hölder's inequality and then the change of variable $y = x + \vartheta\theta\eta^{(\varepsilon)}_{s,z}(x)$, by Corollary 3.4 we get (6.23) for $i = 2$ and $\varepsilon \in (0, \varepsilon_0)$, with ε_0 given in Corollary 3.4, which proves that $R^2_\varepsilon \in \mathbb{V}^{1*}_p$ for $\varepsilon \in (0, \varepsilon_0)$. We can prove in the same way that $R^i_\varepsilon \in \mathbb{V}^{1*}_p$ for $i = 3, 4, 5$ and

$\varepsilon \in (0,\varepsilon_0)$. Similarly, due to the boundedness of ζ, Assumptions 2.2 and 2.4, using Lemma 3.1 and Hölder's inequality we have (6.23) for $i = 6,7$ for $\varepsilon \in (0,\varepsilon_0)$. Hence $F^1_\varepsilon \in \mathbb{V}^{1*}_p$ for $\varepsilon \in (0,\varepsilon_0)$. Due to the boundedness of ζ, $\sigma^{(\varepsilon)r}$ and $\beta^{(\varepsilon)r}$, by Davis' and Hölder's inequalities, we get

$$F^2_\varepsilon(v) \le CE\Big(\int_0^T \sum_r |(\mathcal{M}^{\varepsilon r}_s,\varphi)|^2\,ds\Big)^{1/2} \le C'|v|_{\mathbb{V}^1_p}|\varphi|_{L_q}$$

for $v \in \mathbb{V}^1_p$ with constants C and C' independent of v and ε, which shows $F^2_\varepsilon \in \mathbb{V}^{1*}_p$. By the boundedness of ζ, using Davis' and Hölder's inequalities, we obtain

$$F^3_\varepsilon(v) \le C\big(A_1(v) + A_2(v)\big)$$

for $v \in \mathbb{V}^1_p$ with a constant C independent of ε and v, where

$$A_1(v) = E(\int_0^T \int_Z |(I^{\eta^{(\varepsilon)}} v_s,\varphi)|^2\,\mu(dz)\,ds)^{1/2},$$

and

$$A_2(v) = E(\int_0^T \int_Z |(\gamma^{(\varepsilon)}_{s,z} T^{\eta^{(\varepsilon)}} v_s,\varphi)|^2\,\mu(dz)\,ds)^{1/2}.$$

Due to Assumptions 2.3 and 2.4 by Lemma 3.1 we have

$$A_i(v) \le CE\Big(\int_0^T \int_Z \bar\eta^2(z)|v_s|^2_{H^1_p}|\varphi|^2_{L_q}\,\mu(dz)\,ds\Big)^{1/2} \le C'|v|_{\mathbb{V}^1_p}|\varphi|_{L_q}$$

for $i = 1,2$ with constants C and C' independent of v and ε. Consequently, $F^3_\varepsilon \in \mathbb{V}^{1*}_p$. In the same way we obtain $F^i \in \mathbb{V}^{1*}_p$ for $i = 1,2,3$. □

Lemma 6.4. *For each $i = 1,2,3$*

$$\lim_{\varepsilon\to 0} \sup_{|v|_{\mathbb{V}^1_p}\le 1} |(F^i_\varepsilon - F^i)(v)| = 0. \tag{6.24}$$

Proof. We define the functionals R^i for $i = 1,2,...,7$ in the same way as R^i_ε are defined in (6.22), but with a,b,c, ξ, η, λ^ξ and λ^η in place of $a^\varepsilon, b^{(\varepsilon)}, c^{(\varepsilon)}$, $\xi^{(\varepsilon)}$ $\eta^{(\varepsilon)}$, $(\lambda^\xi)^{(\varepsilon)}$ and $(\lambda^\eta)^{(\varepsilon)}$, respectively. To prove $F^1_\varepsilon \to F^1$ strongly in $\mathbb{V}^{1*}_p$, we notice that

$$|F^1_\varepsilon(v) - F^1(v)| \le \sum_{i=1}^7 |R^i_\varepsilon(v) - R^i(v)| \quad \text{for } v \in \mathbb{V}^1_p.$$

Since ζ is bounded, for a constant N independent of v and ε, we have

$$|R^1_\varepsilon(v) - R^1(v)| \le N\sum_{i=1}^3 Q^i_\varepsilon(v)$$

for all $\varepsilon \ge 0$ with

$$Q^1_\varepsilon(v) := E\int_0^T \int_{\mathbb{R}^d} |D_j v(s,x)||a^{\varepsilon ij}(s,x) - a^{ij}(s,x)||D_i\varphi(x)|\,dx\,ds,$$

$$Q^2_\varepsilon(v) := E\int_0^T \int_{\mathbb{R}^d} |v(s,x)||\bar b^{i(\varepsilon)}(s,x) - \bar b^i(s,x)||D_i\varphi(x)|\,dx\,ds,$$

$$Q^3_\varepsilon(v) := E\int_0^T\int_{\mathbb{R}^d}|v(s,x)||c^{(\varepsilon)}(s,x)-c(s,x)||\varphi(x)|\,dx\,ds.$$

By Hölder's inequality and well-known properties of mollifications

$$\sup_{|v|_{\mathbb{V}^1_p}\le 1} Q^i_\varepsilon(v)\le N\varepsilon|\varphi|_{H^1_q}\quad\text{for } i=1,2,3$$

with a constant $N=N(K,d)$, where $q=p/(p-1)$. Hence we get

$$\lim_{\varepsilon\to 0}\sup_{|v|_{\mathbb{V}^1_p}\le 1}|(R^i_\varepsilon-R^i)(v)|=0. \tag{6.25}$$

for $i=1$. Clearly,

$$|R^2_\varepsilon(v)-R^2(v)|\le H^1_\varepsilon(v)+H^2_\varepsilon(v) \tag{6.26}$$

with

$$H^1_\varepsilon(v):=\int_0^1\int_0^T\int_Z(|\eta^{(\varepsilon)}-\eta||v(s,\tau_{\theta\eta^{(\varepsilon)}})-v(s)|,|D\varphi|)\,\mu(dz)\,ds\,d\theta,$$

$$H^2_\varepsilon(v):=\int_0^1\int_0^T\int_Z(\bar\eta|v(s,\tau_{\theta\eta^{(\varepsilon)}})-v(s,\tau_{\theta\eta})|,|D\varphi|)\,\mu(dz)\,ds\,d\theta.$$

Note that $|\eta^{(\varepsilon)}-\eta|\le\varepsilon\bar\eta$ for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in Z$ and $\varepsilon>0$. Moreover, by Taylor's formula, Minkowski's inequality and Corollary 3.4

$$|v(s,\tau_{\theta\eta^{(\varepsilon)}})-v(s)|_{L_p}\le\bar\eta\int_0^1|Dv(s,\tau_{\vartheta\theta\eta^{(\varepsilon)}})|_{L_p}\,d\vartheta\le N\bar\eta|Dv(s)|_{L_p},$$

$$\begin{aligned}|v(s,\tau_{\theta\eta^{(\varepsilon)}})-v(s,\tau_{\theta\eta})|_{L_p}&\le\int_0^1\big||\eta^{(\varepsilon)}-\eta||Dv(s,(1-\vartheta)\tau_{\theta\eta^{(\varepsilon)}}+\vartheta\tau_{\theta\eta})|\big|_{L_p}\,d\vartheta\\&\le N\varepsilon\bar\eta|Dv(s)|_{L_p}\end{aligned}$$

for $s\in[0,T]$, $z\in Z$, $\omega\in\Omega$ and $\varepsilon\in(0,\varepsilon_0)$, with a constant $N=N(K,d,p)$. Hence by Hölder's inequality we have

$$H^i_\varepsilon\le\varepsilon N|v|_{\mathbb{V}^1_p}|D\varphi|_{L_q}\int_Z\bar\eta^2(z)\mu(dz)=\varepsilon NK^2_\eta|v|_{\mathbb{V}^1_p}|D\varphi|_{L_q}\quad\text{for } i=1,2\text{ and }\varepsilon\in(0,\varepsilon_0),$$

which by virtue of (6.26) proves (6.25) for $i=2$. We can prove similarly (6.25) for $i=3,4,5,6,7$, which proves $F^1_\varepsilon\to F^1$ strongly. By the boundedness of ζ and using Davis' and Hölder's inequalities we have

$$\begin{aligned}|F^2_\varepsilon(v)-F^2(v)|&\le CE\Big(\int_0^T|(\mathcal{M}^{\varepsilon r}_s v_s-\mathcal{M}^r_s v_s,\varphi)\,ds)|^2\Big)^{1/2}\\&\le CE\Big(\int_0^T\sum_{r=1}^\infty(|\sigma^{(\varepsilon)r}_s-\sigma^r_s||Dv_s|,|\varphi|)^2\,ds\Big)^{1/2}\\&\quad+CE\Big(\int_0^T\sum_{r=1}^\infty(|\beta^{(\varepsilon)r}_s-\beta^r_s||v_s|,|\varphi|)^2\,ds\Big)^{1/2}\\&\le C(A^1_\varepsilon(v)+A^2_\varepsilon(v))\end{aligned}$$

for $v\in\mathbb{V}^1_p$ and integers $k\geq 1$ with a constant C independent of v and ε, where

$$A^1_\varepsilon(v):=E\big(\int_0^T|Dv_s|^2_{L_p}||\sigma^{(\varepsilon)}_s-\sigma_s||\varphi||^2_{L_q}\big)^{1/2}$$

and

$$A^2_\varepsilon(v):=E\big(\int_0^T|v_s|^2_{L_p}||\beta^{(\varepsilon)}_s-\beta_s||\varphi||^2_{L_q}\,ds\big)^{1/2}$$

with $q=p/(p-1)$. By standard properties of mollification

$$|\sigma^{(\varepsilon)}_t-\sigma_t|+|\beta^{(\varepsilon)}_t-\beta_t|\leq N\varepsilon$$

for all $\varepsilon>0$ and $(x,t,\omega)\in H_T\times\Omega$ with a constant $N=N(K,d)$. Thus,

$$\sup_{|v|_{\mathbb{V}^1_p}\leq 1}A^i_\varepsilon(v)\leq\varepsilon NT^{(p-2)/2p}|\varphi|_{L_q}\quad\text{for } i=1,2$$

with $q=p/(p-1)$ and a constant $N=N(K,d)$. Consequently, letting here $\varepsilon\to 0$ we obtain (6.24) for $i=2$. By the boundedness of ζ, using Davis' inequality we get

$$|F^3_\varepsilon(v)-F^3(v)|\leq C(B^1_\varepsilon(v)+B^2_\varepsilon(v))$$

for $v\in\mathbb{V}^1_p$ with a constant C independent of ε and v, where

$$B^1_\varepsilon(v)=E\Big(\int_0^T\int_Z|(I^{\eta^{(\varepsilon)}}v_s-I^\eta v_s,\varphi)|^2\,\mu(dz)\,ds\Big)^{1/2}$$

and

$$B^2_\varepsilon(v)=E\Big(\int_0^T\int_Z|(\gamma^{(\varepsilon)}_{s,z}T^{\eta^{(\varepsilon)}}v_s-\gamma_{s,z}T^\eta v_s,\varphi)|^2\,\mu(dz)\,ds\Big)^{1/2}.$$

Notice that by Taylor's formula

$$(I^{\eta^{(\varepsilon)}}v_s-I^\eta v_s,\varphi)=\int_{\mathbb{R}^d}\int_0^1 v_i(\chi^\varepsilon_\theta(s,z,x))(\eta^{(\varepsilon)i}_{s,z}(x)-\eta^i_{s,z}(x))\varphi(x)\,d\theta\,dx$$

where $v_i:=D_iv$ and

$$\chi^\varepsilon_\theta(s,z,x):=x+\theta\eta^{(\varepsilon)}_{s,z}(x)+(1-\theta)\eta_{s,z}(x)$$

for $\theta\in(0,1)$, $\varepsilon>0$ and $(s,z,\omega)\in[0,T]\times Z\times\Omega$. By Corollary 3.4 there are positive constants ε_0 and $M=M(K,d,m)$ such that the function $\chi^\varepsilon_\theta(s,z,\cdot)$ is a $C^{\bar m}$-diffeomorphism on $\mathbb{R}^d$ for every $\varepsilon\in(0,\varepsilon_0)$, $\theta\in(0,1)$, $\omega\in\Omega$, $t\in[0,T]$ $z\in Z$, and

$$|D\chi^\varepsilon_\theta(s,z,x)|\leq M\quad\text{and}\quad M^{-1}\leq|\det D\chi^\varepsilon_\theta(s,z,x)|$$

for all $(x,s,z,\omega)\in H_T\times Z\times\Omega$. Due to Assumption 2.3 we have

$$|\eta^{(\varepsilon)}_{s,z}(x)-\eta_{s,z}(x)|\leq\varepsilon\bar\eta(z)\quad\text{for all }\varepsilon>0\text{ and }(s,x,\omega,z)\in H_T\times\Omega\times Z.$$

Thus using Hölder's inequality we get

$$B^1_\varepsilon(v)\leq E\Big(\int_0^T\int_0^1\int_Z|Dv(\chi^\varepsilon_\theta(s,z))|^2_{L_p}|\varphi|^2_{L_q}\varepsilon^2\bar\eta^2(z)\,\mu(dz)\,d\theta\,ds\Big)^{1/2}$$
$$\leq C\varepsilon|\varphi|_{L_q}|v|_{\mathbb{V}^1_p}|\bar\eta|_{\mathcal{L}_2}\tag{6.27}$$

with a constant C independent of ε and v. Furthermore we notice that by Assumption 2.4

$$|\gamma^{(\varepsilon)}_{s,z}(x)-\gamma_{s,z}(x)|\leq N\varepsilon\bar\eta(z)$$

with a constant $N = N(K, d)$ for all $\varepsilon > 0$ and $(s, z, \omega, z) \in H_T \times \Omega \times Z$. Hence, in a similar way as the estimate of $B^1_\varepsilon(v)$ is obtained, we can show

$$B^2_\varepsilon(v) \le C\varepsilon|\varphi|_{L_q}|v|_{\mathbb{V}^1_p} \tag{6.28}$$

with a constant C independent of ε and v. Hence, combining (6.27) and (6.28) we get (6.24) for $i = 3$. □

Since $F^i_\varepsilon \to F^i$ strongly in $\mathbb{V}^{1*}_p$ as $\varepsilon \to 0$ and $u^{\varepsilon_k} \to u$ weakly in $\mathbb{V}^1_p$ as $\varepsilon_k \to 0$, we have

$$\lim_{k\to\infty} F(u^{\varepsilon_k}) = F(u), \quad \lim_{k\to\infty} F^i_k(u^{\varepsilon_k}) = F^i(u) \quad \text{for} \quad i = 1, 2, 3.$$

By well-known properties of mollifications and using Lemma 3.5 it is easy to show

$$\lim_{k\to\infty} E\int_0^T \zeta_t(\psi^{(\varepsilon_k)}, \varphi)\,dt = E\int_0^T \zeta_t(\psi, \varphi)\,dt,$$

$$\lim_{k\to\infty} E\int_0^T \int_0^t \zeta_t(f^{(\varepsilon_k)}_s, \varphi)\,ds\,dt = E\int_0^T \int_0^t \zeta_t(f_s, \varphi)\,ds\,dt,$$

$$\lim_{k\to\infty} E\int_0^T \zeta_t \int_0^t (g^{(\varepsilon_k)r}_s, \varphi)\,dw^r_s\,dt = E\int_0^T \zeta_t \int_0^t (g^r_s, \varphi)\,dw^r_s\,dt,$$

and

$$\lim_{k\to\infty} E\int_0^T \zeta_t \int_0^t \int_Z (h^{(\varepsilon_k)}_s, \varphi)\,\tilde{\pi}(dz, ds)\,dt = E\int_0^T \zeta_t \int_0^t \int_Z (h_s, \varphi)\,\tilde{\pi}(dz, ds)\,dt.$$

Hence, taking $k \to \infty$ in equation (6.21) we get

$$\begin{aligned} E\int_0^T \zeta_t(u_t, \varphi)\,dt =& E\int_0^T \zeta_t(\psi, \varphi)\,dt + E\int_0^T \zeta_t \int_0^t \langle \mathcal{A}u_s, \varphi\rangle\,ds\,dt \\ &+ E\int_0^T \zeta_t \int_0^t (f_s, \varphi)\,ds\,dt + E\int_0^T \zeta_t \int_0^t (\mathcal{M}^r_s u_s + g^r_s, \varphi)\,ds\,dt \\ &+ E\int_0^T \zeta_t \int_0^t \int_Z (I^\eta u_s + \gamma_{s,z} T^\eta u_s + h_s, \varphi)\,\tilde{\pi}(dz, ds)\,dt \end{aligned}$$

for every bounded well-measurable process ζ and every $\varphi \in C^\infty_0$, which implies that for every $\varphi \in C^\infty_0$ equation (1.1) holds $P \otimes dt$ almost everywhere. Hence, by Lemma 3.10 u has an L_p-valued cadlag modification, denoted also by u, which is a generalised solution to (1.1)-(1.2). Moreover, from (6.19) we obtain

$$|u|_{\mathbb{V}^n_{r,p}} \le \liminf_{\varepsilon_k\to 0} |u^{\varepsilon_k}|_{\mathbb{V}^n_{r,p}} \le N(|\psi|_{\Psi^n_p} + |f|_{\mathbb{H}^n_p} + |g|_{\mathbb{H}^{n+1}_p(l_2)} + |h|_{\mathbb{H}^{n+i}_p(\mathcal{L}_{p,2})})$$

for $n = 0, 1, ..., m$ for every integer $r > 1$ with a constant $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$, where $i = 1$ when $p = 2$ and $i = 2$ for $p > 2$. Letting here $r \to \infty$ we obtain

$$E \operatorname*{ess\,sup}_{t\in[0,T]} |u_t|^p_{H^n_p} \le N(E|\psi|^p_{H^n_p} + E\mathcal{K}^p_{n,p}(T)) \tag{6.29}$$

for $n = 0, 1, 2, ...m$ with a constant $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$. We already know that u is and L_p-valued cadlag process. Hence, applying Lemma 3.12 with $V = H^m_p$, $U = H^0_p$ and we obtain that u is weakly cadlag as an H^m_p-valued process, and we can change the essential supremum into supremum in (6.29), i.e.,

$$E\sup_{t\in[0,T]}|u_t|^p_{H^n_p}\leq N(E|\psi|^p_{H^n_p}+E\mathcal{K}^p_{n,p}(T))\quad\text{for } n=0,1,2,...,m. \tag{6.30}$$

Thus we can also see that u is strongly cadlag as an H^{m-1}_p-valued process. To dispense with Assumption 6.1 we take a non-negative function $\chi\in C^\infty_0(\mathbb{R}^d)$ such that $\chi(x)=1$ for $|x|\leq 1$ and $\chi(x)=0$ for $|x|\geq 2$, and for integers $n\geq 1$ define

$$\psi^n(x)=\psi(x)\chi_n(x),\quad f^n_t(x,z)=f_t(x,z)\chi_n(x),$$
$$g^{nr}_t(x)=g^r_t(x)\chi_n(x),\quad h^n_t(x,z)=h_t(x,z)\chi_n(x)$$

for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in Z$, where $\chi_n(x)=\chi(x/n)$. Then for each n there is a unique generalised solution $u^n=(u^n_t)_{t\in[0,T]}$ to equation (1.1)-(1.2) with ψ^n, f^n, g^n and h^n in place of ψ, f, g and h, respectively. Moreover by (6.30)

$$E\sup_{t\leq T}|u^n_t-u^l_t|^p_{H^m_p}\leq NE|\psi^n-\psi^l|^p_{H^m_p}+NE\int_0^T|f^n_s-f^l_s|^p_{H^m_p}\,ds$$
$$+NE\int_0^T+|g^n_s-g^l|^p_{H^{m+1}_p(l_2)}+|h^n_s-h^l_s|^p_{H^{m+1}_p(\mathcal{L}_{p,2})}+\mathbf{1}_{p>2}|h^n_s-h^l_s|^p_{H^{m+2}_p(\mathcal{L}_{p,2})}\,ds$$

with a constant $N=N(d,p,m,T,K,K_{\bar{\xi}},K_{\bar{\eta}})$. Letting here $l,n\to\infty$ we get

$$\lim_{n,l\to\infty}E\sup_{t\leq T}|u^n_t-u^l_t|^p_{H^m_p}=0.$$

Consequently, there is an H^m_p-valued adapted process $u=(u_t)_{t\in[0,T]}$ such that for a subsequence $n'\to\infty$ we have $\sup_{t\leq T}|u^{n'}_t-u_t|_{H^m_p}\to 0$ almost surely. Hence u is an H^{m-1}_p-valued cadlag process, and it is easy to show that it is a generalised solution to (1.1)-(1.2), such that (6.30) holds and u is weakly cadlag as an H^m_p-valued process.

The next theorem extends the above result to equation (1.1)-(1.2) with $\mathcal{R}$ satisfying Assumption 2.5.

Theorem 6.5. *Let Assumptions 2.1 through 2.6 hold with an integer $m\geq 1$ and a real number $p\geq 2$. Assume also* (6.16). *Then equation* (1.1)-(1.2) *has a unique generalised solution u, such that u is an H^m_p-valued weakly cadlag process, satisfying estimate* (6.30), *and it is cadlag as an H^{m-1}_p-valued process.*

Proof. We use the standard method of continuity, see, e.g., [23]. For $\lambda\in[0,1]$, we consider the equation

$$du_t=(\mathcal{A}^0_tu_t+\lambda\mathcal{R}_tu_{t-}+f_t)\,dt+(\mathcal{M}^r_tu_t+g^r_t)\,dw^r_t$$
$$+\int_Z(I^\eta u_{t-}+\gamma_tT^\eta u_{t-}+h_t)\,\tilde{\pi}(dz,dt) \tag{6.31}$$

for $(t,x)\in H_T$ with initial condition

$$u_0=\psi, \tag{6.32}$$

where $\mathcal{A}^0_t=\mathcal{L}_t+\mathcal{N}^\xi_t+\mathcal{N}^\eta_t$ for every $t\in[0,T]$. We look for a solution u from the space $\mathcal{H}^m_p$ of $\mathcal{F}_t$-adapted H^m_p-valued weakly cadlag processes which are strongly cadlag as H^{m-1}_p-valued processes such that $|u|^p_{\mathcal{H}^m_p}=E\sup_{t\leq T}|u_t|^p_{H^m_p}<\infty$. Notice that $\mathcal{H}^m_p$ is a Banach space. If

$u \in \mathcal{H}_p^m$ is a generalised solution to (6.31), then by Assumption 2.5 and estimate (6.30) we have

$$E \sup_{s \le t} |u_s|^p_{H^n_p} \le NE|\psi|^p_{H^n_p} + NE \int_0^t |\lambda \mathcal{R}_s u_{s-}|^p_{H^n_p} + NE\mathcal{K}^p_{p,n}(T)$$

$$\le NE|\psi|^p_{H^n_p} + NK \int_0^t E \sup_{r \le s} |u_r|^p_{H^n_p} \, ds \quad \text{for } n = 0, 1, ..., m$$

with a constant $N = N(m, d, p, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$. Hence by Gronwall's lemma we have estimate (6.30) for u. Let Λ denote the set of $\lambda \in [0, 1]$ such that for any $\psi \in \Psi^m_p = L_p(\Omega, \mathcal{F}_0; H^m_p)$, $f \in \mathbb{H}^m_p$, $g \in \mathbb{H}^{m+1}_p(l^2)$ and $h \in \mathbb{H}^{m+i}_p(\mathcal{L}_{p,2})$, with $i = 1$ when $p = 2$ and $i = 2$ when $p > 2$, equation (6.31)-(6.32) has a unique generalised solution in $\mathcal{H}^m_p$. Clearly $0 \in \Lambda$, and we need to prove $1 \in \Lambda$. To this end, it suffices to show that there is an $\delta > 0$ such that for any $\lambda_0 \in \Lambda$,

$$[\lambda_0 - \delta, \lambda_0 + \delta] \cap [0, 1] \in \Lambda.$$

Fix $\lambda_0 \in \Lambda$, $\psi \in \Psi^m_p$, $f \in \mathbb{H}^m_p$, $g \in \mathbb{H}^{m+1}_p(l^2)$ and $h \in \mathbb{H}^{m+i}_p(\mathcal{L}_{p,2})$. For $v \in \mathcal{H}^m_p$ and $\lambda \in [0, 1]$ we consider the equation

$$du_t = (\mathcal{A}^0_t u_t + \lambda_0 \mathcal{R}_t u_{t-} + (\lambda - \lambda_0)\mathcal{R}_t v_{t-} + f_t) \, dt + (\mathcal{M}^r_t u_t + g^r_t) \, dw^r_t$$

$$+ \int_Z (I^\eta u_{t-} + \gamma_t T^\eta u_{t-} + h_t) \, \tilde{\pi}(dz, dt)$$

for $(t, x) \in H_T$, with initial condition $u_0 = \psi$. Since $\lambda_0 \in \Lambda$, this problem has a unique generalised solution $u \in \mathcal{H}^m_p$. Define the operator Q_λ by $u = Q_\lambda v$. Then Q_λ maps $\mathcal{H}^m_p$ into itself, and $\lambda \in \Lambda$ if and only if there is a fixed point of Q_λ. If $v^i \in \mathcal{H}^m_p$ and $u^i = Q_\lambda v^i$ for $i = 1, 2$, then for $u := u^2 - u^1$ we have

$$du_t = (\mathcal{A}^0_t u_t + \lambda_0 \mathcal{R}_t u_{t-} + (\lambda - \lambda_0)\mathcal{R}_t(v^2_t - v^1_t)) \, dt + \mathcal{M}^r_t u_t \, dw^r_t$$

$$+ \int_Z (I^\eta u_{t-} + \gamma_t T^\eta u_{t-}) \, \tilde{\pi}(dz, dt), \quad (t, x) \in H_T,$$

with $u_0 = 0$. Hence, using estimate (6.30) for u, due to Assumption 2.5 on $\mathcal{R}$ we get

$$|Q_\lambda v^2 - Q_\lambda v^1|_{\mathcal{H}^m_p} \le N|\lambda - \lambda_0||\mathcal{R}(v^2 - v^1)|_{\mathcal{H}^m_p} \le N'|\lambda - \lambda_0||v^2 - v^1|_{\mathcal{H}^m_p}$$

with constants N and N' depending only on $m, d, p, T, K, K_{\bar{\xi}}$ and $K_{\bar{\eta}}$. Taking $\delta = (2N')^{-1}$ we obtain that Q_λ is a contraction mapping on $\mathcal{H}^m_p$ if $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \cap [0, 1]$. Consequently, (6.31) has a unique solution u in $\mathcal{H}^m_p$, and it satisfies (6.30). □

If $m \ge 1$ is not an integer, then we set $\theta = m - \lfloor m \rfloor$ and by Theorem 4.1 we have

$$[\Psi^{\lfloor m \rfloor}_p, \Psi^{\lceil m \rceil}_p]_\theta = \Psi^m_p, \quad [\mathbb{H}^{\lfloor m \rfloor}_p, \mathbb{H}^{\lceil m \rceil}_p]_\theta = \mathbb{H}^m_p,$$

$$[\mathbb{H}^{\lfloor m \rfloor + 1}_p(l_2), \mathbb{H}^{\lceil m \rceil + 1}_p(l_2)]_\theta = \mathbb{H}^{m+1}_p(l_2), \quad [\mathbb{H}^{\lfloor m \rfloor + i}_p(\mathcal{L}_{p,2}), \mathbb{H}^{\lceil m \rceil + i}_p(\mathcal{L}_{p,2})]_\theta = \mathbb{H}^{m+i}_{p,2}(\mathcal{L}_{p,2})$$

for $i = 1, 2$, and

$$\mathbb{U}^m_{r,p} = [\mathbb{U}^{\lfloor m \rfloor}_{r,p}, \mathbb{U}^{\lceil m \rceil}_{r,p}]_\theta$$

for integers $r > 1$. If Assumptions 2.1 through 2.6 with $m \geq 1$ hold then we have shown above that the solution operator $\mathbb{S}$, which maps the data (ψ, f, g, h) into the generalised solution u of (1.1)-(1.2), is continuous from

$$\Psi_p^{\lfloor m \rfloor} \times \mathbb{H}_p^{\lfloor m \rfloor} \times \mathbb{H}_p^{\lfloor m \rfloor+1}(l_2) \times \mathbb{H}_{p,2}^{\lfloor m \rfloor+i}$$

to $\mathbb{U}_{p,r}^{\lfloor m \rfloor}$, and from

$$\Psi_p^{\lceil m \rceil} \times \mathbb{H}_p^{\lceil m \rceil} \times \mathbb{H}_p^{\lceil m \rceil+1}(l_2) \times \mathbb{H}_{p,2}^{\lceil m \rceil+i}$$

to $\mathbb{U}_{p,r}^{\lceil m \rceil}$, for $i = 1$ when $p = 2$ and for $i = 2$ when $p > 2$, with operator norms bounded by a constant $N = N(d, p, m, T, K, K_{\bar{\xi}}, K_{\bar{\eta}})$. Hence by Theorem 4.1 (i) we have

$$|u|^p_{\mathbb{U}^m_{r,p}} \leq N(E|\psi|^p_{H^m_p} + E\mathcal{K}^p_{m,p}(T))$$

with a constant $N = (p, d, m, T, K, K_{\bar{\eta}})$. In the same way we get

$$|u|^p_{\mathbb{U}^s_{r,p}} \leq N(E|\psi|^p_{H^s_p} + E\mathcal{K}^p_{s,p}(T)) \quad \text{for any } s \in [0, m].$$

Now, like before, letting here $r \to \infty$ we obtain (6.29) for real numbers $s \in [0, m]$, and using Lemma 3.12 we get that u is an H^m_p-valued weakly cadlag process such that (6.30) holds for any $s \in [0, m]$. Taking into account that u is a strongly cadlag L_p-valued process and using the interpolation inequality Theorem 4.1(v) with $A_0 := L_p$ and $A_1 := H^m_p$, we get that u is strongly cadlag as an H^s_p-valued process for every real number $s < m$.

Finally we can prove estimate (2.6) for $q \in (0, p)$ by applying Lemma 3.11 in the same way as it is used in [16] to prove the corresponding supremum estimate therein.

Acknowledgement. The main result of this paper was presented at the conference on "Harmonic Analysis for Stochastic PDEs" in Delft, 10-13 July, 2018 and at the "9th International Conference on Stochastic Analysis and Its Applications" in Bielefeld, 3-7 September, 2018. The authors are grateful to the organisers of these meetings, and to Konstantinos Dareiotis, Stefan Geiss and Mark Veerar for useful discussions and precious remarks. Finally we would like to thank the referees for their thorough work, comments and suggestions, and for pointing out several typos in the manuscript.

References

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2009.

[2] D. Applebaum and S. Blackwood, The Kalman-Bucy filter for integrable Lévy processes with infinite second moment, J. Applied Prob. 52 (2015), 636-648.

[3] O.E. Barndorff-Nielsen, T. Mikosch, S.I. Resnick (Editors), *Lévy Processes. Theory and Applications*, Birkhäuser, 2001.

[4] S. Blackwood, Lévy Processes and Filtering Theory, Thésis, The University Sheffield, 2014.

[5] J. Bergh and J. Löfström, *Interpolation spaces*, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, 1976.

[6] T. Chang and K. Lee, On a stochastic partial differential equation with a fractional Laplacian operator, Stochastic Processes and their Applications, Volume 122, Issue 9, 2012, pp. 3288-3311.

[7] Z. Chen and K. Kim, An L_p-theory for non-divergence form SPDEs driven by Lévy processes, Forum Mathematicum, Volume 26, Issue 5 (2014), 1381-1411.

[8] K. Dareiotis, A Note on Degenerate Stochastic Integro–Differential Equations, arXiv:1406.5649v1, Jun 2014.

[9] M. León-Contreras, I. Gyöngy and S. Wu, On Solvability of Integro-Differential Equations, arXiv:1809.06840.

[10] H. Dong, D. Kim, On L_p-estimates for a class of non-local elliptic equations, J. Func. Anal. 262 (3) (2012), 1166-1199.

[11] B. P. W. Fernando and E. Hausenblas, Nonlinear filtering with correlated Lévy noise characterized by copulas, Brazilian Journal of Probability and Statistics Vol. 32, No. 2 (2018) 374-421.

[12] M. Gerencsér, I. Gyöngy and N. V. Krylov, On the solvability of degenerate stochastic partial differential equations in Sobolev spaces, Stochastic Partial Differential Equations: Analysis and Computations, 3 (2015), no. 1, 52-83.

[13] W. B. Gordon, On the Diffeomorphisms of Euclidean Space, American Mathematical Monthly, Vol. 79, No. 7 (1972), 755-759.

[14] B. Grigelionis, Reduced stochastic equations of nonlinear filtering of random processes, Lithuan. Math. J. 16 (1976), 348-358.

[15] I. Gyöngy, On stochastic equations with respect to semimartingales. III. Stochastics 7 (1982), no. 4, 231-254.

[16] I. Gyöngy and N.V. Krylov, On the rate of convergence of splitting-up approximations for SPDEs, Progress in Probability, Vol. 56, 301-321, 2003 Birkhäuser Verlag.

[17] I. Gyöngy and S. Wu, Itô's formula for jump processes in L_p-spaces, arXiv:1904.128982019, 2019.

[18] T. Hytönen, J. van Neerven, M. Veraar and L. Weis, *Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory*, Springer 2016.

[19] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland, 2011.

[20] K. Kim and P. Kim, An L_p-theory of a class of stochastic equations with the random fractional Laplacian driven by Lévy processes, Stochastic Processes and their Applications, Volume 122, Issue 12, 2012, pp. 3921-3952.

[21] K. Kim and K. Lee, A note on W_p^γ-theory of linear stochastic parabolic partial differential systems, Stochastic Processes and their Applications, 123 (2013), 76-90.

[22] N.V. Krylov, On L_p-theory of stochastic partial differential equations in the whole space, SIAM J. Math. Anal. 27 (1996), no. 2, 313-340.

[23] N.V. Krylov, Lectures on Elliptic and Parabolic equations in Sobolev Spaces, Graduate Studies in Mathematics, Volume 96, AMS, Providence, Rhode Island 2008.

[24] N.V. Krylov, Itô's formula for the L_p-norm of stochastic W_p^1-valued processes, Probab. Theory Relat. Fields 147 (2010), 583–605.

[25] N.V. Krylov and B.L. Rozovskii, Stochastic evolution equations, J. Sov. Math., 16 (1981) 1233–1277.

[26] N.V. Krylov and B.L. Rozovskii, Characteristics of degenerating second-order parabolic Itô equations, J. Soviet Maths., 32 (1986), 336-348. (Translated from Trudy Seminara imeni I.G. Petrovskogo, No. 8. pp.. 153-168, 1982.)

[27] J-M. Leahy and R. Mikulevičius, On degenerate linear stochastic evolution equations driven by jump processes. Stochastic Process. Appl. 125 (2015), no. 10, 3748-3784.

[28] C. Marinelli and M. Röckner, Well-posedness and asymptotic behavior for stochastic reaction-diffusion equations with multiplicative Poisson noise, *Electron. J. Probab.*, 15 (2010) 1528-1555.

[29] R. Mikulevičius, C. Phonsom, On L_p-theory for parabolic and elliptic integro-differential equations with scalable operators in the whole space, Stoch. Partial Differ. Equ. Anal. Comput. 5 (2017), no. 4, 472-519.

[30] R. Mikulevičius, C. Phonsom, On the Cauchy Problem for Stochastic Integro-differential Parabolic Equations in the Scale of L_p-Spaces of Generalised Smoothness, Potential Anal. 50 (2019), no. 3, 467-519.

[31] R. Mikulevičius and H. Pragarauskas, On L_p-theory for stochastic parabolic integro-differential equations. Stoch. Partial Differ. Equ. Anal. Comput. 1 no. 2, (2013), 282-324.

[32] E. Pardoux, Équations aux dériveés partielles stochastiques non linéaries monotones. Étude des solution forte de type Ito. Thése Université de Paris Sud, Orsay, 1975.

[33] Sz. Peszat and J. Zabczyk, *Stochastic partial differential equations with Lévy noise*, Cambridge University Press, Cambridge, 2007.

[34] H. Triebel, *Interpolation Theory - Function Spaces - Differential Operators*, North Holland Publishing Company, Amsterdam-New York-Oxford, 1978.

[35] X. Zhang, L^p-maximal regularity of nonlocal parabolic equations and applications, Ann. I. H. Poincaré, AN 30 (2013) 573-614.

[36] J. Zhu, Z. Brzeźniak and W. Liu, Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with applications to stochastic quasi-geostrophic equations, *Siam J. Math. Anal.*, Vol. 51, No. 3, 2121-2167.

School of Mathematics and Maxwell Institute, University of Edinburgh, Scotland, United Kingdom.

E-mail address: i.gyongy@ed.ac.uk

School of Mathematics, University of Edinburgh, King's Buildings, Edinburgh, EH9 3JZ, United Kingdom

E-mail address: Sizhou.Wu@ed.ac.uk