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ACCELERATED FINITE ELEMENTS SCHEMES FOR PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. For a class of finite elements approximations for linear stochastic parabolic PDEs it is proved that one can accelerate the rate of convergence by Richardson extrapolation. More precisely, by taking appropriate mixtures of finite elements approximations one can accelerate the convergence to any given speed provided the coefficients, the initial and free data are sufficiently smooth.

1. Introduction

We are interested in finite elements approximations for Cauchy problems for stochastic parabolic PDEs of the form of equation (2.1) below. Such kind of equations arise in various fields of sciences and engineering, for example in nonlinear filtering of partially observed diffusion processes. Therefore these equations have been intensively studied in the literature, and theories for their solvability and numerical methods for approximations of their solutions have been developed. Since the computational effort to get reasonably accurate numerical solutions grow rapidly with the dimension d of the state space, it is important to investigate the possibility of accelerating the convergence of spatial discretizations by Richardson extrapolation. About a century ago Lewis Fry Richardson had the idea in [18] that the speed of convergence of numerical approximations, which depend on some parameter h converging to zero, can be increased if one takes appropriate linear combinations of approximations corresponding to different parameters. This method to accelerate the convergence, called Richardson extrapolation, works when the approximations admit a power series expansion in h at $h = 0$ with a remainder term, which can be estimated by a higher power of h. In such cases, taking appropriate mixtures of approximations with different parameters, one can eliminate all other terms but the zero order term and the remainder in the expansion. In this way, the order of accuracy of the mixtures is the exponent $k+1$ of the power h^{k+1}, that estimates the remainder. For various numerical methods applied to solving deterministic partial differential equations (PDEs) it has been proved that such expansions exist and that Richardson extrapolations can spectacularly increase the speed of convergence of the methods, see, e.g., [16], [17] and [20]. Richardson's idea has also been applied to numerical solutions of stochastic equations. It was shown first in [21] that by Richardson extrapolation one can accelerate the weak convergence of Euler approximations of stochastic differential equations. Further results in this direction can be found in [14], [15] and the references therein. For stochastic PDEs the first result on accelerated finite difference schemes appears in [7], where it is shown that

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by Richardson extrapolation one can accelerate the speed of finite difference schemes in the spatial variables for linear stochastic parabolic PDEs to any high order, provided the initial condition and free terms are sufficiently smooth. This result was extended to (possibly) degenerate stochastic PDEs in to [6], [8] and [9]. Starting with [22] finite elements approximations for stochastic PDEs have been investigated in many publications, see, for example, [3], [4], [10], [11], [12] and [23].

Our main result, Theorem 2.4 in this paper, states that for a class of finite elements approximations for stochastic parabolic PDEs given in the whole space an expansion in terms of powers of a parameter h, proportional to the size of the finite elements, exists up to any high order, if the coefficients, the initial data and the free terms are sufficiently smooth. Then clearly, we can apply Richardson extrapolation to such finite elements approximations in order to accelerate the convergence. The speed we can achieve depends on the degree of smoothness of the coefficients, the initial data and free terms; see Corollary 2.5. Note that due to the symmetry we require for the finite elements, in order to achieve an accuracy of order $J+1$ we only need $\lfloor \frac{J}{2} \rfloor$ terms in the mixture of finite elements approximation. As far as we know this is the first result on accelerated finite elements by Richardson extrapolation for stochastic parabolic equations. There are nice results on Richardson extrapolation for finite elements schemes in the literature for some (deterministic) elliptic problems; see, e.g., [1], [2] and the literature therein.

We note that in the present paper we consider stochastic PDEs on the whole space $\mathbb{R}^d$ in the spatial variable, and our finite elements approximations are the solutions of infinite dimensional systems of equations. Therefore one may think that our accelerated finite elements schemes cannot have any practical use. In fact they can be implemented if first we localise the stochastic PDEs in the spatial variable by multiplying their coefficients, initial and free data by sufficiently smooth non-negative "cut-off" functions with value 1 on a ball of radius R and vanishing outside of a bigger ball. Then our finite elements schemes for the "localised stochastic PDEs" are fully implementable and one can show that the results of the present paper can be extended to them. Moreover, by a theorem from [6] the error caused by the localization is of order $\exp(-\delta R^2)$ within a ball of radius $R' < R$. Moreover, under some further constraints about a bounded domain D and particular classes of finite elements such as those described in subsections 6.1-6.2, our arguments could extend to parabolic stochastic PDEs on D with periodic boundary conditions. Note that our technique relies on finite elements defined by scaling and shifting one given mother element, and that the dyadic rescaling used to achieve a given speed of convergence is similar to that of wavelet approximation. We remark that our accelerated finite elements approximations can be applied also to implicit Euler-Maruyama time discretizations of stochastic parabolic PDEs to achieve higher order convergence with respect to the spatial mesh parameter of fully discretised schemes. However, as one can see by adapting and argument from [5], the strong rate of convergence of these fully discretised schemes with respect to the temporal mesh parameter cannot be accelerated by Richardson approximation. Dealing with weak speed of convergence of time discretisations is beyond the scope of this paper.

In conclusion we introduce some notation used in the paper. All random elements are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ equipped with an increasing family $(\mathcal{F}_t)_{t\geq 0}$

of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. The predictable σ-algebra of subsets of $\Omega\times[0,\infty)$ is denoted by $\mathcal{P}$, and the σ-algebra of the Borel subsets of $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$. We use the notation

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_iD_j = \frac{\partial^2}{\partial x_i \partial x_j}, \quad i,j = 1,2,...,d$$

for first order and second order partial derivatives in $x = (x_1,...,x_d) \in \mathbb{R}^d$. For integers $m \ge 0$ the Sobolev space H^m is defined as the closure of C_0^∞, the space of real-valued smooth functions φ on $\mathbb{R}^d$ with compact support, in the norm $|\varphi|_m$ defined by

$$|\varphi|_m^2 = \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} |D^\alpha \varphi(x)|^2 \, dx, \tag{1.1}$$

where $D^\alpha = D_1^{\alpha_1}...D_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for multi-indices $\alpha = (\alpha_1,...,\alpha_d)$, $\alpha_i \in \{0,1,...,d\}$, and D_i^0 is the identity operator for $i = 1,...,d$. Similarly, the Sobolev space $H^m(l_2)$ of l_2-valued functions are defined on $\mathbb{R}^d$ as the closure of the of l_2-valued smooth functions $\varphi = (\varphi_i)_{i=1}^\infty$ on $\mathbb{R}^d$ with compact support, in the norm denoted also by $|\varphi|_m$ and defined as in (1.1) with $\sum_{i=1}^\infty |D^\alpha \varphi_i(x)|^2$ in place of $|D^\alpha\varphi(x)|^2$. Unless stated otherwise, throughout the paper we use the summation convention with respect to repeated indices. The summation over an empty set means 0. We denote by C and N constants which may change from one line to the next, and by $C(a)$ and $N(a)$ constants depending on a parameter a.

For theorems and notations in the L_2-theory of stochastic PDEs the reader is referred to [13] or [19].

2. Framework and some notations

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\ge 0})$ be a complete filtered probability space carrying a sequence of independent Wiener martingales $W = (W^\rho)_{\rho=1}^\infty$ with respect to a filtration $(\mathcal{F}_t)_{t\ge0}$.

We consider the stochastic PDE problem

$$du_t(x) = \big[\mathcal{L}_t u_t(x) + f_t(x)\big]dt + \big[\mathcal{M}_t^\rho u_t(x) + g_t^\rho(x)\big]dW_t^\rho, \quad (t,x) \in [0,T]\times\mathbb{R}^d, \tag{2.1}$$

with initial condition

$$u_0(x) = \phi(x), \quad x \in \mathbb{R}^d, \tag{2.2}$$

for a given $\phi \in H^0 = L_2(\mathbb{R}^d)$, where

$$\begin{aligned} \mathcal{L}_t u(x) &= D_i(a_t^{ij}(x)D_ju(x)) + b_t^i(x)D_iu(x) + c_t(x)u(x), \\ \mathcal{M}_t^\rho u(x) &= \sigma_t^{i\rho}(x)D_iu(x) + \nu_t^\rho(x)u(x) \quad \text{for } u \in H^1 = W_2^1(\mathbb{R}^d), \end{aligned}$$

with $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable real-valued bounded functions a^{ij}, b^i, c, and l_2-valued bounded functions $\sigma^i = (\sigma^{i\rho})_{\rho=1}^\infty$ and $\nu = (\nu^\rho)_{\rho=1}^\infty$ defined on $\Omega\times[0,T]\times\mathbb{R}^d$ for $i,j \in \{1,...,d\}$. Furthermore, $a_t^{ij}(x) = a_t^{ji}(x)$ a.s. for every $(t,x) \in [0,T]\times\mathbb{R}^d$. For $i = 1,2,...,d$ the notation $D_i = \frac{\partial}{\partial x_i}$ means the partial derivative in the i-th coordinate direction.

The free terms f and $g = (g^\rho)_{\rho=1}^\infty$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega\times[0,T]\times\mathbb{R}^d$, with values in $\mathbb{R}$ and l_2 respectively. Let $H^m(l_2)$ denote the H^m space of l_2-valued functions on $\mathbb{R}^d$. We use the notation $|\varphi|_m$ for the H^m-norm of $\varphi \in H^m$ and of $\varphi \in H^m(l_2)$, and $|\varphi|_0$ denotes the L_2-norm of $\varphi \in H^0 = L_2$.

Let $m \ge 0$ be an integer, $K \ge 0$ be a constant and make the following assumptions.

Assumption 2.1. *The derivatives in $x \in \mathbb{R}^d$ up to order m of the coefficients a^{ij}, b^i, c, and of the coefficients σ^i, ν are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable functions with values in $\mathbb{R}$ and in l_2-respectively. For almost every ω they are continuous in x, and they are bounded in magnitude by K.*

Assumption 2.2. *The function ϕ is an H^m-valued $\mathcal{F}_0$-measurable random variable, and f and $g = (g^\rho)_{\rho=1}^\infty$ are predictable processes with values in H^m and $H^m(l_2)$, respectively, such that*

$$\mathcal{K}_m^2 := |\phi|_m^2 + \int_0^T \big(|f_t|_m^2 + |g_t|_m^2\big)\,dt < \infty \text{ (a.s.)}. \tag{2.3}$$

Assumption 2.3. *There exists a constant $\kappa > 0$, such that for $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^d$*

$$\sum_{i,j=1}^d \big(a_t^{ij}(x) - \tfrac{1}{2}\sum_\rho \sigma_t^{i\rho}(x)\sigma_t^{j\rho}(x)\big) z^i z^j \geq \kappa |z|^2 \quad \text{for all } z = (z^1, ..., z^d) \in \mathbb{R}. \tag{2.4}$$

For integers $n \geq 0$ let $\mathbb{W}_2^n(0,T)$ denote the space of H^n-valued predictable processes $(u_t)_{t\in[0,T]}$ such that almost surely

$$\int_0^T |u_t|_n^2\,dt < \infty.$$

Definition 2.1. A continuous L_2-valued adapted process $(u_t)_{t\in[0,T]}$ is a generalised solution to (2.1)-(2.2) if it is in $\mathbb{W}_2^1(0,T)$, and almost surely

$$\begin{aligned}(u_t, \varphi) =& (\phi, \varphi) + \int_0^t \big(a_s^{ij} D_j u_s, D_i^* \varphi) + (b_s^i D_i u_s + c_s u_s + f_s, \varphi\big)\,ds \\ &+ \int_0^t \big(\sigma_s^{i\rho} D^i u_s + \nu_s^\rho u_s + g_s^\rho, \varphi\big)\,dW_s^\rho\end{aligned}$$

for all $t \in [0,T]$ and $\varphi \in C_0^\infty$, where $D_i^* := -D_i$ for $i \in \{1, 2, ..., d\}$, and $(,)$ denotes the inner product in L_2.

For $m \geq 0$ set

$$\mathfrak{K}_m = |\phi|_m^2 + \int_0^T \big(|f_t|_{m-1}^2 + |g_t|_m^2\big)dt. \tag{2.5}$$

Then the following theorem is well-known (see, e.g., [19]).

Theorem 2.1. *Let Assumptions 2.1, 2.2 and 2.3 hold. Then (2.1)-(2.2) has a unique generalised solution $u = (u_t)_{t\in[0,T]}$. Moreover, $u \in \mathbb{W}_2^{m+1}(0,T)$, it is an H^m-valued continuous process, and*

$$E \sup_{t\in[0,T]} |u_t|_m^2 + E\int_0^T |u_t|_{m+1}^2\,dt \leq C E \mathfrak{K}_m,$$

where C is a constant depending only on κ, d, T, m and K.

The finite elements we consider in this paper are determined by a continuous real function $\psi \in H^1$ with compact support, and a finite set $\Lambda \subset \mathbb{Q}^d$, containing the zero vector, such that ψ and Λ are symmetric, i.e.,

$$\psi(-x) = \psi(x) \text{ for all } x \in \mathbb{R}^d, \quad \text{and } \Lambda = -\Lambda. \tag{2.6}$$

We assume that $|\psi|_{L_1}=1$, which can be achieved by scaling. For each $h\neq 0$ and $x\in\mathbb{R}^d$ we set $\psi^h_x(\cdot):=\psi((\cdot-x)/h)$, and our set of finite elements is the collection of functions $\{\psi^h_x : x\in\mathbb{G}_h\}$, where

$$\mathbb{G}_h:=\left\{h\sum_{i=1}^n n_i\lambda_i:\lambda_i\in\Lambda,\ n_i,n\in\mathbb{N}\right\}.$$

Let V_h denote the vector space

$$V_h:=\left\{\sum_{x\in\mathbb{G}_h}U(x)\psi^h_x:(U(x))_{x\in\mathbb{G}_h}\in\ell_2(\mathbb{G}_h)\right\},$$

where $\ell_2(\mathbb{G}_h)$ is the space of functions U on $\mathbb{G}_h$ such that

$$|U|^2_{0,h}:=|h|^d\sum_{x\in\mathbb{G}_h}U^2(x)<\infty. \tag{2.7}$$

Definition 2.2. An $L_2(\mathbb{R}^d)$-valued continuous process $u^h=(u^h_t)_{t\in[0,T]}$ is a finite elements approximation of u if it takes values in V_h and almost surely

$$\begin{aligned}(u^h_t,\psi^h_x)=&(\phi,\psi^h_x)+\int_0^t\left[(a^{ij}_sD_ju^h_s,D^*_i\psi^h_x)+(b^i_sD_iu^h_s+c_su^h_s+f_s,\psi^h_x)\right]ds\\&+\int_0^t(\sigma^{i\rho}_sD_iu^h_s+\nu^\rho_su^h_s+g^\rho_s,\psi^h_x)\,dW^\rho_s,\end{aligned} \tag{2.8}$$

for all $t\in[0,T]$ and ψ^h_x is as above for $x\in\mathbb{G}_h$. The process u^h is also called a V_h-*solution* to (2.8) on $[0,T]$.

Since by definition a V_h-valued solution $(u^h_t)_{t\in[0,T]}$ to (2.8) is of the form

$$u^h_t(x)=\sum_{y\in\mathbb{G}_h}U^h_t(y)\psi^h_y(x),\quad x\in\mathbb{R}^d,$$

we need to solve (2.8) for the random field $\{U^h_t(y):y\in\mathbb{G}_h,t\in[0,T]\}$. Remark that (2.8) is an infinite system of stochastic equations. In practice one should "truncate" this system to solve numerically a suitable finite system instead, and one should also estimate the error caused by the truncation. We will study such a procedure and the corresponding error elsewhere.

Our aim in this paper is to show that for some well-chosen functions ψ, the above finite elements scheme has a unique solution u^h for every $h\neq 0$, and that for a given integer $k\geq 0$ there exist random fields $v^{(0)}$, $v^{(1)}$,...,$v^{(k)}$ and r_k, on $[0,T]\times\mathbb{G}_h$, such that almost surely

$$U^h_t(x)=v^{(0)}_t(x)+\sum_{1\leq j\leq k}v^{(j)}_t(x)\frac{h^j}{j!}+r^h_t(x),\quad t\in[0,T],\,x\in\mathbb{G}_h, \tag{2.9}$$

where $v^{(0)}$,..., $v^{(k)}$ do not depend on h, and there is a constant N, independent of h, such that

$$E\sup_{t\leq T}|h|^d\sum_{x\in\mathbb{G}_h}|r^h_t(x)|^2\leq N|h|^{2(k+1)}E\mathfrak{K}^2_m \tag{2.10}$$

for all $|h|\in(0,1]$ and some $m>\frac{d}{2}$.

To write (2.8) more explicitly as an equation for $(U_t^h(y))_{y\in\mathbb{G}_h}$, we introduce the following notation:

$$R_\lambda^{\alpha\beta} = (D_\beta\psi_\lambda, D_\alpha^*\psi), \quad \alpha,\beta\in\{0,1,...,d\},$$
$$R_\lambda^{\beta} = R_\lambda^{0\beta} := (D_\beta\psi_\lambda, \psi), \quad R_\lambda := R_\lambda^{00} := (\psi_\lambda,\psi), \quad \lambda\in\mathbb{G}, \tag{2.11}$$

where $\psi_\lambda := \psi_\lambda^1$, and $\mathbb{G} := \mathbb{G}_1$.

Lemma 2.2. *For $\alpha,\beta\in\{1,...,d\}$ and $\lambda\in\mathbb{G}$ we have:*

$$R_{-\lambda}^{\alpha\beta} = R_\lambda^{\alpha\beta}, \quad R_{-\lambda}^{\beta} = -R_\lambda^{\beta}, \quad R_{-\lambda} = R_\lambda.$$

Proof. Since $\psi(-x)=\psi(x)$ we deduce that for any $\alpha\in\{1,...,d\}$ we have $D_\alpha\psi(-x) = -D_\alpha\psi(x)$. Hence for any $\alpha,\beta\in\{1,...,d\}$ and $\lambda\in\mathbb{G}$, a change of variables yields

$$\begin{aligned}
R_{-\lambda}^{\alpha\beta} &= \int_{\mathbb{R}^d} D_\beta\psi(z+\lambda)D_\alpha^*\psi(z)dz = \int_{\mathbb{R}^d} D_\beta\psi(-z+\lambda)D_\alpha^*\psi(-z)dz\\
&= \int_{\mathbb{R}^d} D_\beta\psi(z-\lambda)D_\alpha^*\psi(z)dz = R_\lambda^{\alpha\beta},\\
R_{-\lambda}^{\beta} &= \int_{\mathbb{R}^d} D_\beta\psi(-z+\lambda)\psi(-z)dz = -\int_{\mathbb{R}^d} D_\beta\psi(z-\lambda)\psi(z)dz = -R_\lambda^{\beta},\\
R_{-\lambda} &= \int_{\mathbb{R}^d} \psi(-z+\lambda)\psi(-z)dz = \int_{\mathbb{R}^d} \psi(z-\lambda)\psi(z)dz = R_\lambda;
\end{aligned}$$

this concludes the proof. □

To prove the existence of a unique V_h-valued solution to (2.8), and a suitable estimate for it, we need the following condition.

Assumption 2.4. *There is a constant $\delta>0$ such that*

$$\sum_{\lambda,\mu\in\mathbb{G}} R_{\lambda-\mu} z^\lambda z^\mu \ge \delta \sum_{\lambda\in\mathbb{G}} |z^\lambda|^2, \quad \text{for all } (z^\lambda)_{\lambda\in\mathbb{G}}\in\ell_2(\mathbb{G}).$$

Remark 2.1. *Note that since $\psi\in H^1$ has compact support, there exists a constant M such that*

$$|R_\lambda^{\alpha,\beta}|\le M \quad \text{for } \alpha,\beta\in\{0,...,d\} \text{ and } \lambda\in\mathbb{G}.$$

Remark 2.2. *Due to Assumption 2.4 for $h\neq 0$, $u:=\sum_{y\in\mathbb{G}_h} U(y)\psi_y^h$, $U=(U(y))_{y\in\mathbb{G}_h}\in\ell_2(\mathbb{G}_h)$ we have*

$$\begin{aligned}
|u|_0^2 &= \sum_{x,y\in\mathbb{G}_h} U(x)U(y)(\psi_x^h,\psi_y^h)\\
&= \sum_{x,y\in\mathbb{G}_h} R_{(x-y)/h}U(x)U(y)|h|^d \ge \delta\sum_{x\in\mathbb{G}_h} U^2(x)|h|^d = \delta|U|_{0,h}^2.
\end{aligned} \tag{2.12}$$

Clearly, since ψ has compact support, only finitely many $\lambda\in\mathbb{G}$ are such that $(\psi_\lambda,\psi)\neq 0$; hence

$$|u|_0^2 \le \sum_{x,y\in\mathbb{G}_h} |R_{(x-y)/h}|\,|U(x)U(y)||h|^d \le N|h|^d \sum_{x\in\mathbb{G}_h} U^2(x) = N|U|_{0,h}^2,$$

where N is a constant depending only on ψ.

By virtue of this remark for each $h \neq 0$ the linear mapping $\mathbf{\Phi}_h$ from $\ell_2(\mathbb{G}_h)$ to $V_h \subset L_2(\mathbb{R}^d)$, defined by

$$\mathbf{\Phi}_h U := \sum_{x \in \mathbb{G}_h} U(x)\psi^h_x \quad \text{for } U = (U(x))_{x\in\mathbb{G}_h} \in \ell_2(\mathbb{G}_h),$$

is a one-to-one linear operator such that the norms of U and $\mathbf{\Phi}_h U$ are equivalent, with constants independent of h. In particular, V_h is a closed subspace of $L_2(\mathbb{R}^d)$. Moreover, since $D_i\psi$ has compact support, (2.12) implies that

$$|D_i u|_0 \le \frac{N}{|h|}\|u\| \quad \text{for all } u \in V_h, \quad i \in \{1,2,...,d\},$$

where N is a constant depending only on $D_i\psi$ and δ. Hence for any $h > 0$

$$|u|_1 \le N(1+|h|^{-1})|u|_0 \quad \text{for all } u \in V_h \tag{2.13}$$

with a constant $N = N(\psi, d, \delta)$ which does not depend on h.

Theorem 2.3. *Let Assumptions 2.1 through 2.4 hold with $m = 0$. Then for each $h \neq 0$ equation (2.8) has a unique V_h-solution u^h. Moreover, there is a constant $N = N(d, K, \kappa)$ independent of h such that*

$$\begin{aligned} &E \sup_{t\in[0,T]} |u^h_t|_0^2 + E\int_0^T |u^h_t|_1^2\,dt \\ &\qquad \le NE|\pi^h\phi|_0^2 + NE\int_0^T \Big(|\pi^h f_s|_0^2 + \sum_\rho |\pi^h g^\rho_s|_0^2\Big)\,ds \le NE\mathcal{K}_0^2 \end{aligned} \tag{2.14}$$

for all $h \neq 0$, where π^h denotes the orthogonal projection of $H^0 = L_2$ into V_h.

Proof. We fix $h \neq 0$ and define the bilinear forms A^h and $B^{h\rho}$ by

$$\begin{aligned} A^h_s(u,v) &:= (a^{ij}_s D_j u, D^*_i v) + (b^i_s D_i u + c_s u, v) \\ B^{h\rho}_s(u,v) &:= (\sigma^{i\rho}_s D_i u + \nu^\rho_s u, v) \end{aligned}$$

for all $u, v \in V_h$. Using Assumption 2.1 with $m = 0$, by virtue of (2.13) we have a constant $C = C(|h|, K, d, \delta, \psi)$, such that

$$A^h_s(u,v) \le C|u|_0|v|_0 \quad B^{h\rho}_s(u,v) \le C|u|_0|v|_0 \quad \text{for all } u,v \in V_h.$$

Hence, identifying V_h with its dual space $(V_h)^*$ by the help of the $L_2(\mathbb{R}^d)$ inner product in V_h, we can see there exist bounded linear operators $\mathbb{A}^h_s$ and $\mathbb{B}^{h\rho}_s$ on V_h such that

$$A^h_s(u,v) = (\mathbb{A}^h_s u, v), \quad B^{h\rho}_s(u,v) = (\mathbb{B}^{h\rho}_s u, v) \quad \text{for all } u,v \in V_h,$$

and for all $\omega \in \Omega$ and $t \in [0,T]$. Thus (2.8) can be rewritten as

$$u^h_t = \pi^h\phi + \int_0^t (\mathbb{A}^h_s u^h_s + \pi^h f_s)\,ds + \int_0^t (\mathbb{B}^{h\rho}_s u^h_s + \pi^h g^\rho_s)\,dW^\rho_s, \tag{2.15}$$

which is an (affine) linear SDE in the Hilbert space V_h. Hence, by classical results on solvability of SDEs with Lipschitz continuous coefficients in Hilbert spaces we get a unique

V_h-solution $u^h = (u^h_t)_{t\in[0,T]}$. To prove estimate (2.14) we may assume $E\mathcal{K}_0^2 < \infty$. By applying Itô's formula to $|u^h|_0^2$ we obtain

$$|u^h(t)|_0^2 = |\pi^h\phi|_0^2 + \int_0^t I^h_s\,ds + \int_0^t J^{h,\rho}_s\,dW^\rho_s, \tag{2.16}$$

with

$$\begin{aligned} I^h_s :=& 2(\mathbb{A}^h_s u^h_s + \pi^h f_s, u^h_s) + \sum_\rho |\mathbb{B}^{h\rho}_s u^h_s + \pi^h g^\rho_s|_0^2, \\ J^{h\rho}_s :=& 2(\mathbb{B}^{h\rho}_s u^h_s + \pi^h g^\rho_s, u^h_s). \end{aligned}$$

Owing to Assumptions 2.1, 2.2 and 2.3, by standard estimates we get

$$I^h_s \le -\kappa|u^h(s)|_1^2 + N\Big(|u^h_s|_0^2 + |f_s|_0^2 + \sum_\rho |g^\rho_s|_0^2\Big) \tag{2.17}$$

with a constant $N = N(K,\kappa,d)$; thus from (2.16) using Gronwall's lemma we obtain

$$E|u^h_t|_0^2 + \kappa E\int_0^T |u^h_s|_1^2\,ds \le NE\mathcal{K}_0^2 \quad t\in[0,T] \tag{2.18}$$

with a constant $N = N(T,K,\kappa,d)$. One can estimate $E\sup_{t\le T}|u^h_t|_0^2$ also in a standard way. Namely, since

$$\sum_\rho |J^{h\rho}_s|^2 \le N^2\left(|u^h_s|_1^2 + |g_s|_0^2\right)\sup_{s\in[0,T]}|u^h_s|_0^2$$

with a constant $N = N(K,d)$, by the Davis inequality we have

$$\begin{aligned} E\sup_{t\le T}\Big|\int_0^t J^h_s\,dW^\rho_s\Big| \le&\ 3E\Big(\int_0^T\sum_\rho |J^{h,\rho}_s|^2\,ds\Big)^{1/2} \\ \le&\ 3NE\Big(\sup_{s\in[0,T]}|u^h_s|_0^2\int_0^T\left(|u^h_s|_1^2 + |g_s|_0^2\right)ds\Big)^{1/2} \\ \le&\ \frac{1}{2}E\sup_{s\in[0,T]}|u^h_s|_0^2 + 5N^2E\int_0^T\left(|u^h_s|_1^2 + |g_s|_0^2\right)ds. \end{aligned} \tag{2.19}$$

Taking supremum in t in both sides of (2.16) and then using (2.17), (2.18) and (2.19), we obtain estimate (2.14). □

Remark 2.3. An easy computation using the symmetry of ψ imposed in (2.6) shows that for every $x\in\mathbb{R}^d$ and $h\neq 0$ we have $\psi^{-h}_x = \psi^h_x$. Hence the uniqueness of the solution to (2.8) proved in Theorem 2.3 implies that the processes u^{-h}_t and u^h_t agree for $t\in[0,T]$ a.s.

To prove rate of convergence results we introduce more conditions on ψ and Λ.

Notation. Let Γ denote the set of vectors λ in $\mathbb{G}$ such that the intersection of the support of $\psi_\lambda := \psi^1_\lambda$ and the support of ψ has positive Lebesgue measure in $\mathbb{R}^d$. Then Γ is a finite set.

Assumption 2.5. *Let R_λ, R^i_λ and R^{ij}_λ be defined by* (2.11)*; then for $i,j,k,l \in \{1,2,...,d\}$:*

$$\sum_{\lambda\in\Gamma} R_\lambda = 1, \quad \sum_{\lambda\in\Gamma} R^{ij}_\lambda = 0, \tag{2.20}$$

$$\sum_{\lambda\in\Gamma} \lambda_k R^i_\lambda = \delta_{i,k}, \tag{2.21}$$

$$\sum_{\lambda\in\Gamma} \lambda_k\lambda_l R^{ij}_\lambda = \delta_{\{i,j\},\{k,l\}} \quad \text{for } i\neq j, \quad \sum_{\lambda\in\Gamma} \lambda_k\lambda_l R^{ii}_\lambda = 2\delta_{\{i,i\},\{k,l\}}, \tag{2.22}$$

$$\sum_{\lambda\in\Gamma} Q^{ij,kl}_\lambda = 0 \quad \text{and} \quad \sum_{\lambda\in\Gamma} \tilde{Q}^{i,k}_\lambda = 0, \tag{2.23}$$

where

$$Q^{ij,kl}_\lambda := \int_{\mathbb{R}^d} z_k z_l D_j\psi_\lambda(z) D^*_i\psi(z)\,dz, \quad \tilde{Q}^{i,k}_\lambda := \int_{\mathbb{R}^d} z_k D_i\psi_\lambda(z)\psi(z)\,dz,$$

and for sets of indices A and B the notation $\delta_{A,B}$ means 1 when $A=B$ and 0 otherwise.

Note that if Assumption 2.5 holds true, then for any family of real numbers $X_{ij,kl}$, $i,j,k,l \in \{1,...,d\}$ such that $X_{ij,kl} = X_{ji,kl}$ we deduce from the identities (2.22) that

$$\frac{1}{2}\sum_{i,j=1}^{d}\sum_{k,l=1}^{d} X_{ij,kl}\sum_{\lambda\in\Gamma}\lambda_k\lambda_l R^{ij}_\lambda = \sum_{i,j=1}^{d} X_{ij,ij}. \tag{2.24}$$

Our main result reads as follows.

Theorem 2.4. *Let $J \geq 0$ be an integer. Let Assumptions 2.1 and 2.2 hold with $m > 2J + \frac{d}{2} + 2$. Assume also Assumption 2.3 and Assumptions 2.4 and 2.5 on ψ and Λ. Then expansion* (2.9) *and estimate* (2.10) *hold with a constant $N = N(m, J, \kappa, K, d, \psi, \Lambda)$, where $v^{(0)} = u$ is the solution of* (2.1) *with initial condition ϕ in* (2.2)*. Moreover, in the expansion* (2.9) *we have $v^{(j)}_t = 0$ for odd values of j.*

Set

$$\bar{u}^h_t(x) = \sum_{j=0}^{\bar{J}} c_j u^{h/2^j}_t(x) \quad t\in[0,T], \quad x\in\mathbb{G}_h,$$

with $\bar{J} := \lfloor \frac{J}{2} \rfloor$, $(c_0, .., c_{\bar{J}}) = (1, 0..., 0)V^{-1}$, where V^{-1} is the inverse of the $(\bar{J}+1)\times(\bar{J}+1)$ Vandermonde matrix

$$V^{ij} = 2^{-4(i-1)(j-1)}, \quad i,j = 1,2,...,\bar{J}+1.$$

We make also the following assumption.

Assumption 2.6.

$$\psi(0) = 1 \quad \text{and } \psi(\lambda) = 0 \text{ for } \lambda \in \mathbb{G}\setminus\{0\}.$$

Corollary 2.5. *Let Assumption 2.6 and the assumptions of Theorem 2.4 hold. Then*

$$E\sup_{t\in[0,T]}\sum_{x\in\mathbb{G}_h}|u_t(x) - \bar{u}^h_t(x)|^2|h|^d \leq |h|^{2J+2}NE\mathfrak{K}^2_m$$

for $|h| \in (0,1]$, with a constant $N = N(m, K, \kappa, J, T, d, \psi, \Lambda)$ independent of h, where u is the solution of (2.1)-(2.2).

3. Preliminaries

Assumptions 2.1, 2.2 and 2.4 are assumed to hold throughout this section. Recall that $|\cdot|_{0,h}$ denote the norm, and $(\cdot,\cdot)_{0,h}$ denote the inner product in $\ell_2(\mathbb{G}_h)$, i.e.,

$$|\varphi_1|_{0,h}^2 := |h|^d \sum_{x\in\mathbb{G}_h} \varphi_1^2(x)\,, \quad (\varphi_1,\varphi_2)_{0,h} := |h|^d \sum_{x\in\mathbb{G}_h} \varphi_1(x)\varphi_2(x)$$

for functions $\varphi_1, \varphi_2 \in \ell_2(\mathbb{G}_h)$.

Dividing by $|h|^d$, it is easy to see that the equation (2.8) for the finite elements approximation

$$u_t^h(y) = \sum_{x\in\mathbb{G}_h} U_t^h(x)\psi_x(y), \quad t\in[0,T],\ y\in\mathbb{R}^d,$$

can be rewritten for $(U_t^h(x))_{x\in\mathbb{G}_h}$ as

$$\begin{aligned}\mathcal{I}^h U_t^h(x) = \phi^h(x) &+ \int_0^t \big(\mathcal{L}_s^h U_s^h(x) + f_s^h(x)\big)\,ds \\ &+ \int_0^t \big(\mathcal{M}_s^{h,\rho} U_s^h(x) + g_s^{h,\rho}(x)\big)\,dW_s^\rho, \end{aligned} \tag{3.1}$$

$t\in[0,T]$, $x\in\mathbb{G}_h$, where

$$\begin{aligned}\phi^h(x) &= \int_{\mathbb{R}^d} \phi(x+hz)\psi(z)\,dz, \quad f_t^h(x) = \int_{\mathbb{R}^d} f_t(x+hz)\psi(z)\,dz \\ g_t^{h,\rho}(x) &= \int_{\mathbb{R}^d} g_t^\rho(x+hz)\psi(z)\,dz, \end{aligned} \tag{3.2}$$

and for functions φ on $\mathbb{R}^d$

$$\mathcal{I}^h\varphi(x) = \sum_{\lambda\in\Gamma} R_\lambda \varphi(x+h\lambda), \tag{3.3}$$

$$\mathcal{L}^h\varphi(x) = \sum_{\lambda\in\Gamma} \Big[\frac{1}{h^2}A_t^h(\lambda,x) + \frac{1}{h}B_t^h(\lambda,x) + C_t^h(\lambda,x)\Big]\varphi(x+h\lambda), \tag{3.4}$$

$$\mathcal{M}^{h,\rho}\varphi(x) = \sum_{\lambda\in\Gamma} \Big[\frac{1}{h}S_t^{h,\rho}(\lambda,x) + N_t^{h,\rho}(\lambda,x)\Big]\varphi(x+h\lambda), \tag{3.5}$$

with

$$\begin{aligned} A_t^h(\lambda,x) &= \int_{\mathbb{R}^d} a_t^{ij}(x+hz) D_j\psi_\lambda(z) D_i^*\psi(z)\,dz, \\ B_t^h(\lambda,x) &= \int_{\mathbb{R}^d} b_t^i(x+hz) D_i\psi_\lambda(z)\psi(z)\,dz, \quad C_t^h(\lambda,x) = \int_{\mathbb{R}^d} c_t(x+hz)\psi_\lambda(z)\psi(z)\,dz, \\ S_t^{h,\rho}(\lambda,x) &= \int_{\mathbb{R}^d} \sigma_t^{i\rho}(x+hz) D_i\psi_\lambda(z)\psi(z)\,dz, \quad N_t^{h,\rho}(\lambda,x) = \int_{\mathbb{R}^d} \nu_t^\rho(x+hz)\psi_\lambda(z)\psi(z)\,dz. \end{aligned}$$

Remark 3.1. Notice that due to the symmetry of ψ and Λ required in (2.6), equation (3.1) is invariant under the change of h to $-h$.

Remark 3.2. *Recall the definition of* Γ *introduced before Assumption 2.5. Clearly*

$$R_\lambda = 0,\ A^h_t(\lambda, x) = B^h_t(\lambda, x) = C^h_t(\lambda, x) = S^{h,\rho}_t(\lambda, x) = N^{h,\rho}_t(\lambda, x) = 0 \quad \text{for } \lambda \in \mathbb{G} \setminus \Gamma,$$

i.e., the definition of $\mathcal{I}^h$, $\mathcal{L}^h_t$ *and* $\mathcal{M}^{h,\rho}_t$ *does not change if the summation there is taken over* $\lambda \in \mathbb{G}$. *Owing to Assumption 2.1 with* $m = 0$ *and the bounds on* $R^{\alpha\beta}_\lambda$, *the operators* $\mathcal{L}^h_t$ *and* $\mathcal{M}^{h,\rho}_t$ *are bounded linear operators on* $\ell_2(\mathbb{G}_h)$ *such that for each* $h \neq 0$ *and* $t \in [0, T]$

$$|\mathcal{L}^h_t \varphi|_{0,h} \le N_h |\varphi|_{0,h}, \quad \sum_\rho |\mathcal{M}^{h,\rho}_t \varphi|^2_{0,h} \le N^2_h |\varphi|^2_{0,h}$$

for all $\varphi \in \ell_2(\mathbb{G}_h)$, *with a constant* $N_h = N(|h|, K, d, \psi, \Lambda)$. *One can similarly show that*

$$|\mathcal{I}^h \varphi|_{0,h} \le N |\varphi|_{0,h} \quad \text{for } \varphi \in \ell_2(\mathbb{G}_h), \tag{3.6}$$

with a constant $N = N(K, d, \Lambda, \psi)$ *independent of* h. *It is also easy to see that for every* $\phi \in L_2$ *and* ϕ^h *defined as in* (3.2) *we have*

$$|\phi^h|_{0,h} \le N |\phi|_{L_2}$$

with a constant $N = N(d, \Lambda, \psi)$ *which does not depend on* h; *therefore*

$$|\phi^h|^2_{0,h} + \int_0^T \Big(|f^h_t|^2_{0,h} + \sum_\rho |g^{h,\rho}_t|^2_{0,h} \Big)\, dt \le N^2 \mathcal{K}^2_0.$$

Lemma 3.1. *The inequality* (3.6) *implies that the mapping* $\mathcal{I}^h$ *is a bounded linear operator on* $\ell_2(\mathbb{G}_h)$. *Owing to Assumption 2.4 it has an inverse* $(\mathcal{I}^h)^{-1}$ *on* $\ell_2(\mathbb{G}_h)$, *and*

$$|(\mathcal{I}^h)^{-1} \varphi|_{0,h} \le \frac{1}{\delta} |\varphi|_{0,h} \quad \text{for } \varphi \in \ell_2(\mathbb{G}_h). \tag{3.7}$$

Proof. For $\varphi \in \ell_2(\mathbb{G}_h)$ and $h \neq 0$ we have

$$\begin{aligned}
(\varphi, \mathcal{I}^h \varphi)_{0,h} &= |h|^d \sum_{x \in \mathbb{G}_h} \varphi(x) \mathcal{I}^h \varphi(x) = |h|^d \sum_{x \in \mathbb{G}_h} \sum_{\lambda \in \mathbb{G}} \varphi(x) (\psi_\lambda, \psi) \varphi(x + h\lambda) \\
&= |h|^d \sum_{x \in \mathbb{G}_h} \sum_{y - x \in \mathbb{G}_h} \varphi(x) (\psi_{\frac{y-x}{h}}, \psi) \varphi(y) = |h|^d \sum_{\lambda, \mu \in \mathbb{G}} \varphi(h\mu) R_{\lambda - \mu} \varphi(h\lambda) \\
&\ge \delta |h|^d \sum_{\lambda \in \mathbb{G}} |\varphi(h\lambda)|^2 = \delta |\varphi|^2_{0,h}.
\end{aligned}$$

Together with (3.6), this estimate implies that $\mathcal{I}^h$ is invertible and that (3.7) holds. □

Remark 3.2 and Lemma 3.1 imply that equation (3.1) is an (affine) linear SDE in the Hilbert space $\ell_2(\mathbb{G}_h)$, and by well-known results on solvability of SDEs with Lipschitz continuous coefficients in Hilbert spaces, equation (3.1) has a unique $\ell_2(\mathbb{G}_h)$-valued continuons solution $(U_t)_{t \in [0,T]}$, which we call an ℓ_2-*solution* to (3.1).

Now we formulate the relationship between equations (2.8) and (3.1).

Theorem 3.2. *Let Assumption 2.4 hold. Then the following statements are valid.*

(i) Let Assumptions 2.1 and 2.2 be satisfied with $m = 0$, *and*

$$u^h_t = \sum_{x \in \mathbb{G}_h} U^h_t(x) \psi^h_x, \quad t \in [0, T] \tag{3.8}$$

be the unique V_h-solution of (2.8); then $(U^h_t)_{t\in[0,T]}$ is the unique ℓ_2-solution of (3.1).

(ii) Let Assumption 2.1 hold with $m=0$. Let Φ be an $\ell_2(\mathbb{G}_h)$-valued $\mathcal{F}_0$-measurable random variable, and let $F=(F_t)_{t\in[0,T]}$ and $G^\rho=(G^\rho_t)_{[0,T]}$ be $\ell_2(\mathbb{G}_h)$-valued adapted processes such that almost surely

$$\mathcal{K}^2_{0,h}:=|\Phi|^2_{0,h}+\int_0^T\Big(|F_t|^2_{0,h}+\sum_\rho|G^\rho_t|^2_{0,h}\Big)\,dt<\infty.$$

Then equation (3.1) with Φ, F and G^ρ in place of ϕ^h, f^h and $g^{\rho,h}$, respectively, has a unique $\ell_2(\mathbb{G}_h)$-solution $U^h=(U^h_t)_{t\in[0,T]}$. Moreover, if Assumption 2.3 also holds then

$$E\sup_{t\in[0,T]}|U^h_t|^2_{0,h}\le NE\mathcal{K}^2_{0,h} \tag{3.9}$$

with a constant $N=N(K,d,\kappa,\delta,\Lambda,\psi)$ which does not depend on h.

Proof. (i) Substituting (3.8) into equation (2.8), then dividing both sides of the equation by $|h|^d$ we obtain equation (3.1) for U^h by simple calculation. Hence by Remark 3.2 we can see that U^h is the unique $\ell_2(\mathbb{G})$-valued solution to (3.1).

To prove (ii) we use Remark 3.1 on the invertibility of $\mathcal{I}^h$ and a standard result on solvability of SDEs in Hilbert spaces to see that equation (3.1) with Φ, F and G^ρ has a unique $\ell_2(\mathbb{G})$-valued solution U^h. We claim that $u^h_t(\cdot)=\sum_{y\in\mathbb{G}}U^h_t(y)\psi^h_y(\cdot)$ is the V_h-valued solution of (2.8) with

$$\phi(\cdot):=\sum_{y\in\mathbb{G}_h}(\mathcal{I}^h)^{-1}\Phi(y)\psi^h_y(\cdot),\quad f_t(\cdot):=\sum_{y\in\mathbb{G}_h}(\mathcal{I}^h)^{-1}F_t(y)\psi^h_y(\cdot),$$

and

$$g^\rho_t(\cdot):=\sum_{y\in\mathbb{G}_h}(\mathcal{I}^h)^{-1}G^\rho_t(y)\psi^h_y(\cdot),$$

respectively. Indeed, (3.3) yields

$$\begin{aligned}|h|^{-d}(\phi,\psi^h_x)&=|h|^{-d}\sum_{y\in\mathbb{G}_h}(\psi^h_y,\psi^h_x)(\mathcal{I}^h)^{-1}\Phi(y)=\sum_{y\in\mathbb{G}_h}R_{\frac{y-x}{h}}(\mathcal{I}^h)^{-1}\Phi(y)\\&=\mathcal{I}^h\{(\mathcal{I}^h)^{-1}\Phi\}(x)=\Phi(x),\quad x\in\mathbb{G}_h.\end{aligned}$$

In the same way we have

$$|h|^{-d}(f_t,\psi^h_x)=F_t(x),\quad |h|^{-d}(g^\rho_t,\psi^h_x)=G^\rho_t(x)\quad\text{for } x\in\mathbb{G}_h,$$

which proves the claim. Using Remarks 2.2 and 3.1 we deduce

$$\|\phi\|\le N|(\mathcal{I}^h)^{-1}\Phi|_{0,h}\le\frac{N}{\delta}|\Phi|_{0,h},\quad \|f_t\|\le N|(\mathcal{I}^h)^{-1}F_t|_{0,h}\le\frac{N}{\delta}|F_t|_{0,h},$$

$$\sum_\rho\|g^\rho_t\|^2\le N^2\sum_\rho|(\mathcal{I}^h)^{-1}G^\rho_t|^2_{0,h}\le\frac{N^2}{\delta^2}\sum_\rho|G^\rho_t|^2_{0,h}$$

with a constant $N=N(\psi,\Lambda)$. Hence by Theorem 2.3

$$E\sup_{t\le T}\|u^h_t\|^2\le NE|\phi|^2_{0,h}+NE\int_0^T\Big(|F_t|^2_{0,h}+\sum_\rho|G^\rho_t|^2_{0,h}\Big)\,dt$$

with $N = N(K, T, \kappa, d, \psi, \Lambda, \delta)$ independent of h, which by virtue of Remark 2.2 implies estimate (3.9). □

4. Coefficients of the expansion

Notice that the lattice $\mathbb{G}_h$ and the space V_h can be "shifted" to any $y \in \mathbb{R}^d$, i.e., we can consider $\mathbb{G}_h(y) := \mathbb{G}_h + y$ and

$$V_h(y) := \Big\{ \sum_{x \in \mathbb{G}_h(y)} U(x)\psi^h_x : (U(x))_{x \in \mathbb{G}_h(y)} \in \ell_2(\mathbb{G}_h(y)) \Big\}.$$

Thus equation (2.8) for $u^h = \sum_{x \in \mathbb{G}_h(y)} U(x)\psi^h_x$ should be satisfied for ψ_x, $x \in \mathbb{G}_h(y)$. Correspondingly, equation (3.1) can be considered for all $x \in \mathbb{R}^d$ instead of $x \in \mathbb{G}_h$.

To determine the coefficients $(v^{(j)})_{j=1}^k$ in the expansion (2.9) we differentiate formally (3.1) in the parameter h, j times, for $j = 1, 2, ..., J$, and consider the system of SPDEs we obtain for the formal derivatives

$$u^{(j)} = D^j_h U^h\big|_{h=0}, \tag{4.1}$$

where D_h denotes differentiation in h. To this end given an integer $n \geq 1$ let us first investigate the operators $\mathcal{I}^{(n)}$, $\mathcal{L}^{(n)}_t$ and $\mathcal{M}^{(n)\rho}_t$ defined by

$$\begin{aligned} &\mathcal{I}^{(n)}\varphi(x) = D^n_h \mathcal{I}^h \varphi(x)\big|_{h=0}, \quad \mathcal{L}^{(n)}_t \varphi(x) = D^n_h \mathcal{L}^h_t \varphi(x)\big|_{h=0}, \\ &\mathcal{M}^{(n)\rho}_t \varphi(x) = D^n_h \mathcal{M}^{h,\rho}_t \varphi(x)\big|_{h=0} \end{aligned} \tag{4.2}$$

for $\varphi \in C^\infty_0$.

Lemma 4.1. *Let Assumption 2.1 hold with $m \geq n + l + 2$ for nonnegative integers l and n. Let Assumption 2.5. also hold. Then for $\varphi \in C^\infty_0$ and $t \in [0, T]$ we have*

$$|\mathcal{I}^{(n)}\varphi|_l \leq N|\varphi|_{l+n}, \quad |\mathcal{L}^{(n)}_t \varphi|_l \leq N|\varphi|_{l+2+n}, \quad \sum_\rho |\mathcal{M}^{(n)\rho}_t \varphi|^2_l \leq N^2 |\varphi|_{l+1+n} \tag{4.3}$$

with a constant $N = N(K, d, l, n, \Lambda, \Psi)$ which does not depend on h.

Proof. Clearly, $\mathcal{I}^{(n)} = \sum_{\lambda \in \Gamma} R_\lambda \partial^n_\lambda \varphi$, where

$$\partial_\lambda \varphi := \sum_{i=1}^d \lambda^i D_i \varphi. \tag{4.4}$$

This shows the existence of a constant $N = N(\Lambda, \psi, d, n)$ such that the first estimate in (4.3) holds. To prove the second estimate we first claim the existence of a constant $N = N(K, d, l, \Lambda, \psi)$ such that

$$\Big| D^n_h \Phi_t(h, \cdot)\big|_{h=0} \Big|_l \leq N|\varphi|_{l+n+2} \tag{4.5}$$

for

$$\Phi_t(h, x) := h^{-2} \sum_{\lambda \in \Gamma} \varphi(x + h\lambda) \int_{\mathbb{R}^d} a^{ij}_t(x + hz) D_j \psi_\lambda(z) D^*_i \psi(z)\, dz.$$

Recall the definition of R^{ij}_λ given in (2.11). To prove (4.5) we write $\Phi_t(h,x)=\sum_{i=1}^3\Phi^{(i)}_t(h,x)$ for $h\neq 0$ with

$$\begin{aligned}\Phi^{(1)}_t(h,x)&=h^{-2}\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d}a^{ij}_t(x)D_j\psi_\lambda(z)D^*_i\psi(z)\,dz\\&=h^{-2}a^{ij}_t(x)\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)R^{ij}_\lambda,\\ \Phi^{(2)}_t(h,x)&=h^{-1}\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d}\sum_{k=1}^d D_ka^{ij}_t(x)z_kD_j\psi_\lambda(z)D^*_i\psi(z)\,dz,\\&=h^{-1}\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)Da^{ij}_t(x)S^{ij}_\lambda,\end{aligned}$$

for

$$S^{ij}_\lambda:=\int_{\mathbb{R}^d}zD_j\psi_\lambda(z)D^*_i\psi(z)\,dz\in\mathbb{R}^d,$$

and

$$\Phi^{(3)}_t(h,x)=\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d}\int_0^1(1-\vartheta)D_{kl}a^{ij}_t(x+h\vartheta z)z^kz^lD_j\psi_\lambda(z)D^*_i\psi(z)\,d\vartheta\,dz,$$

where $D_{kl}:=D_kD_l$. Here we used Taylor's formula

$$f(h)=\sum_{i=0}^n\frac{h^i}{i!}f^{(i)}(0)+\frac{h^{n+1}}{n!}\int_0^1(1-\theta)^nf^{(n+1)}(h\theta)\,d\theta \tag{4.6}$$

with $n=1$ and $f(h):=a^{ij}_t(x+h\lambda)$.

Note that Lemma 2.2 and (2.20) in Assumption 2.5 imply

$$\begin{aligned}\Phi^{(1)}_t(h,x)&=\frac12a^{ij}_t(x)\sum_{\lambda\in\Gamma}R^{ij}_\lambda h^{-2}(\varphi(x+h\lambda)-2\varphi(x)+\varphi(x-h\lambda))\\&=\frac12a^{ij}_t(x)\sum_{\lambda\in\Gamma}R^{ij}_\lambda\int_0^1\int_0^1\partial^2_\lambda\varphi(x+h\lambda(\theta_1-\theta_2))\,d\theta_1d\theta_2.\end{aligned}\tag{4.7}$$

To rewrite $\Phi^{(2)}_t(h,x)$ note that $S^{ij}_{-\lambda}=-S^{ij}_\lambda$; indeed since $\psi(-x)=\psi(x)$ the change of variables $y=-z$ implies that

$$\begin{aligned}S^{ij}_{-\lambda}&=\int_{\mathbb{R}^d}zD_j\psi(z+\lambda)D^*_i\psi(z)dz=-\int_{\mathbb{R}^d}yD_j\psi(-y+\lambda)D^*_i\psi(-y)dy\\&=-\int_{\mathbb{R}^d}yD_j\psi(y-\lambda)D^*_i\psi(y)dy=-S^{ij}_\lambda.\end{aligned}\tag{4.8}$$

Furthermore, an obvious change of variables, (4.8) and Lemma 2.2 yield

$$\begin{aligned}S^{ji}_\lambda&=\int_{\mathbb{R}^d}zD_i\psi(z-\lambda)D^*_j\psi(z)dz=\int_{\mathbb{R}^d}(z+\lambda)D_i\psi(z)D^*_j\psi(z+\lambda)dz\\&=\int_{\mathbb{R}^d}zD^*_i\psi(z)D_j\psi_{-\lambda}(z)dz+\lambda\int_{\mathbb{R}^d}D^*_i\psi(z)D_j\psi_{-\lambda}(z)dz\end{aligned}$$

$$= S^{ij}_{-\lambda} + \lambda R^{ij}_{-\lambda} = -S^{ij}_{\lambda} + \lambda R^{ij}_{\lambda}.$$

This implies

$$S^{ji}_{\lambda} + S^{ij}_{\lambda} = \lambda R^{ij}_{\lambda}, \quad i,j = 1,...,d.$$

Note that since $a^{ij}_t(x) = a^{ji}_t(x)$, we deduce

$$Da^{ij}_t(x)S^{ij}_{\lambda} = Da^{ij}_t(x)S^{ji}_{\lambda} = \frac{1}{2}Da^{ij}_t(x)\lambda R^{ij}_{\lambda} = \frac{1}{2}R^{ij}_{\lambda}\partial_{\lambda}a^{ij}_t(x), \tag{4.9}$$

for $\partial_\lambda F$ defined by (4.4). Thus the equations (4.8) and (4.9) imply

$$\begin{aligned}\Phi^{(2)}_t(h,x) &= \frac{1}{2}\sum_{\lambda\in\Gamma} h^{-1}(\varphi(x+h\lambda) - \varphi(x-h\lambda))Da^{ij}_t(x)S^{ij}_{\lambda} \\ &= \frac{1}{4}\sum_{\lambda\in\Gamma} R^{ij}_{\lambda}\partial_{\lambda}a^{ij}_t(x)\; 2\int_0^1 \partial_{\lambda}\varphi(x+h\lambda(2\theta-1))\,d\theta. \end{aligned} \tag{4.10}$$

From (4.7) and (4.10) we get

$$D^n_h\Phi^{(1)}_t(h,x)\big|_{h=0} = \frac{1}{2}a^{ij}_t(x)\sum_{\lambda\in\Gamma} R^{ij}_{\lambda}\int_0^1\int_0^1 \partial^{n+2}_{\lambda}\varphi(x)(\theta_1-\theta_2)^n\,d\theta_1 d\theta_2,$$

$$D^n_h\Phi^{(2)}_t(h,x)\big|_{h=0} = \frac{1}{2}\sum_{\lambda\in\Gamma} R^{ij}_{\lambda}\partial_{\lambda}a^{ij}\int_0^1 \partial^{n+1}_{\lambda}\varphi(x)(2\theta-1)^n\,d\theta.$$

Furthermore, the definition of $\Phi^{(3)}_t(h,x)$ yields

$$\begin{aligned} &D^n_h\Phi^{(3)}_t(h,x)\big|_{h=0} \\ &= \sum_{\lambda\in\Gamma}\sum_{k=0}^{n}\binom{n}{k}\partial^{n-k}_{\lambda}\varphi(x)\int_{\mathbb{R}^d}\int_0^1 (1-\theta)\theta^k\partial^k_z D_{kl}a^{ij}_t(x)z^k z^l D_j\psi_{\lambda}(z)D^*_i\psi(z)\,d\theta\,dz. \end{aligned}$$

Using Assumption 2.1 and Remark 2.1, this completes the proof of (4.5).

Taylor's formula (4.6) with $n=0$ and $f(h) := b^i_t(x+hz)$ implies

$$\begin{aligned}\tilde\Phi_t(h,x) :&= h^{-1}\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d} b^i_t(x+hz)D_i\psi_{\lambda}(z)\psi(z)dz \\ &= \Phi^{(4)}_t(h,x) + \Phi^{(5)}_t(h,x),\end{aligned}$$

with

$$\Phi^{(4)}_t(h,x) = h^{-1}b^i_t(x)\sum_{\lambda\in\Gamma}\varphi(x+h\lambda)R^i_{\lambda},$$

$$\Phi^{(5)}_t(h,x) = \sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d}\int_0^1(1-\theta)\sum_{k=1}^{d} D_k b^i_t(x+h\theta z)z_k D_i\psi_{\lambda}(z)\psi(z)d\theta dz.$$

Using Lemma 2.2 and computations similar to those used to prove (4.5) we deduce that

$$\Phi^{(4)}_t(h,x) = \frac{1}{2}\sum_{\lambda\in\Gamma} h^{-1}\big[\varphi(x+h\lambda) - \varphi(x-h\lambda)\big]b^i_t(x)R^i_{\lambda}$$

$$= b_t^i(x)\sum_{\lambda\in\Gamma} R_\lambda^i \int_0^1 \partial_\lambda\varphi\big(x+h\lambda(2\theta-1)\big)d\theta,$$

which yields

$$D_h^n\Phi_t^{(4)}(h,x)\big|_{h=0} = b_t^i(x)\sum_{\lambda\in\Gamma} R_\lambda^i\partial_\lambda^{n+1}\varphi(x)\int_0^1(2\theta-1)^n d\theta.$$

Furthermore, the definition of $\Phi^{(5)}(h,x)$ implies

$$D_h^n\Phi_t^{(5)}(h,x)\big|_{h=0}$$
$$= \sum_{\lambda\in\Gamma}\sum_{\alpha=0}^n \binom{n}{\alpha}\int_{\mathbb{R}^d}\int_0^1(1-\theta)\partial_\lambda^{n-\alpha}\varphi(x)\theta^\alpha\partial_z^\alpha D_\alpha b_t^i(x)z^\alpha D_i\psi_\lambda(z)\psi(z)\,d\theta\,dz.$$

This implies the existence of a constant $N=N(K,d,l,\Lambda,\psi)$ which does not depend on h such that

$$\Big|D_h^n\tilde{\Phi}_t(h,\cdot)\big|_{h=0}\Big|_l \le N|\varphi|_{l+n+1}. \tag{4.11}$$

Finally, let

$$\Phi_t^{(6)}(h,x) := \sum_{\lambda\in\Gamma}\varphi(x+h\lambda)\int_{\mathbb{R}^d} c_t(x+hz)\psi_\lambda(z)\psi(z)dz.$$

Then we have

$$D_h^n\Phi_t^{(6)}(h,x)\big|_{h=0} = \sum_{\lambda\in\Gamma}\sum_{\alpha=0}^n\binom{n}{\alpha}\partial_\lambda^{n-\alpha}\varphi(x)\int_{\mathbb{R}^d}\partial_z^\alpha c_t(x)\psi_\lambda(z)\psi(z)\,dz.$$

so that

$$\Big|D_h^n\Phi_t^{(6)}(h,\cdot)\big|_{h=0}\Big|_l \le N|\varphi|_{l+n} \tag{4.12}$$

for some constant N as above.

Since $\mathcal{L}_t^h\varphi(x) = \Phi_t(h,x)+\tilde{\Phi}_t(h,x)+\Phi_t^{(6)}(h,x)$, the inequalities (4.5), (4.11) and (4.12) imply that $\mathcal{L}_t^{(n)}$ satisfies the estimate in (4.3); the upper estimates of $\mathcal{M}_t^{(n),\rho}$ can be proved similarly. □

For an integer $k\ge 0$ define the operators $\hat{L}_t^{(k)h}$, $\hat{M}_t^{(k)h,\rho}$ and $\hat{I}^{(k)h}$ by

$$\hat{L}_t^{(k)h}\varphi = \mathcal{L}_t^h\varphi - \sum_{i=0}^k\frac{h^i}{i!}\mathcal{L}_t^{(i)}\varphi, \quad \hat{M}_t^{(k)h,\rho}\varphi = \mathcal{M}_t^{h,\rho}\varphi - \sum_{i=0}^k\frac{h^i}{i!}\mathcal{M}_t^{(i)\rho}\varphi,$$
$$\hat{I}^{(k)h}\varphi = \mathcal{I}^h\varphi - \sum_{i=0}^k\frac{h^i}{i!}\mathcal{I}^{(i)}\varphi, \tag{4.13}$$

where $\mathcal{L}_t^{(0)} := \mathcal{L}_t$, $\mathcal{M}_t^{(0),\rho} := \mathcal{M}_t^\rho$, and $\mathcal{I}^{(0)}$ is the identity operator.

Lemma 4.2. *Let Assumption 2.1 hold with $m\ge k+l+3$ for nonnegative integers k and n. Let Assumption 2.5 also hold. Then for $\varphi\in C_0^\infty$ we have*

$$|\hat{L}_t^{(k)h}\varphi|_l \le N|h|^{k+1}|\varphi|_{l+k+3}, \quad \sum_\rho|\hat{M}_t^{(k)h,\rho}\varphi|_l^2 \le N^2|h|^{2k+2}|\varphi|_{l+k+2}^2.$$

$$|\hat{I}^{(k)h}\varphi|_l \leq N|h|^{k+1}|\varphi|_{k+1},$$

for a constant N which does not depend on h.

Proof. We obtain the estimate for $\hat{L}_t^{(k)h}$ by applying Taylor's formula (4.6) to $f(h) := \Phi_t^{(i)}(h,x)$ for $i=1,...,6$ defined in the proof of Lemma 4.1, and by estimating the remainder term

$$\frac{h^{k+1}}{k!}\int_0^1 (1-\theta)^k f^{(k+1)}(h\theta)\,d\theta$$

using the Minkowski inequality. Recall that $\mathcal{L}_t\varphi(x) = \mathcal{L}_t^{(0)}\varphi(x)$. Using Assumption 2.5 we prove that $\mathcal{L}_t^{(0)}\varphi(x) = \lim_{h\to 0}\mathcal{L}_t^h\varphi(x)$. We have $\mathcal{L}_t^h\varphi(x) = \sum_{i=1}^6 \Phi_t^{(i)}(h,x)$ for $h\neq 0$. The proof of Lemma 4.1 shows that $\tilde{\Phi}_t^{(i)}(0,x) := \lim_{h\to 0}\Phi_t^{(i)}(h,x)$ exist and we identify these limits. Using (4.7), (4.10) and (2.24) with $X_{ij,kl} = a_t^{ij}(x)D_{kl}\varphi(x)$ (resp. $X_{ij,kl} = \partial_k a_t^{ij}(x)\partial_l\varphi(x)$) we deduce

$$\begin{aligned}
\tilde{\Phi}_t^{(1)}(0,x) =& \sum_{i,j}\frac{1}{2}a_t^{ij}(x)\sum_{k,l}D_kD_l\varphi(x)\sum_{\lambda\in\Gamma}\lambda_k\lambda_l R_\lambda^{ij} = \sum_{i,j}a_t^{ij}(x)D_{ij}\varphi(x)\\
\tilde{\Phi}_t^{(2)}(0,x) =& \frac{1}{2}\sum_{i,j}\sum_{k,l}\partial_k a_t^{ij}(x)\partial_l\varphi(x)\sum_{\lambda\in\Gamma}\lambda_k\lambda_l R_\lambda^{ij} = \sum_{i,j}\partial_i a_t^{ij}(x)\partial_j\varphi(x),
\end{aligned}$$

which implies that $\tilde{\Phi}_t^{(1)}(0,x)+\tilde{\Phi}_t^{(2)}(0,x) = D_i\big(a_t^{ij}D_j\varphi\big)(x)$. The first identity in (2.23) (resp. (2.21), the second identity in (2.23) and the first identity in (2.20)) imply

$$\begin{aligned}
\tilde{\Phi}_t^{(3)}(0,x) =& \frac{1}{2}\varphi(x)\sum_{k,l}\sum_{i,j}D_{kl}a_t^{ij}(x)\sum_{\lambda\in\Gamma}Q_\lambda^{ij,kl} = 0,\\
\tilde{\Phi}_t^{(4)}(0,x) =& \sum_i b_t^i(x)\sum_k\partial_k\varphi(x)\sum_{\lambda\in\Gamma}R_\lambda^i\lambda_k = \sum_i b_t^i(x)\partial_i\varphi(x),\\
\tilde{\Phi}_t^{(5)}(0,x) =& \frac{1}{2}\varphi(x)\sum_k\sum_i D_kb_t^i(x)\sum_{\lambda\in\Gamma}\tilde{Q}_\lambda^{i,k} = 0,\\
\tilde{\Phi}_t^{(6)}(0,x) =& \varphi(x)c_t(x)\sum_{\lambda\in\Gamma}R_\lambda = \varphi(x)c_t(x).
\end{aligned}$$

This completes the identification of $\mathcal{L}_t$ as the limit of $\mathcal{L}_t^h$. Using once more the Minkowski inequality and usual estimates, we prove the upper estimates of the H^l norm of $\hat{L}_t^{(k)h}\varphi$. The other estimates can be proved similarly. □

Assume that Assumption 2.2 is satisfied with $m\geq J+1$ for an integer $J\geq 0$. A simple computation made for differentiable functions in place of the formal ones introduced in (4.1) shows the following identities

$$\phi^{(i)}(x) = \int_{\mathbb{R}^d}\partial_z^i\phi(x)\psi(z)\,dz,\ \ f_t^{(i)}(x) = \int_{\mathbb{R}^d}\partial_z^i f_t(x)\psi(z)\,dz,\ \ g_t^{(i)\rho}(x) = \int_{\mathbb{R}^d}\partial_z^i g_t^\rho(x)\psi(z)\,dz,$$

where $\partial_z^i\varphi$ is defined iterating (4.4), while ϕ^h, f_t^h and $g_t^{h,\rho}$ are defined in (3.2). Set

$$\hat\phi^{(J)h} := \phi^h - \sum_{i=0}^J \frac{h^i}{i!}\phi^{(i)},\ \hat f_t^{(J)h} := f_t^h - \sum_{i=0}^J \frac{h^i}{i!} f_t^{(i)} \text{ and } \hat g_t^{(J)h\rho} := g_t^{h,\rho} - \sum_{i=0}^J g_t^{(i)\rho}\frac{h^i}{i!}. \quad (4.14)$$

Lemma 4.3. *Let Assumption 2.1 holds with $m \ge l+J+1$ for nonnegative integers J and l. Then there is a constant $N = N(J,l,d,\psi)$ independent of h such that*

$$|\hat\phi^{(J)h}|_l \le |h|^{J+1}N|\phi|_{l+1+J}, \quad |\hat f_t^{(J)h}|_l \le N|h|^{J+1}|f_t|_{l+1+J}, \quad |\hat g_t^{(J)h\rho}|_l \le N|h|^{J+1}|g_t^\rho|_{l+1+J}.$$

Proof. Clearly, it suffices to prove the estimate for $\hat\phi^{(J)h}$, and we may assume that $\phi \in C_0^\infty$. Applying Taylor formula (4.6) to $f(h) = \phi^h(x)$ for the remainder term we have

$$\hat\phi^{(J)h}(x) = \frac{h^{J+1}}{J!}\int_0^1 \int_{\mathbb{R}^d} (1-\theta)^J \partial_z^{J+1}\phi(x+\theta hz)\psi(z)\,dz.$$

Hence by Minkowski's inequality and the shift invariance of the Lebesgue measure we get

$$|\hat\phi^{(J)h}(x)| \le \frac{h^{J+1}}{J!}\int_0^1 \int_{\mathbb{R}^d} (1-\theta)^J |\partial_z^{J+1}\phi(\cdot+\theta hz)|_l|\psi(z)|\,dz \le Nh^{J+1}|\phi|_{l+J+1}$$

with a constant $N = N(J,m,d,\psi)$ which does not depend on h. □

Differentiating formally equation (3.1) with respect to h at 0 and using the definition of $\mathcal{I}^{(i)}$ in (4.2), we obtain the following system of SPDEs:

$$dv_t^{(i)} + \sum_{1\le j\le i}\binom{i}{j}\mathcal{I}^{(j)}dv_t^{(i-j)} = \Big\{\mathcal{L}_t^{(0)}v_t^{(i)} + f_t^{(i)} + \sum_{1\le j\le i}\binom{i}{j}\mathcal{L}_t^{(j)}v_t^{(i-j)}\Big\}\,dt$$

$$+ \Big\{\mathcal{M}_t^{(0)\rho}v_t^{(i)} + g_t^{(i)\rho} + \sum_{1\le j\le i}\binom{i}{j}\mathcal{M}_t^{(j)\rho}v_t^{(i-j)}\Big\}\,dW_t^\rho, \quad (4.15)$$

$$v_0^{(i)}(x) = \phi^{(i)}(x), \quad (4.16)$$

for $i = 1,2,....,J$, $t\in[0,T]$ and $x\in\mathbb{R}^d$, where $\mathcal{L}_t^{(0)} = \mathcal{L}_t$, $\mathcal{M}_t^{(0)\rho} = \mathcal{M}_t^\rho$, and $v^{(0)} = u$ is the solution to (2.1)-(2.2).

Theorem 4.4. *Let Assumptions 2.1 and 2.2 hold with $m \ge J+1$ for an integer $J\ge 1$. Let Assumptions 2.3 through 2.5 be also satisfied. Then (4.15)-(4.16) has a unique solution $(v^{(0)},...,v^{(J)})$ such that $v^{(n)}\in\mathbb{W}_2^{m+1-n}(0,T)$ for every $n = 0,1,...,J$. Moreover, $v^{(n)}$ is a H^{m-n}-valued continuous adapted process, and for every $n = 0,1,...,J$*

$$E\sup_{t\le T}|v_t^{(n)}|_{m-n}^2 + E\int_0^T |v_t^{(n)}|_{m+1-n}^2\,dt \le NE\mathfrak{K}_m^2 \quad (4.17)$$

with a constant $N = N(m,J,d,T,\Lambda,\psi,\kappa)$ independent of h, and $\mathfrak{K}_m$ defined in (2.5).

Proof. The proof is based on an induction argument. We can solve this system consecutively for $i = 1,2,...,J$, by noticing that for each $i = 1,2,...,k$ the equation for $v^{(i)}$ does not contain $v^{(n)}$ for $n = i+1,...,J$. For $i = 1$ we have $v_0^{(1)} = \phi^{(1)}$ and

$$dv_t^{(1)} + \mathcal{I}^{(1)}du_t = \{\mathcal{L}_t v_t^{(1)} + f_t^{(1)} + \mathcal{L}_t^{(1)}u_t\}\,dt$$

$$+ \{\mathcal{M}_t^\rho v_t^{(1)} + g_t^{(1)\rho} + \mathcal{M}_t^{(1)\rho} u_t\}\, dW_t^\rho,$$

i.e.,

$$dv_t^{(1)} = (\mathcal{L}_t v_t^{(1)} + \bar{f}_t^{(1)})\, dt + (\mathcal{M}_t^\rho v_t^{(1)} + \bar{g}_t^{(1)\rho})\, dW_t^\rho,$$

with

$$\begin{aligned} \bar{f}_t^{(1)} &:= f_t^{(1)} - \mathcal{I}^{(1)} f_t + (\mathcal{L}_t^{(1)} - \mathcal{I}^{(1)}\mathcal{L}_t) u_t, \\ \bar{g}_t^{(1)\rho} &:= g_t^{(1)\rho} - \mathcal{I}^{(1)} g_t^\rho + (\mathcal{M}_t^{(1)\rho} - \mathcal{I}^{(1)}\mathcal{M}_t^\rho) u_t. \end{aligned}$$

By virtue of Theorem 2.1 this equation has a unique solution $v^{(1)}$ and

$$\begin{aligned} &E \sup_{t\le T} |v_t^{(1)}|_{m-1}^2 + E\int_0^T |v_t^{(1)}|_m^2\, dt \\ &\qquad \le NE|\phi^{(1)}|_{m-1}^2 + NE\int_0^T \big(|\bar{f}_t^{(1)}|_{m-2}^2 + |\bar{g}_t^{(1)}|_{m-1}^2\big)\, dt. \end{aligned}$$

Clearly, Lemma 4.1 implies

$$|\phi^{(1)}|_{m-1}^2 \le N|\phi|_m^2, \quad |f_t^{(1)}|_{m-2} + |\mathcal{I}^{(1)} f_t|_{m-2} \le N|f_t|_{m-1}, \quad |g_t^{(1)\rho} - \mathcal{I}^{(1)} g_t^\rho|_{m-1} \le N|g_t^\rho|_m,$$
$$|(\mathcal{L}_t^{(1)} - \mathcal{I}^{(1)}\mathcal{L}_t)u|_{m-2} \le N|u|_{m+1}, \quad \sum_\rho |(\mathcal{M}_t^{(1)\rho} - \mathcal{I}^{(1)}\mathcal{M}_t^\rho)u|_{m-1}^2 \le N^2|u|_{m+1}^2,$$

with a constant $N = N(d, K, \Lambda, \psi, m)$ which does not depend on h. Hence for $m \ge 1$

$$\begin{aligned} &E \sup_{t\le T} |v_t^{(1)}|_{m-1}^2 + E\int_0^T |v_t^{(1)}|_m^2\, dt \\ &\qquad \le NE|\phi|_m^2 + NE\int_0^T \big(|f_t|_{m-1}^2 + |g_t|_m^2 + |u_t|_{m+1}^2\big)\, dt \le NE\mathfrak{K}_m^2. \end{aligned}$$

Let $j \ge 2$. Assume that for every $i < j$ the equation for $v^{(i)}$ has a unique solution such that (4.15) holds and that its equation can be written as $v_0^{(i)} = \phi^{(i)}$ and

$$dv_t^{(i)} = (\mathcal{L}_t v_t^{(i)} + \bar{f}_t^{(i)})\, dt + (\mathcal{M}_t^\rho v_t^{(i)} + \bar{g}_t^{(i)\rho})\, dW_t^\rho$$

with adapted processes $\bar{f}^{(i)}$ and $\bar{g}^{(i)\rho}$ taking values in H^{m-i-1} and H^{m-i} respectively, such that

$$E\int_0^T \big(|\bar{f}_t^{(i)}|_{m-i-1}^2 + |\bar{g}_t^{(i)}|_{m-i}^2\big)\, dt \le NE\mathfrak{K}_m^2 \tag{4.18}$$

with a constant $N = N(K, J, m, d, T, \kappa, \Lambda, \psi)$ independent of h. Hence

$$E\Big(\sup_{t\in[0,T]} |v_t^{(i)}|_{m-i}^2 + \int_0^T |v_t^{(i)}|_{m+1-i}^2 dt\Big) \le NE\mathfrak{K}_m^2, \quad i = 1, ..., j-1. \tag{4.19}$$

Then for $v^{(j)}$ we have

$$dv_t^{(j)} = (\mathcal{L}_t v_t^{(j)} + \bar{f}_t^{(j)})\, dt + (\mathcal{M}_t^\rho v_t^{(j)} + \bar{g}_t^{(j)\rho})\, dW_t^\rho, \quad v_0^{(j)} = \phi^{(j)}, \tag{4.20}$$

with

$$\bar{f}_t^{(j)} := f_t^{(j)} + \sum_{1\le i\le j} \binom{j}{i} \big(\mathcal{L}_t^{(i)} - \mathcal{I}^{(i)}\mathcal{L}_t\big) v_t^{(j-i)} - \sum_{1\le i\le j} \binom{j}{i} \mathcal{I}^{(i)} \bar{f}_t^{(j-i)},$$

$$\bar{g}_t^{(j)\rho} := g_t^{(j)\rho} + \sum_{1\le i\le j} \binom{j}{i} \big(\mathcal{M}_t^{(i)\rho} - \mathcal{I}^{(i)}\mathcal{M}_t^{\rho}\big) v_t^{(j-i)} - \sum_{1\le i\le j} \binom{j}{i} \mathcal{I}^{(i)} \bar{g}_t^{(j-i)\rho}.$$

Note that $|f_t^{(j)}|_{m-1-j} \le N|f_t|_{m-1}$; by virtue of Lemma 4.1, and by the inequalities (4.18) and (4.19) we have

$$E\int_0^T |(\mathcal{L}_t^{(i)} - \mathcal{I}^{(i)}\mathcal{L}_t)v_t^{(j-i)}|^2_{m-j-1}\,dt \le NE\int_0^T |v_t^{(j-i)}|^2_{m-j+1+i}\,dt \le NE\mathfrak{K}_m^2,$$

$$E\int_0^T |\mathcal{I}^{(i)}\bar{f}_t^{(j-i)}|^2_{m-j-1}\,dt \le NE\int_0^T |\bar{f}_t^{(j-i)}|_{m-j+i-1}\,dt \le NE\mathfrak{K}_m^2,$$

where $N = N(K, J, d, T, \kappa, \psi, \Lambda)$ denotes a constant which can be different on each occurrence. Consequently,

$$E\int_0^T |\bar{f}_t^{(j)}|^2_{m-j-1}\,dt \le NE\mathfrak{K}_m^2,$$

and we can get similarly

$$E\int_0^T |\bar{g}_t^{(j)}|^2_{m-j}\,dt \le NE\mathfrak{K}_m^2.$$

Hence (4.20) has a unique solution $v^{(j)}$, and Theorem 2.1 implies that the estimate (4.17) holds for $v^{(j)}$ in place of $v^{(n)}$. This completes the induction and the proof of the theorem. □

Recall that the norm $|\cdot|_{0,h}$ has been defined in (2.7).

Corollary 4.5. *Let Assumptions 2.1 and 2.2 hold with $m > \frac{d}{2}+J+1$ for an integer $J \ge 1$. Let Assumptions 2.3 through 2.5 be also satisfied. Then almost surely $v^{(i)}$ is continuous in $(t,x) \in [0,T]\times\mathbb{R}^d$ for $i \le J$, and its restriction to $\mathbb{G}_h$ is an adapted continuous $\ell_2(\mathbb{G}_h)$-valued process. Moreover, almost surely (4.15)-(4.16) hold for all $x \in \mathbb{R}^d$ and $t \in [0,T]$, and*

$$E\sup_{t\in[0,T]}\sup_x |v_t^{(j)}(x)|^2 + E\sup_{t\le T}|v_t^{(j)}|^2_{0,h} \le NE\mathfrak{K}_m^2, \quad j = 1,2,...,J.$$

for some constant $N = N(m, J, d, T, \Lambda, \psi, \kappa)$ independent of h.

One can obtain this corollary from Theorem 4.4 by a standard application of Sobolev's embedding of H^m into C_b^2 for $m > 2 + d/2$ and by using the following known result; see, for example [7], Lemma 4.2.

Lemma 4.6. *Let $\varphi \in H^m$ for $m > d/2$. Then there is a constant $N = N(d,\Lambda)$ such that*

$$|I\varphi|^2_{0,h} \le N|\varphi|^2_m,$$

where I denotes the Sobolev embedding operator from H^m into $C_b(\mathbb{R}^d)$.

5. Proof of Theorem 2.4

Define a random field r^h by

$$r^h_t(x) := u^h_t(x) - \sum_{0\le i\le J} v^{(i)}_t(x)\frac{h^i}{i!}, \tag{5.1}$$

where $(v^{(1)}, ..., v^{(J)})$ is the solution of (4.15) and (4.16).

Theorem 5.1. *Let Assumptions 2.1 and 2.2 hold with $m > \frac{d}{2} + 2J + 2$ for an integer $J \ge 0$. Let Assumptions 2.3 through 2.5 be also satisfied. Then r^h is an $\ell_2(\mathbb{G}_h)$-solution of the equation*

$$\mathcal{I}^h dr^h_t(x) = \big(\mathcal{L}^h_t r^h_t(x) + F^h_t(x)\big)\,dt + \big(\mathcal{M}^{h,\rho}_t r^h_t(x) + G^{h,\rho}_t(x)\big)\,dW^\rho_t, \tag{5.2}$$

$$r^h_0(x) = \hat\phi^{(J)h}(x), \tag{5.3}$$

where F^h and G^h are adapted $\ell_2(\mathbb{G}_h)$-valued such that for all $h \ne 0$ with $|h| \le 1$

$$E\int_0^T \big(|F^h_t|^2_{\ell_2(\mathbb{G}_h)} + |G^h_t|^2_{\ell_2(\mathbb{G}_h)}\big)\,dt \le N|h|^{2(J+1)} E\mathfrak{K}^2_m, \tag{5.4}$$

where $N = N(m, K, J, d, T, \kappa, \Lambda, \psi)$ is a constant which does not depend on h.

Proof. Using (5.1), the identity $u^h_t(x) = U^h_t(x)$ for $x \in \mathbb{G}_h$ which is deduced from Assumption 2.6 and equation (3.1), we deduce that for $x \in \mathbb{G}_h$,

$$\begin{aligned}
d\big(\mathcal{I}^h r^h_t(x)\big) &= d\mathcal{I}^h U^h_t - \sum_{i=0}^J \frac{h^i}{i!}\,\mathcal{I}^h dv^{(i)}_t(x)\\
&= \big[\mathcal{L}^h_t U^h_t(x) + f^h_t(x)\big]dt + \big[\mathcal{M}^{h,\rho}_t U^h_t(x) + g^{h,\rho}_t(x)\big]dW^\rho_t - \sum_{i=0}^J \frac{h^i}{i!}\,\mathcal{I}^h dv^{(i)}_t(x)\Big).\\
&= \mathcal{L}^h_t r^h_t(x)dt + \Big[\mathcal{L}^h_t \sum_{i=0}^J \frac{h^i}{i!}\,v^{(i)}_t(x) + f^h_t(x)\Big]dt + \mathcal{M}^{h,\rho}_t r^h_t(x)dW^\rho_t\\
&\quad + \Big[\mathcal{M}^{h,\rho}_t \sum_{i=0}^J \frac{h^i}{i!}\,v^{(i)}_t(x) + g^{h,\rho}_t(x)\Big]dW^\rho_t - \sum_{i=0}^J \frac{h^i}{i!}\,\mathcal{I}^h dv^{(i)}_t(x).
\end{aligned} \tag{5.5}$$

Taking into account Corollary 4.5, in the definition of $dv^{(i)}_t(x)$ in (4.15) we set

$$dv^{(i)}_t(x) = \big[B(i)_t(x) + F(i)_t(x)\big]dt + \big[\sigma(i)^\rho_t(x) + G(i)^\rho_t(x)\big]dW^\rho_t, \tag{5.6}$$

where $B(i)_t$ (resp. $\sigma(i)^\rho_t$) contains the operators $\mathcal{L}^{(j)}$ (resp. $\mathcal{M}^{(j)\rho}_t$) for $0 \le j \le i$ while $F(i)_t$ (resp. $G(i)^\rho_t$) contains all the free terms $f^{(j)}_t$ (resp. $g^{(j)\rho}_t$) for $1 \le j \le i$. We at first focus on the deterministic integrals. Using the recursive definition of the processes $v^{(i)}$ in (4.15), it is easy to see that

$$B(i)_t + \sum_{1\le j\le i}\binom{i}{j}\mathcal{I}^{(j)}B(i-j)_t = \sum_{j=0}^i \binom{i}{j}\mathcal{L}^{(j)}_t v^{(i-j)}_t, \tag{5.7}$$

$$F(i)_t + \sum_{1\le j\le i} \binom{i}{j} \mathcal{I}^{(j)} F(i-j)_t = f_t^{(i)}. \tag{5.8}$$

In the sequel, to ease notations we will not mention the space variable x. Using the expansion of $\mathcal{L}_t^h$, $\mathcal{I}^h$ and the definitions of $\hat{L}_t^{(J),h}$ and $\hat{I}^{(J),h}$ in (4.13), the expansion of f_t^h and the definition of $\hat{f}_t^{(J)h}$ given in (4.14) together with the definition of $dv_t^{(i)}$ in (5.6), we deduce

$$\Big[\mathcal{L}_t^h \sum_{i=0}^J \frac{h^i}{i!} v_t^{(i)} + f_t^h\Big] dt - \sum_{i=0}^J \frac{h^i}{i!} \mathcal{I}^h \big[B(i)_t^h + F_t^{(i)}\big] = \sum_{j=1}^6 \mathcal{T}_t^h(i) dt,$$

where

$$\mathcal{T}_t^h(1) = \sum_{i=0}^J \sum_{j=0}^i \frac{h^j}{j!} \frac{h^{i-j}}{(i-j)!} \big[\mathcal{L}_t^{(j)} v_t^{(i-j)} - \mathcal{I}^{(j)} B(i)_t\big],$$

$$\mathcal{T}_t^h(2) = \sum_{i=0}^J \sum_{\substack{0\le j\le J\\ i+j\ge J+1}} \frac{h^i}{i!} \frac{h^j}{j!} \big[\mathcal{L}_t^{(i)} v_t^{(j)} - \mathcal{I}^{(i)} B(j)_t\big],$$

$$\mathcal{T}_t^h(3) = \hat{L}_t^{(J),h} \sum_{i=0}^J \frac{h^i}{i!} v_t^{(i)} - \hat{I}^{(J),h} \sum_{i=0}^J \frac{h^i}{i!} B(i)_t,$$

$$\mathcal{T}_t^h(4) = \sum_{i=0}^J \frac{h^i}{i!} f_t^{(i)} - \sum_{i=0}^J \sum_{j=0}^i \frac{h^j}{j!} \frac{h^{i-j}}{(i-j)!} \mathcal{I}^{(j)} F(i-j)_t,$$

$$\mathcal{T}_t^h(5) = -\sum_{i=0}^J \sum_{\substack{0\le j\le J\\ i+j\ge J+1}} \frac{h^i}{i!} \frac{h^j}{j!} \mathcal{I}^{(j)} F(i)_t,$$

$$\mathcal{T}_t^h(6) = \hat{f}_t^{(J)h} - \sum_{i=0}^J \frac{h^i}{i!} \hat{I}^{(J)h} f_t^{(i)}.$$

Equation (4.15) implies

$$\mathcal{T}_t^h(1) = \sum_{i=0}^J \frac{h^i}{i!} \Big[\mathcal{L}_t^{(0)} v_t^{(i)} + \sum_{j=1}^i \binom{i}{j} \mathcal{L}_t^{(j)} v_t^{(i-j)} - B(i)_t - \sum_{j=1}^i \binom{i}{j} \mathcal{I}^{(j)} B(i-j)_t\Big].$$

Using the recursive equation (5.7) we deduce that for every $h>0$ and $t\in[0,T]$,

$$\mathcal{T}_t^h(1) = 0. \tag{5.9}$$

A similar computation based on (5.8) implies

$$\mathcal{T}_t^h(4) = 0. \tag{5.10}$$

In $\mathcal{T}_t^h(2)$ all terms have a common factor h^{J+1}. We prove an upper estimate of

$$E\int_0^T |\mathcal{L}_t^{(i)} v_t^{(j)}|_{0,h}^2\, dt$$

for $0 \le i,j \le J$. Let I denote the Sobolev embedding operator from H^k to $C_b(\mathbb{R}^d)$ for $k > d/2$. Lemma 4.6, inequalities (4.3) and (4.17) imply that for $k > d/2$,

$$E\int_0^T |I\mathcal{L}_t^{(i)} v_t^{(j)}|_{0,h}^2 dt \le NE\int_0^T |\mathcal{L}_t^{(i)} v_t^{(j)}|_k^2 dt \le NE\int_0^T |v_t^{(j)}|_{i+k+2}^2 dt \le NE\mathfrak{K}_{i+j+k+1}^2,$$

where the constant N does not depend on h and changes from one upper estimate to the next. Clearly, for $0 \le i,j \le J$ with $i+j \ge J+1$, we have $i+j+k+1 > 2J+1+\frac{d}{2}$. Similar computations prove that for $i,j \in \{0,...,J\}$ with $i+j \ge J+1$ and $k > \frac{d}{2}$,

$$\begin{aligned} E\int_0^T \big|I\mathcal{I}^{(i)} B(j)_t\big|_{0,h}^2\, dt &\le N\sum_{l=0}^{j} E\int_0^T \big|\mathcal{L}_t^{(l)} v_t^{(j-l)}\big|_{k+i}^2\, dt \\ &\le N\sum_{l=0}^{j} E\int_0^T \big|v_t^{(j-l)}\big|_{k+i+l+2}^2\, dt \\ &\le NE\mathfrak{K}_{k+i+j+1}^2. \end{aligned}$$

These upper estimates imply the existence of some constant N independent of h such that for $|h| \in (0,1]$ and $k > \frac{d}{2}$

$$E\int_0^T |\mathcal{T}_t^h(2)|_{0,h}^2\, ds \le N|h|^{2(J+1)} E\mathfrak{K}_{k+2J+1}^2. \tag{5.11}$$

We next find an upper estimate of the $|\cdot|_{0,h}$ norm of both terms in $\mathcal{T}_t^h(3)$. Using Lemmas 4.6, 4.2 and (4.17) we deduce that for $k > \frac{d}{2}$

$$\begin{aligned} E\int_0^T \Big|I\hat{L}_t^{(J),h}\sum_{i=0}^{J}\frac{h^i}{i!}v_t^{(i)}\Big|_{0,h}^2 dt \le& NE\int_0^T \Big|\hat{L}_t^{(J),h}\sum_{i=0}^{J}\frac{h^i}{i!}v_t^{(i)}\Big|_k^2 dt \\ \le& N|h|^{2(J+1)}\sum_{i=0}^{J}\int_0^T \big|v_t^{(i)}\big|_{k+J+3}^2 dt \\ \le& N|h|^{2(J+1)}E\mathfrak{K}_{k+2J+2}^2, \end{aligned}$$

where N is a constant independent of h with $|h| \in (0,1]$. Furthermore, similar computations yield for $k > \frac{d}{2}$ and $|h| \in (0,1]$

$$\begin{aligned} E\int_0^T \Big|I\hat{I}^{(J),h}\sum_{i=0}^{J}\frac{h^i}{i!}B(i)_t\Big|_{0,h}^2 dt \le& NE\int_0^T \Big|\sum_{i=0}^{J}\frac{h^i}{i!}\hat{I}^{(J),h}B(i)_t\Big|_k^2 dt \\ \le& N|h|^{2(J+1)}E\int_0^T \sum_{i=0}^{J}\Big|\sum_{l=0}^{i}\binom{i}{l}\mathcal{L}_t^{(l)} v_t^{(i-l)}\Big|_{k+J+1}^2 dt \\ \le& N|h|^{2(J+1)}\sum_{i=0}^{J}\sum_{l=0}^{i}|v_t^{(i-l)}|_{k+J+l+3}^2 dt \\ \le& N|h|^{2(J+1)}E\mathfrak{K}_{k+2J+2}^2. \end{aligned}$$

Hence we deduce the existence of a constant N independent of h such that for $|h|\in(0,1]$,

$$E\int_0^T |\mathcal{T}_t^h(3)|_{0,h}^2 dt \le N|h|^{2(J+1)} E\mathfrak{K}_{k+2J+2}^2, \tag{5.12}$$

where $k>\frac{d}{2}$.

We next compute an upper estimate of the $|\cdot|_{0,h}$ norm of $\mathcal{T}_t^h(5)$. All terms have a common factor $h^{(J+1)}$. Recall that $\mathcal{I}^{(0)}=Id$. The induction equation (5.8) shows that $F(i)_t$ is a linear combination of terms of the form $\Phi(i)_t := \big(\mathcal{I}^{(a_1)}\big)^{k_1}...\big(\mathcal{I}^{(a_i)}\big)^{k_i} f_t$ for $a_p,k_p\in\{0,...,i\}$ for $1\le p\le i$ with $\sum_{p=1}^i a_p k_p = i$, and of terms of the form $\Psi(i)_t := \big(\mathcal{I}^{(b_1)}\big)^{l_1}...\big(\mathcal{I}^{(b_{i-j})}\big)^{l_{i-j}} f_t^{(j)}$ for $1\le j\le i$, $b_p,l_p\in\{0,...,i-j\}$ for $1\le p\le i-j$ with $\sum_{p=1}^{i-j} b_p l_p + j = i$. Using Lemmas 4.6 and 4.1 we obtain for $k>\frac{d}{2}$, $i,j=1,...J$

$$\begin{aligned}E\int_0^T |I\mathcal{I}^{(j)}\Phi(i)_t|_{0,h}^2 dt \le& NE\int_0^T |\mathcal{I}^{(j)}\Phi(i)_t(x)|_k^2\, dt\\ \le& NE\int_0^T |\Phi(i)_t|_{k+j}^2 dt\\ \le& NE\int_0^T |f_t|_{k+j+a_1k_1+\cdots a_ik_i}^2 dt\\ \le& NE\int_0^T |f_t|_{k+i+j}^2 dt \le NE\mathfrak{K}_{k+i+j}^2.\end{aligned}$$

A similar computation yields

$$\begin{aligned}E\int_0^T |I\mathcal{I}^{(j)}\Psi(i)_t|_{0,h}^2\, dt \le& NE\int_0^T |f_t^{(i)}|_{k+j+b_1l_1+\cdots+b_{i-j}l_{i-j}}^2\, dt\\ \le& NE\int_0^T |f_t|_{k+j+(i-j)+j}^2\, dt\\ \le& NE\mathfrak{K}_{k+i+j}^2.\end{aligned}$$

These upper estimates imply that for $k>\frac{d}{2}$, there exists some constant N independent on h such that for $|h|\in(0,1)$

$$E\int_0^T |\mathcal{T}_t^h(5)|_{0,h}^2\, dt \le N|h|^{2(J+1)} E\mathfrak{K}_{k+2J}^2. \tag{5.13}$$

We finally prove an upper estimate of the $|\cdot|_{0,h}$-norm of both terms in $\mathcal{T}_t^h(6)$. Using Lemmas 4.6 and 4.3, we obtain for $k>\frac{d}{2}$,

$$\begin{aligned}E\int_0^T \big|I\hat{f}_t^{(J)h}\big|_{0,h}^2\, dt \le& NE\int_0^T \big|\hat{f}_t^{(J)h}\big|_k^2 dt\\ \le& N|h|^{2(J+1)}E\int_0^T |f_t|_{k+J+1}^2 dt\\ \le& N|h|^{2(J+1)}E\mathfrak{K}_{k+J+1}^2,\end{aligned}$$

where N is a constant which does not depend on h. Furthermore, Lemmas 4.6 and 4.2 yield for $k > \frac{d}{2}$ and $|h| \in (0,1]$,

$$\begin{aligned} E\int_0^T \Big| I \sum_{i=0}^J \frac{h^i}{i!} \hat{I}^{(J)h} f_t^{(i)} \Big|_{0,h}^2 dt \leq & NE \int_0^T \Big| \sum_{i=0}^J \frac{h^i}{i!} \hat{I}^{(J)h} f_t^{(i)} \Big|_k^2 dt \\ \leq & N|h|^{2(J+1)} E \int_0^T \sum_{i=0}^J |f_t^{(i)}|_{k+J+1}^2 dt \\ \leq & N|h|^{2(J+1)} E\mathfrak{K}_{k+2J+1}^2, \end{aligned}$$

for some constant N independent of h. Hence we deduce that for some constant N which does not depend on h and $k > \frac{d}{2}$, we have for $|h| \in (0,1]$

$$E\int_0^T |\mathcal{T}_t^h(6)|_{0,h}^2 \, dt \leq N|h|^{2(J+1)} E\mathfrak{K}_{k+2J+1}^2. \tag{5.14}$$

Similar computations can be made for the coefficients of the stochastic integrals. The upper bounds of the corresponding upper estimates in (5.11) and (5.12) are still valid because the operators $\mathcal{M}_t^\rho$ are first order operators while the operator $\mathcal{L}_t$ is a second order one. This implies that all operators $\mathcal{M}_t^{h,\rho}$, $\mathcal{M}_t^{(i)\rho}$ and $\hat{M}_t^{(J)h}$ contain less derivatives than the corresponding ones deduced from $\mathcal{L}_t$.

Using the expansion (5.5), the upper estimates (5.9)-(5.14) for the coefficients of the deterministic and stochastic integrals, we conclude the proof. ☐

We now complete the proof of our main result.

Proof of Theorem 2.4. By virtue of Theorem 3.2 and Theorem 5.1 we have for $|h| \in (0,1]$

$$E\sup_{t\in[0,T]} |r_t^h|_{0,h}^2 \leq NE|\hat{\phi}^{(J)h}|_{0,h}^2 + NE\int_0^T \big(|F^h|_{0,h}^2 + |G_h|_{0,h}^2\big)\, dt \leq |h|^{2(J+1)} NE\mathfrak{K}_m^2.$$

Using Remark 3.1 we have $U_t^{-h} = U_t^h$ for $t \in [0,T]$ a.s. Hence from the expansion (2.9) we obtain that $v^{(j)} = -v^{(j)}$ for odd j, which completes the proof of Theorem 2.4. ☐

6. Some examples of finite elements

In this section we propose three examples of finite elements which satisfy Assumptions 2.4, 2.5 and 2.6.

6.1. **Linear finite elements in dimension 1.** Consider the following classical linear finite elements on $\mathbb{R}$ defined as follows:

$$\psi(x) = \big(1-|x|\big)\, 1_{\{|x|\leq 1\}}. \tag{6.1}$$

Let $\Lambda = \{-1,0,1\}$; clearly ψ and Λ satisfy the symmetry condition (2.6). Recall that Γ denotes the set of elements $\lambda \in \mathbb{G}$ such that the intersection of the support of $\psi_\lambda := \psi_\lambda^1$ and of the support of ψ has a positive Lebesgue measure. Then $\Gamma = \{-1,0,1\}$, the function ψ is clearly non negative, $\int_{\mathbb{R}} \psi(x)dx = 1$, $\psi(x) = 0$ for $x \in \mathbb{Z}\setminus\{0\}$ and Assumption 2.6 clearly holds.

Simple computations show that

$$R_0 = 2\int_0^1 x^2 dx = \frac{2}{3}, \quad R_{-1} = R_1 = \int_0^1 x(1-x)dx = \frac{1}{6}.$$

Hence $\sum_{\lambda\in\Gamma} R_\lambda = 1$. Furthermore, given any $z = (z_n) \in \ell_2(\mathbb{Z})$ we have using the Cauchy-Schwarz inequality:

$$\sum_{n\in\mathbb{Z}} \Big(\frac{2}{3}z_n^2 + \frac{1}{6}z_n z_{n-1} + \frac{1}{6}z_n z_{n+1}\Big) \ge \frac{2}{3}\|z\|^2 - \frac{1}{6}\sum_{n\in\mathbb{Z}} \big(z_n^2 + z_{n+1}^2\big) = \frac{1}{3}\|z\|^2.$$

Hence Assumption 2.4 is satisfied. Easy computations show that for $\epsilon \in \{-1,1\}$ we have

$$R_0^{11} = -2, \quad R_\epsilon^{11} = 1, \quad R_0^1 = 0 \quad \text{and } R_\epsilon^1 = \frac{\epsilon}{2}.$$

Hence $\sum_{\lambda\in\Gamma} R_\lambda^{11} = 0$, which completes the proof of (2.20). Furthermore, $\sum_{\lambda\in\Gamma} \lambda R_\lambda^1 = 1$, which proves (2.21) while $\sum_{\lambda\in\Gamma} \lambda^2 R_\lambda^{11} = 2$, which proves (2.22).

Finally, we have for $\epsilon \in \{-1,1\}$

$$Q_0^{11,11} = -\frac{2}{3}, \quad Q_\epsilon^{11,11} = \frac{1}{3}, \quad \tilde{Q}_0^{11} = 0 \quad \text{and } \tilde{Q}_\epsilon^{11} = -\frac{\epsilon}{6}.$$

This clearly implies $\sum_{\lambda\in\Gamma} Q_\lambda^{11,11} = 0$ and $\sum_{\lambda\in\Gamma} \tilde{Q}_\lambda^{11} = 0$, which completes the proof of (2.23); therefore, Assumption 2.5 is satisfied.

The following example is an extension of the previous one to any dimension.

6.2. **A general example.** Consider the following finite elements on $\mathbb{R}^d$ defined as follows: let ψ be defined on $\mathbb{R}^d$ by $\psi(x) = 0$ if $x \notin (-1,+1]^d$ and

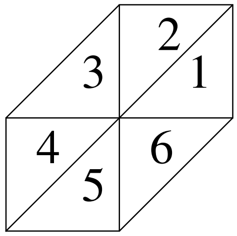
$$\psi(x) = \prod_{k=1}^d \big(1 - |x_k|\big) \text{ for } x = (x_1, ..., x_d) \in (-1,+1]^d. \tag{6.2}$$

The function ψ is clearly non negative and $\int_{\mathbb{R}^d} \psi(x)dx = 1$. Let $\Lambda = \{0,\, \epsilon_k e_k,\, \epsilon_k \in \{-1,+1\},\, k = 1, ..., d\}$. Then ψ and Λ satisfy the symmetry condition (2.6). Furthermore, $\psi(x) = 0$ for $x \in \mathbb{Z}^d \setminus \{0\}$; Assumption 2.6 clearly holds.

These finite elements also satisfy all requirements in Assumptions 2.4–2.5. Even if these finite elements are quite classical in numerical analysis, we were not able to find a proof of these statements in the literature. To make the paper self-contained the corresponding easy but tedious computations are provided in an Appendix.

6.3. **Linear finite elements on triangles in the plane.** We suppose that $d = 2$ and want to check that the following finite elements satisfy Assumptions 2.4-2.6. For $i = 1, ..., 6$, let τ_i be the triangles defined as follows:

$$\begin{aligned}
&\tau_1 = \{x \in \mathbb{R}^2 : 0 \le x_2 \le x_1 \le 1\},\ \tau_2 = \{x \in \mathbb{R}^2 : 0 \le x_1 \le x_2 \le 1\},\\
&\tau_3 = \{x \in \mathbb{R}^2 : 0 \le x_2 \le 1, x_2 - 1 \le x_1 \le 0\},\ \tau_4 = \{x \in \mathbb{R}^2 : -1 \le x_1 \le x_2 \le 0\},\\
&\tau_5 = \{x \in \mathbb{R}^2 : -1 \le x_2 \le x_1 \le 0\},\ \tau_6 = \{x \in \mathbb{R}^2 : 0 \le x_1 \le 1, x_1 - 1 \le x_2 \le 0\}.
\end{aligned} \tag{6.3}$$



Let ψ be the function defined by:

$$\psi(x) = 1 - |x_1| \text{ on } \tau_1 \cup \tau_4, \;\; \psi(x) = 1 - |x_2| \text{ on } \tau_2 \cup \tau_5,$$
$$\psi(x) = 1 - |x_1 - x_2| \text{ on } \tau_3 \cup \tau_6, \text{ and } \;\psi(x) = 0 \text{ otherwise.} \tag{6.4}$$

It is easy to see that the function ψ is non negative and that $\int_{\mathbb{R}^2} \psi(x)dx = 1$. Set $\Lambda = \{0, e_1, -e_1, e_2, -e_2\}$; the function ψ and the set Λ fulfill the symmetry condition (2.6).
Furthermore, $\Gamma = \big\{\epsilon_1 e_1 + \epsilon_2 e_2 : (\epsilon_1, \epsilon_2) \in \{-1, 0, 1\}^2, \; \epsilon_1\epsilon_2 \in \{0, 1\}\big\}$. Hence ψ satisfies Assumption 2.6.

For $\mathbf{i} = (i_1, i_2) \in \mathbb{Z}^2$, let $\psi_{\mathbf{i}}$ the function defined by

$$\psi_{\mathbf{i}}(x_1, x_2) = \psi\big((x_1, x_2) - \mathbf{i}\big).$$

For $\gamma = 1, 2, ..., 6$, we denote by $\tau_\gamma(\mathbf{i}) = \big\{(x_1, x_2) : (x_1, x_2) - \mathbf{i} \in \tau_\gamma\big\}$. Then

$$D_1\psi_{\mathbf{i}} = -1 \text{ on } \tau_1(\mathbf{i}) \cup \tau_6(\mathbf{i}) \text{ and } D_1\psi_{\mathbf{i}} = 1 \text{ on } \tau_3(\mathbf{i}) \cup \tau_4(\mathbf{i}),$$
$$D_2\psi_{\mathbf{i}} = -1 \text{ on } \tau_2(\mathbf{i}) \cup \tau_3(\mathbf{i}) \text{ and } D_2\psi_{\mathbf{i}} = 1 \text{ on } \tau_5(\mathbf{i}) \cup \tau_6(\mathbf{i}),$$
$$D_1\psi_{\mathbf{i}} = 0 \text{ on } \tau_2(\mathbf{i}) \cup \tau_5(\mathbf{i}) \text{ and } D_2\psi_{\mathbf{i}} = 0 \text{ on } \tau_1(\mathbf{i}) \cup \tau_4(\mathbf{i}).$$

Easy computations show that for $\mathbf{i} \in \mathbb{Z}^2$, and $\mathbf{k} \in \{\mathbf{i} + \lambda : \lambda \in \Gamma\}$

$$(\psi_{\mathbf{i}}, \psi_{\mathbf{i}}) = \frac{1}{2}, \quad (\psi_{\mathbf{i}}, \psi_{\mathbf{k}}) = \frac{1}{12},$$

and $(\psi_{\mathbf{i}}, \psi_{\mathbf{j}}) = 0$ otherwise. Thus

$$\sum_{\lambda\in\Gamma} R_\lambda = \sum_{\lambda\in\Gamma} (\psi, \psi_\lambda) = \frac{1}{2} + 6 \times \frac{1}{12} = 1,$$

which proves the first identity in (2.20). First we check that given any $\alpha \in (0, 1)$, for some positive constants C_1 and C_2 we have for every $(U_{\mathbf{i}}) \in \ell_2(\mathbb{Z}^2)$

$$\begin{aligned}
|\sum_{\mathbf{i}} U_{\mathbf{i}}\psi_{\mathbf{i}}|^2_{L^2} \geq & \sum_{\mathbf{i}} \int_0^\alpha dx_1 \int_0^{x_1} \big|(1 - x_1)U_{\mathbf{i}} + (x_1 - x_2)U_{\mathbf{i}+e_1} + x_2 U_{\mathbf{i}+e_1+e_2}\big|^2 dx_2 \\
& + \sum_{\mathbf{i}} \int_0^\alpha dx_2 \int_0^{x_2} \Big|(1 - x_2)U_{\mathbf{i}} + (x_2 - x_1)U_{\mathbf{i}+e_2} + x_1 U_{\mathbf{i}+e_1+e_2}\Big|^2 dx_1 \\
\geq & \|U\|^2 \big(\alpha^2 - C_1\alpha^3 - C_2\alpha^4\big) \geq \frac{\alpha^2}{2} \|U\|^2;
\end{aligned}$$

the last lower estimates follow from the Cauchy-Schwarz inequality and from the fact that when $\alpha \in (0, 1)$ is small enough. Therefore, we have $1 - C_1\alpha - C_2\alpha^2 \geq \frac{1}{2}$. This proves that Assumption 2.4 is satisfied.

We next check the compatibility conditions in Assumption 2.5. Easy computations prove that for $k=1,2$ and $l\in\{1,2\}$ with $l\neq k$, $\epsilon_k,\epsilon_l\in\{-1,1\}$ we have

$$\begin{aligned}(D_k\psi,D_k\psi)=&2,\quad (D_k\psi,D_k\psi_{\epsilon_k e_k})=-1,\quad (D_k\psi,D_k\psi_{\epsilon_l e_l})=0,\\ (D_k\psi,D_k\psi_\lambda)=&0 \text{ for } \lambda=\epsilon_1e_1+\epsilon_2e_2,\ \epsilon_1\epsilon_2=1,\end{aligned}$$

while

$$\begin{aligned}(D_k\psi,D_l\psi)=-1,\quad (D_k\psi,D_l\psi_{\epsilon_k e_k})=(D_k\psi,D_l\psi_{\epsilon_l e_l})=\frac12,\\ (D_k\psi,D_l\psi_\lambda)=-\frac12 \text{ for } \lambda=\epsilon_1e_1+\epsilon_2e_2,\ \epsilon_1\epsilon_2=1.\end{aligned}$$

Hence for any $k,l=1,2$ and $l\neq k$ we have

$$\sum_{\lambda\in\Gamma}(D_k\psi,D_k\psi_\lambda)=2+2\times(-1)=0,\quad \sum_{\lambda\in\Gamma}(D_k\psi,D_l\psi_\lambda)=-1+4\times\frac12+2\times\big(-\frac12\big)=0.$$

This completes the proof of equation $\sum_{\lambda\in\Gamma}R^{ij}_\lambda=0$ and hence of equation (2.20). Furthermore, given $k,l=1,2$ with $k\neq l$ we have for $\alpha=k$ or $\alpha=l$:

$$\begin{aligned}&\sum_{\lambda\in\Gamma}R^{kk}_\lambda\lambda_k\lambda_k=-\sum_{\lambda\in\Gamma}(D_k\psi,D_k\psi_\lambda)\lambda_k\lambda_k=2\times1^2=2,\\ &\sum_{\lambda\in\Gamma}R^{kk}_\lambda\lambda_l\lambda_l=-\sum_{\lambda\in\Gamma}(D_k\psi,D_k\psi_\lambda)\lambda_l\lambda_l=0,\\ &\sum_{\lambda\in\Gamma}R^{kk}_\lambda\lambda_k\lambda_l=-\sum_{\lambda\in\Gamma}(D_k\psi,D_k\psi_\lambda)\lambda_k\lambda_l=0,\\ &\sum_{\lambda\in\Gamma}R^{kl}_\lambda\lambda_k\lambda_l=-\sum_{\lambda\in\Gamma}(D_k\psi,D_l\psi_\lambda)\lambda_k\lambda_l=\frac12\times1^2+\frac12(-1)^2=1,\\ &\sum_{\lambda\in\Gamma}R^{kl}_\lambda\lambda_\alpha\lambda_\alpha=-\sum_{\lambda\in\Gamma}(D_k\psi,D_l\psi_\lambda)\lambda_\alpha\lambda_\alpha=0.\end{aligned}$$

The last identities come from the fact that $(D_k\psi,D_l\psi_{\epsilon e_k})$, $(D_k\psi,D_l\psi_{\epsilon e_l})$ or $(D_k\psi,D_l\psi_{\epsilon(e_1+e_2)}$ agree for $\epsilon=-1$ and $\epsilon=1$. This completes the proof of equation (2.22).

We check the third compatibility condition. Using Lemma 2.2 we deduce for $k,l=1,2$ with $k\neq l$ and $\epsilon\in\{-1,+1\}$

$$\begin{aligned}(D_k\psi,\psi)=0,\quad (D_k\psi_{\epsilon e_k},\psi)=\frac{\epsilon}{3},\\ (D_k\psi_{\epsilon e_l},\psi)=-\frac{\epsilon}{6},\quad (D_k\psi_{\epsilon(e_1+e_2)},\psi)=\frac{\epsilon}{6}.\end{aligned}$$

Therefore, using Lemma 2.2 we deduce that

$$\begin{aligned}&\sum_{\lambda\in\Gamma}(D_k\psi_\lambda,\psi)\lambda_k=\frac13+(-1)\times\big(-\frac13\big)+\frac16+(-1)\times\big(-\frac16\big)=1,\\ &\sum_{\lambda\in\Gamma}(D_k\psi_\lambda,\psi)\lambda_l=-\frac16+\frac16\times(-1)+\frac16-\frac16\times(-1)=0.\end{aligned}$$

This completes the proof of equation (2.21).

Let us check the first identity in (2.23). Recall that

$$Q_\lambda^{ij,kl} = -\int_{\mathbb{R}^2} z_k z_l D_i\psi(z) D_j\psi_\lambda(z)dz$$

and suppose at first that $i=j$. Then we have for $k\neq i$, $\alpha\neq i$, $k\neq l$ and $\epsilon\in\{-1,+1\}$

$$Q_0^{ii,ii} = -\frac{2}{3},\quad Q_{\epsilon e_i}^{ii,ii} = \frac{1}{3},\quad Q_{\epsilon e_\alpha}^{ii,ii} = Q_{\epsilon(e_i+e_\alpha)}^{ii,ii} = 0,$$
$$Q_0^{ii,kk} = -\frac{1}{3},\quad Q_{\epsilon e_i}^{ii,kk} = \frac{1}{6},\quad Q_{\epsilon e_k}^{ii,kk} = Q_{\epsilon(e_i+e_k)}^{ii,ii} = 0,$$
$$Q_0^{ii,kl} = -\frac{1}{6},\quad Q_{\epsilon e_i}^{ii,kl} = \frac{1}{12},\quad Q_{\epsilon e_\alpha}^{ii,kl} = Q_{\epsilon(e_i+e_\alpha)}^{ii,ii} = 0.$$

Suppose that $i\neq j$; then for $k\neq l$ and $\epsilon\in\{-1,+1\}$ we have

$$Q_0^{ij,ii} = \frac{1}{6},\quad Q_{\epsilon e_j}^{ij,ii} = -\frac{1}{12},\quad Q_{\epsilon e_i}^{ij,ii} = -\frac{1}{4},\quad Q_{\epsilon(e_i+e_j)}^{ij,ii} = \frac{1}{4},$$
$$Q_0^{ij,jj} = \frac{1}{6},\quad Q_{\epsilon e_i}^{ij,jj} = -\frac{1}{12},\quad Q_{\epsilon e_j}^{ij,jj} = -\frac{1}{12},\quad Q_{\epsilon(e_i+e_j)}^{ij,jj} = \frac{1}{12},$$
$$Q_0^{ij,kl} = -\frac{1}{12},\quad Q_{\epsilon e_j}^{ij,kl} = \frac{1}{24},\quad Q_{\epsilon e_i}^{ij,kl} = -\frac{1}{8},\quad Q_{\epsilon(e_i+e_j)}^{ij,kl} = \frac{1}{8}.$$

The above equalities prove $\sum_{\lambda\in\Gamma} Q_\lambda^{ij,kl} = 0$ for any choice of $i,j,k,l=1,2$. Hence the first identity in (2.23) is satisfied.

We finally check the second identity in (2.23). Recall that $\tilde{Q}_\lambda^{i,k} = \int_{\mathbb{R}^2} z_k D_i\psi_\lambda(z)\psi(z)dz$. For $i=k\in\{1,2\}$, $j\in\{1,2\}$ with $i\neq j$ and $\epsilon\in\{-1,+1\}$ we have

$$\tilde{Q}_0^{i,i} = -\frac{3}{12},\quad \tilde{Q}_{\epsilon e_i}^{i,i} = \frac{3}{24},\quad \tilde{Q}_{\epsilon e_j}^{i,i} = -\frac{1}{24},\quad \tilde{Q}_{\epsilon(e_i+e_j)}^{i,i} = \frac{1}{24}.$$

Hence $\sum_{\lambda\in\Gamma}\tilde{Q}_\lambda^{i,i} = 0$. Let $i\neq k$; then for $\epsilon\in\{-1,+1\}$ we have

$$\tilde{Q}_0^{i,k} = \tilde{Q}_{\epsilon e_i}^{i,k} = 0,\quad \tilde{Q}_{\epsilon e_k}^{i,k} = -\frac{1}{12},\quad \tilde{Q}_{\epsilon(e_i+e_k)}^{i,k} = \frac{1}{12}.$$

Hence $\sum_{\lambda\in\Gamma}\tilde{Q}_\lambda^{i,k} = 0$ for any choice of $i,k=1,2$, which concludes the proof of (2.23) Therefore, the function ψ defined by (6.4) satisfies all Assumptions 2.4-2.6.

7. Appendix

The aim of this section is to prove that the example described in 6.2 satisfies Assumptions 2.4 and 2.5.

For $k=1,...,d$, let $e_k\in\mathbb{Z}^d$ denote the k-th unit vector of $\mathbb{R}^d$; then $\mathbb{G}=\mathbb{Z}^d$ and

$$\Gamma = \Big\{ \sum_{k=1}^d \epsilon_k e_k \ :\ \epsilon_k\in\{-1,0,1\} \text{ for } k=1,...,d \Big\}.$$

For fixed $k=1,...,d$ (resp. $k\neq l\in\{1,...,d\}$) let

$$\mathcal{I}(k) = \{1,...,d\}\setminus\{k\},\quad \text{resp. } \mathcal{I}(k,l) = \{1,...,d\}\setminus\{k,l\}. \tag{7.1}$$

Note that in the particular case $d=1$, the functions ψ gives rise to the classical linear finite elements. Then for $\mathbf{i}\in\mathbb{Z}^d$, we have for $k=0,1,...,d$:

$$R_{\mathbf{i}} := (\psi_{\mathbf{i}},\psi) = \binom{d}{k}\Big(\frac{1}{6}\Big)^k\Big(\frac{2}{3}\Big)^{d-k} \quad \text{if} \quad \sum_{l=1}^d |i_l| = k. \tag{7.2}$$

Furthermore, given $k=0,1,...,d$, there are 2^k elements $\mathbf{i}\in\mathbb{Z}^d$ such that $\sum_{l=1}^d |i_l|=k$. Therefore, we deduce

$$\sum_{\mathbf{i}\in\mathbb{Z}^d}(\psi_{\mathbf{i}},\psi) = \sum_{k=0}^d 2^k\binom{d}{k}\Big(\frac{1}{6}\Big)^k\Big(\frac{2}{3}\Big)^{d-k} = \binom{d}{k}\Big(\frac{2}{6}\Big)^k\Big(\frac{2}{3}\Big)^{d-k} = 1,$$

which yields the first compatibility conditon in (2.20).

We at first check that Assumption 2.4 holds true, that is

$$\delta\sum_{\mathbf{i}\in\mathbb{Z}^d} U_{\mathbf{i}}^2 = \delta|U|^2_{\ell_2(\mathbb{Z}^d)} \le \Big|\sum_{\mathbf{i}\in\mathbb{Z}^d} U_{\mathbf{i}}\psi_{\mathbf{i}}\Big|^2_{L^2} = \sum_{\mathbf{i},\mathbf{j}\in\mathbb{Z}^d} R_{\mathbf{i}-\mathbf{j}}U_{\mathbf{i}}U_{\mathbf{j}}, \quad U\in\ell_2(\mathbb{Z}^d).$$

for some $\delta>0$. For $U\in\ell_2(\mathbb{Z}^d)$ and $k=1,...,d$, let $T_kU=U_{e_k}$, where e_k denotes the k-th vector of the canonical basis.

For $U\in\ell_2(\mathbb{Z}^d)$ we have

$$\begin{aligned}\Big|\sum_{\mathbf{i}} U_{\mathbf{i}}\psi_{\mathbf{i}}\Big|^2_{L^2} = \sum_{\mathbf{i}\in\mathbb{Z}^{\mathbf{d}}}\int_{[0,1]^d}\Big[U_{\mathbf{i}}\prod_{j=1}^d(1-x_j) + \sum_{k=1}^d (T_kU)_{\mathbf{i}}\, x_k \prod_{j\in\mathcal{I}(k)}(1-x_j)\\ + \sum_{1\le k_1<k_2\le d}(T_{k_1}\circ T_{k_2}U)_{\mathbf{i}}\, x_{k_1}x_{k_2}\prod_{j\in\mathcal{I}(k_1,k_2)}(1-x_j) + \cdots + (T_1\circ T_2\cdots\circ T_dU)_{\mathbf{i}}\prod_{k=1}^d x_k\Big]^2 dx,\end{aligned}$$

Given $\alpha\in(0,1)$ if we let

$$I(\alpha)=\int_0^\alpha(1-x)^2dx = \alpha-\alpha^2+\frac{\alpha^3}{3},\ J(\alpha)=\int_0^\alpha x(1-x)dx = \frac{\alpha^2}{2}-\frac{\alpha^3}{3},\ K(\alpha)=\int_0^\alpha x^2dx = \frac{\alpha^3}{3},$$

restricting the above integral on the set $[0,\alpha]^d$, expanding the square and using the Cauchy Schwarz inequality we deduce the existence of some constants $C(\gamma_1,\gamma_2,\gamma_3)$ defined for $\gamma_i\in\{0,1,...,d\}$ such that

$$\begin{aligned}\Big|\sum_{\mathbf{i}} U_{\mathbf{i}}\psi_{\mathbf{i}}\Big|^2_{L^2} \ge& \sum_{\mathbf{i}}|U_{\mathbf{i}}|^2\Big[I(\alpha)^d + \binom{d}{1}K(\alpha)\,I(\alpha)^{d-1} + \binom{d}{2}K(\alpha)^2\,I(\alpha)^{d-2} + \cdots + K(\alpha)^d\Big]\\ &- 2\Big(\sum_{\mathbf{i}}|U_{\mathbf{i}}|^2\Big)\sum_{\gamma_1+\gamma_2+\gamma_3=d,\gamma_2+\gamma_3\ge1} C(\gamma_1,\gamma_2,\gamma_3)\,I(\alpha)^{\gamma_1}\,J(\alpha)^{\gamma_2}\,K(\alpha)^{\gamma_3}\\ \ge& |U|^2_{\ell_2(\mathbb{Z}^d)}\Big(\alpha^d - \sum_{l=d+1}^{3d} C_l\alpha^l\Big),\end{aligned}$$

where C_l are some positive constants. Choosing α small enough, we have $\big|\sum_{\mathbf{i}} U_{\mathbf{i}}\psi_{\mathbf{i}}\big|^2_{L^2} \ge \frac{\alpha^d}{2}|U|^2_{\ell_2(\mathbb{Z}^d)}$, which implies the invertibility Assumption 2.4.

We now prove that the compatibility Assumption 2.5 holds true. For $l = 1, ..., d$, $n = 0, ..., d-1$:

$$(D_l\psi_{\mathbf{i}}, D_l\psi) = -\,2^{d-1-n}\left(\frac{1}{6}\right)^n\left(\frac{1}{3}\right)^{d-1-n} \quad \text{for } |i_l| = 1, \sum_{r\neq l, 1\leq n\leq d} |i_r| = n, \tag{7.3}$$

$$(D_l\psi_{\mathbf{i}}, D_l\psi) = +\,2^{d-n}\left(\frac{1}{6}\right)^n\left(\frac{1}{3}\right)^{d-1-n} \quad \text{for } |i_l| = 0, \sum_{r\neq l, 1\leq r\leq d} |i_r| = n. \tag{7.4}$$

For $n = 1, ..., d-1$ and $k_1 < k_2 < ... < k_n$ with $k_r \in \mathcal{I}(l)$, where $\mathcal{I}(l)$ is defined in (7.1), let

$$\Gamma_l(k_1, ..., k_n) = \Big\{ \sum_{r=1}^n \epsilon_{k_r} e_{k_r} \ : \ \epsilon_{k_r} \in \{-1, 1\}, r = 1, ..., n \Big\},$$

$$\Gamma_l(l; k_1, ..., k_n) = \Big\{ \epsilon_l e_l + \sum_{r=1}^n \epsilon_{k_r} e_{k_r} \ : \ \epsilon_l \in \{-1, 1\} \text{ and } \epsilon_{k_r} \in \{-1, 1\}, r = 1, ..., n \Big\}.$$

Then $|\Gamma_l(k_1, ..., k_n)| = 2^n$ while $|\Gamma_l(l; k_1, ..., k_n)| = 2^{n+1}$. For $l = 1, ..., d$, the identities (7.3) and (7.4) imply

$$\begin{aligned}
\sum_{\lambda\in\Gamma} (D_l\psi, D_l\psi_\lambda) &= \Big[(D_l\psi, D_l\psi) + \sum_{\epsilon_l\in\{-1,+1\}} (D_l\psi, D_l\psi_{\epsilon_l e_l})\Big] \\
&\quad + \sum_{n=1}^{d-1} \sum_{k_1<k_2<...<k_n, k_j\in\mathcal{I}(l)} \Big[\sum_{\lambda\in\Gamma_l(k_1,...,k_n)} (D_l\psi, D_l\psi_\lambda) + \sum_{\lambda\in\Gamma_l(l;k_1,...,k_n)} (D_l\psi, D_l\psi_\lambda)\Big] \\
&= \Big[2^d\Big(\frac{1}{3}\Big)^{d-1} - 2\times 2^{d-1}\Big(\frac{1}{3}\Big)^{d-1}\Big] \\
&\quad + \sum_{n=1}^{d-1} 2^n \binom{d-1}{n} \Big[2^{d-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n} - 2\times 2^{d-1-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n}\Big] = 0.
\end{aligned}$$

This proves the second identity in (2.20) when $i = j$. Furthermore, (7.3) implies

$$\begin{aligned}
\sum_{\lambda\in\Gamma} R^{ll}_\lambda \lambda_l\lambda_l &= -\sum_{\lambda\in\Gamma} (D_l\psi, D_l\psi_\lambda)\lambda_l\lambda_l = -\sum_{\epsilon_l\in\{-1,1\}} \big(D_l\psi, D_l\psi_{\epsilon_l e_l}\big) \\
&\quad - \sum_{n=1}^{d-1} \sum_{k_1<k_2<...<k_n, k_j\in\mathcal{I}(l)} \sum_{\lambda\in\Gamma_l(l;k_1,...,k_n)} (D_l\psi, D_l\psi_\lambda) \\
&= 2\times 2^{d-1}\Big(\frac{1}{3}\Big)^{d-1} + \sum_{d=1}^{n} \binom{d-1}{n} 2^{n+1}\times 2^{d-1-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n} \\
&= 2\sum_{n=0}^{d-1} \binom{d-1}{n} \Big(\frac{2}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-1-n} = 2, \quad l = 1, ..., d.
\end{aligned}$$

Furthermore, given $k \neq l \in \{1, ..., d\}$,

$$\sum_{\lambda\in\Gamma} R^{ll}_\lambda \lambda_k\lambda_k = -\sum_{\lambda\in\Gamma} (D_l\psi, D_l\psi_\lambda)\lambda_k\lambda_l = 0.$$

Indeed, for $n=1,...,d-1$, $k_1<...<k_n$ where $k_r\in\mathcal{I}(l)$ and at least one of the indices k_r is equal to k for $r=1,...,n$, given $\lambda\in\Gamma_l(k_1,...,k_n)$ we have using (7.3) and (7.4)

$$\sum_{\epsilon_l\in\{-1,1\}}(D_l\psi,D_l\psi_{\epsilon_l e_l+\lambda})\lambda_k\lambda_l=-2^{d-1-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n}\times(-1+1)=0.$$

This proves the second identity in (2.22) when both derivatives agree.

Also note that for $k\neq l\in\{1,...,d\}$ we have $\sum_{\lambda\in\Gamma}R^{kl}_\lambda=0$. Indeed, for λ as above

$$\begin{aligned}(D_k\psi,D_l\psi_\lambda)&+\sum_{\epsilon_l\in\{-1,1\}}(D_k\psi,D_l\psi_{\epsilon_l e_l+\lambda})\\&=2^{d-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n}-2\times2^{d-1-n}\Big(\frac{1}{6}\Big)^n\Big(\frac{1}{3}\Big)^{d-1-n}=0,\end{aligned}$$

while $R^{kl}_\lambda=0$ for other choices of $\lambda\in\Gamma$.

We now study the case of mixed derivatives. Given $k\neq l\in\{1,...,d\}$ recall that $\mathcal{I}(k,l)=\{1,...,d\}\backslash\{k,l\}$. Then for $k\neq l\in\{1,...,d\}$ and $\mathbf{i}\in\mathbb{Z}^d$ we have for $n=0,...,d-2$

$$(D_k\psi_{\mathbf{i}},D_l\psi)=0\ \text{ if } |i_k\,i_l|\neq1,\tag{7.5}$$

$$(D_k\psi_{\mathbf{i}},D_l\psi)=-\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-n-2}\ \text{ if }\ i_k\,i_l=1,\quad\sum_{r\in\mathcal{I}(k,l)}|i_r|=n,\tag{7.6}$$

$$(D_k\psi_{\mathbf{i}},D_l\psi)=+\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-n-2}\ \text{ if }\ i_k\,i_l=-1,\quad\sum_{r\in\mathcal{I}(k,l)}|i_r|=n.\tag{7.7}$$

For $n=1,...,d-2$ and $k_1<...<k_n$ with $k_r\in\mathcal{I}(k,l)$ for $r=1,...,n$, set

$$\Gamma_{k,l}(k_1,...,k_n)=\Big\{\sum_{r=1}^n\epsilon_{k_r}e_{k_r}\ :\ \epsilon_r\in\{-1,1\}\Big\}.$$

For $n=0$ there is no such family of indices $k_1<...<k_n$ and we let $\Gamma_{k,l}(\emptyset)=\{0\}$. Thus for $n=0,...,d-2$, $|\Lambda_{k,l}(k_1,...,k_n)|=2^n$. Using the identities (7.5)-(7.7) we deduce

$$\begin{aligned}\sum_{\lambda\in\Gamma}(D_k\psi,D_l\psi_\lambda)&=\sum_{n=0}^{d-2}\sum_{k_1<k_2<...<k_n,k_r\in\mathcal{I}(k,l)}\sum_{\lambda\in\Gamma_{k,l}(k_1,...,k_n)}\big[(D_k\psi,D_l\psi_{e_k+e_l+\lambda})\\&\quad+(D_k\psi,D_l\psi_{e_k-e_l+\lambda})+(D_k\psi,D_l\psi_{-e_k+e_l+\lambda})+(D_k\psi,D_l\psi_{-e_k-e_l+\lambda})\big]\\&=\sum_{n=0}^{d-2}\binom{d-2}{n}2^n\Big[-\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-2-n}+\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-2-n}\\&\quad+\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-2-n}-\Big(\frac{1}{2}\Big)^2\Big(\frac{1}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-2-n}\Big]=0,\quad k\neq l.\end{aligned}\tag{7.8}$$

This completes the proof of the second identity in (2.20) when $i\neq j$, and hence (2.20) holds true. Furthermore, the identities (7.6) and (7.7) imply for $i\neq j\in\{1,...,d\}$ and $\{i,j\}=\{k,l\}$

$$\sum_{\lambda\in\Gamma}(D_k\psi,D_l\psi_\lambda)\lambda_k\lambda_l=\sum_{n=0}^{d-2}\sum_{k_1<k_2<...<k_n,k_r\in\mathcal{I}(k,l)}\sum_{\lambda\in\Gamma_{k,l}(k_1,...,k_n)}\big[(D_k\psi,D_l\psi_{e_k+e_l+\lambda})$$

$$- (D_k\psi, D_l\psi_{e_k-e_l+\lambda}) - (D_k\psi, D_l\psi_{-e_k+e_l+\lambda}) + (D_k\psi, D_l\psi_{-e_k-e_l+\lambda})\Big]$$

$$= -4\Big(\frac{1}{2}\Big)^2 \sum_{n=0}^{d-2} \binom{d-2}{n} 2^n \Big(\frac{1}{6}\Big)^n \Big(\frac{2}{3}\Big)^{d-2-n} = -\sum_{n=0}^{d-2}\binom{d-2}{n}\Big(\frac{2}{6}\Big)^n\Big(\frac{2}{3}\Big)^{d-2-n} = -1.$$

Equation (7.5) proves that $(D_k\psi, D_l\psi_\lambda) = 0$ if $|\lambda_k\lambda_l| \neq 1$. Hence using (7.8) we deduce that for any $r = 1, ..., d$,

$$\sum_{\lambda\in\Gamma}(D_k\psi, D_l\psi_\lambda)\lambda_r\lambda_r = 0.$$

Let $r \in \mathcal{I}(k,l)$ and for $n = 1, ..., d-3$, let $k_1 < ... < k_n$ be such that $k_j \in \{1, ..., d\}\backslash\{k,l,r\}$ and $\lambda = \sum_{j=1}^n \epsilon_{k_j} e_{k_j}$ for $\epsilon_{k_j} \in \{-1, 1\}$, $j = 1, ..., n$. Then for any choice of ϵ_k and ϵ_l in $\{-1, 1\}$ the equalities (7.6) and (7.7) imply that

$$(D_k\psi, D_l\psi_{\lambda+\epsilon_k e_k+\epsilon_l e_l+e_r}) = (D_k\psi, D_l\psi_{\lambda+\epsilon_k e_k+\epsilon_l e_l-e_r}).$$

This clearly yields that for $r \in \mathcal{I}(k,l)$ we have

$$\sum_{\lambda\in\Gamma}(D_k\psi, D_l\psi_\lambda)\lambda_k\lambda_r = \sum_{\lambda\in\Gamma}(D_k\psi, D_l\psi_\lambda)\lambda_l\lambda_r = 0.$$

Finally, given $n = 2, ..., d$ and $k_1 < ... < k_n$ where the terms $k_j \in \mathcal{I}(k,l)$, then given any choice of ϵ_k and ϵ_l in $\{-1, 1\}$, the value of $(D_k\psi, D_l\psi_{\epsilon_k e_k+\epsilon_l e_l+\lambda})$ does not depend on the value of $\lambda \in \Gamma_{k,l}(k_1, ..., k_n)$. Therefore, if we fix $r_1 \neq r_2$ in the set $\mathcal{I}(k,l)$, for fixed n there are as many choices of indices $k_1 < ... < k_n$ such that $\epsilon_{r_1}\epsilon_{r_2} = 1$ that of such indices such that $\epsilon_{r_1}\epsilon_{r_2} = -1$. This proves

$$\sum_{\lambda\in\Gamma}(D_k\psi, D_l\psi_\lambda)\lambda_{r_1}\lambda_{r_2} = 0,$$

which completes the proof of the first identity in (2.22) for mixed derivatives; hence (2.22) holds true.

We now check the compatibility condition (2.21). Fix $j \in \{1, ..., d\}$; then

$$(D_j\psi, \psi) = 2^{d-1}\Big(\prod_{k\neq j}\int_0^1 (1-x_k)^2 dx_k\Big)\Big[\int_0^1 (-1)(1-x_j)dx_j + \int_{-1}^0 (1+x_j)dx_j\Big] = 0, \quad (7.9)$$

while

$$(D_j\psi, \psi_{e_j}) = 2^{d-1}\Big(\prod_{k\neq j}\int_0^1 (1-x_k)^2 dx_k\Big)\int_0^1 (-1)\big(1+(x_j-1)\big)dx_j = -\frac{1}{2}\Big(\frac{2}{3}\Big)^{d-1},$$

$$(D_j\psi, \psi_{-e_j}) = 2^{d-1}\Big(\prod_{k\neq j}\int_0^1 (1-x_k)^2 dx_k\Big)\int_{-1}^0 \big(1-(x_j+1)\big)dx_j = \frac{1}{2}\Big(\frac{2}{3}\Big)^{d-1}. \quad (7.10)$$

For $n = 1, ..., d-1$ and $k_1 < ... < k_n$ where the indexes k_r, $r = 1, ..., n$ are different from j we have for any $\lambda \in \Gamma_j(k_1, ..., k_n)$

$$(D_j\psi, \psi_\lambda) = 2^{d-(n+1)}\Big(\prod_{k\in\Gamma\backslash\{j,k_1,...,k_n\}}\int_0^1 (1-x_k)^2 dx_k\Big) \times \Big(\prod_{r=1}^n \int_0^1 x_{k_r}(1-x_{k_r})dx_{k_r}\Big)$$

$$\times \Big[\int_0^1 (-1)(1-x_j)dx_j + \int_{-1}^0 (1+x_j)dx_j \Big] = 0, \tag{7.11}$$

while

$$\begin{aligned}(D_j\psi, \psi_{e_j+\lambda}) =& 2^{d-(n+1)} \Big(\prod_{k\in\Gamma\setminus\{j,k_1,\ldots,k_n\}} \int_0^1 (1-x_k)^2 dx_k \Big) \times \Big(\prod_{r=1}^n \int_0^1 x_{k_r}(1-x_{k_r})dx_{k_r} \Big) \\ & \times \int_0^1 (-1)\big(1+(x_j-1)\big)dx_j = -\frac{1}{2}\,\Big(\frac{2}{3}\Big)^{d-(n+1)}\,\Big(\frac{1}{6}\Big)^n, \end{aligned} \tag{7.12}$$

and

$$\begin{aligned}(D_j\psi, \psi_{-e_l+\lambda}) =& 2^{d-(n+1)} \Big(\prod_{k\in\Gamma\setminus\{j,k_1,\ldots,k_n\}} \int_0^1 (1-x_k)^2 dx_k \Big) \times \Big(\prod_{r=1}^n \int_0^1 x_{k_r}(1-x_{k_r})dx_{k_r} \Big) \\ & \times \int_{-1}^0 \big(1-(x_j+1)\big)dx_j = \frac{1}{2}\,\Big(\frac{2}{3}\Big)^{d-(n+1)}\,\Big(\frac{1}{6}\Big)^n. \end{aligned} \tag{7.13}$$

Note that the number of terms $(D_j\psi, \psi_{\epsilon_l e_l+\lambda})$ with $\epsilon_l = -1$ or $\epsilon_l = +1$ is equal to $\binom{d-1}{n}2^n$. Therefore, the identities (7.9)-(7.13) imply that for any $j = 1, \ldots, d$ we have

$$\sum_{\lambda\in\Gamma} \lambda_j R^j_\lambda = -\sum_{\lambda\in\Gamma} (D_j\psi, \psi_\lambda)\ \lambda_j = \frac{1}{2}\Big(\frac{2}{3}\Big)^{d-1} - \frac{1}{2}\Big(\frac{2}{3}\Big)^{d-1}(-1) \tag{7.14}$$

$$+\frac{1}{2}\sum_{n=1}^{d-1} \binom{d-1}{n} 2^n \Big(\frac{2}{3}\Big)^{d-1-n}\Big(\frac{1}{6}\Big)^n - \frac{1}{2}\sum_{n=0}^{d-1} \binom{d-1}{n} 2^n \Big(\frac{2}{3}\Big)^{d-1-n}\Big(\frac{1}{6}\Big)^n \times (-1)$$

$$= \sum_{n=0}^{d-1} \binom{d-1}{n} \Big(\frac{2}{6}\Big)^n \Big(\frac{2}{3}\Big)^{d-1-n} = 1. \tag{7.15}$$

This proves (2.21) when $i = k$.

Let $k \neq j \in \{1, \ldots, d\}$ and given $n = 1, \ldots, d-1$ let $k_1 < \ldots < k_n$ be indices that belong to $\mathcal{I}(j)$ such that one of the indices $k_r, r = 1, \ldots, n$ is equal to k. Given any $\lambda \in \Gamma_j(k_1, \ldots, k_n)$ we deduce that

$$(D_j\psi, \psi_{e_l+\lambda})\lambda_k + (D_j\psi, \psi_{-e_l+\lambda})\lambda_k = 0.$$

This completes the proof of the identity (2.21).

In order to complete the proof of the validity of Assumption 2.5, it remains to check that the identities in (2.23) hold true. Recall that for $\lambda \in \Gamma$ and $i, j, k, l \in \{1, \ldots, d\}$ we have

$$Q^{ij,kl}_\lambda = \int_{\mathbb{R}^d} z_k z_l D_j\psi_\lambda(z) D_i^*\psi(z)dz = -\int_{\mathbb{R}^d} z_k z_l D_j\psi_\lambda(z) D_i\psi(z)dz.$$

For $p = 1, \ldots, 4$, $n = 1, \ldots, d-p$ and $i_1, \ldots, i_p \in \{1, \ldots, d\}$ with $i_1, \ldots, i_p$ pairwise different let

$$\begin{aligned}\mathcal{I}_n(i_1, \ldots, i_p) := \Big\{ \sum_{\alpha=1}^n \epsilon_\alpha e_{k_\alpha}; \epsilon_\alpha \in \{-1, +1\},\ 1 \le k_1 < \ldots < k_n \le d, \\ k_\alpha \notin \{i_1, \ldots, i_p\} \text{ for } \alpha = 1, \ldots, n \Big\}, \end{aligned}$$

and $\mathcal{I}_0(i_1, \ldots, i_p) = \{0\}$.

First suppose that $i = j$.
First let $k = l = i$; then for $n = 0, ..., d-1$ and $\mu \in \mathcal{I}_n(i)$ we have

$$Q_{\mu}^{ii,ii} + Q_{\mu+e_i}^{ii,ii} + Q_{\mu-e_i}^{ii,ii} = 0.$$

Let $k = l$ with $k \neq i$; then then for $n = 0, ..., d-1$ and $\mu \in \mathcal{I}_n(i)$ we have

$$Q_{\mu}^{ii,kk} + Q_{\mu+e_i}^{ii,kk} + Q_{\mu-e_i}^{ii,kk} = 0.$$

Let $l = i$ and $k \neq i$; then for $n = 0, ..., d-2$, $\epsilon \in \{-1, +1\}$ and $\mu \in \mathcal{I}_n(i,k)$ we have

$$Q_{\mu+\epsilon e_i+e_k}^{ii,ki} + Q_{\mu+\epsilon e_i-e_k}^{ii,ki} = 0.$$

A similar result holds for $k = i$ and $l \neq i$. Furthermore, $Q_{\lambda}^{ii,ki} = 0$ is λ is not equal to $\mu + \epsilon e_i + e_k$ or $\mu + \epsilon e_i - e_k$ for $\mu \in \mathcal{I}_n(i,k)$ for some n.

Let $k \neq l$ with $k \neq i$ and $l \neq i$; then for $n = 0, ..., d-2$, $\epsilon \in \{-1, +1\}$ and $\mu \in \mathcal{I}_n(k,l)$ we have

$$Q_{\mu+\epsilon e_k+e_l}^{ii,kl} + Q_{\mu+\epsilon e_k-e_l}^{ii,kl} = 0,$$

while $Q_{\lambda}^{ii,kl} = 0$ is λ is not equal to $\mu + \epsilon e_i + e_k$ or $\mu + \epsilon e_i - e_k$ for $\mu \in \mathcal{I}_n(i,k)$ for some n.

We now suppose that $i \neq j$.
First suppose that $k = i$ and $l = j$; then for $n = 0, ..., d-1$ and $\mu \in \mathcal{I}_n(i)$ we have

$$Q_{\mu}^{ij,ij} + Q_{\mu+e_j}^{ij,ij} + Q_{\mu-e_j}^{ij,ij} = 0.$$

Let $k = l = i$; then for $n = 0, ..., d-2$, $\epsilon \in \{-1+1\}$ and $\mu \in \mathcal{I}_n(i,j)$ we have

$$Q_{\mu+\epsilon e_i+e_j}^{ij,ii} + Q_{\mu+\epsilon e_i-e_j}^{ij,ii} = 0,$$

while $Q_{\lambda}^{ij,ii} = 0$ is λ is not equal to $\mu + \epsilon e_i + e_j$ or $\mu + \epsilon e_i - e_j$ where $\mu \in \mathcal{I}_n(i,j)$ for some n. A similar result holds exchanging i and j for $k = l = j$.
Let $k = l$ with $k \notin \{i,j\}$ and $l \notin \{i,j\}$; then for $n = 0, ..., d-2$, $\epsilon \in \{-1, +1\}$ and $\mu \in \mathcal{I}_n(i,j)$ we have

$$Q_{\mu+\epsilon e_i+e_j}^{ij,kk} + Q_{\mu+\epsilon e_i-e_j}^{ij,kk} = 0,$$

while $Q_{\lambda}^{ij,kk} = 0$ is λ is not equal to $\mu + \epsilon e_i + e_j$ where $\mu + \epsilon e_i - e_j$ for $\mu \in \mathcal{I}_n(i,j)$ for some n.
Let $l = i$ and $k \notin \{i,j\}$; then for $n = 0, ..., d-2$, $\epsilon \in \{-1+1\}$ and $\mu \in \mathcal{I}_n(i,k)$ we have

$$Q_{\mu+\epsilon e_i+e_k}^{ij,ki} + Q_{\mu+\epsilon e_i-e_k}^{ij,ki} = 0,$$

while $Q_{\lambda}^{ij,ki} = 0$ is λ is not equal to $\mu + \epsilon e_i + e_k$ or $\mu + \epsilon e_i - e_k$ where $\mu \in \mathcal{I}_n(i,k)$ for some n. A similar result holds exchanging i and j for $k = l = j$.
Finally, let $k \neq l$ with $k \notin \{i,j\}$ and $l \notin \{i,j\}$; then for $n = 0, ..., d-4$, $\epsilon_i, \epsilon_j, \epsilon_k \in \{-1+1\}$ and $\mu \in \mathcal{I}_n(i,j,k,l)$ we have

$$Q_{\mu+\epsilon_i e_i+\epsilon_j e_j+\epsilon_k e_k+e_l}^{ij,kl} + Q_{\mu+\epsilon_i e_i+\epsilon_j e_j+\epsilon_k e_k-e_l}^{ij,kl} = 0,$$

while $Q_{\lambda}^{ij,kl} = 0$ is λ is not equal to $\mu + \epsilon_i e_i + \epsilon_j e_j + \epsilon_k e_k + e_l$ or $\mu + \epsilon_i e_i + \epsilon_j e_j + \epsilon_k e_k - e_l$ where $\mu \in \mathcal{I}_n(i,j,k,l)$ for some n. These computations complete the proof of the first identity in (2.23). Recall that for $i, k \in \{1, ..., d\}$ and $\lambda \in \Gamma$ we let

$$\tilde{Q}_{\lambda}^{i,k} := \int_{\mathbb{R}^d} z_k D_i \psi_\lambda(z) \psi(z) dz.$$

Let $k=i$; for $n=0,...,d-1$ and $\mu\in\mathcal{I}_n(i)$ we have

$$\tilde{Q}^{i,i}_{\mu}+\tilde{Q}^{i,i}_{\mu+e_i}+\tilde{Q}^{i,i}_{\mu-e_i}=0.$$

Let $k\neq i$; for $n=0,...,d-2$, $\epsilon\in\{-1,0,+1\}$ and $\mu\in\mathcal{I}_n(i,k)$ we have

$$\tilde{Q}^{i,k}_{\mu+\epsilon e_i+e_k}+\tilde{Q}^{i,k}_{\mu+\epsilon e_i-e_k}=0$$

while $\tilde{Q}^{i,k}_{\lambda}=0$ if λ is not equal to $\mu+\epsilon e_i+e_k$ or $\mu+\epsilon e_i-e_k$ where $\mu\in\mathcal{I}_n(i,k)$ for some n. This completes the proof of the second identity in (2.23); therefore Assumption 2.5 is satisfied for these finite elements. This completes the verification of the validity of Assumptions 2.4-2.5 for the function ψ defined by (6.2).

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