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Textbook accounts of the rules of indices with rational exponents

Christopher James Sangwin[a]

[a]School of Mathematics, University of Edinburgh, Edinburgh, EH9 3FD, UK

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ABSTRACT
The rules of indices, e.g. $a^n b^n = (ab)^n$, are a particularly important part of elementary algebra. This paper reports results from a textbook analysis which examined how the shift from integer to rational exponents in the rules of indices is discussed in school textbooks. The analysis also considered related issues, such as notation and the introduction of complex numbers. A selection of popular textbooks from the period 1800–2000 was examined and the nature of the justification given for the extension of meaning to rational indices considered. In both the definition and computational rules, when extending the domain of n in a^n to rational numbers the (potential) contraction of the domain of a to positive numbers was often quietly ignored. A wide variety of approaches are used in choosing what is to be a definition, what is to follow, and how this is justified. The difference between computational rules for practical algebraic manipulation and a formal definition was often blurred.

KEYWORDS
Modes of reasoning; Algebra; Textbooks; Secondary mathematics

Maths subject classification: 97H20

1. Background

In this paper I report the results of a content analysis I undertook to examine how the rules of indices are discussed in school textbooks. I pay particular attention to how authors explain the shift from integer to rational exponents. The notation a^n is usually defined as repeated multiplication of a with itself n times. For natural numbers n and m the following rules of indices can then be derived

$$a^n a^m = a^{n+m}, \quad (a^n)^m = a^{nm}, \quad a^n b^n = (ab)^n. \tag{1}$$

When finding square roots we have an extension of the definition of the notation a^n where n is now a rational number, e.g. $a^{\frac{1}{2}}$, normally in such a way that seeks to preserve the rules of indices (1).

We also have parallel, but much older, notation $\sqrt{a}$ which may, or may not, be defined to be exactly the same as $a^{\frac{1}{2}}$. At this moment a number of interrelated mathematical issues arise. How many roots do we expect? Sometimes $\sqrt{a}$ is taken to mean the positive square root of $a \geq 0$. This chooses a principal value and a restricted

orcid.org/0000-0002-3725-8625. Email: C.J.Sangwin@ed.ac.uk

domain $a \geq 0$. In other situations we take $\pm\sqrt{a}$, effectively making use of a multi-function. If $a > 0$ then $\sqrt{-a}$ may, or may not, be defined or permitted. For example, $\sqrt[3]{-8}$ is often taken to be -2, since $(-2)^3 = -8$. The cube root function is defined on the whole of the real line, and multi-functions only become necessary when we extend to the complex domain to return all three complex solutions of $x^3 = -8$.

There are further extensions of the meaning of (1), e.g. when n is real, when n is complex and when a is complex. An important related concept is the inverse of the map $n \to a^n$, i.e. the logarithm. Each student is likely to encounter these extensions of meaning, and the associated issues, over many years. The particular extension of meaning to rational powers is very likely to occur at a stage before students are used to a significant level of formality. Therefore, it may not appear to be an extension of meaning or a definition at all. The modern definition of $a^n = e^{n\log(a)}$ in a real analysis course (e.g. Hardy [1, §209], or Spivak [2, p. 288]) is only encountered by specialist university mathematics students, after many years of practice and familiarity with (1) as computational rules. This raises important educational questions about when and how these shifts are addressed and the nature of the justifications given.

I examine, and compare, the meaning given to rational powers and examine the justifications provided for this extension of meaning, with the following research questions.

(1) Are surd quantities, e.g. $\sqrt{3}$, discussed before general rational powers?
(2) What role do the computational rules (1) play? If any are derived what justifications are given for the domain enlargement from n, m in the natural numbers ($\mathbb{N}$) to the rational numbers ($\mathbb{Q}$)?
(3) What is the definition of a^n for n a rational number? What, if anything, is said about the case $a < 0$?
E.g. is $(-2)^{\frac{1}{2}}$ discussed? What is said about $(-8)^{\frac{1}{3}}$? Is $(-2)^{\frac{1}{2}}(-3)^{\frac{1}{2}}$ considered?
(4) Are complex numbers introduced?
(5) Are the differences between rational numbers and their representation as fractions discussed? In particular, does the representation of a rational number as a particular fraction affect the result of a computation? Are examples such as $\sqrt{9}$ or $(-9)^{\frac{2}{4}}$ considered?

By comparing mathematical texts, I hope to identify a comprehensive range of approaches, and the issues/compromises which arise with each. I shall therefore also consider when authors quietly pass over some issue, or what is said by one author but not by another. For example, some books deny the possibility of taking the square root of a negative number.

To investigate these research questions I consider the official published accounts found in algebra and mathematics textbooks. The research reported in this paper is undertaken as a preliminary to identify how these rather subtle issues have been addressed historically as a precursor to studies of contemporary conceptions. Vinner, [3], for example, surveyed university students who he *'expected to identify the defining formulae of exponentiation'*. In particular, one of his defining formulae was

$$a^{n/m} = \sqrt[m]{a^n}, \text{ where } a \neq 0, \text{ and } m, n \text{ are whole numbers.}$$

Fractional exponents are therefore defined in terms of roots of powers. This presupposes the discussion of roots, by some other means, before introduction of rational powers. It is quite possible that rational powers could be introduced first, and motivate a discussion of roots, in which case a discussion of the (historic) notation $\sqrt[m]{a^n} := a^{n/m}$

would be more appropriate. I am interested in whether the equations taken by Vinner [3] are uniformly used to define the meaning of rational powers in mathematics texts. If they are not, then this could help to explain why his students demonstrated such a variety of different views.

Levenson [4] used the zero exponent to investigate teachers' understanding of the difference between definition and theorem and found some confusion on the distinction between the two. I am also interested in whether any of the formulae in (1) are considered to be a definition or derived theorem in textbooks. What, if anything, is the relationship between them? If one is given as a definition, are the others derived? If (1) are taken as a definition they are certainly not arbitrary, indeed they preempt what will follow. In this sense, for n and m rational numbers, (1) fall part way between an arbitrary definition and a theorem which can be deduced. Bernardo [5] interviewed year 10 students about the meaning of $\sqrt[6]{(-8)^2}$ and concluded that *'the review of textbooks shows that there are substantial differences in dealing with the $\surd$ sign'.* In this paper I report the results of a review of historic textbooks to examine how the shift from the original definition of notation a^n, as repeated multiplication of a with itself n times, is extended to rational powers.

The use of exponential notation enables multiplication to be rephrased in terms of addition of indices. This reveals an underlying structure, and hence exponential notation is used more widely to represent repeated processes such as differentiation and function composition. Although I do not consider these other processes here they are indicative of the importance and strength of the rules (1). I am interested in how textbook authors have chosen to sequence the introduction of the mathematical ideas associated with (1) in the context of fractional/rational powers, and address the consequences. The choices made by textbook authors are clearly intended, and very likely, to have an effect on how students view algebra.

1.1. Motivation from computer algebra

I was surprised by the following recent observation:

> [...] most publications containing the Cardano solution of a cubic equation do not mention that his formula is not always correct for non-real coefficients. Consequently this formula has been misused by many people, including some computer algebra implementers, such as me. The consequences can be disastrous. [6]

By using a linear shift to place the y-axis through the point of inflexion, every cubic equation can be written in the reduced form

$$x^3 + 3px - 2q = 0.$$

Cardano's formula, [7, Chapt. XII], can then be written as

$$x = \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}}. \tag{2}$$

The difficulty arises in knowing which root to take. Importantly $a^{\frac{1}{2}}$ cannot be defined on the complex numbers as a single-valued function with reference to purely computational rules, without contradictions. For example, when considering how to apply the rule $a^n b^n = (ab)^n$ to $\sqrt{-a} \times \sqrt{-b}$, there are two distinct options. Either

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{-a \times -b} = \sqrt{ab} \tag{3}$$

or

$$\left.\begin{aligned} \sqrt{-a} \times \sqrt{-b} &= (\sqrt{-1} \times \sqrt{a}) \times (\sqrt{-1} \times \sqrt{b}) &&= (\sqrt{-1})^2 \sqrt{a}\sqrt{b} &&= -\sqrt{ab}. \\ \sqrt{-a} \times \sqrt{-b} &= i\sqrt{a} \times i\sqrt{b} &&= i^2 \sqrt{a}\sqrt{b} &&= -\sqrt{ab}. \end{aligned}\right\} \tag{4}$$

The option (3) is more direct and therefore perhaps appears more natural, whereas (4) actually leads to fewer contradictions later in the subject. Neither take account of the $\pm$ issue. Perhaps I need the 'or' between (3) and (4) to be taken as both, leading to a multifunction. In this case how should I compute? In particular, what should I do in (2)?

As an immediate response, I used contemporary computer algebra systems (CAS) to compute $\sqrt[3]{-8}$. Maxima returns -2 whereas Derive and Wolfram Alpha return $1 + i\sqrt{3}$. Maple currently has three separate commands, `(-8)^(1/3)`, `root(-8,3)` and `surd(-8,3)` which all return different answers. In the past, discourse about such variety has often been in terms of 'CAS giving wrong answers' and various solutions have been proposed, e.g. [6]. These solutions include better tracking of domain conventions, principal values, winding numbers, and multi-functions. While CAS developers have thought deeply about these issues authors, such as Jeffrey and Norman [8], complain that the wider mathematical community has few ways of learning about the conclusions reached during those debates. These issues might appear to matter only in advanced mathematics, particularly once complex numbers are adopted, but they impinge on the most elementary algebra in the form of basic calculation such as (1).

1.2. Use of historic textbooks

My hypothesis is that when a student or teacher is faced with a difficult issue or contradiction, such as whether $\sqrt{-2}\sqrt{-3}$ is taken to be $\sqrt{6}$ or $-\sqrt{6}$, or both, or neither, then a textbook would be consulted as authoritative. This hypothesis is supported by previous work, such as that of Pehkonen [9] who found that teachers trusted textbook authors as experts. Conversely, Kajander and Lovric [10] suggested that some textbooks might actually create and support misconceptions. It is therefore entirely appropriate to consider the official published accounts, and hence undertake a textbook analysis.

I have investigated the explanation given in textbooks using a methodology of content analysis. I accept the limitations of analysing the content of textbooks without investigating how the particular textbooks are actually used by students and teachers.

> Previous research has suggested that textbooks have the potential to be agents of change to transform the curriculum (e.g. [...]), so we wanted to analyse the textbooks in their 'rawest' form, without consideration of how they might be used in the actual classroom situation. [11, p. 188]

Rezat [12] developed a model for textbook use, framed within activity theory, and acknowledged that *'the textbook is a historically and culturally formed mediating artefact'*. Later developments of this model emphasise the use to which textbooks are put by students and concluded with insights into student's dispositions towards mathematics.

> Learning mathematics comprises mainly learning rules, applying rules and worked examples to tasks, and developing proficiency in tasks that are similar to teacher mediated tasks. [13, p. 1267]

(1) are precisely the kinds of rules to which Rezat [13] refers. Textbooks are therefore important authoritative sources of knowledge. I have concentrated on the forms of argument they contain.

I argue that it is appropriate to take a historical selection of texts from the period 1800–2000. Contemporary textbooks are tied to particular curricula or targeted at a specific examination in one jurisdiction, e.g. Howson comments on this issue in [14, p. 652]. They are also often written for a particular year group in school which splits algebra over a number books, possibly by different authors. Each textbook is written in its own historic and sociocultural context, and therefore a historic approach also provides an opportunity for a wider variety of conceptions to be evident than might be the case with a selection of contemporary books.

I have chosen not to look at algebra texts prior to 1800 because what is now elementary algebra was then the subject of mathematical research. In 1799 Gauss published a criticism of d'Alembert's proof of the Fundamental Theorem of Algebra (FTA), and provided a proof of his own. Regardless of whether contemporary mathematicians accept this as the first satisfactory proof, between 1759–1800 the FTA was an active topic of mathematics research by some of the most pre-eminent mathematicians, see [15]. Indeed, after 1800 mathematical research continues to develop what are now considered elementary topics. The quadratic formula is well known, and Cardano's method leads to the formula (2) for a root of the cubic in terms of the coefficients. The search for general methods for solving classes of equations was still an area of active research prior to this period. The work of Évariste Galois, who died in 1832, is now used to show that there is no corresponding formula for equations of degree 5 and higher. The first algebra book in my corpus by Euler, [16], does not envisage this negative result.

> §780. This is the greatest length to which we have yet arrived in the resolution of algebraic equations. All the pains that have been taken in order to resolve equations of the fifth degree, and those of higher dimensions, in the same manner, or, at least, to reduce them to inferior degrees, have been unsuccessful: so that we cannot give any general rules for finding the roots of equations, which exceed the fourth degree. [16]

It is interesting to speculate if Euler suspected the impossibility of the solution of the quintic by radicals, but he provides no hint to the reader here. Wessel gave the first geometric interpretation of complex numbers in 1797 and Argand in 1806, and these ideas were extended and developed by Gauss and others. The period (1814–1851) saw rapid and profound research in complex analysis, [17]. Prior to 1800 complex numbers were not well understood or even accepted. As late as 1872 Dedekind [18, p. 22] remarked that $\sqrt{2}\sqrt{3} = \sqrt{6}$ had not been proved rigorously. He was criticising a reliance on formal rules of calculation, rather than using a definition of real number. The formal algebraic rules, and the justifications which Dedekind criticises, are precisely those I consider in this paper and these rules remain as a core component of practical elementary algebraic manipulation.

Another reason I decided not to consider earlier works on algebra is that mathematical notation differed significantly. For example, one of the most popular Seventeenth century English algebra texts was William Oughtred's (1574 – 1660) *Clavis Mathematicae* (Key to Mathematics), see [19] and [20]. Oughtred wrote Q for the square of the unknown and C for the cube of the unknown, hence it was impossible for him to write rules of indices such as (1). For this reason I have not included algebra texts from the Seventeenth century and earlier. A history of early algebra is given by Stedall [21], and the classic history of mathematical notations is Cajori's [22], but see also Heeffer [23].

2. Methodology: selection of texts

I have chosen to look at textbooks which either seek to give a comprehensive and complete treatment of elementary algebra or those intended for use by advanced school students, working at the transition from school to university, i.e. those people who are being prepared for analysis and complex numbers in later education. I have therefore not considered (i) introductory arithmetic books, practical book keeping texts, or (ii) real analysis books. I have only considered textbooks published in English. In choosing specific texts I used the following criteria.

1. Popularity, i.e. sales;
2. Longevity (duration in print & number of editions);
3. Influence, e.g. acknowledgements from subsequent authors;
4. Experts in both history of algebra and mathematics education were consulted for suggestions of texts.

I only accepted the recommendations of experts when other corroborating evidence warranted the inclusion of a text. Indeed, a number of texts which were recommended to me had only a single edition and I could find no evidence of extensive sales, or subsequent citations. These may have much to commend them, but it is hard to find any evidence that they had a widespread influence and so these have been excluded.

To keep the corpus of texts to a manageable size, only books published in the United Kingdom have been included. Many of the authors of the selected texts cite international influences, and some of the texts in the corpus are direct translations. In turn there is evidence that the texts I have included influenced writers in other countries. A discussion of texts from the United States of America is given by [24], for example. Given the historic nature of the texts, I have decided it is not appropriate to preserve their anonymity with a coding scheme, as was done for example by Stacey [25] with contemporary texts.

The UK *School Mathematics Project* (SMP) books created a particular challenge. This collection of work is a distinct departure from traditional textbooks as they were based on the work of teachers in schools, see [26, p. 139–140], over a period of many years. The publication history of the SMP books is complex, and the lack of explicit authorship deliberately chosen to not clearly attribute individuals. For this study I have taken *SMP New Books 1–5* [27] and *SMP Advanced Mathematics Books 1–4* [28] as representative, treating the two sets as separate works.

3. Methodology: analysis of texts

I have chosen content analysis to address the research questions posed in Section 1. Fan, [29], published a framework for comparing recent textbook research. Within Fan's framework, this study is *'Textbooks as the subject of research'*, so the results are descriptive in nature.

The number of texts in this study is small relative to that often used in content analysis. However, I have taken a qualitative approach to very large individual works reading and isolating passages relating to specific topics. In some texts the relevant subject is discussed more than once, which increases the number of specific instances for analysis. Other works have a large number of volumes, notably the SMP books, and for some it was necessary to cross reference a number of editions. My goal was not to read the whole book in detail. Imagine a situation in which a student or teacher encountered false reasoning, or a tricky case such as (3) and looks in the book for an explanation or discussion. Can they resolve the situation satisfactorily, either with the written declarative text, or by finding an appropriate worked example of a similar type? My goal in this study is not to judge the text, but rather to investigate how rules of exponents are introduced, justified and discussed. My goal was also not to undertake a review of the exercises (e.g. as did [30], or [31]).

The methodology for addressing the research questions in Section 1 for each work was as follows.

- Read a late edition of the whole book lightly identifying all relevant sections. Draw up a general *narrative account*.
- Make detailed notes on the relevant sections. Actually *type out*, rather than photograph, passages to *force very close reading*.
- Cross reference the first and last editions, for the relevant sections and consult other editions where appropriate.

Looking at the research questions in Section 1 it might appear strange to ask if surd quantities are discussed before general rational powers. Introducing the concept of a square root of a number is, strictly speaking, independent of notation used to represent it. If this concept has already been introduced, perhaps with notation $\sqrt{a}$, then this will naturally affect the justification given for the domain enlargement in notation a^n.

To answer the second research question I needed to classify the form of the justification, if any, provided by the book authors. Many previous writers have considered forms of argument, including for example Harel [32]. My analysis initially followed the model used by Stacey [25], who examined the reasoning presented in seven topics in nine Australian eighth-grade textbooks and identified several modes of reasoning. This included, deduction using a general case, concordance of a rule with a model and experimental demonstration. Stacey [25] was studying year 8 textbooks, so had no need to use a code for *formal proof.* This is included by Blum [33], by Harel [32] within their broader *deductive proof,* and by Sierpinska [34] as *scientific explanation.*

Initially I compiled excerpts from 12 textbooks together with a coding protocol based on [25] and invited 6 colleagues to code the data. We were unable to code the data satisfactorily. With hindsight, the coding scheme of [25] is designed for very fine-grained work. For example, in coding the justification of an individual step in an argument, or a single well-defined theorem. The data we looked at could not be isolated into a specific proof and so is not amenable to such a micro analysis. In any one book, a discussion of the issues of rules of indices may occur in a number of places, and there may be a lack of clear distinction between a definition or theorem. What may be a theorem in one section becomes a definition in another. Furthermore, what does general and specific mean? E.g. when coding and argument based on $a^{\frac{1}{2}}$ the a is general, but the $\frac{1}{2}$ is specific. Arguments based on this are both general in one sense and specific in another, and so it is not clear here if [25]'s deduction using a general case or deduction using a specific case should be used. What I can say is that both are broadly deductive in reasoning, as opposed to experimental or a formal proof. Having looked at the data returned in the light of these difficulties the scheme shown in Table 1 could be recovered from the coder's comments and data. Table 1 shows a remarkable

Identified reasoning	**Code**
No argument or external authority	NL
Use of real-world physical model	MO
Empirical, or use of specific cases	EM
General deductive argument	DI
Formal Proof	PF
Extension of meaning	EX

Table 1. Modes of reasoning in explanatory text

similarity to the broader classifications used in Harel [32]. This close correspondence adds confidence that these classifications are a useful compromise between specificity of codes and robustness of implementation and interpretation.

4. Results: corpus of texts

The texts selected for inclusion in this study are shown in Figure 1. The horizontal axis is the date, and the solid horizontal bars give the known period of publication, i.e. the period between first edition and reference to the last published edition/printing as found in the catalogues of copyright libraries, such as that of the Bodleian library, Oxford. A dashed continuation line indicates historical reprints. E.g. the American Mathematical Society reprinted Chrystal's books [35,36] in 1999, indicating its continuing value as a text. The complete text of the majority of books published prior to 1900 is now readily available online, obviating the need for such historic reprints.

Dots with numbers above/below indicate edition numbers. For Euler's *Algebra*, [16], letters indicate the language of publication. It was written in German (D), first published in Russian (Ru), but it was the French edition which was translated into English (En) and published in 1797. [37] suggests that *'one popular German edition from Reclam Verlag sold no less that 108,000 copies between 1883 to 1943'*, indicating its enduring international popularity and potential influence throughout the period considered. Acknowledgements of influence by textbook authors are shown as arrows.

Most of the texts considered here had numerous published editions over more than 25 years, with some in print for over 75 years. Publication of online editions facilitates cross-referencing of early and later editions and where significant differences are identified in the relevant topics this will be noted below.

In some cases I question whether a book is a single publication or closer to a brand. The symbol † on the diagram indicates the death of one author. The authorship of Wood/Lund/Todhunter is especially complex, e.g. see [53, p. 197–198] and [54] for a discussion of the similarities between Todhunter's Algebra [43] and Wood's Algebra [50]. I treat Wood [50] and Lund [51] as separate publications, even though they form a continuous publication history.

In many cases the work is a single book but many textbooks are published as a series, in multiple volumes. In these cases it is appropriate to treat more than one book as a single work for the purposes of this study. For example, [35] (3rd edition) was published before the second volume [36] was written. It is clear to me these two books form a single work, indeed later editions are combined (see [55]). However, the text which was published as volume I of [39] was originally envisaged as forming volume IV of [40]. I have included all six volumes of [40] and [39] as a single work. I have therefore used some discretion in deciding what constitutes a single work. While

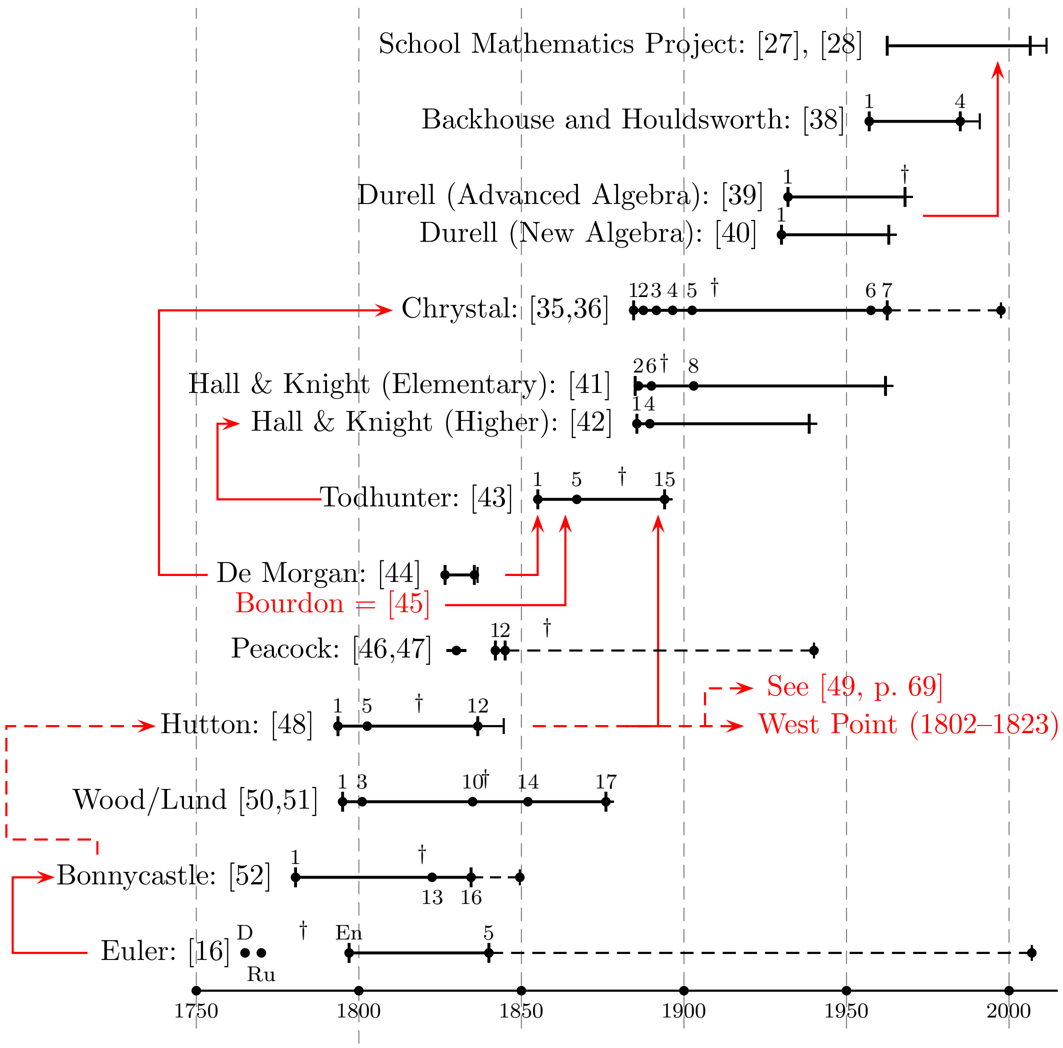

Figure 1. Selected English Algebra Texts, and their acknowledged interdependencies 1800–2000. Solid horizontal bars indicate continuous publication, with numbers indicating edition numbers and letters to language of publication, e.g. D means German, Ru Russian and En English. Dashed horizontal lines indicate subsequent historic reprints. Interconnecting lines and arrows indicate acknowledged influences. † denotes the death of one author.

		$\surd$ 1st?	$\mathbb{N} \to \mathbb{Q}$	R-of-S?	$a < 0$ excl.	$\mathbb{C}$ used?
[16]	Euler	Y	EM	N	N	Pre-alg
[52]	Bonnycastle	N	NL	Y	Y	Denied
[50]	Wood(1–13)	N	DI	Y	Y	No
[51]	Lund (14–)	N	DI+PF	Y	Y	Tentative
[48]	Hutton	N	NL	Y	Y	No
[44]	De Morgan	N	EX	N	N	Yes + (trig)
[46,47]	Peacock	Y	EX	N	N	Complete
[43]	Todhunter	N	DI+EX	Y	N	Yes
[41]	Hall & Knight (E)	Y	NL+EX	Y	N	No
[42]	Hall & Knight (H)	Y	EX	Y	N	Yes, later
[35,36]	Chrystal	N	NL+PF	N	N	Complete
[39,40]	Durell	N	EX	N	N	(Trig: [56])
[38]	Backhouse	Y	EX	N	N	Complete
[27]	SMP (New books)	Y	MO	N	N	No
[28]	SMP (Advanced)	Y	EM	N	N	Yes

Table 2. Summary of the data

[40] and [39] contain no acknowledgment of their sources, Durell was a prolific author of school textbooks which were widely used for many years. His books are cited by the *School Mathematics Project* (SMP) as being influential. This is the only case of a negative influence.

> We set out to create exercises where no two questions looked the same so that students were faced with new challenges all the time. This was a reaction to the Durell type texts which had long exercises of very repetitive questions. [26, p. 139–140]

5. Results: rules of indices

My first research question was to consider if surd quantities, e.g. $\sqrt{3}$, are discussed before general rational powers. A summary of the data is shown in table 2, with the first column indicating which books discuss surds first.

When considering roots many of the texts have a case-by-case analysis, and Lund's treatment is typical.

> §146. If the root to be extracted be expressed by an odd number, the sign of the root will be the same with the sign of the proposed quantity, as appears by Art. 137.
>
> $$\text{Thus } \sqrt{8} \text{ is } 2;\ \sqrt[3]{-8} \text{ is } -2; \&c.$$
>
> [sic: first surd should be $\sqrt[3]{}$.]
>
> §147. If the root to be extracted be expressed by an even number, and the quantity proposed be positive, the root may either be positive or negative. Because either a positive or negative quantity raised to such a power is positive (Art. 137).
>
> $$\text{Thus } \sqrt{a^2} \text{ is } \pm a;\ \sqrt[4]{(a+x)^8} \text{ is } \pm (a+x)^2; \&c.$$
>
> §148. If the root proposed to be extracted be expressed by an even number, and the sign of the proposed quantity be negative, the root cannot be extracted; because no quantity raised to an even power can produce a negative result. Such roots are called

impossible. [51, p. 70]

This, rather long, quotation is typical of the style of many of the earlier books. I call such a discussion the *rules of signs* (R-of-S). Other books do not contain such a discussion. Which books do and do not contain such a discussion is shown in the third column of table 2.

Over the period of time considered, complex numbers are increasingly accepted and used. For example Lund [51], in his text *"Designed for the use of students in the University"*, warns students not to make use of complex numbers.

> The student is recommended to have as little as possible to do with *imaginary* quantities, that is, with quantities which have no meaning either as to *number* or *magnitude*. He need not wonder that the difficulties are likely to be introduced by the use of them, when he considers that $\sqrt{-1}$ signified an operation to be performed which is absolutely impossible. Any discussion upon the interpretation which *may* be give to such symbols, and the uses to which they may be applied, would be quite out of place in an Elementary Treatise like the present. Ed. [57, Footnote, p. 75]

This footnote also appears as a footnote to §173 of many of the subsequent editions. I have listed his use of complex numbers as tentative, because despite this warning later editions of this book do make some use of complex numbers, e.g. see §178 of the 1841 edition [57].

The second column in table 2 provides data on the second research question: what justifications are given for the domain enlargement from n, m in the natural numbers ($\mathbb{N}$) to the rational numbers ($\mathbb{Q}$)? In his introduction to complex numbers De Morgan [44, p. 110–111] is the first in my corpus to discuss a progression of extensions in the meaning of number from negative, through rational and surds to real numbers and finally the meaning of $\sqrt{-1}$ (see also [58, Chapt IV]). De Morgan includes a geometrical interpretation of complex numbers, amounting to the Argand diagram, and he is the first author in my corpus to do so.

Rather than extending the meaning, Chrystal takes the laws as *axiomatic*, see [35, p. 29]. Students are expected to undertake mechanical manipulation without interpretation. I code this approach as an appeal to authority. When Chrystal returns to this subject he provides a proof, working from the definition, that the rational exponent is related to the idea of roots. See [35, p. 181]. The formal definition is used to prove the new notation corresponds to the concept of qth roots. Hence, I code this as a proof. Chrystal also appeals to continuity of the graph which is unusual. He fully accepts complex numbers, and deals with these in [35, Chap. XII]. This discussion also includes the Argand diagram.

Durell's works on algebra provide the typical extension of meaning, see [40, v. 3, p. 2], however his *definition* of complex numbers is part of *trigonometry*, [56]. Durell defines a complex number as an ordered pair $[a, b]$ and then defines multiplication and addition as formal rules. He then derives various algebraic and geometric properties. Durell is the only author in my corpus of texts who does not attempt to define complex numbers as a direct extension of the number system, thereby giving meaning to the square root of minus one. Durell's approach is abstract and thoroughly modern. The subsequent report, [59, p. 16], questioned the efficacy of this for all students: *'Such ideas call for powers of abstraction not to be expected of such young mathematicians'*.

The SMP New Books are the only books to work from a continuous model and this avoids the algebraic extensions of their predecessors.

2. Rational Powers

If the Megalogean tree grows steadily and doubles its height every year, by what factor is its height multiplied in half a year? In n years its height is multiplied by 2^n, so in half a year its height ought to be multiplied by $2^{\frac{1}{2}}$, but what does this mean? [27, 4(2), p. 65–66]

In appealing to a physical system, i.e. the size of a tree, the justification makes use of a model. The SMP Advanced Mathematics Books also represent a significant departure. These books ask the student to undertake sequences of exercises *before* sumarising the rules of indices as a synopsis of the previous work. See [28, p. 167–170]. The justification, provided by the students in this case, is experimental.

During the extension of meaning of the domain of n to be rational in a^n, some authors omit any discussion of the case $a < 0$ whereas other authors do address the case $a < 0$ directly, as shown in the fourth column of table 2. The fourth column indicates whether the case $a < 0$ is excluded in the definition of rational powers. That is, the fourth column indicates if the authors explicitly contract the domain to $a \geq 0$. For example, Euler adopts (3) in his elementary algebra text.

§148 Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$ we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$ or 2, for the value of the product of $\sqrt{-1}$ and $\sqrt{-4}$. Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one. [16]

De Morgan took the opposite view, adopting (4).

The only case which requires notice is that of forming the symbol which is to represent $\sqrt{-a} \times \sqrt{-b}$. This, by common rules, may be either $\sqrt{-a \times -b}$ or $\sqrt{ab}$, or $\sqrt{a} \times \sqrt{-1}$ multiplied by $\sqrt{b} \times \sqrt{-1}$ or $\sqrt{ab} \times -1$, that is $-\sqrt{ab}$. For reasons hereafter to appear, let the student always take the later, that is let

$$\sqrt{-a} \times \sqrt{-b} \text{ be } -\sqrt{ab} \text{ not } +\sqrt{ab}.$$

[44, p. 122]

Neither Euler or Dr Morgan exclude $a < 0$, so do not contract the domain, indeed they address what to do in this situation.

Many other authors simply do not discuss the case $a < 0$ when the domain of powers are taken to be rational. For example, Hall & Knight's elementary book [41] does discuss the rules of signs in §117, but later when the domain of powers are taken to be rational the question of $a < 0$ being excluded is omitted.

§236. These are the fundamental laws of combination of indices, and they are proved directly from a definition which is intelligible only on the supposition that the indices are *positive* and *integral*.

But it is found convenient to use fractional and negative indices, such as $a^{\frac{4}{5}}$, or a^{-7}, or more generally $a^{\frac{p}{q}}$, or a^{-n}; and these have at present no intelligible meaning. For it is plain that the definition of a^m, [Art. 232], upon which we based the three propositions just proved, is no longer applicable when m is *fractional* or *negative*.

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws. We therefore determine meanings for symbols such as $a^{\frac{p}{q}}$, or a^{-n}, in the following way: we assume that they conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads us. It will be found that the symbols so interpreted will also obey the other laws enunciated in Props. II and III. [41, p. 258–259]

The corresponding parts in the advanced text are

The interpretations for $a^{\frac{p}{q}}$, a^0, a^{-n} thus derive from the first law are found to be in strict conformity with the other two laws; and henceforth the laws of indices can be applied consistently and with perfect generality. [42, p. 431]

Hence Hall & Knight at this point do not exclude $a < 0$, so in Table 2 this is coded as "N" to indicate they do not exclude $a < 0$, i.e. they do not contract the domain.

Euler [16] is unique (in the corpus selected) in introducing complex numbers before algebra, e.g. see §145 of [16]. It is only some pages later that a general discussion of rational powers takes place. Euler extends the notation through a discussion of doing and undoing. See §196 of [16]. Euler introduces numerous repetitive examples and works inductively from them. I could find no clear statement of rules such as (1). All we have are examples, such as §203 and §205, with an expectation that *'Practice will render similar reductions easy.'* Since complex numbers have already been introduced, there is no reason to specially highlight the case $a < 0$, and Euler does not do so. He therefore also does not need to discuss what I termed rules of signs. However Euler's choice for $\sqrt{-a}\sqrt{-b} = \sqrt{ab}$ does not correspond to modern views of complex numbers which is intriguing. Unusually, Euler does discuss fractions not in lowest terms, see §204.

6. Discussion

I think it is remarkable how consistent algebraic notation is between texts throughout the period 1800–2000, even the choice of letters a, n and m in $(a^n)^m$ is ubiquitous. I think it is also remarkable that parallel notation $\sqrt{a}$ and $a^{\frac{1}{2}}$ has persisted[1]. For many authors $\sqrt{a}$ and $a^{\frac{1}{2}}$ are the same, but for others, including De Morgan [44], the notation is defined to be different.

Having two symbols to indicate the nth root of a, namely $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$, we shall employ the first in the simple arithmetical sense, and the second to denote any one of the algebraical roots, that is, any one we please, unless some particular root be specified. Thus $\sqrt{4}$ is 2, without reference to sign; but $(4)^{\frac{1}{2}}$ may be either $+2$ or -2. [44, p. 122]

Having two notations for the same thing is potentially confusing, and results in many exercises simply converting from one form to the other without apparently adding any new concepts. Euler, [16, §200], suggested that *'We may therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have just explained'*. He preferred exponential notation *'because it manifestly corresponds with the nature of the thing'*. However, he continued to use surds regularly *'as we have been long accustomed to those signs, and meet with them in most books of Algebra, it might be wrong to banish them entirely from calculations'*.

Many of the books have no diagrams whatsoever, and from a modern perspective this is odd. Functions are currently a central concept in elementary mathematics. It would now be unusual when discussing solutions of quadratic equations, for example, not to mention the connection between roots, factors and x-axis intercepts on the graph of the function.

The majority of books articulated the need to define a^n for $n \in \mathbb{Q}$. To do this, the rules of indices (1) are a theorem on $\mathbb{N}$ but then $a^n a^m = a^{n+m}$ becomes a defining axiom for rational n, m. This definition is not an arbitrary convention but is chosen

[1]The notation $\sqrt{x+a}$ is a combination of the radical sign $\surd$ and the vinculum. For example, [51] uses both parentheses and the vinculum for grouping, sometimes in the same expression. [52] writes, e.g. $\sqrt[3]{}(2x+3)$. Historic typography separated these $\sqrt[3]{}\overline{x+a}$ which is now written as $\sqrt[3]{x+a}$.

to preserve a structural property. This is what Peacock calls the *'principle of the permanence of equivalent forms'*, see [47, p. 59]. For a discussion of Peacock's attempt to create symbolic algebra through analogy, see [60], [61], [62] and [63]. The majority of modern definitions are not arbitrary conventions but preempt the theorems and proofs which follow. Hence definitions are far from arbitrary and are not purely social conventions. *'It is a legitimate part of this process to "look forward", exploring the consequences of alternative possible conventions before deciding which one to adopt.'* [64]. As another example consider the definition of (logical) *or* in which mathematical convention assumes that 'or' could mean both. This definition is precisely that required for deductions such as $AB = 0 \;\Rightarrow A = 0$ or $B = 0$. Further, using this choice De Morgan's laws become a theorem and give an elegant symmetry between 'and' ($\wedge$) and 'or' ($\vee$) when negating ($\neg$) formal statements.

$$\neg(A \wedge B) \equiv \neg A \vee \neg B \quad \text{and} \quad \neg(A \vee B) \equiv \neg A \wedge \neg B.$$

The definition looks ahead and preempts the use to which it is put. A similar argument may be made for $0! = 1$, or choosing to define default angular measure (quantity of rotation) in radians. There are cultural reasons within mathematics why discussions about definitions are hidden. In his criticism of the argument between Bernoulli and Leibnitz about the correct definition of the logarithm of a negative number, Euler acknowledges this freely:

> If at times this disagreement is not expressed strongly the reason is clearly that people do not want the certainties of pure mathematics in general to come under suspicion by revealing in public the difficulties and even contradictions that mathematicians find in this area. [65]

The contemporary discussion amongst CAS designers, discussed in Section 1.1, is a continuation of such debates.

When a concept is extended a particular choice has been made about the extension of the definition in such a way as to maximise the resulting usefulness (e.g. in computation, proving significant theorems), or to minimise the potential for later contradictions (or indeed both). Some of the books examined did discuss extensions of meaning in this way, notably De Morgan's work [44, p. 110–111]. However other books provided no justification, nor any indication of the difference in epistemological status between the equation $a^n a^m = a^{n+m}$ for natural number and rational exponents.

These books are intended for school students who are not yet exposed to formal mathematics. Furthermore, many of the books were written before the modern formal mathematics tradition of the early 20th century. Previous studies have commented on students' difficulties with formal definitions and how these relate to their concept image, e.g. Moore in [66]. Indeed, Edwards [67] made a distinction between kinds of definitions as follows: *'extracted definitions report usage, while stipulated definitions create usage, indeed create concepts, by decree.'*. Is the definition of the notation $a^{\frac{1}{2}}$ simply *stipulated* or somehow *extracted*? Many textbook authors blur this distinction, stipulating a formal rule and then justifying why this is necessary, i.e. extracted from the previous rule for natural numbers.

What many of the textbooks actually do is give rules for computation with fractional exponents. That is, they take a representation of a rational number as a fraction and explain how to compute with formal rules. In doing this they claim that whatever meaning is given to (1) for rational exponents, when those rational numbers which are also natural numbers are used the new meaning should preserve the original meaning of (1) for natural numbers. I.e. given a representation of a rational number as a fraction $\frac{p}{q}$, whatever definition is given to $a^{\frac{p}{q}}$ and whatever computational rules are applicable, whenever $\frac{p}{q}$ is equivalent to $\frac{n}{1}$ then the interpretation of $a^{\frac{p}{q}}$ and result of computation with $a^{\frac{p}{q}}$ should be identical with that for $a^n = a \times a \times \cdots \times a$.

The separation of interpretation or meaning from the computational rules is clear and conscious in some books. For example, Chrystal [35] takes the laws as axiomatic.

> §3. In so far as positive integral indices are concerned, the above laws are a deduction from the definition and from the laws of algebra. The use of indices is not confined to this case, however, and the above are *laid down* as the laws of indices generally. The laws of indices regarded in this way become in reality part of the general laws of algebra, and might have been enumerated in the Synoptic Table already given. In this respect they are subject to the remarks in chap. i,§27. The question of *meaning* of fractional and negative indices is deferred till a later chapter, but the student will have no difficulty in working the exercises given below. All he has to do is use the above laws whenever it is necessary, without regard to any restriction on the value of the indices. [35, p. 29]

In others books, a mixture of formal computational rules, their meaning and any justification is confused. This contributed to the difficulty in assigning a single classification code to the justifications from a fine-grained scheme.

It is common to find statements which are simply too general, and hence not true. E.g. 'these rules can be applied without error'. There is a paradoxical inconsistency on the focus of increasing one domain, but ignoring the other. I find this particularly surprising. In a^n we are changing the domain of n from the natural numbers to (at least) the rational numbers. To preserve previous computational rules of indices, the extension of the domain of n to rational coincides with a (potential) contraction of the domain of a to non-negative numbers. Did authors exclude $a < 0$? The most common answer, 'N', in Table 2 indicates they did not contract the domain to exclude $a < 0$ potentially allowing negative square roots and essentially condoning (3). This potentially awkward double negative (N="fail to exclude $a < 0$") is most commonly just an omission to say anything about a at this point.

My last research question asked if there are the differences between how rational numbers and their representation as fractions are discussed? In particular, does the representation of a rational number as a particular fraction affect the result of a computation? Are examples such as $(-9)^{\frac{2}{4}}$ considered? Few of the books examined contained a discussion of the problematic case with fractions not in lowest terms when $a < 0$. For example,

$$-2 = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \left[(-8)^2\right]^{\frac{1}{6}} = 2.$$

This illustrates why the extension of meaning of a^n for $n \in \mathbb{Q}$ is so epistemologically difficult: the result of computation can depend on the fractional representation of n, which is highly unsatisfactory. Acknowledgement of this issue does occur in Day's book [68, Note F]. Tirosh, [69], based in part on Goell [70], discussed the treatment of fractional powers in a small sample of textbooks and concludes that $(-8)^{\frac{1}{3}}$ is probably best left undefined, just as $\frac{a}{0}$ and $\frac{0}{0}$ are indeterminate forms. However the designers of contemporary CAS strongly disagree with this, as users consistently expect $(-8)^{\frac{1}{3}}$ to be transformed at least to $2(-1)^{\frac{1}{3}}$, if not to -2. CAS users certainly expect the system to *do something*! The contemporary approach is to avoid symbolic expressions $\sqrt{-3}$, replacing them immediately with the symbolic expression $i\sqrt{3}$. Recall that (4) has two forms, but this rather subtle notational move somewhat sidesteps the delicate issue enabling a wider range of formal algebraic computation to proceed without error or contradiction. That is, what appears to be a convention, i.e. to replace $\sqrt{-3}$ with $i\sqrt{3}$, looks ahead to accommodate computational difficulties. In this sense the notation $2i$ is therefore better designed than $\sqrt{-4}$, but it does not resolve all issues. For example, the desire to provide definite rules for how to compute obscures the multiplicity of solutions of $x^n = a$.

7. Conclusion

There is a significant variety of approaches to extending the meaning of a^n to include rational powers, including appeal to authority, deduction from specific cases, deduction from general cases and formal proof. Many books provide more than one justification, and many books blur the difference between a definition and a theorem. The most common approach was to accept rules of indices, as given, and to derive meaning by an extension. In doing this most texts quietly failed to exclude the case $a < 0$, or to discuss potential issues which arise for negative values of a.

There is a remarkable longevity in many of the texts considered, however there is gradual change over the period 1800–2000 with complex numbers increasingly accepted, and integrated. A comparison of the tables of contents will also show very similar overall structural approaches to algebra as a broader subject. Algebraic notation is also remarkably stable.

In these texts there is also an increasing recognition of the interconnected roles of *algebra*, *analysis* and *trigonometry*. A modern treatment of complex numbers requires a substantial amount of trigonometry, and the modern approach to the exponential function forms part of analysis and usually comes relatively late in a university mathematics course. It also seems remarkable that only Euler introduces complex numbers first, and really tries to use them effectively before algebra is introduced at all. Only Durell, with his ordered pair $[0, 1]$ for i, and SMP with their continuous model, are significantly different from the main corpus on the issue of rational powers.

It seems particularly strange that surds continue to be introduced and manipulated after rational powers. Some books contain extensive practice of converting from one notation to the other. Why do these authors retain two notations for the same thing? Sometimes the notation $\sqrt{a}$ and $a^{\frac{1}{2}}$ are not defined to be exactly the same, which seems to be an especially subtle distinction and one which is certainly not universally accepted in mathematical practice. Whether or not the mathematics exists in some external Platonic sense, the notation used to represent it is certainly a social construct. In particular notation, e.g. i vs $\sqrt{-1}$, is the result of design: the design of effective notation is an engineering problem.

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