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# Polygon-circle and word-representable graphs

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**Abstract**

We describe work on the relationship between the independently-studied polygon-circle graphs and word-representable graphs.

A graph $G = (V, E)$ is *word-representable* if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$ form a subword of the form $xyxy\cdots$ or $yxyx\cdots$ iff $xy$ is an edge in $E$. Word-representable graphs generalise several well-known and well-studied classes of graphs [2,3]. It is known that any word-representable graph is $k$-word-representable, that is, can be represented by a word having exactly $k$ copies of each letter for some $k$ dependent on the graph. Recognising whether a graph is word-representable is NP-complete ([3, Theorem 4.2.15]). A *polygon-circle graph* (also known as a *spider graph*) is the intersection graph of a set of polygons inscribed in a circle [4]. That is, two vertices of a graph are adjacent if their respective polygons have a non-empty intersection, and the set of polygons that correspond to vertices in this way are said to *represent* the graph. Recognising whether an input graph is a polygon-circle graph is NP-complete [5]. We show that neither of these two classes is included in the other one by showing that the word-representable Petersen graph and crown graphs are not polygon-circle, while the non-word-representable wheel graph $W_5$ is polygon-circle. We also provide a more refined result showing that for any $k \geq 3$, there are $k$-word-representable graphs which are neither $(k-1)$-word-representable nor polygon-circle.

*Keywords:* polygon-circle graph, word-representable graph, Petersen graph

## 1 Introduction

Polygon-circle graphs are the intersection graphs of polygons inscribed in a circle, and are one of many well-studied classes of intersection graphs. While recognising whether a graph is a polygon-circle graph is NP-complete, many problems that are NP-Complete in general can be resolved in polynomial time given a polygon-circle representation of a graph [5,6]. This algorithmic advantage, as well as the position of the polygon-circle graphs in the broad hierarchy of intersection classes, has been the focus of much of the study on this class. Polygon-circle graphs can also be defined in terms of a string that has a direct correspondence to a polygon-circle representation.

Word-representable graphs are defined primarily in terms of a string representation: we say that two letters $x, y$ *alternate* in a string $S$ if, when we delete all letters other than $x$ and $y$, the string is either of the form $xyxy...$ or $yxyx...$, of either even or odd length. Word-representable graphs can be characterised by a *semi-transitive orientation* of their edges (as described in [2], not to be defined here), and generalise the *comparability graphs* (which are graphs admitting transitive orientations), and the 3-*colourable graphs*.

It is often of interest to consider polygon-circle graphs of limited polygons, or word-representable graphs of limited numbers of copies of each letter: if a graph is representable by polygons of at most $k$ corners inscribed in a circle, then it is a *$k$-polygon-circle* graph, and if a graph is word-representable by a string in which each letter appears at most $k$ times, then it is *$k$-word-representable*. We note that originally $k$-word-representable graphs are defined as graphs admitting representation by a string in which each letter appears *exactly* $k$ times, but "exactly" here is equivalent to "at most", and in fact for any word-representable graph $G$ there exists $k$ such that $G$ is $k$-word-representable [2,3].

2-polygon-circle graphs are the intersection graphs of chords in a circle, and are also known as *circle graphs*. These coincide exactly with the

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2-word-representable graphs, and so 2-polygon-circle graphs are exactly 2-word-representable graphs [2]. Beyond this result, little further was known on interrelations between these two classes of graphs. In this note, we outline progress in understanding the relationship between polygon-circle and word-representable graphs, showing that neither is a subset of the other.

## 2 Definitions and preliminaries

### *2.1 Polygon-circle graphs*

A *polygon-circle representation* of a graph $G = (V, E)$ is a mapping $\mathcal{P}$ of vertices in $V$ to polygons in a circle such that $(u, v) \in E$ if and only if $\mathcal{P}(u)$ intersects $\mathcal{P}(v)$. Sometimes for convenience we will use subscripting, as in $P_i$, to denote the polygon $\mathcal{P}(v_i)$. A *corner-string* of a polygon-circle representation is a string produced by starting at an arbitrary point on the circle that hosts the representation, and proceeding around the circle, adding a letter denoting the vertex represented by a polygon each time a corner of a polygon encountered (example in Figure 1). Note that a single polygon-circle representation has many possible corner-strings, depending on the starting point. It is known [6] that two polygons representing vertices $u$ and $v$ intersect if any only if, when we delete all corners other than $u$ and $v$ from the corner-string, the result has as a substring either $uvuv$ or $vuvu$, in which case we say that $u$ and $v$ *loosely alternate*, and that the segments *overlap*. Note that this loose alternation is less strict than the notion of alternation used to define word-representable graphs: in the string $xyxxyx$ the letters $x$ and $y$ loosely alternate, but they do not alternate in the word-representation sense because there are two copies of $x$ between two consecutive copies of $y$.

If a vertex $v_i$ is represented by a polygon $P_i$ in a polygon-circle representation that gives a corner-string $\mathcal{S}$, then the *segment* of $u$ (denoted $s(u)$) in $\mathcal{S}$ is the minimal contiguous substring of $\mathcal{S}$ that contains all occurrences of $u$.

We say that a segment $s(u)$ is *contained in* $s(v)$ (denoted $s(u) \subset s(v)$) if $s(u)$ is entirely between two consecutive occurrences of $v$. Contrast this to the idea of a subsegment: we say $s(u)$ is a *subsegment* of $s(v)$ if $s(u)$ is a substring of $s(v)$. Segment containment is transitive and antisymmetric. Given a corner-string $\mathcal{S}$, we say that a segment $s$ is *outermost* if it is not contained in any other segment. A *path* from vertex $a$ to $b$ is a sequence of vertices $a, ..., b$ in which vertex is adjacent to the one before it in the sequence.

We use several relatively straightforward lemmas (proofs omitted for space):

**Lemma 2.1** *If two segments intersect but their corresponding polygons do* *not, then one is contained in the other.*

**Lemma 2.2** *If $s(u) \subset s(v)$ and $s(v) \subset s(w)$, then all paths from $u$ to $w$ must pass through the neighbourhood of $v$.*

**Lemma 2.3** *In a corner-string from a polygon-circle representation of a chordless cycle on $\geq 4$ vertices at least one vertex's segment contains another.*

### *2.2 Word-representable graphs*

The *wheel graph* $W_n$ is the graph on $n+1$ vertices obtained from the cycle graph $C_n$ by adding an all-adjacent vertex.

**Lemma 2.4** **([3])** *Wheel graphs $W_{2n+1}$ are non-word-representable for $n \geq 2$.*

**Lemma 2.5** **([2,3])** *The Petersen graph (shown in Figure 2) is 3-word-representable but not 2-word-representable.*

A *crown graph* $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. See Figure 2 for $H_{4,4}$.

**Lemma 2.6** **([1])** *For $n \geq 5$, the crown graph $H_{n,n}$ is $\lceil n/2 \rceil$-word-representable but not $(\lceil n/2 \rceil - 1)$-word-representable.*

## 3 Polygon-circle graphs vs word-representable graphs

In this section we will show that the word-representable Petersen graph and crown graphs are not polygon-circle, while the non-word-representable wheel graph $W_5$ is polygon-circle. We also provide a more refined result showing that for any $k \geq 3$, there are $k$-word-representable graphs which are neither $(k-1)$-word-representable nor polygon-circle.

**Lemma 3.1** *The wheel graph $W_n$ is a polygon-circle graph for $n \geq 3$.*

**Proof.** Essentially, for a wheel $W_n$ we represent the vertices on the large outer cycle $v_1 ... v_n$ with chords on the circle. We can place their corners starting at an arbitrary point on the circle in the sequence $v_0, v_n, v_1, v_0, v_2, v_1, v_3, ... v_{n-1}, v_n$. Then the central high-degree vertex is represented by a polygon with corners between the first corner of $v_2$ and the second corner of $v_1$, the first corner of $v_4$ and the second corner of $v_3$, etc - essentially between every second pair of chord corners, giving this polygon representing the central vertex of the wheel on $n$ vertices $\lceil n/2 \rceil$ corners, producing a $\lceil n/2 \rceil$-polygon-circle representation. An example of this representation for $W_5$ in particular is given in Figure 1. □

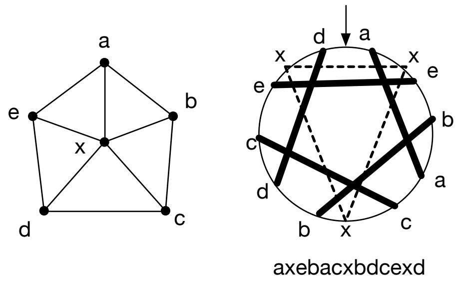


Fig. 1. The wheel $W_5$ with a polygon-circle representation. All vertices except for $x$ are represented with lines, and the vertex $x$ is represented by the dashed triangle. The corner-string below the representation is produced by starting at the top of the circle (indicated by the arrow) and reading clockwise around the circle.

**Theorem 3.2** *There are non-word-representable polygon-circle graphs.*

**Proof.** This is an immediate corollary of Lemmas 2.4 and 3.1. □

**Lemma 3.3** *The Petersen graph is not a polygon-circle graph.*

**Proof.** We proceed by contradiction. Let $\mathcal{S}$ be a corner-string of a representation of the Petersen graph $P = (V_P, E_P)$, as in Figure 2: we denote the segment corresponding to vertex $v$ as $s(v)$. By Lemma 2.1 there is at least one segment on each of the inner and outer cycles of five vertices that contains all segments corresponding to non-neighbours. Without loss of generality, let vertex $a$ be a vertex on the outer cycle represented by a segment that is outermost and contains the segments corresponding to $d$ and $c$.

Let $N[a]$ be the closed neighbourhood of $a$. We now argue that no segment corresponding to a vertex in $V \backslash N[a]$ can contain any other: let $x, y$ be non-adjacent vertices in $V \backslash N[a]$. Both $s(x) \subset s(a)$ and $s(y) \subset s(a)$. If $s(x) \subset s(y)$, then by Lemma 2.2 every path from $x$ to $a$ must pass through the neighbourhood of $y$, a contradiction to the fact that every member of $V \backslash N[a]$ has a path to $a$ avoiding the neighbourhood of each other member of $V \backslash N[a]$.

Then no segment representing a vertex in $V \backslash N[a]$ contains any other. However, the graph $P[V \backslash N[a]]$ is a chordless cycle on six vertices, which by Lemma 2.1 requires containment, a contradiction. □

**Theorem 3.4** *There are word-representable non-polygon-circle graphs.*

**Proof.** This is an immediate corollary of Lemmas 3.3 and 2.5. □

**Lemma 3.5** *Crown graphs $H_{n,n}$ for $n \geq 4$ are not polygon-circle graphs.*

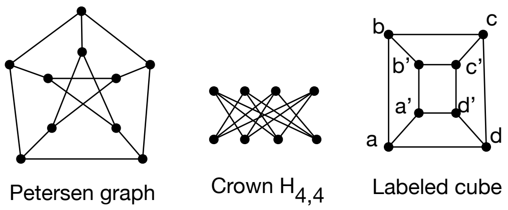


Fig. 2. Three graphs that we show are not polygon-circle graphs, some labeled for convenience. Note that the crown and cube on the right are isomorphic.

**Proof.** Because $H_{n+1,n+1}$ contains $H_{n,n}$ as an induced subgraph, it suffices to show that $H_{4,4}$ (isomorphic to the cube) is not a polygon-circle graph. It is already known that the cube is not a member of a superclass of the polygon-circle graphs [6]. A direct argument in the polygon-circle setting is also possible, and hinges on the repeated use of Lemma 2.1 along with the antisymmetry of segment containment to show that representing the many 4-cycles in the cube leads to contradictory constraints. □

**Theorem 3.6** *For every $k \geq 3$, there are $k$-word-representable graphs that are neither $(k-1)$-word-representable nor polygon-circle.*

**Proof.** The desired result follows from Lemmas 2.6 and 3.5, since $H_{2k,2k}$ is $k$-word-representable but not $(k-1)$-word-representable for $k \geq 3$. □

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