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The rectangular representation of the double affine Hecke algebra via elliptic Schur-Weyl duality

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Given a module M for the algebra $\mathcal{D}_\mathsf{q}(G)$ of quantum differential operators on G, and a positive integer n, we may equip the space $F_n^G(M)$ of invariant tensors in $V^{\otimes n} \otimes M$, with an action of the double affine Hecke algebra of type A_{n-1}. Here $G = SL_N$ or GL_N, and V is the N-dimensional defining representation of G. In this paper we take M to be the basic $\mathcal{D}_\mathsf{q}(G)$-module, i.e. the quantized coordinate algebra $M = \mathcal{O}_\mathsf{q}(G)$. We describe a weight basis for $F_n^G(\mathcal{O}_\mathsf{q}(G))$ combinatorially in terms of walks in the type A weight lattice, and standard periodic tableaux, and subsequently identify $F_n^G(\mathcal{O}_\mathsf{q}(G))$ with the irreducible "rectangular representation" of height N of the double affine Hecke algebra.

1 Introduction

This article furthers the study of "elliptic Schur-Weyl duality" – a functorial relationship between modules for the algebra $\mathcal{D}_\mathsf{q}(G)$ of quantum differential operators on $G = GL_N$ or SL_N, and modules for type A double affine Hecke algebras. This functor has proved very useful because, while the representation theory of double affine Hecke algebras is rather well-understood in terms of type A algebraic combinatorics, the representation theory of $\mathcal{D}_\mathsf{q}(G)$ is much less well-understood. Let us now recall the basic construction in more detail, before stating our main results.

On the first side of the duality, an important role is played by the ad-equivariant quantized algebra $\mathcal{O}_\mathsf{q}(G)$, a deformation of $\mathcal{O}(G)$ along the so-called Semenov-Tian-Shansky Poisson bracket. Likewise $\mathcal{D}_\mathsf{q}(G)$ is a simultaneous q-deformation of the algebra $D(G)$ of differential operators on G, and of functions on $G \times G$, with respect to the Heisenberg double Poisson bracket. The quantized coordinate algebra $\mathcal{O}_\mathsf{q}(G)$ is naturally a module for $\mathcal{D}_\mathsf{q}(G)$, which we call the basic $\mathcal{D}_\mathsf{q}(G)$-module.

On the other side of the duality lie Cherednik's double affine Hecke algebras of type A_{n-1}, $\mathbb{H}_{q,t}$, which we will also refer to as the DAHA.[*] There is a variant for $G = GL_n$, and for $G = SL_n$; we will consider both. Each is a universal deformation of the semi-direct product, $\mathbb{C}[\mathfrak{S}_n] \rtimes \mathcal{D}_{\mathsf{q}}(H)$, of the group algebra of the symmetric group $\mathfrak{S}_n$ and the algebra of *difference operators* on the Cartan subgroup $H \subset G$. The t parameter deforms the reflection action of $\mathbb{C}[\mathfrak{S}_n]$ to an action of the finite Hecke algebra, while the parameter q corresponds to the step for the difference action on H.

In [15], the first author constructed functors

$$F_n^{SL} : \mathcal{D}_{\mathsf{q}}(SL_N)\text{-mod} \to \mathbb{H}_{\boldsymbol{q},t}(SL_n)\text{-mod} \tag{1.1}$$

from the category of $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules to the category of modules for $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, specialized at $t = \mathsf{q}$, $\boldsymbol{q} = \mathsf{q}^{1/N}$. The underlying vector space on which $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ acts is

$$F_n^{SL}(M) = (V^{\otimes n} \otimes M)^{inv} \tag{1.2}$$

of $U_{\mathsf{q}}(\mathfrak{sl}_N)$-invariants in the tensor product, where we make M a module over $U_{\mathsf{q}}(\mathfrak{sl}_N)$ via the quantum moment map, and we take invariants with respect to that action. The action of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ is given by certain explicit operators, which we recall in Section 7.1. Here n and N are independent integer parameters, though in this paper we restrict to the case where $n = kN$, for some positive integer k.

The functor F_n^{SL} has a natural modification to a functor

$$F_n^{GL} : \mathcal{D}_{\mathsf{q}}(GL_N)\text{-mod} \to \mathbb{H}_{q,t}(GL_n)\text{-mod},$$

where the parameters are specialized at $t = \mathsf{q}$, $q = \mathsf{q}^{-2k}$.

While the SL version of the functor is simpler to define, the output is simpler to analyze in the GL case. The main results are essentially the same in the two cases, but with subtle minor modifications. For this reason, in this paper we construct and analyze the functor in both settings in parallel, paying special care to what happens when we pass between the two settings.

Our main results all concern the case when M is the basic $\mathcal{D}_{\mathsf{q}}(G)$-module $M = \mathcal{O}_{\mathsf{q}}(G)$; we compute the outputs $F_n^{GL}(\mathcal{O}_{\mathsf{q}}(GL_N))$ and $F_n^{SL}(\mathcal{O}_{\mathsf{q}}(SL_N))$ of the functor in this case. This is easily seen to lie in Category $\mathcal{O}$ for $\mathbb{H}_{q,t}$, a particularly nice subcategory of $\mathbb{H}_{q,t}$-modules, upon which a commutative subalgebra of $\mathbb{H}_{q,t}$ acts locally finitely (see Section 5). Our main result (given in more detail in Theorem 7.11) is:

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[*] Because the parameters q in $U_{\mathsf{q}}(\mathfrak{g})$, q in $\mathbb{H}_{q,t}(GL_n)$ and $\boldsymbol{q}$ in $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ do not coincide, yet each notation is well-established, we distinguish them by using different typeface.

Theorem. We have isomorphisms:

$$F_n^{GL}(\mathcal{O}_{\mathsf{q}}(GL_N)) \cong L(k^N), \qquad F_n^{SL}(\mathcal{O}_{\mathsf{q}}(SL_N)) \cong \overline{L}(k^N),$$

where $L(k^N)$ and $\overline{L}(k^N)$ denote the so-called rectangular representations of $\mathbb{H}_{q,t}(GL_n)$ and $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, respectively. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We note that if $k = \frac{n}{N}$ is not a positive integer, then $F_n^G(\mathcal{O}_{\mathsf{q}}(G)) = 0$, see Remark 7.5.

First, let us discuss the $G = GL$ case. The rectangular representations are irreducible modules in Category $\mathcal{O}$ which are moreover $\mathcal{Y}$-semisimple, where $\mathcal{Y}$ is the commutative subalgebra appearing in the definition of Category $\mathcal{O}$, see Section 6. The proof of the main theorem relies on identifying the weights of each module and then appealing to the structure of $\mathcal{Y}$-semisimple modules, to yield the isomorphism. In particular, these modules have simple spectrum and so come with distinguished basis up to scaling. This lets us focus on a combinatorial analysis of the $\mathcal{Y}$-weights to fully understand the module.

The computation of the $\mathcal{Y}$-weights involves three well-known elements of type A combinatorics: walks on the weight lattice, tableaux on skew diagrams, and periodic tableaux on an $N \times \infty$ strip. Schur-Weyl duality considerations, together with the Peter-Weyl decomposition, lead us naturally to a $\mathcal{Y}$-weight basis of $F_n^{GL}(\mathcal{O}_{\mathsf{q}}(GL_N))$ indexed by so-called *looped walks* in the dominant chamber of the weight lattice. In fact, each Peter-Weyl component is naturally a simple submodule upon restriction to the affine Hecke algebra.

We compute the $\mathcal{Y}$-weights, following a technique of Orellana and Ram [19], involving well-known formulas for the action of the ribbon element of the quantum group. The most natural combinatorial expression for the $\mathcal{Y}$-weights comes after we identify the set of looped walks with the set of standard tableaux on size n skew diagrams with k boxes in each of its N rows. Via this bijection, the $\mathcal{Y}$-weights can be read off as the values of certain "diagonal labels" on the tableaux (a variant of what is sometimes called "content" in the literature).

Finally, we introduce a procedure called "periodization", which turns a tableau on such a skew diagram D into a periodic tableau on the $N \times \infty$ strip. Roughly, D gets identified with a fundamental domain in the $N \times \infty$ strip, under horizontal shifting, and then the periodization map sends a standard tableau on D to a standard periodic tableau on the $N \times \infty$ strip, by extending the entries according to a periodicity rule with respect to the shifting action. We appeal to Cherednik's classification [9] of irreducible $\mathcal{Y}$-semisimple $\mathbb{H}_{q,t}(GL_n)$-modules via periodic Young diagrams, and in particular to the enumeration of the $\mathcal{Y}$-weights via periodic tableaux on such diagrams, described combinatorially in the follow-up paper, [24]. In this framework, the set of standard periodic tableaux on the $N \times \infty$ strip index a weight basis for a unique irreducible representation $L(k^N)$, hence allowing us to prove the theorem.

For $G = SL$, the above story holds with minor modifications. The role of the commutative algebra $\mathcal{Y}$ is played instead by an algebra $\mathcal{Z}$, which we may regard as a quotient of $\mathcal{Y}$ by a determinant-equals-one relation. We consider periodic tableaux on the $N \times \infty$ strip modulo a natural equivalence relation of horizontal shifting

by k steps. The diagonal labelling of these must be modified so as to be compatible with this periodicity. Once this is done, analogous bijections hold as in the GL case, and the $\mathcal{Z}$-weights can once again be read off from the (modified) diagonal labelling. Cherednik's classification no longer holds on the nose, as the construction does not distinguish irreducible $\mathcal{Z}$-semisimple modules from those obtained by scaling the action of the shifting generator π by a root of unity. Once we account for this, the SL statement in the theorem follows.

1.1   Rational degeneration of F_n^{SL}

The functor F_n^{SL} is a q-deformation of a similar functor,

$$F_n : D(\mathfrak{sl}_N)\text{-mod} \rightarrow \text{RCA}_n(c)\text{-mod},$$

introduced by Calaque, Enriquez and Etingof in [7]. Here $\text{RCA}_n(c)$ denotes the rational Cherednik algebra of type A_{n-1} with parameter $c = \frac{N}{n}$. Their functor in turn builds on a similar construction of Arakawa and Suzuki [1] for the degenerate affine Hecke algebra. In [7, Theorem 8.8], the authors compute the image of their functor on the basic $\mathcal{D}$-module $M = \mathcal{O}(\mathfrak{sl}_N)$, and identify it as a unique simple quotient of an induced rectangular module. The techniques we use here in the non-degenerate case are, however, completely different than in the degenerate setting.

One can connect these theorems more directly, following [15, Section 6] and [6, Section 6]. One can define a suitable degeneration $\mathcal{D}_q(SL_N) \rightsquigarrow D(\mathfrak{sl}_N)$, and a degeneration $\mathbb{H}_{q,t} \rightsquigarrow \text{RCA}_n(c)$, compatible in such a way that we obtain a degeneration of the functors $F_n^{SL} \rightsquigarrow F_n$. Tracing through these degenerations, one may obtain from Theorem 7.11 an independent proof Calaque, Enriquez, and Etingof's description of $F_n(\mathcal{O}(\mathfrak{sl}_N))$.

Category $\mathcal{O}$ for $\text{RCA}_n(c)$ is a highest weight category and its simples can be obtained as unique simple quotients of induced modules. While the construction of the DAHA-module $L(k^N)$ (resp. $\overline{L}(k^N)$) combinatorially via its weight basis was a key ingredient in the proof of the main theorems, we also give an alternate construction as a quotient of an induced module, motivated by the parallel construction for the rational Cherednik algebra. Another motivation for this alternate construction is that $\mathcal{O}_q(G)$ is most naturally constructed as an induced $\mathcal{D}_q(G)$-module.

More precisely, we construct $L(k^N)$ as the unique simple quotient of the module $M(k^N)$, induced from the $N \times k$ rectangular representation for the affine Hecke algebra in Corollary 6.11. In Theorem 6.14, we modify the construction of $\overline{L}(k^N)$ as a quotient of the module $\overline{M}(k^N)$, induced from the $N \times k$ rectangular representation for the appropriate quotient $H(\mathcal{Z})$ of the affine Hecke algebra. It is no longer the unique simple quotient, as $\overline{M}(k^N)$ also has quotients that are obtained from $\overline{L}(k^N)$ by twisting by an automorphism of $\mathbb{H}_{q,t}(SL_n)$, see also Remark 6.15.

Let us also mention here an interesting parallel approach to elliptic Schur-Weyl duality, initiated in [2], and continued in [23], [26], [18], in which the role of (classical, or quantum) differential operators on G is played

rather by the (classical, quantum) affine Lie algebra, but double affine Hecke algebras appear (at present, only in their rational and trigonometric degenerations).

While we are unaware of a precise connection between those constructions and the present framework, the basic structure of the functor – extending the ordinary Schur-Weyl duality on $V^{\otimes n}$ by taking instead group covariants from $V^{\otimes n}$ into a module for some auxiliary algebra – is the same, and it seems to be a very interesting question how to relate the two approaches, in particular how to relate quantum differential operators to affine Lie algebras.

1.2   Relation to factorization homology

The action of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ on $F_n^{SL}(M)$ was initially defined via generators and relations. Following [3, 4], it may be re-cast as a consequence of a much more general construction of the algebra $\mathcal{D}_{\mathsf{q}}(G)$, via the factorization homology of surfaces valued in braided tensor categories. Since such a description does not yet appear explicitly in the literature, we outline it here. This context is not technically necessary for any of the proofs in this paper, so we will keep the discussion informal.

Recall from [3], we have the following equivalence of categories:

$$\int_{T^2 \setminus D^2} \mathrm{Rep}_{\mathsf{q}}(SL_N) \simeq \mathcal{D}_{\mathsf{q}}(SL_N)\text{-mod}_{\mathrm{Rep}_{\mathsf{q}} SL_N},$$

where the left hand side denotes the factorization homology of a once-punctured torus with coefficients in the braided tensor category $\mathrm{Rep}_{\mathsf{q}}(SL_N)$ of integrable $U_{\mathsf{q}}(\mathfrak{sl}_N)$-modules, and the right hand side denotes the category of $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules equipped with an equivariance structure. In analogy with the theory of D-modules, these are called "weakly equivariant" $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules.

Recall from [4], we have a further equivalence of categories:

$$\int_{T^2} \mathrm{Rep}_{\mathsf{q}}(SL_N) \simeq \mathcal{D}_{\mathsf{q}}(\frac{SL_N}{SL_N})\text{-mod},$$

where now the righthand side denotes the category of $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules such that the quantum adjoint action, obtained via the "quantum moment map,"

$$\mu_{\mathsf{q}} : U_{\mathsf{q}}(\mathfrak{sl}_N)^{lf} \to \mathcal{D}_{\mathsf{q}}(SL_N),$$

makes $\mathcal{D}_{\mathsf{q}}(SL_N)$ into an integrable $U_{\mathsf{q}}(\mathfrak{sl}_N)$-module. Here $U_{\mathsf{q}}(\mathfrak{sl}_N)^{lf}$ denotes the sub-algebra consisting of elements which are locally finite for the quantum adjoint representation, see Section 3 for more details. In analogy with the theory of D-modules, these are called "strongly equivariant" $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules: a weakly equivariant $\mathcal{D}_{\mathsf{q}}(G)$-module is strong if its weak equivariance structure coincides with that coming from the

quantum moment map. Alternatively, a $\mathcal{D}_{\mathsf{q}}(G)$-module without a specified equivariance structure is a strong $\mathcal{D}_{\mathsf{q}}(G)$-module if the $U_{\mathsf{q}}(\mathfrak{g})$-action given by the quantum moment map is integrable.

It is an immediate consequence of formula (1.2) that F_n^{SL} factors through the functor $M \to M^{lf}$, of taking locally finite vectors for the quantum adjoint action, hence we may just as well regard it as a functor from $\mathcal{D}_{\mathsf{q}}(\frac{SL_N}{SL_N})$-mod. Indeed, it is more natural to expect the factorization homology of the closed torus to appear, since we regard the double affine Hecke algebra as a quotient of the braid group of the closed torus (also known as the elliptic braid group) by the quadratic Hecke relations.

The $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ action in the definition of F_n^{SL} most naturally arises as an action of the braid group of the closed torus, which descends to its quotient $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ in important special cases. Let us now explain the origins of the elliptic braid group action in factorization homology terms. Given an embedding $\iota^n : (D^2)^{\sqcup n} \to T^2$, i.e. a collection of n disjoint discs in the torus, factorization homology produces a functor,

$$\iota_*^n : \mathrm{Rep}_{\mathsf{q}}(SL_N)^{\boxtimes n} \to \int_{T^2} \mathrm{Rep}_{\mathsf{q}}(SL_N) \simeq \mathcal{D}_{\mathsf{q}}(\frac{SL_N}{SL_N})\text{-mod.}$$

This functor moreover carries an action, by natural isomorphisms, of the elliptic braid group B_n^{Ell}. The functors ι_*^n have been identified in [3] and [4] with free module functors to their respective categories of $\mathcal{D}_{\mathsf{q}}(SL_N)$-modules. It follows from these descriptions that, for any strongly equivariant $\mathcal{D}_{\mathsf{q}}(SL_N)$-module M, we may identify

$$F_n^{SL}(M) \cong \mathrm{Hom}\left(\iota_*^n\left((^*V)^{\boxtimes n}\right), M\right),$$

where $V \in \mathrm{Rep}_{\mathsf{q}}(SL_N)$ denotes the defining representation, and $(^*V)$ denotes its left dual. The action of the torus braid group B_n^{Ell} – and hence of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ – on $F_n^{SL}(M)$ is that induced by the action of B_n^{Ell} on the functor ι_*^n itself.

From this perspective the $\mathcal{D}_{\mathsf{q}}(G)$-module $\mathcal{O}_{\mathsf{q}}(G)$ can be understood to arise from the solid torus 3-manifold $D^2 \times S^1$, with boundary the two-dimensional torus. For this reason $\mathcal{O}_{\mathsf{q}}(G)$ can be expected to play an important role in the associated four-dimensional topological field theory, more precisely in computing the invariants associated to 3-manifolds obtained by surgery.

In the GL case, we modify the construction above by instead fixing an inclusion ι^{n+1}, of $n + 1$ discs into the torus. Instead of applying ι_*^n to $(^*V)^{\boxtimes n}$, we apply ι_*^{n+1} to $\det_{\mathsf{q}}(V)^{\otimes k} \boxtimes {^*V} \boxtimes \cdots \boxtimes {^*V}$, where $\det_{\mathsf{q}}(V) := \bigwedge_{\mathsf{q}}^N (V)$. Instead of B_n^{Ell}, the relevant braid group becomes $B_{n,1}^{Ell}$. See Section 4.2.

In future work, we plan to extend the results of the present paper to understand the structure of $\mathcal{D}_{\mathsf{q}}(\frac{G}{G})$-mod more generally, through the functor F_n^G. As a starting point, we will introduce the so-called quantum Springer sheaf, of which $\mathcal{O}_{\mathsf{q}}(G)$ is a distinguished simple quotient, and in fact is a simple summand occurring with multiplicity one.

1.3   Outline

In Section 2, we collect some preliminaries about the weight lattices for type A, their GL and SL variants: in particular, the set of walks in the dominant chamber, and a bijection to an appropriate set of standard skew tableaux. In Section 3, we recall the quantum algebras we will study: $U_{\mathsf{q}}(\mathfrak{g})$, $\mathcal{O}_{\mathsf{q}}(G)$, and $\mathcal{D}_{\mathsf{q}}(G)$, and we recall some facts from the representation theory of $U_{\mathsf{q}}(\mathfrak{g})$.

In Section 4, we recall the elliptic braid group, the double affine Hecke algebras of GL and SL type, and how these all relate to one another. In Section 5, we recall some structural results about Category $\mathcal{O}$ for the double affine Hecke algebra, and in particular the $\mathcal{Y}$-semisimple modules. In Section 6, we recall the classification of irreducible $\mathcal{Y}$-semisimple modules in Category $\mathcal{O}$ in terms of periodic skew diagrams and their bases, indexed by standard tableaux; for expedience of exposition, we only focus on the case we will need, of the so-called "rectangular representations." We also discuss the construction of the rectangular representation as a quotient of an induced module. In Section 7, we state and prove our main results: we give the GL modification of the functor F_n^G, we identify a weight basis for $F_n^G(\mathcal{O}_{\mathsf{q}}(G))$, for $G = GL_N$ and $G = SL_N$, with the set of periodic standard tableaux on an infinite strip, and we subsequently identify $F_n^G(\mathcal{O}_{\mathsf{q}}(G))$ as an $\mathbb{H}_{q,t}$-module with the rectangular representation.

2   Combinatorics in type A: Lattice Walks and Skew Tableaux

In this section we review a number of well-known combinatorial constructions related to the weight lattice in type A.

2.1   The GL_N root system

Let $\mathfrak{g} = \mathfrak{gl}_N = \mathfrak{gl}(N, \mathbb{C})$ be the Lie algebra of the general linear algebraic group $G = GL_N = GL(N, \mathbb{C})$. Let $\mathbb{E}_{\mathfrak{gl}_N} = \mathbb{R}^N$, with standard basis,

$$\mathcal{E} = \mathcal{E}_{\mathfrak{gl}_N} = \{\epsilon_i \quad | \quad i = 1, \ldots, N\}.$$

and symmetric form $\langle\,,\,\rangle$ with respect to which $\mathcal{E}$ is an orthonormal basis[†]. The weight lattice of $\mathfrak{gl}_N$ is

$$\Lambda_{\mathfrak{gl}_N} = \bigoplus_{i=1}^N \mathbb{Z}\epsilon_i = \mathbb{Z}^N.$$

Elements of $\Lambda_{\mathfrak{gl}_N}$ are called integral weights. The dominant integral weights are

$$\Lambda_{\mathfrak{gl}_N}^+ = \{m_1\epsilon_1 + \cdots + m_N\epsilon_N \mid m_i \in \mathbb{Z}, m_1 \geq \cdots \geq m_N\}.$$

[†] Technically we should be using the pairing $\mathbb{E}^* \times \mathbb{E} \to \mathbb{R}$, particularly if we were to consider non-simply laced type. But for convenience we identify $\mathbb{E}^*$ with $\mathbb{E}$ and express our formulas with respect to the symmetric form we thus chose to denote $\langle\,,\,\rangle$ over the more standard $(\,|\,)$.

Let us denote $\mathbf{d} := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_N$, which is the weight of the trace representation in $\mathfrak{gl}_N$, corresponding to the the character of the determinant representation of GL_N.

Definition 2.1. Given a dominant integral weight $\lambda = \sum_i m_i \epsilon_i \in \Lambda^+_{\mathfrak{gl}_N}$ we denote by $\mathrm{YD}(\lambda)$ the diagram (or integer partition) with fewer than N parts,

$$\mathrm{YD}(\lambda) = (m_1 - m_N, m_2 - m_N, \ldots, m_{N-1} - m_N, 0).$$

$\square$

The construction of $\mathrm{YD}(\lambda)$ highlights a representative of $\lambda \bmod \mathbf{d}$ with $m_N = 0$. We can recover λ by remembering an additional datum, that of a *diagonal labelling* on $\mathrm{YD}(\lambda)$. The diagonals of a Young diagram run down and to the right through the NW and SE corners of its boxes; and a diagonal labelling is an assignment of an integer to each diagonal. The diagonal through the upper left box of $\mathrm{YD}(\lambda)$ is called the *principal diagonal*, and we decree that this diagonal is labelled with m_N. The other diagonals are labelled consecutively, so that the next diagonal to the right is labelled $m_N + 1$, etc. Equivalently, we can say that the upper left box is in row 1 and column $m_N + 1$, and then the diagonal is the column number minus the row number. These labels will be discussed further in Section 2.4.

Note the diagram $\mathrm{YD}(\lambda + r\mathbf{d})$ is that of $\mathrm{YD}(\lambda)$ shifted r units right, and so its diagonal labels are incremented $+r$. Hence, although we draw the same diagram for λ as well as $\lambda + r\mathbf{d}$, they are distinguished by their diagonal labels.

Note that if we think of the "main" diagonal of an integer partition $\gamma = (m_1, \ldots, m_N)$ with $m_N \geq 0$ as the diagonal through the box in its upper leftmost corner, that is the one in its first row and first column, then we traditionally assign its label or "content" to be 0. This is consistent with our labeling as the $(m_N + 1)$st column of γ agrees with the leftmost, i.e. $(m_N + 1)$st, column of $\mathrm{YD}(\sum_i m_i \epsilon_i)$. Or in other words the first column of $\mathrm{YD}(\sum_i m_i \epsilon_i)$ is also the first column of γ. Further, our conventions let us make sense of partitions with negative parts. See Figure 14.

Given $\lambda = \sum_i m_i \epsilon_i$, its dual weight is $\lambda^* := \sum_i -m_i \epsilon_{N+1-i}$, in terms of coordinate vectors $(m_1, \ldots, m_N)^* = (-m_N, \ldots, -m_1)$. Observe therefore that if one takes $\mathrm{YD}(\lambda^*)$ and rotates it 180 degrees, then it is the complement to $\mathrm{YD}(\lambda)$ in a $N \times (m_1 - m_N)$ rectangle. See Figure 1.

Let us describe the diagonal labels in terms of the inner product on $\Lambda^+_{\mathfrak{gl}_N}$. Consider λ as compared to $\lambda + \epsilon_i$. The diagram has one extra box and we claim the diagonal of that box is labeled

$$\langle \lambda, \epsilon_i \rangle + 1 - i = \langle \lambda + \epsilon_i, \epsilon_i \rangle - i. \tag{2.1}$$

The new box in the ith row. Note that $m_i = \langle \lambda, \epsilon_i \rangle$. The ith row of λ has "length" $m_i - m_N$, which is to say it ends $m_i - m_N$ units to the right of the leftmost column, so the new box is in column $m_i + 1 = (1 + m_N) + (m_i - m_N)$,

yielding that it is on the $m_i + 1 - i$ diagonal.

We introduce a special weight ρ given by

$$\rho = \frac{1}{2}((N-1)\epsilon_1 + (N-3)\epsilon_2 + (N-5)\epsilon_3 + \cdots + (1-N)\epsilon_N).$$

Observe $2\rho \in \Lambda^+_{\mathfrak{gl}_N}$, although ρ might not be depending on the parity of N. We also note

$$\langle 2\rho, \epsilon_i \rangle = N + 1 - 2i. \tag{2.2}$$

Hence another way to describe the diagonal label of the box above is by

$$m_i + 1 - i = \langle \lambda, \epsilon_i \rangle + \langle \rho, \epsilon_i \rangle - \langle \rho, \epsilon_1 \rangle. \tag{2.3}$$

Remark 2.2. We remark that $\mathcal{E}$ are the weights of the N-dimensional defining representation $V = V_{\epsilon_1}$ of G. $\square$

2.2 The SL_N root system

Let $\mathfrak{g} = \mathfrak{sl}_N = \mathfrak{sl}(N, \mathbb{C})$ be the Lie algebra of the special linear algebraic group $G = SL_N = SL(N, \mathbb{C})$. Here

$$\mathbb{E}_{\mathfrak{sl}_N} = \mathbb{E}_{\mathfrak{gl}_N} \Big/ \mathbb{R} \cdot (\epsilon_1 + \cdots + \epsilon_N)$$

and the weight lattice is

$$\Lambda_{\mathfrak{sl}_N} = \sum_{i=1}^{N} \mathbb{Z}\epsilon_i \Big/ \mathbb{Z} \cdot (\epsilon_1 + \cdots + \epsilon_N)$$

which is a free $\mathbb{Z}$-module of rank $N - 1$. Let

$$\mathcal{E} = \mathcal{E}_{\mathfrak{sl}_N} = \{\varepsilon_i \quad | \quad 1 \leq i \leq N\}$$

where $\varepsilon_i = \epsilon_i + \mathbb{Z} \cdot (\epsilon_1 + \cdots + \epsilon_N)$. As before these are the weights of $V = V_{\varepsilon_1}$. Note $\varepsilon_1 + \cdots + \varepsilon_N = 0$.

$\mathbb{E}_{\mathfrak{sl}_N}$ has three bases we consider of which the first two are also $\mathbb{Z}$-bases of $\Lambda_{\mathfrak{sl}_N}$. The first basis is $\{\varepsilon_i \quad | \quad 1 \leq i < N\} \subseteq \mathcal{E}$. The second basis is $\{\omega_i \mid 1 \leq i < N\}$ consisting of the fundamental weights $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. The weight lattice can also be expressed as

$$\Lambda_{\mathfrak{sl}_N} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\omega_i = \bigoplus_{i=1}^{N-1} \mathbb{Z}\varepsilon_i.$$

The set of dominant integral weights is

$$\Lambda^+_{\mathfrak{sl}_N} = \sum_{i=1}^{N-1} \mathbb{Z}_{\geq 0}\omega_i = \{m_1\varepsilon_1 + \cdots + m_{N-1}\varepsilon_{N-1} \mid m_i \in \mathbb{Z}, \, m_1 \geq \cdots \geq m_{N-1} \geq 0\}.$$

Our third basis is the set of simple roots

$$\Pi = \Pi^{A_{N-1}} = \{\alpha_i \mid 1 \leq i < N\}$$

where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The root lattice is $Q = Q^{A_{N-1}} = \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i$ and it is a sublattice of index $N = [\Lambda_{\mathfrak{sl}_N} : Q^{A_{N-1}}]$. The set of roots, and the set of positive roots, are

$$\Delta = \Delta^{A_{N-1}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq N, i \neq j\}, \qquad \Delta_+ = \Delta_+^{A_{N-1}} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq N\}.$$

This is the root system of type A_{N-1}.

We have a symmetric form $\langle \, , \, \rangle$ on $\mathbb{E}_{\mathfrak{sl}_N}$ for which $[\langle \alpha_i, \alpha_j \rangle]_{i,j=1}^{N-1}$ yields the Cartan matrix of type A_{N-1}. It is useful to note

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} - \frac{1}{N}. \tag{2.4}$$

We have $\rho \in \Lambda^+_{\mathfrak{sl}_N}$ given by

$$\rho = \frac{1}{2} \sum_{\beta \in \Delta_+} \beta = \frac{1}{2} \left((N-1)\varepsilon_1 + (N-3)\varepsilon_2 + (N-5)\varepsilon_3 + \cdots + (1-N)\varepsilon_N \right) = \sum_{i=1}^{N-1} \omega_i.$$

We note

$$\langle \varepsilon_i, 2\rho \rangle = N - (2i - 1). \tag{2.5}$$

The Young diagram $\mathrm{YD}(\lambda)$ is defined the same way as for GL, and the only difference is in the diagonal labelling. We say the size of $\lambda \in \Lambda^+_{\mathfrak{sl}_N}$ is

$$|\lambda| = \sum_{i=1}^{N} (m_i - m_N)$$

which corresponds to the number of boxes in $\mathrm{YD}(\lambda)$. We label the principal diagonal $-\frac{|\lambda|}{N}$. Hence the diagonal labels will lie in $-\frac{p}{N} + \mathbb{Z}$ if λ is in the coset $\omega_p + Q$.

Note that while $|\lambda|$ is well-defined, i.e. $|\lambda| = |\lambda + \mathbf{d}|$, the sum $\sum_i m_i = |\lambda| + Nm_N$ is not. Another natural choice of representative for λ would be to choose the (possibly fractional) representative with $\sum_i m_i = 0$, as follows

$$\lambda = (m_1 - \frac{\sum_i m_i}{N}, \ldots, m_{N-1} - \frac{\sum_i m_i}{N}, m_N - \frac{\sum_i m_i}{N}).$$

One may consider its upper left corner to lie in row 1, but column $1 + (m_N - \frac{\sum_i m_i}{N}) = 1 - \frac{|\lambda|}{N}$. This explains in part the appearance of fractional diagonal labels.

Another useful calculation is

$$\langle \varepsilon_j, \lambda \rangle = m_j - \frac{\sum_i m_i}{N} = m_j - m_N - \frac{1}{N}|\lambda|. \tag{2.6}$$

2.3   Walks on the weight lattice

For the following definition we may take $G = GL_N, SL_N$. Correspondingly, we let $\Lambda^+ = \Lambda^+_{\mathfrak{gl}_N}$ or $\Lambda^+_{\mathfrak{sl}_N}$. Since ϵ_i is a representative of ε_i, we will use ϵ_i below to denote either one.

Definition 2.3. A *walk in Λ^+ of length n*, from weight λ to weight μ is a finite sequence,

$$\underline{u} = (\lambda = u_0, u_1, \ldots, u_n = \mu),$$

where each $u_i \in \Lambda^+$, and each difference $u_i - u_{i-1}$ lies in $\mathcal{E}$. We denote by $\delta_i(\underline{u})$ the index of $u_i - u_{i-1} \in \mathcal{E}$, so that $u_i - u_{i-1} = \epsilon_{\delta_i(\underline{u})}$. $\square$

Remark 2.4. The condition that each $u_i \in \Lambda^+$ means that we could call these "dominant walks" or "walks in the dominant chamber." Since these are the only kinds of walks we will consider, we drop mention of the adjective dominant henceforth. $\square$

Definition 2.5. A walk in $\Lambda^+_{\mathfrak{sl}_N}$ which begins and ends at the same λ is called a *looped walk* at λ. A walk in $\Lambda^+_{\mathfrak{gl}_N}$ which begins at λ and ends at $\lambda + k\mathbf{d}$ is also called a *looped walk* at λ. We denote by $\mathcal{W}^{N,k}_\lambda$ the set of all looped walks at λ of length $n = Nk$. $\square$

See Figures 3, 4, and 5 for examples of looped walks.

Remark 2.6. Since $\mathbf{d} = 0 \in \Lambda^+_{\mathfrak{sl}_N}$, we can say in either case that a looped walk begins at λ and ends at $\lambda + k\mathbf{d}$. $\square$

Remark 2.7. Note that the multiset $\{\epsilon_{\delta_i(\underline{u})} \mid 1 \le i \le n\}$ of steps taken on any looped walk $\underline{u}$ consists of $\mathcal{E}$ with integer multiplicity $k = n/N$. $\square$

2.4   Skew tableaux from walks on the weight lattice

For the rest of the paper, unless otherwise noted, we let $k \in \mathbb{Z}_{>0}$ and $n = Nk$.

We shall now recall an alternative combinatorial description of the set $\mathcal{W}^{N,k}_\lambda$ of looped walks at λ in terms of skew tableaux. To begin, we associate to a weight $\lambda \in \Lambda^+$ a skew diagram $D^{N,k}_\lambda$ constructed in either of the following clearly equivalent ways (See Figure 1):

- as the skew diagram $D^{N,k}_\lambda = (\mathrm{YD}(\lambda) + (k^N))/\mathrm{YD}(\lambda)$.

- as the skew diagram $D_\lambda^{N,k}$ obtained by removing $YD(\lambda)$ from the upper left, and $YD(\lambda^*)$, rotated 180 degrees, from the lower right, of the $N \times (k + m_1 - m_N)$ rectangular diagram.

The skew diagram $D_\lambda^{N,k}$ inherits diagonal labels from $YD(\lambda)$ as well as choice of principal diagonal.
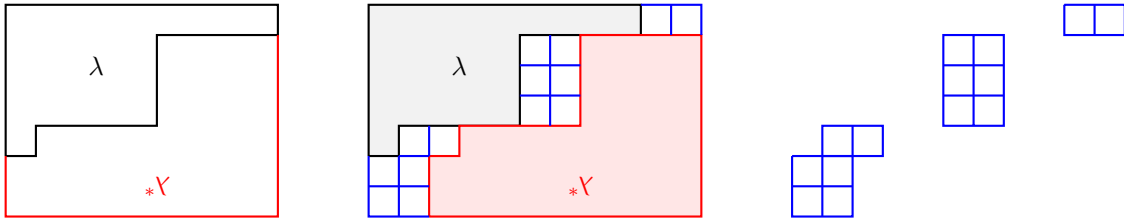

Fig. 1. The skew diagram $D_\lambda^{7,2}$ in the case $N = 7$, $k = 2$, $n = 14$.

Recall for a (skew) diagram with n boxes that a standard tableau is a filling of its boxes with $\{1, 2, \ldots, n\}$ such that entries increase across rows and down columns.

Definition 2.8. Given a weight $\lambda \in \Lambda^+$, we denote by $\mathcal{SK}_\lambda^{N,k}$ the set of all standard tableaux of diagonal-labeled skew shape $D_\lambda^{N,k}$. □
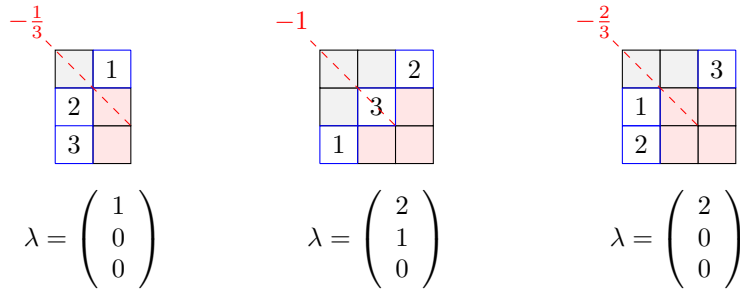

Fig. 2. $G = SL_3$, $k = 1$. We list $\mathcal{T} \in \mathcal{SK}_\lambda^{N,k}$ with principal diagonal labeled $-\frac{|\lambda|}{3}$. Once periodized via $\overline{\mathcal{P}er}(\mathcal{T})$, these tableaux form a π orbit, see Figure 12.
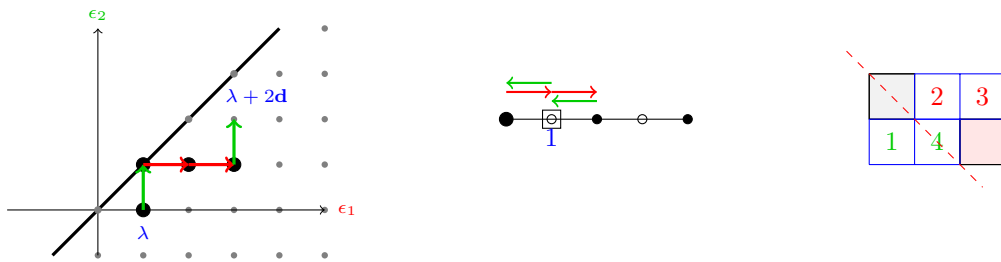

Fig. 3. A looped walk $\underline{u}$ at $\lambda = \epsilon_1 \in \Lambda_{\mathfrak{gl}_N}^+$ with 4 steps, its projection in $\Lambda_{\mathfrak{sl}_N}^+$ for $N = 2$, and the skew tableau $\mathcal{T}ab(\underline{u})$ with the principal diagonal indicated by a dashed line. For $\Lambda_{\mathfrak{gl}_N}^+$ the principal diagonal is labelled 0, while for $\Lambda_{\mathfrak{sl}_N}^+$ it is labelled $-\frac{1}{2}$.

Definition 2.9. Define the map $\mathcal{T}ab : \mathcal{W}_\lambda^{N,k} \to \mathcal{SK}_\lambda^{N,k}$, from length $n = Nk$ looped walks at $\lambda \in \Lambda^+$ to standard skew tableaux of shape $D_\lambda^{N,k}$ as follows: for each $i = 1, \ldots, n$ fill the leftmost vacant box in the $\delta_i(\underline{u})$-th row of $D_\lambda^{N,k}$ with the symbol i. □

We note that the map $\mathcal{T}ab$ doesn't reference the extra data of the diagonal labelling. When it is important to differentiate the SL setting, we will use $\overline{\mathcal{T}ab}(\lambda)$ for $\lambda \in \Lambda^+_{\mathfrak{sl}_N}$. The following is essentially proved in [19]:

Proposition 2.10. The map $\mathcal{T}ab : \mathcal{W}^{N,k}_\lambda \xrightarrow{\sim} \mathcal{SK}^{N,k}_\lambda$ is a bijection. $\square$
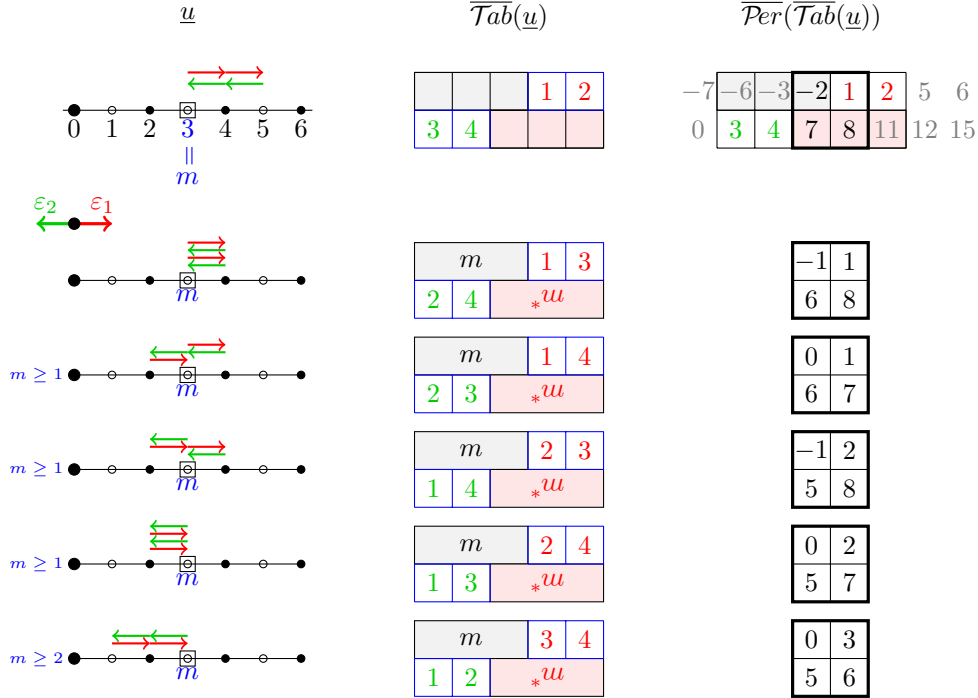

Fig. 4. The case $G = SL_2$, $n = 4$. The first column lists the four-step walks in $\mathcal{W}^{2,2}_m$, of which there are six when m is large. The second column lists the standard skew tableaux $\mathcal{SK}^{2,2}_m$. The final column lists fundamental domains of the six corresponding standard periodic tableaux in $\mathrm{P}_2\mathrm{SYT}(2^2)/\pi^4$ (see Section 6.4), in the case $m = 3$. For purposes of illustration, we also take $m = 3$ across the entire first row; otherwise m is free. The placement of rows indicates the bijections $\overline{\mathcal{T}ab}$ of Definition 2.9 and $\overline{\mathcal{P}er}$ of Definition 7.3.

Example 2.11. For the first walk in Figure 4,

$$\underline{u} = (m, m + \varepsilon_1, m + \varepsilon_1 + \varepsilon_1, m + \varepsilon_1 + \varepsilon_1 + \varepsilon_2, m + \varepsilon_1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_2 = m),$$

and so the sequence

$$(\delta_1(\underline{u}), \delta_2(\underline{u}), \delta_3(\underline{u}), \delta_4(\underline{u})) = (1, 1, 2, 2).$$

Compare this to the first skew tableau $\mathcal{T} = \mathcal{T}ab(\underline{u})$ in Figure 4 which places 1 and 2 in the first row, 3 and 4 in the second row.

See Figure 6 to see how leaving the dominant chamber results in the tableau becoming non-standard. In particular, if $m > 1$, then the walk would not leave the chamber and the 3 would not be directly above the 2.
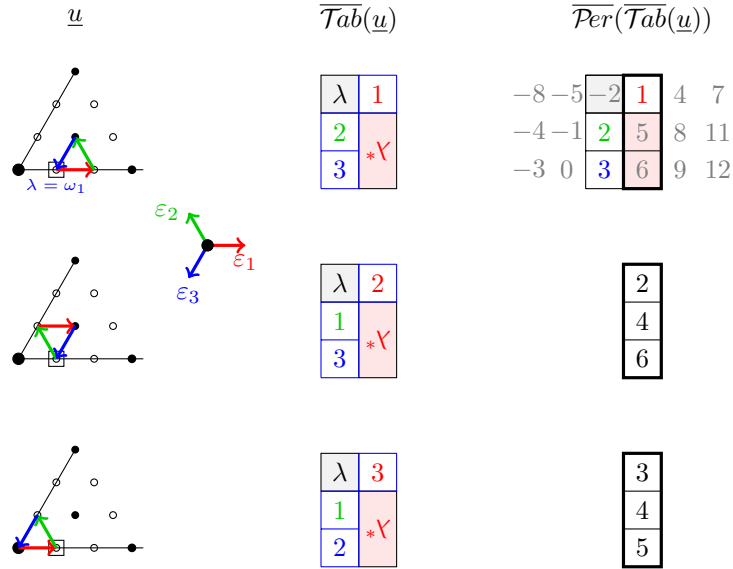
$\square$

Fig. 5.   The set $\mathcal{W}^{3,1}_{\omega_1}$ of three-step looped walks in $\Lambda^+_{\mathfrak{sl}_N}$ at ω_1. The allowed steps ε_1, ε_2 and ε_3 are shown to the right of the first column. As in Figure 4 the second and third columns list the corresponding tableaux in $\mathcal{SK}^{3,1}_{\omega_1}$ and $\mathrm{P}_3\mathrm{SYT}(3^1)/\pi^3$ under $\overline{\mathcal{T}ab}$ from Definition 2.9 and $\overline{\mathcal{P}er}$ from Definition 7.3.
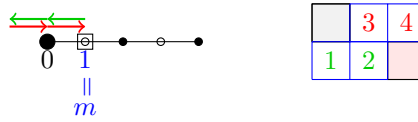

Fig. 6.   The above is not a walk as it exits the dominant chamber when $m = 1$; likewise the corresponding skew tableau assigned by $\overline{\mathcal{T}ab}$ is not standard.

3   Quantum algebras: $U_{\mathsf{q}}(\mathfrak{g})$, $\mathcal{O}_{\mathsf{q}}(G)$ and $\mathcal{D}_{\mathsf{q}}(G)$

3.1   The quantum groups $U_{\mathsf{q}}(\mathfrak{gl}_N)$, $U_{\mathsf{q}}(\mathfrak{sl}_N)$

We will consider the braided tensor categories $\mathrm{Rep}_{\mathsf{q}}(GL_N)$ and $\mathrm{Rep}_{\mathsf{q}}(SL_N)$ of integrable $U_{\mathsf{q}}(\mathfrak{gl}_N)$-modules (resp. $U_{\mathsf{q}}(\mathfrak{sl}_N)$-modules). We refer to [17] for detailed definitions, in particular the Serre presentation of the quantum groups $U_{\mathsf{q}}(\mathfrak{gl}_N)$ and $U_{\mathsf{q}}(\mathfrak{sl}_N)$, the formulas for R-matrices, and the Peter-Weyl theorem. Recall that a $U_{\mathsf{q}}(\mathfrak{g})$-module is called integrable if the Cartan generators K_i act diagonalizably with eigenvalues in $\mathsf{q}^{\mathbb{Z}}$, and each vector lies in a finite-dimensional submodule.

For each $\lambda \in \Lambda^+_{\mathfrak{gl}_N}$ or $\Lambda^+_{\mathfrak{sl}_N}$, we denote by V_λ the unique simple module of highest weight λ. Note that we have an isomorphism $(V_\lambda)^* \cong V_{\lambda^*}$.

3.2   The vector representation

The representation $V_{\epsilon_1} \cong \mathbb{C}^N$ for either $U_{\mathsf{q}}(\mathfrak{gl}_N)$ and $U_{\mathsf{q}}(\mathfrak{sl}_N)$ will be simply denoted V. We fix $e_1, \ldots, e_N$ to be the standard basis for V, and we denote by ρ_V the associated homomorphism to $\mathrm{End}(V)$. Recall that the GL_N

R-matrix for the vector representation can be expressed explicitly:

$$R := (\rho_V \otimes \rho_V)(\mathcal{R}) = \left(\mathsf{q} \sum_i E_i^i \otimes E_i^i + \sum_{i \neq j} E_i^i \otimes E_j^j + (\mathsf{q} - \mathsf{q}^{-1}) \sum_{i > j} E_i^j \otimes E_j^i \right). \tag{3.1}$$

We define $R_{jl}^{ik}, (R^{-1})_{jl}^{ik} \in \mathbb{C}$, for $i, j, k, l = 1, \ldots, N$ by:

$$R(e_i \otimes e_j) = \sum_{k,l} R_{ij}^{kl}(e_k \otimes e_l), \quad R^{-1}(e_i \otimes e_j) = \sum_{k,l} (R^{-1})_{ij}^{kl}(e_k \otimes e_l).$$

We can write the coefficients explicitly as follows:

$$R_{ij}^{kl} = \begin{cases} \mathsf{q}, & i = j = k = l; \\ 1, & i = k \neq j = l; \\ \mathsf{q} - \mathsf{q}^{-1}, & i = l < j = k; \\ 0, & \text{otherwise}; \end{cases} \quad (R^{-1})_{ij}^{kl} = \begin{cases} \mathsf{q}^{-1}, & i = j = k = l; \\ 1, & i = k \neq j = l; \\ \mathsf{q}^{-1} - \mathsf{q}, & i = l < j = k; \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Let $\tau : V \otimes V$ denote the tensor flip, $\tau(v \otimes w) = w \otimes v$. The braiding, $\sigma_{V,V} = \tau \circ R$, for $U_\mathsf{q}(\mathfrak{gl}_N)$ satisfies a Hecke relation,

$$(\sigma_{V,V} - \mathsf{q})(\sigma_{V,V} + \mathsf{q}^{-1}) = 0.$$

The SL_N R-matrix on the vector representation is equal to the GL_N R-matrix, multiplied by a factor of $\mathsf{q}^{-\frac{1}{N}}$. To avoid confusion, we will reserve the notation R for the GL_N R-matrix, and write $\mathsf{q}^{-\frac{1}{N}} R$ to reference the SL_N R-matrix. Hence the resulting braiding $\sigma_{V,V} = \tau \circ (\mathsf{q}^{-\frac{1}{N}} R)$ for $U_\mathsf{q}(\mathfrak{sl}_N)$ satisfies the shifted Hecke relation,

$$(\sigma_{V,V} - \mathsf{q}^{-\frac{1}{N}} \mathsf{q})(\sigma_{V,V} + \mathsf{q}^{-\frac{1}{N}} \mathsf{q}^{-1}) = 0.$$

3.3 The determinant representation

We will denote by $\det_\mathsf{q}(V) = \bigwedge_q^N (V)$ the one-dimensional determinant representation of $U_\mathsf{q}(\mathfrak{gl}_N)$, and we will abbreviate

$$\det_\mathsf{q}^k(V) = \begin{cases} (\det_\mathsf{q}(V))^{\otimes k}, & k \geq 0, \\ (\det_\mathsf{q}(V)^*)^{\otimes k}, & k < 0. \end{cases}$$

3.4 The ribbon element

Recall the ribbon element[‡] $\nu \in U_\mathsf{q}(\mathfrak{g})$. The ribbon element is central and satisfies the identity

$$\Delta(\nu) = R_{21} R_{12}(\nu \otimes \nu). \tag{3.3}$$

[‡]Note that axioms for the ribbon element and its inverse are swapped between [15] and [19]; we will follow the conventions of [19].

It acts on any irreducible representation V_λ, and hence on any isotypic component $X[\lambda]$, by the scalar

$$\nu|_{X[\lambda]} = \mathsf{q}^{\langle \lambda + 2\rho, \lambda \rangle}. \tag{3.4}$$

3.5   The quantum coordinate algebra

Definition 3.1. The reflection equation algebra of type GL_N, denoted $\mathcal{O}_\mathsf{q}(Mat_N)$, is the algebra generated by symbols a^i_j, for $i, j = 1, \ldots N$, subject to the relations,

$$R_{21} A_1 R_{12} A_2 = A_2 R_{21} A_1 R_{12},$$

where $A := \sum_{i,j} a^i_j E^j_i$ is a matrix with entries the generators a^i_j, and for a matrix X, we write $X_1 = X \otimes \mathrm{Id}_V$, $X_2 = \mathrm{Id}_V \otimes X$, so that the matrix equation above is equivalent to the list of relations, for $i, j, n, r \in \{1, \ldots, N\}$:

$$\sum_{k,l,m,p} R^{ij}_{kl} a^l_m R^{mk}_{np} a^p_r = \sum_{s,t,u,v} a^i_s R^{sj}_{tu} a^u_v R^{vs}_{nr} \tag{3.5}$$

$\square$

Remark 3.2. We note that, since R appears quadratically on both sides of the defining relation, the relations are unchanged by replacing $R \rightsquigarrow \mathsf{q}^{-\frac{1}{N}} R$. $\square$

Proposition 3.3 ([16])**.** The element,

$$\det_\mathsf{q}(A) := \sum_{\sigma \in \mathfrak{S}_N} (-\mathsf{q})^{\ell(\sigma)} \cdot \mathsf{q}^{e(\sigma)} a^1_{\sigma(1)} \cdots a^N_{\sigma(N)},$$

is central in $\mathcal{O}_\mathsf{q}(Mat_N)$. $\square$

Here $\ell(\sigma)$ denotes the length, i.e. the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$, and $e(\sigma)$ denotes the excedence, i.e. the number of elements i such that $\sigma(i) > i$.

Definition 3.4. The quantum coordinate algebras $\mathcal{O}_\mathsf{q}(GL_N)$ and $\mathcal{O}_\mathsf{q}(SL_N)$ are the algebras obtained from $\mathcal{O}_\mathsf{q}(Mat_N)$, by inverting, respectively specializing to one, the central element $\det_\mathsf{q}$. That is,

$$\mathcal{O}_\mathsf{q}(GL_N) = \mathcal{O}_\mathsf{q}(Mat_N)[\det_\mathsf{q}(A)^{-1}], \qquad \mathcal{O}_\mathsf{q}(SL_N) = \mathcal{O}_\mathsf{q}(Mat_N)/\langle \det_\mathsf{q}(A) - 1 \rangle.$$

$\square$

Theorem 3.5 (Peter-Weyl decomposition)**.** As a module for $U_\mathsf{q}(\mathfrak{gl}_N)$ and $U_\mathsf{q}(\mathfrak{sl}_N)$, respectively, we have isomorphisms:

$$\mathcal{O}_\mathsf{q}(GL_N) \cong \bigoplus_{\lambda \in \Lambda^+_{\mathfrak{gl}_N}} V^*_\lambda \otimes V_\lambda, \qquad \mathcal{O}_\mathsf{q}(SL_N) \cong \bigoplus_{\lambda \in \Lambda^+_{\mathfrak{sl}_N}} V^*_\lambda \otimes V_\lambda.$$

□

3.6   Quantum differential operators

The algebra of quantum differential operators on G, which we denote by $\mathcal{D}_{\mathsf{q}}(G)$, was studied in many different settings. The presentation below as a twisted tensor product is adapted from the paper [25] (see also [6]), and hence matches the conventions of [15] (see footnote 3 there, however).

Definition 3.6. For $G = GL_N$, or SL_N, the algebra $\mathcal{D}_{\mathsf{q}}(G)$ is the twisted tensor product,

$$\mathcal{D}_{\mathsf{q}}(G) = \mathcal{O}_{\mathsf{q}}(G)\widetilde{\otimes}\mathcal{O}_{\mathsf{q}}(G), \tag{3.6}$$

with cross relations,

$$D_2 R_{21} A_1 = R_{21} A_1 R_{12} D_2 R_{21}, \qquad \text{if } G = GL_N$$

$$D_2 R_{21} A_1 = R_{21} A_1 R_{12} D_2 R_{21}\mathsf{q}^{-2/N}, \qquad \text{if } G = SL_N$$

where $A = \sum_{i,j} a^i_j E^j_i$ and $D = \sum_{i,j} \partial^i_j E^j_i$ denote matrices of generators of each tensor factor, so that the matrix equation above is equivalent to the list of cross relations, for $i, k, l, n \in \{1, \ldots, N\}$:

$$\sum_{j,m} \partial^i_j R^{jk}_{lm} a^m_n = \sum_{p,r,s,t,u,v} R^{ik}_{pr} a^r_s R^{sp}_{tu} \partial^u_v R^{vt}_{ln}.$$

□

We denote by ℓ and $\partial_{\triangleleft}$ the inclusions into the first and second tensor factor of (3.6), so that $\ell \otimes \partial_{\triangleleft} :$ $\mathcal{O}_{\mathsf{q}}(G) \otimes \mathcal{O}_{\mathsf{q}}(G) \to \mathcal{D}_{\mathsf{q}}(G)$ is the tautological isomorphism of $U_{\mathsf{q}}(\mathfrak{g})$-modules (however it is not an algebra homomorphism). This is a q-deformation of the tensor decomposition $\mathcal{D}(G) \cong \mathcal{O}(G) \otimes U(\mathfrak{g})$ into functions on G, and the vector fields of left-translation, hence the notation.

4   Double affine Hecke algebras of type A

Let $\mathcal{K}$ denote a field of characteristic zero, and let $q, t \in \mathcal{K}^{\times}$, and assume neither q nor t is a root of unity. Typical instances are $\mathcal{K} = \mathbb{C}$, $\mathbb{C}(t)$, or $\mathbb{C}(q,t)$.

4.1 The extended affine symmetric group

Definition 4.1. The extended affine symmetric group is[§]

$$\widehat{\mathfrak{S}}_n = \left\langle \pi, s_i, \ i \in \mathbb{Z}/n\mathbb{Z} \ \middle| \ \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } i \in \mathbb{Z}/n\mathbb{Z}, \\ s_i s_j = s_j s_i & \text{for } j \not\equiv i \pm 1 \bmod n, \\ \pi s_i = s_{i+1} \pi & \text{for } i \in \mathbb{Z}/n\mathbb{Z}, \\ s_i^2 = 1 & \text{for } i \in \mathbb{Z}/n\mathbb{Z} \end{array} \right\rangle.$$

$\square$

We can associate to SL_n the quotient $\overline{\mathfrak{S}}_n = \widehat{\mathfrak{S}}_n / \langle \pi^n \rangle$. Then we can think of the image, $\overline{\pi}$, of π as the Dynkin diagram automorphism or the generator of the cyclic group $\Lambda_{\mathfrak{sl}_N}/Q$. Note both $\widehat{\mathfrak{S}}_n$ and $\overline{\mathfrak{S}}_n$ have as a subgroup the affine symmetric group $\langle s_i \mid i \in \mathbb{Z}/n\mathbb{Z} \rangle$.

We recall that $\widehat{\mathfrak{S}}_n$ acts on $\mathbb{Z}$ by n-periodic permutations, i.e. bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(i+n) = \sigma(i) + n$. It also acts on the set $(\mathcal{K}^{\times})^n$ via:

$$s_i \cdot (a_1, \ldots, a_i, a_{i+1}, \ldots a_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$$
$$s_0 \cdot (a_1, a_2, \ldots, a_{n-1}, a_n) = (q a_n, a_2, \ldots, a_{n-1}, q^{-1} a_1) \tag{4.1}$$
$$\pi \cdot (a_1, \ldots, a_n) = (q a_n, a_1, a_2, \ldots, a_{n-1})$$

This action is relevant to the GL_n double affine Hecke algebra (see Section 4.3). We modify the action of $\widehat{\mathfrak{S}}_n$ on $(\mathcal{K}^{\times})^n$ for SL_n as follows, so that it will factor through the quotient $\overline{\mathfrak{S}}_n$. Let $\boldsymbol{q} \in \mathcal{K}^{\times}$ be another constant which we assume is not a root of unity.

$$s_i \cdot (z_1, \ldots, z_i, z_{i+1}, \ldots z_n) = (z_1, \ldots, z_{i+1}, z_i, \ldots z_n),$$
$$s_0 \cdot (z_1, z_2, \ldots, z_{n-1}, z_n) = (\boldsymbol{q}^{-2n} z_n, z_2, \ldots, z_{n-1}, \boldsymbol{q}^{2n} z_1), \tag{4.2}$$
$$\pi \cdot (z_1, \ldots, z_n) = (\boldsymbol{q}^{-2n+2} z_n, \boldsymbol{q}^2 z_1, \boldsymbol{q}^2 z_2, \ldots, \boldsymbol{q}^2 z_{n-1}).$$

It is easy to check that in this case the action of the generators satisfy the defining relations of $\widehat{\mathfrak{S}}_n$, as well as the additional relation $\pi^n = \text{Id}$.

See Section 4.5 for a discussion of the relationship between $\boldsymbol{q}^{-2n}$ and q that unifies the action of s_0 in (4.1) with that in (4.2).

[§]We drop the first relation when $n = 2$.

4.2   The elliptic braid group

Definition 4.2. The *elliptic braid group* B_n^{Ell} is the fundamental group of the configuration space of n points on the torus T^2. The *marked elliptic braid group* $B_{n,1}^{Ell}$ is the fundamental group of the configuration space of n points on the punctured torus $T^2 \backslash D^2$. $\square$

Proposition 4.3 ([5, 22]). The elliptic braid group B_n^{Ell} is presented by:

- Pairwise commuting generators $\mathsf{X}_1, \ldots, \mathsf{X}_n$,
- Pairwise commuting generators $\mathsf{Y}_1, \ldots, \mathsf{Y}_n$,
- The braid group of the plane,

$$B_n = \left\langle \quad \mathsf{T}_1, \ldots, \mathsf{T}_{n-1} \quad \left| \quad \begin{array}{ll} \mathsf{T}_i\mathsf{T}_{i+1}\mathsf{T}_i = \mathsf{T}_{i+1}\mathsf{T}_i\mathsf{T}_{i+1}, & i = 1, \ldots n - 2, \\ \mathsf{T}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{T}_i, & |i - j| \geq 2 \end{array} \right. \right\rangle,$$

with the cross relations:

$$\mathsf{T}_i\mathsf{X}_i\mathsf{T}_i = \mathsf{X}_{i+1}, \ \mathsf{T}_i\mathsf{Y}_i\mathsf{T}_i = \mathsf{Y}_{i+1}, \quad i = 1, \ldots, n - 1,$$

$$\mathsf{X}_1\mathsf{Y}_2 = \mathsf{Y}_2\mathsf{X}_1\mathsf{T}_1^2, \qquad (\prod_i \mathsf{X}_i)\mathsf{Y}_j = \mathsf{Y}_j(\prod_i \mathsf{X}_i), \quad j = 1, \ldots, n,$$

$$\mathsf{X}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{X}_i, \ \mathsf{Y}_i\mathsf{T}_j = \mathsf{T}_j\mathsf{Y}_i, \quad \text{for } |i - j| > 1.$$

$\square$

It is well-known that $B_{n,1}^{Ell}$ embeds into B_{n+1}^{Ell}, as the subgroup generated by $\mathsf{X}_1, \ldots, \mathsf{X}_n$, $\mathsf{Y}_1, \ldots, \mathsf{Y}_n$, $\mathsf{T}_1, \ldots, \mathsf{T}_{n-1}$, and by T_n^2.

4.3   Double affine Hecke algebra for GL_n

Definition 4.4. The GL_n double affine Hecke algebra $\mathbb{H}_{q,t}(GL_n)$ is the $\mathcal{K}$-algebra presented by generators:

$$T_0, T_1, \ldots T_{n-1}, \pi^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1},$$

subject to relations[¶]:

$$(T_i - t)(T_i + t^{-1}) = 0 \quad (i = 0, \ldots, n-1), \tag{4.3}$$

$$T_i T_j T_i = T_j T_i T_j \quad (j \equiv i \pm 1 \bmod n), \qquad\qquad T_i T_j = T_j T_i \quad (\text{otherwise}), \tag{4.4}$$

$$\pi T_i \pi^{-1} = T_{i+1} \quad (i = 0, \ldots, n-2), \qquad\qquad \pi T_{n-1} \pi^{-1} = T_0,$$

$$T_i Y_i T_i = Y_{i+1} \quad (i = 1, \ldots, n-1), \qquad\qquad T_0 Y_n T_0 = q^{-1} Y_1$$

$$T_i Y_j = Y_j T_i \quad (j \not\equiv i, i+1 \bmod n),$$

$$\pi Y_i \pi^{-1} = Y_{i+1} \quad (i = 1, \ldots, n-1), \qquad\qquad \pi Y_n \pi^{-1} = q^{-1} Y_1$$

$$\square$$

Any $\sigma \in \widehat{\mathfrak{S}}_n$ has a canonical lift $T_\sigma \in \mathbb{H}_{q,t}(GL_n)$, defined as follows: if σ is written as a reduced word $\sigma = \pi^r s_{i_1} \cdots s_{i_k}$ of the generators, then we set $T_\sigma = \pi^r T_{i_1} \cdots T_{i_k}$. This expression for T_σ is well-defined because the T_i satisfy the same braid relations as $s_i \in \widehat{\mathfrak{S}}_n$. We abuse notation and abbreviate $\pi = T_\pi$. For $\beta = (b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n$ we denote by $Y^\beta = Y_1^{b_1} \cdots Y_n^{b_n}$ the corresponding monomial. We note that $\mathbb{H}_{q,t}(GL_n)$ has basis $\{T_\sigma Y^\beta \mid \sigma \in \widehat{\mathfrak{S}}_n, \beta \in \mathbb{Z}^n\}$.

Given the combinatorial viewpoint of this paper, the presentation above involving π is the most convenient for us. However, it is sometimes desirable to define

$$X_1 = \pi T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1}, \qquad X_{i+1} = T_i X_i T_i \quad (i = 1, \ldots, n-1).$$

Then it is not hard to show $X_i X_j = X_j X_i$ and that these elements generate a Laurent polynomial subalgebra $\mathcal{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \subseteq \mathbb{H}_{q,t}(GL_n)$. It is also easy to show that $X_1 X_2 \cdots X_n = \pi^n$, and that this element q-commutes with each Y_i.

We thus have two commutative sub-algebras,

$$\mathcal{X} = \mathcal{K}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \qquad \text{and} \qquad \mathcal{Y} = \mathcal{K}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}],$$

of $\mathbb{H}_{q,t}(GL_n)$, each isomorphic to a Laurent polynomial ring.

Further, the DAHA $\mathbb{H}_{q,t}(GL_n)$ has two distinguished subalgebras

$$H(\mathcal{Y}) = \langle T_1, \ldots, T_{n-1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}\rangle, \qquad H(\mathcal{X}) = \langle T_0, T_1, \ldots, T_{n-1}, \pi^{\pm 1}\rangle,$$

each of which is isomorphic to the extended affine Hecke algebra of type A.

[¶]As with $\widehat{\mathfrak{S}}_n$, we drop the relations on the second line when $n = 2$.

Finally, we will denote by H_n the finite Hecke algebra, generated by T_i and subject to the relations (4.3) and (4.4). The finite Hecke algebra sits as a common subalgebra of $H(\mathcal{X})$ and $H(\mathcal{Y})$, but it is also naturally realized as a quotient of $H(\mathcal{Y})$ via the homomorphism $H(\mathcal{Y}) \to H_n$ determined by

$$T_i \mapsto T_i, \qquad Y_1 \mapsto 1. \tag{4.5}$$

In this way we can inflate any H_n-module to be a $H(\mathcal{Y})$-module.

We note that the center $Z(H(\mathcal{Y})) = \mathcal{K}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]^{\mathfrak{S}_n}$ consists of the symmetric Laurent polynomials. In particular the product $Y_1 Y_2 \cdots Y_n$ commutes with all T_i. (It even commutes with T_0 which is not in $H(\mathcal{Y})$, but does not commute with π.)

Proposition 4.5. There exists a unique isomorphism,

$$\phi : \mathcal{K}[B_{n,1}^{Ell}] \Big/ \big\langle \mathsf{T}_n^2 = q, \qquad (\mathsf{T}_i - t)(\mathsf{T}_i + t^{-1}) = 0 \quad (i = 1, \ldots, n-1) \big\rangle \to \mathbb{H}_{q,t}(GL_n),$$

such that

$$\phi(\mathsf{T}_i) = T_i, \quad (i = 1, \ldots n-1), \qquad \phi(\mathsf{X}_1) = \pi T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1}, \qquad \phi(\mathsf{Y}_1) = Y_1.$$

$\square$

Proof. We shall mostly leave this to the reader, but let us explain the relation $\mathsf{T}_n^2 = q$ here. In B_{n+1}^{Ell}, we have: $\mathsf{X}_{n+1} \mathsf{Y}_n = \mathsf{T}_n^2 \mathsf{Y}_n \mathsf{X}_{n+1}$. Recall too that the X_i commute with each other and further T_n commutes with $\mathsf{X}_n \mathsf{X}_{n+1}$. Hence

$$
\begin{aligned}
\mathsf{X}_{n+1} \mathsf{Y}_n (\mathsf{X}_1 \cdots \mathsf{X}_n) &= \mathsf{T}_n^2 \mathsf{Y}_n \mathsf{X}_{n+1} (\mathsf{X}_1 \cdots \mathsf{X}_n) \\
&= \mathsf{T}_n^2 \mathsf{X}_1 \cdots \mathsf{X}_n \mathsf{X}_{n+1} \mathsf{Y}_n \\
&= \mathsf{X}_1 \cdots \mathsf{X}_n \mathsf{X}_{n+1} \mathsf{T}_n^2 \mathsf{Y}_n \\
&\qquad \text{multiplying both sides on the left by } \mathsf{X}_{n+1}^{-1} \text{ gives} \\
\mathsf{Y}_n (\mathsf{X}_1 \cdots \mathsf{X}_n) &= (\mathsf{X}_1 \cdots \mathsf{X}_n) \mathsf{T}_n^2 \mathsf{Y}_n.
\end{aligned}
$$

It is easy to check $\phi(\mathsf{X}_1 \cdots \mathsf{X}_n) = \pi^n$. Hence setting $\phi(\mathsf{T}_n^2) = q$ is consistent with our $\pi^n q Y_n \pi^{-n} = Y_n$ relation. We leave it to the reader to check the other relations. ∎

4.4   Double affine Hecke algebra for SL_n

Let us fix further constants $\boldsymbol{r}, \boldsymbol{q} \in \mathcal{K}^\times$, which we assume are not roots of unity. However, for convenience, we will assume $\mathcal{K}^\times$ contains primitive nth roots of unity.

Definition 4.6. The SL_n double affine Hecke algebra $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ is presented by generators:

$$T_0, T_1, \ldots T_{n-1}, \pi^{\pm 1}, Z_1^{\pm 1}, \ldots, Z_n^{\pm 1},$$

subject to relations:

$(T_i - t)(T_i + t^{-1}) = 0 \quad (i = 0, \ldots, n-1),$

$T_i T_j T_i = T_j T_i T_j \quad (j \equiv i \pm 1 \bmod n), \qquad T_i T_j = T_j T_i \quad \text{(otherwise)},$

$\pi T_i \pi^{-1} = T_{i+1} \quad (i = 0, \ldots, n-2), \qquad \pi T_{n-1} \pi^{-1} = T_0,$

$T_i Z_i T_i = Z_{i+1} \quad (i = 1, \ldots, n-1), \qquad T_i Z_j = Z_j T_i \quad (j \not\equiv i, i+1 \bmod n),$

$T_0 Z_n T_0 = \boldsymbol{q}^{2n} Z_1, \qquad Z_1 Z_2 \cdots Z_n = \boldsymbol{q}^{n(n-1)} \boldsymbol{r}^n, \qquad \pi^n = 1,$

$\pi Z_i \pi^{-1} = \boldsymbol{q}^{-2} Z_{i+1} \quad (i = 1, \ldots, n-1), \qquad \pi Z_n \pi^{-1} = \boldsymbol{q}^{2n-2} Z_1.$

$\square$

Similar to the GL case, we have two commutative subalgebras,

$$\mathcal{X} = \langle X_1^{\pm 1}, \ldots, X_n^{\pm 1} \rangle \qquad \text{and} \qquad \mathcal{Z} = \langle Z_1^{\pm 1}, \ldots, Z_n^{\pm 1} \rangle$$

of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, each isomorphic to the quotient of a Laurent polynomial ring by a single relation on the product of generators, as in Proposition 4.9 below. We let $H(\mathcal{Z})$ denote the subalgebra generated by $\mathcal{Z}$ and H_n. We can realize $H(\mathcal{Z})$ as a quotient of the extended affine Hecke algebra by the relation $Y_1 \cdots Y_n = \boldsymbol{q}^{n(n-1)} \boldsymbol{r}^n$,

$$H(\mathcal{Y}) \to H(\mathcal{Z}),$$

$$T_i \mapsto T_i \qquad\qquad\qquad (i = 1, \ldots, n-1)$$

$$Y_i \mapsto Z_i \qquad\qquad\qquad (i = 1, \ldots, n).$$

Definition 4.7. Let $a \in \mathcal{K}^\times$ satisfy $a^n = 1$. Given an $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$-module M, denote by M^a the *twist* of M by the automorphism of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ that sends $T_i \mapsto T_i$, $Z_i \mapsto Z_i$, $\pi \mapsto a\pi$. $\square$

It may happen for some modules M that $M \cong M^a$. See Section 6.4.

Remark 4.8. For any $a, b \in \mathcal{K}^\times$, simultaneously rescaling all $X_i \mapsto aX_i$ and all $Y_i \mapsto bY_i$ defines an automorphism of $\mathbb{H}_{q,t}(GL_n)$, and $\mathcal{K}[B_n^{Ell}]$, compatible with ϕ.

For $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, however, rescaling $\pi \rightsquigarrow a\pi$ and $Z_i \rightsquigarrow bZ_i$'s changes the relations by $(\pi^n = 1) \rightsquigarrow (\pi^n = a^n)$, and $\boldsymbol{r} \rightsquigarrow b\boldsymbol{r}$. Hence, in order to define an isomorphism a must be an nth root of unity as in Definition 4.7 above.

On the other hand, this shows that the parameter $\boldsymbol{r}$ is inessential, and can be chosen at will by re-scaling the Z_i's. In Section 6, we will take n to be a multiple of a fixed integer N, and will fix $\boldsymbol{r} = \boldsymbol{q}^{1-N^2} = \mathsf{q}^{1/N-N}$, the inverse value of the ribbon element on the defining representation of SL_N. This is a matter of combinatorial convenience. One could also introduce such an inessential parameter for the value of π^n, but $\pi^n = 1$ is the most natural choice combinatorially, so we fix this convention now. This forces the scalar factor in front of $\phi(\mathsf{X}_1)$ below. $\qquad\square$

Proposition 4.9. There exists a unique isomorphism,

$$\phi : \mathcal{K}[B_n^{Ell}] \Big/ \left\langle \begin{array}{c} (\mathsf{T}_i - \boldsymbol{q}^{-1}t)(\mathsf{T}_i + \boldsymbol{q}^{-1}t^{-1}) = 0 \quad (i = 1, \ldots, n-1), \\[1mm] \mathsf{X}_1 \cdots \mathsf{X}_n = \boldsymbol{r}^n, \quad \mathsf{Y}_1 \cdots \mathsf{Y}_n = \boldsymbol{r}^n \end{array} \right\rangle \to \mathbb{H}_{\boldsymbol{q},t}(SL_n),$$

such that

$$\phi(\mathsf{T}_i) = \boldsymbol{q}^{-1} T_i, \quad (i = 1, \ldots n-1), \qquad \phi(\mathsf{X}_1) = \boldsymbol{r} \cdot \boldsymbol{q}^{n-1} \cdot \pi T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1}, \qquad \phi(\mathsf{Y}_1) = Z_1.$$

$$\square$$

We note that this implies $\phi(\mathsf{Y}_i) = \boldsymbol{q}^{2(1-i)} Z_i$.

Remark 4.10. In [15], a variant of Proposition 4.9 is taken as the definition of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, because of the central role played in the paper by the elliptic braid group. $\qquad\square$

4.5 Relationship between DAHAs for GL_n and SL_n

The relationship between the GL_n and SL_n DAHA is not entirely straightforward: the SL_n DAHA may be realized as a sub-quotient of a degree n-extension of GL_n DAHA. This can be found in [10], but we review it here in the notation of this paper.

First we note that in $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, we have the relation $\pi^n = 1$, while in $\mathbb{H}_{q,t}(GL_n)$, we cannot simply set $\pi^n = 1$, because of the relation $\pi^n Y_i \pi^{-n} = q^{-1} Y_i$. However, we note that π^n commutes with the T_i's and with the Laurent polynomials in the Y_i's of total degree zero. We enlarge the degree zero part of $\mathcal{Y}$, by adjoining an element,

$$\widetilde{Y} \quad \text{``} = \text{''} \quad (Y_1 \cdots Y_n)^{-1/n}.$$

More precisely, this means we define a new algebra $\widetilde{\mathbb{H}_{q,t}}(GL_n)$, by adding to $\mathcal{K}$, if necessary, a $2n$th root of q, and adjoining to $\mathbb{H}_{q,t}(GL_n)$ an element $\widetilde{Y}$ in degree -1, such that $\widetilde{Y}$ commutes with all T_i, $(i = 0 \ldots n-1)$, and all Y_i, $(i = 1, \ldots, n)$, and subject to the further relations:

$$\widetilde{Y}^n = (Y_1 \cdots Y_n)^{-1}, \qquad \pi \widetilde{Y} \pi^{-1} = q^{1/n} \widetilde{Y}.$$

We let $\widetilde{\mathcal{Y}}$ denote the resulting subalgebra of degree 0 polynomials, i.e., the subalgebra generated by $(Y_i\widetilde{Y})^{\pm 1}$. The algebra generated by $\pi, T_1, \widetilde{\mathcal{Y}}$ now has $\pi^n - 1$ in its center, so we may quotient by the ideal it generates (and below mark elements of the quotient with a bar). We can identify $\mathbb{H}_{q^{-1/2n},t}(SL_n)$ with this quotient via

$$T_i \mapsto \overline{T_i}, \qquad Z_i \mapsto q^{\frac{1-n}{2n}}\, r\overline{Y_i\widetilde{Y}}, \qquad \pi \mapsto \overline{\pi}.$$

Hence we can identify the parameter q of $\mathbb{H}_{q,t}(SL_n)$ with $q^{-1/2n}$ of $\mathbb{H}_{q,t}(GL_n)$. To avoid fractional exponents, we chose to make q a separate parameter.

5   Category $\mathcal{O}$ and $\mathcal{Y}$-semisimple Representations

In his landmark paper [9], Cherednik gave a complete classification of irreducible $\mathcal{Y}$-semisimple representations, i.e. those $\mathbb{H}_{q,t}$-modules for which the $\mathcal{Y}$-action can be diagonalized. His classification builds on the parallel story for the affine Hecke algebra [12, 11, 13], [20, 21]. Subsequently, the paper [24] built on Cherednik's classification via periodic skew diagrams combinatorially, connecting standard tableaux on the diagrams to $\mathcal{Y}$-weights. We recall these constructions below.

We will give basic definitions in parallel for both GL and SL. Theorems 5.3 and 5.4 were proved in [12, 11, 13], [20] for affine Hecke algebras and in [8, 9], [24] for $\mathbb{H}_{q,t}(GL_n)$. We will also state without proof the analogous results for SL_N, which follow by straightforward modifications of the proofs, keeping track of the additional relation $\pi^n = 1$. We will write $\mathbb{H}_{q,t}$ in this section when we do not need to distinguish between the GL or SL case.

Definition 5.1. Category $\mathcal{O}$ for $\mathbb{H}_{q,t}(GL_n)$ (resp. $\mathbb{H}_{q,t}(SL_n)$) is the full subcategory of finitely generated $\mathbb{H}_{q,t}$-modules M, such that for each vector $m \in M$, its orbit $\mathcal{Y} \cdot m$ (resp. $\mathcal{Z} \cdot m$) is finite dimensional. $\qquad\square$

A tuple $\underline{z} = (z_1, \ldots, z_n) \in (\mathcal{K}^\times)^n$ will be called a weight for $\mathcal{Y}$. A tuple $\underline{z}$ satisfying further that $\prod_i z_i = q^{n(n-1)}r^n$ will be called a $\mathcal{Z}$-weight. We will make use of the actions (4.1) and (4.2) of the extended affine symmetric group on each type of weights.

A $\mathcal{Y}$-weight $\underline{z}$ defines a homomorphism $\mathcal{Y} \to \mathcal{K}$, sending $Y_i \mapsto z_i$, and hence a one-dimensional representation of $\mathcal{Y}$. Similarly a $\mathcal{Z}$-weight $\underline{z}$ defines a homomorphism $\mathcal{Z} \to \mathcal{K}$, sending $Z_i \mapsto z_i$, and hence a one-dimensional representation of $\mathcal{Z}$.

Let M be an $\mathbb{H}_{q,t}$-module. We define its $\underline{z}$-weight space to be,

$$M[\underline{z}] = \{v \in M \mid hv = \underline{z}(h)v,\ \forall h \in \mathcal{Y}\ (\text{resp. } \mathcal{Z})\}.$$

A non-zero $v \in M[\underline{z}]$ is called a weight vector, or $\underline{z}$-weight vector. Its generalized weight space is

$$M^{gen}[\underline{z}] = \{v \in M \mid (h - \underline{z}(h))^m v = 0,\ \text{for all } h \in \mathcal{Y}\ (\text{resp. } \mathcal{Z}), m \gg 0\}$$

Assume for any $M \in \mathcal{O}$ that M splits over $\mathcal{K}$. Then we have an isomorphism,

$$M \cong \bigoplus_{\underline{z}} M^{gen}[\underline{z}],$$

as vector spaces, and in fact as $\mathcal{Y}$- or $\mathcal{Z}$-modules.

We will call

$$\mathrm{supp}(M) = \{\underline{z} \mid M^{gen}[\underline{z}] \neq 0\} = \{\underline{z} \mid M[\underline{z}] \neq 0\},$$

the support of M.

Definition 5.2. We say that M is $\mathcal{Y}$-*semisimple* (resp. $\mathcal{Z}$-semisimple) if we have an isomorphism

$$M \cong \bigoplus_{\underline{z}} M[\underline{z}],$$

as $\mathcal{Y}$- (resp. $\mathcal{Z}$-) modules, i.e. if $\mathrm{Res}_{\mathcal{Y}}^{\mathbb{H}_{q,t}}(M)$ is semisimple as a $\mathcal{Y}$- (resp. $\mathcal{Z}$-) module. We will write "$\mathcal{Y}$-$\mathcal{Z}$-semisimple" to mean either $\mathcal{Y}$- or $\mathcal{Z}$-semisimple, depending on G. □

Such M are called calibrated in [20]. Note that if M is $\mathcal{Y}$-$\mathcal{Z}$-semisimple, it has a *weight basis*: a basis consisting of weight vectors.

The notion of $\mathcal{Y}$-$\mathcal{Z}$-semisimplicity makes sense whether M is a representation of the affine Hecke algebra or of the double affine Hecke algebra. The structure of $\mathcal{Y}$-$\mathcal{Z}$-semisimple modules in both cases is extremely rigid. Given a $\mathcal{Y}$-semisimple module, one can read its composition factors directly from its support. Further if M is both simple and $\mathcal{Y}$-semisimple, one need only determine a single $\underline{z} \in \mathrm{supp}(M)$ in order to determine all of $\mathrm{supp}(M)$, and hence the isomorphism type of M. For simple and $\mathcal{Z}$-semisimple $\mathbb{H}_{q,t}(SL_n)$-modules, similar statements hold, once we allow twisting by an automorphism that rescales π by a root of unity.

Theorem 5.3 ([12, 11, 13, 8], [20],[24])**.** Let M be a simple and $\mathcal{Y}$-$\mathcal{Z}$-semisimple $\mathbb{H}_{q,t}$-module. Then for all $\underline{z} \in \mathrm{supp}(M)$, we have $\dim M[\underline{z}] = 1$. □

Hence, a simple and $\mathcal{Y}$-$\mathcal{Z}$-semisimple $\mathbb{H}_{q,t}$-module has a weight basis that is unique up to rescaling each vector individually, i.e. the underlying vector space has a canonical decomposition as a direct sum of lines.

For $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ one must modify the proof of Theorem 5.3 for $\mathbb{H}_{\boldsymbol{q},t}(GL_n)$ to include the case that the stabilizer of $\underline{z}$ via the action (4.2) contains an element outside of $\langle s_0, s_1, \ldots, s_{n-1} \rangle$. In that case it contains a subgroup conjugate to $\langle \pi^r \rangle$ for some $r \mid n$. One may then appeal to minimal idempotents for the cyclic group $\langle \pi^r \rangle / \langle \pi^n \rangle$ to get the required multiplicity one result.

The supports of simple $\mathcal{Y}$-$\mathcal{Z}$-semisimple modules have a very nice combinatorial structure. It is easy to show that if M is simple, then all its support is contained in a single $\widehat{\mathfrak{S}}_n$-orbit. If additionally M is $\mathcal{Y}$-$\mathcal{Z}$-semisimple, we can say *exactly* what subset of the $\widehat{\mathfrak{S}}_n$-orbit we get, i.e. we can completely determine $\mathrm{supp}(M)$. More precisely,

given $\underline{z} \in \mathrm{supp}(M)$, one can determine the set $S \subset \widehat{\mathfrak{S}}_n$ such that

$$\mathrm{supp}(M) = \{w \cdot \underline{z} \mid w \in S\}.$$

The following theorem uniquely characterizes the set S, which depends on choice of $\underline{z}$:

Theorem 5.4 ([12, 11, 13, 9], [20])**.** Let M be a simple and $\mathcal{Y}$-$\mathcal{Z}$-semisimple $\mathbb{H}_{q,t}$-module. Let $\underline{z} \in \mathrm{supp}(M)$. We have:

1. For $1 \le i < n$, we have $M[s_i \cdot \underline{z}] = 0$ if, and only if, $z_i/z_{i+1} \in \{t^2, t^{-2}\}$. Further, we have:

$$T_i M[\underline{z}] \subset M[\underline{z}] \oplus M[s_i \cdot \underline{z}].$$

2. We have $M[s_0 \cdot \underline{z}] = 0$ if, and only if $q z_n/z_1 \in \{t^2, t^{-2}\}$ for $\mathbb{H}_{q,t}(GL_n)$, $z_n/\boldsymbol{q}^{2n} z_1 \in \{t^2, t^{-2}\}$ for $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$. Further, we have:

$$T_0 M[\underline{z}] \subset M[\underline{z}] \oplus M[s_0 \cdot \underline{z}].$$

3. We have $M[\pi \cdot \underline{z}] \ne 0$, and $\pi M[\underline{z}] = M[\pi \cdot \underline{z}]$.

$\square$

Remark 5.5. Conditions (1) and (2) of Theorem 5.4 may be written more uniformly as $\frac{(s_i \cdot \underline{z})_{i+1}}{z_{i+1}} \in \{t^2, t^{-2}\}$, but not much clarity is gained from this reformulation. $\square$

Note that Theorem 5.4 allows us to precisely describe the action of the $\mathbb{H}_{q,t}(GL_n)$-generators on a weight basis, once we have chosen a sensible normalization/scaling. The proof of this theorem uses the theory of "intertwiners" [9], for which the reader may also consult [24]. For $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, this is true once we pin down the action of π (see Corollary 5.7).

As a consequence of this rigid structure, we also have the following:

Corollary 5.6. Let M, N be simple $\mathbb{H}_{q,t}(GL_n)$-modules, which are both $\mathcal{Y}$-semisimple. Then either $M \cong N$, or $\mathrm{supp}(M) \cap \mathrm{supp}(N) = \emptyset$. $\square$

Corollary 5.7. Let M, N be simple $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$-modules, which are both $\mathcal{Z}$-semisimple. Then either $M \cong N^a$ for some $a \in \mathcal{K}^\times$ with $a^n = 1$, or $\mathrm{supp}(M) \cap \mathrm{supp}(N) = \emptyset$. $\square$

We will use these corollaries to identify the module $F_n^G(\mathcal{O}_{\mathsf{q}}(G))$ in Section 7.

6 The Rectangular Representations

In this section we will detail a very special case of Cherednik's construction, when the Young diagram indexing the irreducible module is an $N \times k$ rectangle,

$$\mu = (k^N) = (\underbrace{k, \ldots, k}_{N}).$$

and the periodicity is purely horizontal, so that the shape is not actually skew. We begin, however, by recalling the simpler setting of the affine Hecke algebra.

6.1 The rectangle and the affine Hecke algebra

One can associate to the partition $\mu = (k^N)$ a finite dimensional irreducible representation of the finite Hecke algebra H_n, with $n = kN$. A basis for this representation is indexed by the set of standard Young tableaux of shape (k^N). We denote by $\mathrm{Rect}(N, k)$ the $H(\mathcal{Y})$-module obtained by inflating this module via the homomorphism (4.5). It is well-known for generic t (i.e. away from small roots of unity) that $\mathrm{Rect}(N, k)$ is $\mathcal{Y}$ semisimple. This is stated precisely in Proposition 6.4 below. Identifying the $N \times k$ rectangular diagram (k^N) with $\mathrm{D}_0^{N,k}$, i.e. $\lambda = 0$, we assign diagonal labels as in Section 2. Hence its principal diagonal is labeled 0, and a box in the jth row and mth column is on the $m - j$ diagonal. This agrees with the traditional notion of content.

Definition 6.1. Given a standard tableau R of shape (k^N), we define a map $\mathrm{diag}_R : \{1, \ldots, n\} \to \mathbb{Z}$ such that $\mathrm{diag}_R(i)$ is the label of the diagonal on which $\boxed{i}$ lies. □
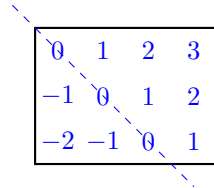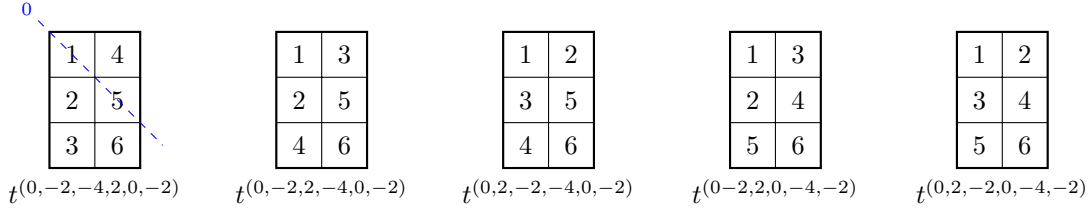
See Figure 7.

Fig. 7.   The diagonal labels of a rectangular tableau. The dashed line traverses the principal diagonal.

Definition 6.2. The *weight* $\mathrm{wt}(R) \in (\mathcal{K}^\times)^n$ of $R \in \mathrm{SYT}(k^N)$, is the tuple,

$$\mathrm{wt}(R) = \left(t^{2\mathrm{diag}_R(1)}, t^{2\mathrm{diag}_R(2)}, \ldots, t^{2\mathrm{diag}_R(n)} \right) = t^{(2\mathrm{diag}_R(1), 2\mathrm{diag}_R(2), \ldots, 2\mathrm{diag}_R(n))}.$$

□

Example 6.3. Let $N = 3, k = 2$. There are 5 standard tableau of shape (2^3). We list them below with their corresponding weights.

Note that for all $R \in \mathrm{SYT}(k^N)$, box $\boxed{1}$ is always in the upper left corner, so $\mathrm{diag}_R(1) = 0$ and $\boxed{n}$ is always in the lower right corner, so $\mathrm{diag}_R(n) = k - N$.

Proposition 6.4. We have:

1. The irreducible $H(\mathcal{Y})$-module $\mathrm{Rect}(N, k)$ has a $\mathcal{Y}$-weight basis,

$$\{v_R \mid R \in \mathrm{SYT}(k^N)\}$$

when t is generic, such that each v_R has weight $\mathrm{wt}(R)$, i.e,

$$Y_i v_R = t^{2\mathrm{diag}_R(i)} v_R.$$

In particular the central element $\prod_{i=1}^{n} Y_i$ acts as the scalar $t^{n(k-N)}$.

2. Set $\boldsymbol{r} = \boldsymbol{q}^{1-n} t^{k-N}$. Then $\mathrm{Rect}(N, k)$ descends to a representation of $H(\mathcal{Z})$, such that

$$Z_i v_R = t^{2\mathrm{diag}_R(i)} v_R.$$

In Section 7.1 we will consider $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$, with the specialization $t = \boldsymbol{q}^N = \mathsf{q}$. In light of Proposition 6.4, we will therefore further specialize $\boldsymbol{r} = \boldsymbol{q}^{1-n} t^{(k-N)} = \boldsymbol{q}^{1-N^2} = \mathsf{q}^{1/N-N}$ in the definition of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$.

6.2   Induction of the rectangular representation to the DAHA for GL

In this section, we prove that when we induce $\mathrm{Rect}(N, k)$ from the affine Hecke algebra to $\mathbb{H}_{q,t}(GL_n)$ under the specialization $q = t^{-2k}$ it has unique simple quotient.

Let

$$M(k^N) = \mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_n)} \mathrm{Rect}(N, k)$$

denote the induction of $\mathrm{Rect}(N, k)$ from $H(\mathcal{Y})$ to $\mathbb{H}_{q,t}(GL_n)$. This module has basis

$$\{T_\sigma \otimes v_R \mid \sigma \in \widehat{\mathfrak{S}}_n / \mathfrak{S}_n, R \in \mathrm{SYT}(k^N)\}$$

which can be ordered (refining weak Bruhat order) so that with respect to this basis $\mathcal{Y}$ acts triangularly. $M(k^N)$ thus has support

$$\mathrm{supp}\left(M(k^N)\right) = \{\sigma \cdot \mathrm{wt}(R) \mid \sigma \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n, R \in \mathrm{SYT}(k^N)\}. \tag{6.1}$$

Theorem 6.5. Let $q = t^{-2k}$. Then $M(k^N)$ has unique simple quotient. $\square$

In the next section we will explicitly construct this unique simple quotient and thereby show it is $\mathcal{Y}$-semisimple.

Proof. We will show below that, for any fixed $R \in \mathrm{SYT}(k^N)$, the $\mathrm{wt}(R)$-weight space of $M(k^N)$ is one-dimensional; hence any weight vector of weight $\mathrm{wt}(R)$ generates the induced module $M(k^N)$ and likewise any proper submodule must avoid it. This shows it has a unique maximal submodule and hence unique simple quotient. Alternatively, the following argument is rather standard for induced modules. Suppose that L is simple and we have a nonzero map

$$M(k^N) = \mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_n)} \mathrm{Rect}(N,k) \to L.$$

By adjointness of induction and restriction (Frobenius reciprocity) L contains a weight vector of weight $\mathrm{wt}(R)$. If there were another simple quotient L', then $L \oplus L'$ would also be a quotient but with 2-dimensional $\mathrm{wt}(R)$-weight space, a contradiction to the $\mathrm{wt}(R)$-weight space of $M(k^N)$ being one-dimensional.

So let us now compute the $\mathrm{wt}(R)$-weight space of $M(k^N)$. Because $\mathrm{Rect}(N,k)$ is $\mathcal{Y}$-semisimple, it suffices to determine which $\sigma \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ stabilize $\mathrm{wt}(R)$; the size of the stabilizer is the dimension of the weight space. On the one hand, the weights of $M(k^N)$ are given in (6.1). On the other hand, each double coset $\mathfrak{S}_n\backslash\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ contains "translation by a dominant weight"—that is to say, any weight in the support of $M(k^N)$ has the form

$$\tau \cdot (q^{\gamma_1}t^{2a_1}, \ldots, q^{\gamma_n}t^{2a_n}) = \tau \cdot (t^{2a_1-2k\gamma_1}, \ldots, t^{2a_n-2k\gamma_n}), \tag{6.2}$$

where $\tau \in \mathfrak{S}_n$, $\gamma_i \in \mathbb{Z}$, $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$, and $t^{(2a_1,\ldots,2a_n)} = \mathrm{wt}(R)$ for some $R \in \mathrm{SYT}(k^N)$. All such R satisfy:

$$a_1 = 0, \qquad 1 - N \le a_i \le k - 1 \quad (i = 2, \ldots n), \qquad a_n = k - N; \tag{6.3}$$

see Figure 8.

We now claim that the only possible γ which can appear in equation (6.2) is the trivial one,
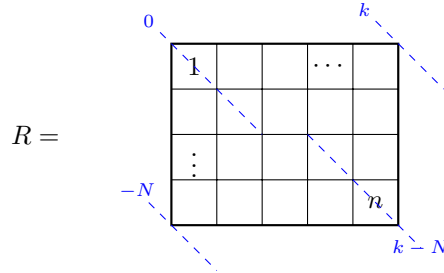
$$\gamma_1 = \gamma_2 = \cdots = \gamma_n = 0.$$

Fig. 8.   The contents of the corners of the rectangle imply the restrictions on $a_i = \mathrm{diag}_R(i)$ in equation (6.3).

Since γ is dominant, it suffices to show that $\gamma_1 \geq 0$ and $\gamma_n \leq 0$. Indeed, suppose for the sake of contradiction that $\gamma_1 < 0$, then we have $a_1 - k\gamma_1 \geq k > k - 1$, hence $a_1 - k\gamma_1 \neq a_i$ for any i, by the restrictions (6.3). Likewise if we suppose $\gamma_n > 0$, we have $a_n - k\gamma_n \leq -N < 1 - N$ so $a_n - k\gamma_n \neq a_i$ for any i, again by (6.3).

Given that $\sigma \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ is taken to be of minimal length, this implies $\sigma = \mathrm{Id}$. In other words, for any $R \in \mathrm{SYT}(k^N)$ the $\mathrm{wt}(R)$ weight space of $M(k^N)$ is one-dimensional. ∎

Let us remark that the technique used in the proof of Theorem 6.5 is more general: for an irreducible representation A of $H(\mathcal{Y})$, if one shows the (generalized) $\mathcal{Y}$-weight space of A of weight $\underline{z}$ has the same dimension as that of $\mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_n)} A$, then by the same Frobenius reciprocity argument as above, the latter must have unique simple quotient. To illustrate, below we give a non-example, that is, an example of an induced representation that has two distinct simple quotients.

Example 6.6. In contrast to the situation in Theorem 6.5 consider the following example of a representation of $\mathbb{H}_{q,t}(GL_n)$ at $n = N = 2$, $k = 1$, $q = t^{-2}$. It is an induced representation with two distinct simple quotients, and furthermore one of those quotients is not $\mathcal{Y}$-semisimple.

Let $\lambda = 2\epsilon_1 = (2,0)$, and as in Section 2.4 consider the skew shape $\mathrm{D}_\lambda^{N,k} = \mathrm{D}_{(2,0)}^{2,1}$ and the two skew tableaux $\mathcal{SK}_{(2,0)}^{2,1}$ of that shape which correspond to the $\mathcal{Y}$-weights (t^2, t^{-4}) and (t^{-4}, t^2). This data is attached to a 2-dimensional irreducible representation A of $H(\mathcal{Y})$. In particular, $A \simeq \mathrm{Ind}_{\mathcal{Y}}^{H(\mathcal{Y})} t^2 \boxtimes t^{-4}$. Let

$$M := \mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_2)} A \simeq \mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}_{q,t}(GL_2)} t^2 \boxtimes t^{-4}.$$

We claim M has two distinct simple quotients. In fact M is isomorphic to the direct sum of two irreducible $\mathbb{H}_{q,t}(GL_2)$-modules. We leave the details of most computations to the reader, but summarize them below. The interested reader can verify the following assertions. While the (t^2, t^{-4})-weight space of A is 1-dimensional, the corresponding weight space of M is 2-dimensional. It is easy to verify this by considering the $\widehat{\mathfrak{S}}_2$-orbit of (t^2, t^{-4}) when $q = t^{-2}$. Hence our example does not exhibit the properties or hypotheses used in the proof of Theorem 6.5.

One simple quotient is our $L(1^2) \simeq M(1^2)$, whose $\mathcal{Y}$-weights are computed in Theorem 6.10 below. (In fact, comparing with equation (6.1) this computation also implies at $k = 1$ that $M(1^2)$ is simple and coincides

with $L(1^2)$.) Its (t^2, t^{-4})-weight space is 1-dimensional, yielding that it occurs as a quotient of M by Frobenius reciprocity. Another simple quotient is $K := \mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_2)} B$, where B is the 1-dimensional "trivial" module on which $Y_1 - t^{-2}$, $Y_2 - t^0$, $T_1 - t$ each vanish. An appropriate name for this K would be $L(2) = M(2)$, except its spectrum is slightly shifted from what one might expect, as B is inflated from the trivial H_2 module along a different homomorphism than (4.5). (Nonetheless, by abuse of notation we will refer to K as $L(2)$ here and in Example 6.7 below.) Note $L(2)$ is *not* $\mathcal{Y}$-semisimple. For instance, its (t^{-2}, t^0)-weight space is 2-dimensional. Again the computation of its (t^2, t^{-4})-weight space gives it as a quotient of M. Further, on closer examination of $\mathcal{Y}$-weight spaces of all the above representations, it in fact follows $M \simeq L(1^2) \bigoplus L(2)$. □

Example 6.7. As another example, in a forthcoming paper we construct an isomorphism,

$$F_2^{GL}(\mathtt{Spr}) \simeq \mathrm{Ind}_{\mathcal{Y}}^{\mathbb{H}_{q,t}(GL_2)} t^0 \boxtimes t^{-2} \simeq L(1^2) \oplus L(2), \tag{6.4}$$

where $\mathtt{Spr}$ is the so-called quantum Springer-Hotta-Kashiwara sheaf, a $\mathcal{D}_{\mathsf{q}}(G)$-module of considerable interest, which q-deforms the Hotta-Kashiwara presentation [14] of the classical Springer sheaf. Unlike in Example 6.6 above, the 2-dimensional $H(\mathcal{Y})$-representation $\mathrm{Ind}_{\mathcal{Y}}^{H(\mathcal{Y})} t^0 \boxtimes t^{-2}$ is reducible; we have a non-split short exact sequence

$$0 \longrightarrow B \longrightarrow \mathrm{Ind}_{\mathcal{Y}}^{H(\mathcal{Y})} t^0 \boxtimes t^{-2} \longrightarrow \mathrm{Rect}(2,1) \longrightarrow 0, \tag{6.5}$$

where B is as in Example 6.6 above. It is very interesting therefore that upon further induction by the functor $\mathrm{Ind}_{H(\mathcal{Y})}^{\mathbb{H}_{q,t}(GL_2)}$, we obtain the splitting (6.4). Thus the $\mathbb{H}_{q,t}(GL_2)$-endomorphism algebra of $F_2^{GL}(\mathtt{Spr})$ is isomorphic to the finite Hecke algebra H_2. More generally we obtain analogous results for quantum Springer-Hotta-Kashiwara sheaves associated to GL_N for any N. □

6.3   The periodic rectangle and the double affine Hecke algebra

The construction of irreducible $\mathcal{Y}$-semisimple $\mathbb{H}_{q,t}(GL_n)$-modules starts with the notion of a periodic skew diagram [9], and the subsequent notion of a periodic skew tableau. Let us recall the construction in [24] here, modified to conform to the conventions of this paper.

Given a skew Young diagram μ/λ, we make it periodic as follows. We embed the n cells of a skew diagram μ/λ into $\mathbb{Z}^2$ using matrix style coordinates (row, column).[‖]

For each r, we have the r-shifted diagram,

$$(\mu/\lambda)[r] = \mu/\lambda + r(-\ell, k)$$

[‖] This is a standard choice for Young diagrams in English notation, in place of Cartesian coordinates.

where $k = \mu_1 =$ the number of columns of μ, and ℓ is determined by $t = q^{-2(k+\ell)}$. The condition that $\mu/\lambda[0] \cup \mu/\lambda[1]$ again forms a skew diagram forces

$$l(\mu) - \mathrm{mult}(\mu_1) \le \ell, \qquad l(\lambda) \le \ell, \tag{6.6}$$

where l denotes the number of rows of the diagram. The case we consider in this paper is very special: we have $\lambda = \emptyset$ and $\mu = (k^N)$, so the first condition in (6.6) is vacuous, as $l(\mu) - \mathrm{mult}(\mu_1) = N - N = 0$. Further, for $\mathbb{H}_{q,t}(GL_n)$ we specialize $q = t^{-2k}$ and so $\ell = 0$. In other words

$$(k^N)[r] = (k^N) + r(0, k)$$

so the boxes of the periodic (skew) diagram form an $N \times \infty$ strip, see Figure 9.
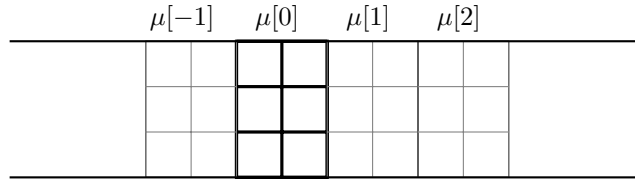

Fig. 9.   The diagram $\mu = (2^3)$ is made periodic by shifting horizontally.

In the GL_n case, we always consider the fundamental domain $\mu[0]$ to be anchored on the 0-diagonal. In other words, the $(1,1)$ cell is always the upper left corner of $\mu[0]$.

Definition 6.8. Let $n = kN$. An n-periodic standard tableaux of shape $\mu = (k^N)$ is a bijection $R : \mathbb{Z} \to \{$boxes of $N \times \infty$ strip$\}$ such that:

- fillings increase across rows and down columns,
- the fillings of $\mu[0]$ are distinct mod n,
- the fillings of $\mu[r]$ are those of $\mu[0] + nr$.

We will denote the set of all such tableaux $\mathrm{P}_n\mathrm{SYT}(k^N)$. $\qquad \square$

An $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$ is completely determined by the fillings of $\mu[0]$, see Figure 10. However it may happen that the filling of $\mu[0]$ is row- and column-increasing, but its periodization is not standard, see Figure 11.
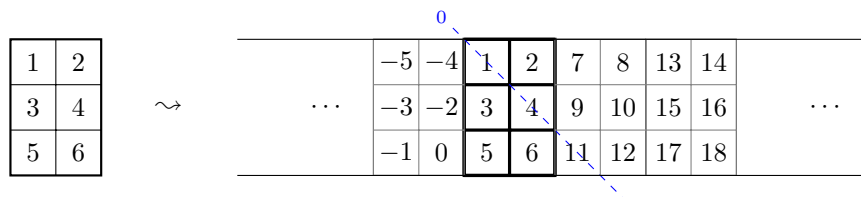

Fig. 10.   The filling of $\mu[0]$ completely determines the filling of $R \in \mathrm{P}_6\mathrm{SYT}(2^3)$, via the periodicity constraint.

1	8
3	10
5	12

$\rightsquigarrow$

$\cdots$

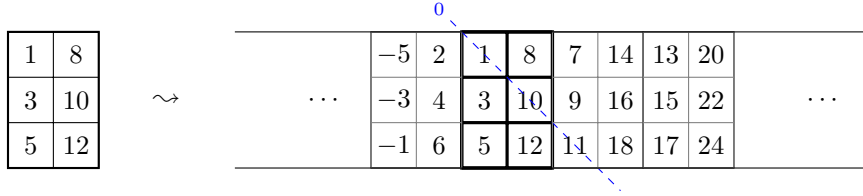| −5	2	1	8	7	14	13	20
−3	4	3	10	9	16	15	22
−1	6	5	12	11	18	17	24

$\cdots$

Fig. 11. The filling of $\mu[0]$ is row- and column-increasing, but its periodization is not standard.

The diagonal and weight functions are defined for periodic skew tableaux $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$ similarly to those for standard Young tableaux, in Definitions 6.1 and 6.2. The diagonal function is $\mathrm{diag}_R(i) = m - j$, when $\boxed{i}$ lies in row j and column m. Note this is defined for all $i \in \mathbb{Z}$ and satisfies $\mathrm{diag}_R(i+n) = \mathrm{diag}_R(i) + k$. The weight, $wt(R)$, of R is the tuple $(t^{2\mathrm{diag}_R(1)}, t^{2\mathrm{diag}_R(2)}, \ldots, t^{2\mathrm{diag}_R(n)})$.

The $\widehat{\mathfrak{S}}_n$ action on $\mathbb{Z}$ descends to an action on periodic tableaux, as follows. We set $\sigma \cdot R$ to be the tableau where $\boxed{i}$ is replaced with $\boxed{\sigma(i)}$.

The function diag_R is compatible with this action:

$$\mathrm{diag}_{\sigma \cdot R}(\sigma(i)) = \mathrm{diag}_R(i)$$

for any $\sigma \in \widehat{\mathfrak{S}}_n$. Furthermore the action intertwines the action of $\widehat{\mathfrak{S}}_n$ on $(\mathcal{K}^\times)^n$ described in (4.1): we have $wt(\sigma \cdot R) = \sigma \cdot wt(R)$. We note that $\sigma \cdot R$ need not be standard, even if R is.

Remark 6.9. We note that any domain for the n-periodicity in Definition 6.8 is also a domain for the π^n-action. Note that π^n shifts the $N \times \infty$ strip k steps horizontally. $\qquad\square$

Theorem 6.10 ([9], [24])**.** Let $L(k^N)$ denote the linear span over $\mathcal{K}$ of

$$\{v_R \mid R \in \mathrm{P}_n\mathrm{SYT}(k^N)\}.$$

When $q = t^{-2k}$ there exists a unique irreducible representation of $\mathbb{H}_{q,t}(GL_n)$ on $L(k^N)$, such that each v_R is a $\mathcal{Y}$-weight vector of weight $wt(R)$, i.e.

$$Y_i v_R = t^{2\mathrm{diag}_R(i)} v_R, \quad (i = 1, \ldots, n).$$

$\qquad\square$

Corollary 6.11. When $q = t^{-2k}$, the unique simple quotient of $M(k^N)$ is $L(k^N)$; in particular it is $\mathcal{Y}$-semisimple and its support is given in Theorem 6.10. $\qquad\square$

Proof. It follows by Frobenius reciprocity that there exists a non-zero map from $M(k^N)$ to $L(k^N)$, hence $L(k^N)$ is indeed the unique simple quotient guaranteed by Theorem 6.5. $\qquad\blacksquare$

Both Theorem 6.10 and Corollary 6.11 justify associating the irreducible $\mathbb{H}_{q,t}(GL_n)$-module with the $N \times k$ rectangle.

6.4 SL modifications

When considering SL versus GL, we need to make the following modifications both to Theorems 6.5 and 6.10 and the underlying combinatorics. Further we specialize $t = \boldsymbol{q}^N$ and $\boldsymbol{r} = \boldsymbol{q}^{1-N^2}$. They are consistent with the modified $\widehat{\mathfrak{S}}_n$ action on $(\mathcal{K}^\times)^n$ described in (4.2).

The role of periodic skew diagram $\mathrm{P}_n\mathrm{SYT}(k^N)$ in the GL case is now played by the set $\mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n$ of equivalence classes: we impose the equivalence relation $R \sim \pi^n \cdot R$, and denote equivalence classes as $\overline{R}$. For $\overline{R} \in \mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n$ we modify the function diag_R to $\mathrm{diag}_{\overline{R}}$ as follows.

When a fundamental domain such as $\mu[0]$ is filled with any $R_0 \in \mathrm{SYT}(k^N)$, it has "filling sum,"

$$\mathfrak{s} = \sum_{i=1}^n i.$$

For $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$, any of its fundamental domains (under π^n-shifts, see Remark 6.9) has filling sum $\mathfrak{s} + np$ for some $p \in \mathbb{Z}$ (even if the domain is not of the form $\mu[r]$). For any $N \times k$ rectangle of $\overline{R}$ we label with $\frac{p}{N}$ the diagonal through its northwest corner; we will call this its *NW diagonal*.

To see that this gives a well-defined diagonal labelling to all of R, we note that if a domain has filling sum $\mathfrak{s} + np$, then the domain that is shifted one unit right has filling sum $\mathfrak{s} + np + nN$ and $\frac{p+N}{N} = \frac{p}{N} + 1$. More precisely, if $\boxed{i}$ sits in the domain $\mu[r]$ in the jth row and $(m + rk)$th column,

$$\mathrm{diag}_{\overline{R}}(i) = m - j + \frac{p}{N} \tag{6.7}$$

where p is determined as above with respect to $\mu[r]$. In particular $\mathrm{diag}_{\overline{R}}(i)$ is independent of r, and $\mathrm{diag}_{\overline{R}}(i) = \mathrm{diag}_{\pi^n \cdot \overline{R}}(i)$ since $\boxed{i}$ now sits in domain $\mu[r+1]$ of $\pi^n \cdot \overline{R}$ but the local information of p stays the same. The function $\mathrm{diag}_{\overline{R}}$ is defined for all $i \in \mathbb{Z}$ and satisfies $\mathrm{diag}_{\overline{R}}(i+n) = \mathrm{diag}_{\overline{R}}(i) + k$.

The weight of $\overline{R} \in \mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n$ is the tuple $(t^{2\mathrm{diag}_{\overline{R}}(1)}, t^{2\mathrm{diag}_{\overline{R}}(2)}, \ldots, t^{2\mathrm{diag}_{\overline{R}}(n)})$.

Example 6.12. See Figure 12 for an example of how the diagonal labels change within a π orbit. One should also compare this to Figure 2, where only the locations of filling by {1,2,3} are marked. We let $N = 3, k = 1$, so $\mathfrak{s} = 6$. In the first periodic tableau R, $1 + 5 + 6 = \mathfrak{s} + 3(2)$ so the chosen fundamental domain has NW diagonal labeled $\frac{2}{3}$. In the second, $2 + 6 + 7 = \mathfrak{s} + 3(3)$, and so on. In the fourth periodic tableau the chosen domain has diagonal labeled $\frac{5}{3}$ and passes through $\boxed{4}$, but the diagonal one step left has label $\frac{2}{3}$ and passes through $\boxed{1}$; and indeed $R \sim \pi^3 \cdot R$. In other words, $\mathrm{diag}_{\overline{R}}(1) = \frac{2}{3} = \mathrm{diag}_{\overline{\pi^3 \cdot R}}(1)$.

We also list $\mathrm{wt}(\overline{R})$ in Figure 12. Note that

$$\pi \cdot (t^{4/3}, t^{-8/3}, t^{-14/3}) = (\boldsymbol{q}^{-2(3)+2}t^{-14/3}, \boldsymbol{q}^2 t^{4/3}, \boldsymbol{q}^2 t^{-8/3}) = (t^{-6}, t^2, t^{-2}),$$
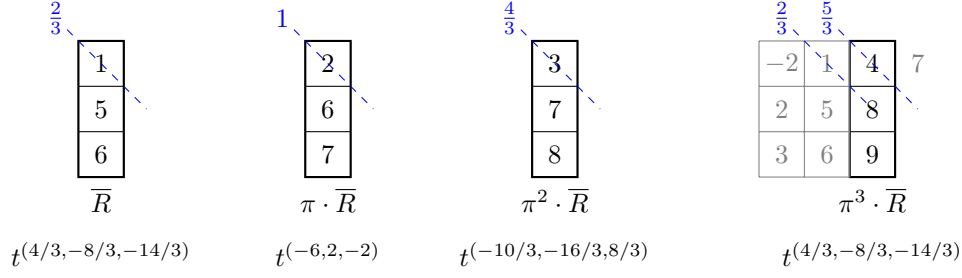
as $t = \boldsymbol{q}^3$.

Fig. 12. The label of the NW diagonal of a fundamental rectangle depends on its filling sum in type SL.

$\square$

Just as in the GL_n case, $\widehat{\mathfrak{S}}_n$ acts on $\mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n$. It is no longer true that $\mathrm{diag}_{\overline{\sigma \cdot R}}(\sigma(i))$ agrees with $\mathrm{diag}_{\overline{R}}(i)$ for $\sigma \in \widehat{\mathfrak{S}}_n$ (in particular for $\sigma = \pi$). However, we do still have the intertwining property

$$\mathrm{wt}(\sigma \cdot \overline{R}) = \sigma \cdot \mathrm{wt}(\overline{R})$$

using the SL_n modified action of $\widehat{\mathfrak{S}}_n$ on $(\mathcal{K}^\times)^n$ described in (4.2).

Let $\overline{L}(k^N)$ denote the linear span over $\mathcal{K}$ of

$$\{v_{\overline{R}} \mid \overline{R} \in \mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n\}.$$

Theorem 6.13. When $\boldsymbol{q}^N = t$ and $\boldsymbol{r} = \boldsymbol{q}^{1-N^2}$ there exists a unique irreducible representation of $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$ on $\overline{L}(k^N)$ such that each $v_{\overline{R}}$ is a $\mathcal{Z}$-weight vector of weight $\mathrm{wt}(\overline{R})$, i.e.

$$Z_i v_{\overline{R}} = t^{2\mathrm{diag}_{\overline{R}}(i)} v_{\overline{R}}, \quad (i = 1, \ldots, n),$$

and such that

$$\pi^N v_{\overline{R}_0} = v_{\overline{R}_0},$$

where $\overline{R}_0$ is as in Figure 13 below. $\square$

The existence of the module $\overline{L}(k^N)$ has been established above, constructing it combinatorially via equivalence classes of standard periodic tableaux. Unlike the GL case, $\overline{L}(k^N)$ is not the unique simple quotient of the induced module $\overline{M}(k^N)$. Rather we have:

Theorem 6.14. Let $\boldsymbol{q}^N = t$ and $\boldsymbol{r} = \boldsymbol{q}^{1-N^2}$ Let $a \in \mathcal{K}^\times$ be a primitive nth root of unity.

1. The induced module $\overline{M}(k^N) = \mathrm{Ind}_{H(\mathcal{Z})}^{\mathbb{H}_{\boldsymbol{q},t}(SL_n)} \mathrm{Rect}(N,k)$ has maximal semisimple quotient

$$\overline{L}(k^N) \oplus \left(\overline{L}(k^N)\right)^a \oplus \cdots \oplus \left(\overline{L}(k^N)\right)^{a^{k-1}}.$$

2. We have $\overline{L}(k^N)^a \cong \overline{L}(k^N)^b$ if and only if $a^N = b^N$.

3. These k twists are all the representations of $\mathbb{H}_{q,t}(SL_n)$ that have the same support as $\overline{L}(k^N)$. Their isomorphism type is distinguished by the extra data of $\pi^N v_{\overline{R}_0} = a^N v_{\overline{R}_0}$.

$\square$

Proof. Along the lines of Theorem 6.5, one sees that for $\overline{R}_0$ depicted in Figure 13, $\mathrm{wt}(\overline{R}_0)$ has stabilizer $\langle \pi^N \rangle$ via the action (4.2), and hence the $\mathrm{wt}(\overline{R}_0)$-weight space of $\overline{M}(k^N)$ has dimension $k = n/N$.

$$\overline{R}_0 \quad = \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & N+1 & & \cdots & & \\ \hline 2 & & & & & \\ \hline \vdots & & & & & n-1 \\ \hline N & 2N & & & & n \\ \hline \end{array}$$

Fig. 13. A periodic skew tableau $\overline{R}_0$ stabilized by π^N.

Let us denote by v_a a basis vector of the one-dimensional $\mathrm{wt}(\overline{R}_0)$ weight space of $(\overline{L}(k^N))^a$. Note that π^N preserves the weight space, hence it scales v_a by some scalar. Since any homomorphism $\varphi : (\overline{L}(k^N))^a \to (\overline{L}(k^N))^b$ commutes with the action of Z_i, it sends v_a to some multiple of v_b, say $\varphi(v_a) = c v_b$. Then

$$a^N c v_a = \varphi(a^N v_a) = \varphi(\pi^N v_a) = \pi^N \varphi(v_a) = c b^N v_b.$$

If $\varphi \neq 0$ then $a^N = b^N$. This proves the "only if" part of (2).

As in the GL case, Frobenius reciprocity gives a map,

$$\overline{M}(k^N) = \mathrm{Ind}_{H(\mathcal{Z})}^{\mathbb{H}_{q,t}(SL_n)} \mathrm{Rect}(N,k) \to \left(\overline{L}(k^N) \right)^a,$$

as the restriction of $\left(\overline{L}(k^N) \right)^a$ to $H(\mathcal{Z})$ contains $\mathrm{Rect}(N,k)$. As each of these quotients has a one-dimensional $\mathrm{wt}(\overline{R}_0)$-weight space, and $\overline{M}(k^N)$ has k-dimensional $\mathrm{wt}(\overline{R}_0)$-weight space, $\overline{M}(k^N)$ can have at most k simple quotients. Now using the pigeon-hole principle, we complete the proof of (2), as well as (1). In particular the discussion on φ shows (3) as well.

$\blacksquare$

Remark 6.15. Alternatively, we could have replaced $\mathbb{H}_{q,t}(SL_n)$ by its subalgebra generated by $\langle T_0, T_1, \ldots, T_{n-1}, Z_1^{\pm 1} \rangle$, so that the weight $\mathrm{wt}(\overline{R}_0)$ (and indeed the weight of any $\overline{R} \in \mathrm{SYT}(k^N)$) would once again have trivial stabilizer. Hence, defining $\overline{M}(k^N)$ by induction in the same way would yield a module with unique simple quotient $\overline{L}(k^N)$. We note that $\mathbb{H}_{q,t}(SL_n)$ is a free module of rank n over this subalgebra with basis $\{1, \pi, \ldots, \pi^{n-1}\}$. This subalgebra is in some ways more similar to the RCA of type SL.

$\square$

7 The Functor, and the Isomorphism

7.1 The functor F_n^G

Let $G = SL_N$ or GL_N, and let M be a $\mathcal{D}_{\mathsf{q}}(G)$-module. Let $n \in \mathbb{N}$, and consider the space,

$$F_n^{SL}(M) = \left(\underset{n}{V} \otimes \cdots \otimes \underset{1}{V} \otimes M \right)^{U_{\mathsf{q}}(\mathfrak{sl}_N)}, \quad \text{and}$$

$$F_n^{GL}(M) = \left(\det_{\mathsf{q}}^{-k}(V) \otimes \underset{n}{V} \otimes \cdots \otimes \underset{1}{V} \otimes M \right)^{U_{\mathsf{q}}(\mathfrak{gl}_N)},$$

of invariants, respectively $\det_{\mathsf{q}}^k$-variants, in the tensor product of M with the n-fold tensor power of the defining N-dimensional representation, $V = V_{\epsilon_1}$ of $U_{\mathsf{q}}(\mathfrak{g})$. In order to match various conventions (while breaking others), we index the tensor factors from right to left, as indicated in the subscripts above. We note that $-k$ is the unique possible tensor exponent of $\det_{\mathsf{q}}(V)$ such that $F_n^{GL}(\mathcal{O}_{\mathsf{q}}(GL_N))$ is non-zero, and that similarly n must be an integer multiple of N in the SL case, see Remark 7.5.

In [15], an action of the double affine Hecke algebra was constructed on the space F_n^{SL}. Let us recall the construction here, and formulate its GL-modification.

Theorem 7.1 ([15]). Let $G = SL_N$, or GL_N, respectively. Let M be a module for $\mathcal{D}_{\mathsf{q}}(G)$, let k be a positive integer, and let $n = kN$. There is a unique representation of B_n^{Ell} (resp. $B_{n,1}^{Ell}$) on $F_n^G(M)$ such that:

1. Each generator T_i $(i = 1, \ldots, n-1)$ acts by the braiding $\sigma_{V,V}$ on the $\underset{i+1}{V} \otimes \underset{i}{V}$ factors.

2. For GL_N, T_n^2 acts by the double braiding on $\det_{\mathsf{q}}^{-k}(V) \otimes \underset{n}{V}$.

3. The operator Y_1 acts only in the rightmost two tensor factors, $\underset{1}{V} \otimes M$, via

$$\mathsf{Y}_1 = \sigma_{M_\lhd, V} \circ \sigma_{V, M_\lhd}, \quad \text{(the double-braiding of } V \text{ and } M, \text{ using } \partial_\lhd).$$

4. The operator X_1 acts only in the rightmost two tensor factors $\underset{1}{V} \otimes M$, via

$$\mathsf{X}_1 = V \otimes M \xrightarrow{\Delta_V \otimes \mathrm{Id}_M} V \otimes \mathcal{O}_{\mathsf{q}}(G) \otimes M \xrightarrow{\mathrm{Id}_V \otimes \mathrm{act}_M} V \otimes M,$$

where in the second arrow, $\mathcal{O}_{\mathsf{q}}(G)$ acts on M via the homomorphism $\ell : \mathcal{O}_{\mathsf{q}}(G) \to \mathcal{D}_{\mathsf{q}}(G)$.

$\square$

Remark 7.2. Since B_n^{Ell} and $B_{n,1}^{Ell}$ are generated by the operators listed above, their action is uniquely determined. That these operators satisfy the defining relations listed in Proposition 4.3 is established in [15, Theorem 22 and Corollary 24]. $\square$

Corollary 7.3. The B_n^{Ell}-action on $F_n^{SL}(M)$ is natural in M and V, satisfies the additional relations

$$(\mathsf{T}_i - \mathsf{q}^{-1/N}\mathsf{q})(\mathsf{T}_i + \mathsf{q}^{-1/N}\mathsf{q}^{-1}) = 0, \qquad \mathsf{Y}_1 \cdots \mathsf{Y}_n = \mathsf{q}^{n(1/N-N)}.$$

Hence by Proposition 4.9 we have an exact functor,

$$F_n^{SL} : \mathcal{D}_{\mathsf{q}}(SL_N)\text{-mod} \to \mathbb{H}_{\boldsymbol{q},t}(SL_n)\text{-mod},$$

for $\boldsymbol{q} = \mathsf{q}^{1/N}$, $t = \mathsf{q}$, and $\boldsymbol{r} = \mathsf{q}^{1/N-N}$. $\square$

Proof. In [15], Proposition 30, it is proved that $\mathsf{Y}_1 \cdots \mathsf{Y}_n$ acts as $(\nu^{-1}|_V)^n$. We compute, using (3.4), that

$$\nu^{-1}|_V = \mathsf{q}^{-\langle \varepsilon_1 + 2\rho, \varepsilon_1 \rangle} = \mathsf{q}^{1/N-N},$$

as desired.   ∎

Corollary 7.4. The $B_{n,1}^{Ell}$-action on $F_n^{GL}(M)$ is natural in M and V, satisfies the additional relations

$$(\mathsf{T}_i - \mathsf{q})(\mathsf{T}_i + \mathsf{q}^{-1}) = 0, \qquad \mathsf{T}_n^2 = \mathsf{q}^{-2k}. \tag{7.1}$$

Hence by Proposition 4.5 we have an exact functor,

$$F_n^{GL} : \mathcal{D}_{\mathsf{q}}(GL_N)\text{-mod} \to \mathbb{H}_{q,t}(GL_n)\text{-mod},$$

for $q = \mathsf{q}^{-2k}$ and $t = \mathsf{q}$. $\square$

Proof. We outline how to modify the proof from [15], to obtain relations (7.1) when SL_N is replaced by GL_N. The differing form of the Hecke relation for $\mathsf{T}_1, \ldots, \mathsf{T}_{n-1}$ is clear from the quantum R-matrix for GL_N compared to SL_N. The quantum determinant representation $\det_{\mathsf{q}}(V)$ and all its tensor powers are invertible, which means that $\det_{\mathsf{q}}^{-k}(V) \otimes V$ is irreducible, so that T_n^2 must act as a scalar; this scalar can be computed using the ribbon element to be q^{-2k} as in the proof of Theorem 7.4 above.   ∎

7.2   The case $M = \mathcal{O}_{\mathsf{q}}(G)$

In the special case $M = \mathcal{O}_{\mathsf{q}}(G)$, the action is compatible with Peter-Weyl decomposition, in the following sense. Let

$$W_\lambda^n = \begin{cases} (V^{\otimes n} \otimes V_\lambda^* \otimes V_\lambda)^{U_{\mathsf{q}}(\mathfrak{g})}, & G = SL_N \\ (\det_{\mathsf{q}}^{-k}(V) \otimes V^{\otimes n} \otimes V_\lambda^* \otimes V_\lambda)^{U_{\mathsf{q}}(\mathfrak{g})}, & G = GL_N \end{cases},$$

and consider the vector space decomposition,

$$F_n^G(\mathcal{O}_{\mathsf{q}}(G)) \cong \bigoplus_{\lambda \in \Lambda^+} W_\lambda^n,$$

induced by the Peter-Weyl decomposition, Theorem 3.5.

Remark 7.5. Recall that we have restricted to the case $n = kN$ for a positive integer k. It is elementary to see that otherwise all spaces W_λ^n, and hence $F_n^G(\mathcal{O}_\mathsf{q}(G))$, are zero. $\qquad\qquad\qquad\qquad\qquad\square$

It follows from the definition of Y_1 and T_i as braiding operators that they preserve the Peter-Weyl decomposition, and the passage to invariants. Hence, each subspace $W_\lambda^n \subset F_n^G(\mathcal{O}_\mathsf{q}(G))$ is a finite-dimensional submodule for the action of the affine Hecke algebra $\mathcal{H}(\mathcal{Y})$.

7.3 Walks, skew diagrams, and a weight basis of the invariants

In this section we identify the $\mathcal{H}(\mathcal{Y})$-modules W_λ^n with those constructed in [19].

Let $\lambda \in \Lambda_{\mathfrak{sl}_N}^+$ or $\Lambda_{\mathfrak{gl}_N}^+$. To each walk $\underline{u} \in \mathcal{W}_\lambda^{N,k}$, we associate a unique line $L_{\underline{u}}$ in W_λ^n as follows. First, in the SL case we have natural isomorphisms,

$$W_\lambda^n = \mathrm{Hom}_{U_\mathsf{q}(\mathfrak{sl}_N)}(\mathbf{1}, V^{\otimes n} \otimes V_\lambda^* \otimes V_\lambda)$$

$$\cong \mathrm{Hom}_{U_\mathsf{q}(\mathfrak{sl}_N)}(\mathbf{1}, V^{\otimes n} \otimes V_\lambda \otimes V_\lambda^*)$$

$$\cong \mathrm{Hom}_{U_\mathsf{q}(\mathfrak{sl}_N)}\left(V_\lambda, V^{\otimes n} \otimes V_\lambda\right),$$

where we have first applied the braiding $\sigma_{V^*,V}$, and then the canonical isomorphism $Hom(\mathbf{1}, X \otimes Y^*) \cong Hom(Y, X)$. In the GL case, a similar series of isomorphisms

$$W_\lambda^n \cong \mathrm{Hom}_{U_\mathsf{q}(\mathfrak{gl}_N)}(\mathrm{det}_\mathsf{q}^k(V) \otimes V_\lambda, V^{\otimes n} \otimes V_\lambda).$$

The Pieri rule gives a multiplicity-free decomposition,

$$V \otimes V_\alpha \cong \bigoplus_\beta V_\beta,$$

where each β is a dominant weight, which differs from α by an ϵ_i. Hence $(V \otimes V_\alpha)[\beta] \cong V_\beta$, where the notation $X[\lambda]$ denotes the λ-isotypic component of a representation X. In particular $\dim \mathrm{Hom}(V_\beta, V \otimes V_\alpha) = 1$. Hence, given a looped walk $\underline{u}$ of length n from λ to $\lambda + k\mathbf{d}$, the space

$$\mathrm{Hom}(V_{\lambda+k\mathbf{d}}, \bigcap_{i=0}^n V^{\otimes(n-i)} \otimes \left((V^{\otimes i} \otimes V_\lambda)[u_i]\right)) \subset \mathrm{Hom}(V_{\lambda+k\mathbf{d}}, V^{\otimes n} \otimes V_\lambda),$$

is also one-dimensional. We define $L_{\underline{u}}$ to be this line. In other words, $L_{\underline{u}}$ is the subspace of $\mathrm{Hom}(V_{\lambda+k\mathbf{d}}, V^{\otimes n} \otimes V_\lambda)$, consisting of vectors whose component in each tensor subfactor $V^{\otimes k} \otimes V_\lambda$ has isomorphism type u_k.

By construction, we have an isomorphism of vector spaces,

$$W_\lambda^n \cong \bigoplus_{\underline{u} \in \mathcal{W}_\lambda^{N,k}} L_{\underline{u}}.$$

Theorem 7.6. Let $G = GL_N$ or SL_N. For any $v \in L_{\underline{u}}$ we have

$$\mathsf{Y}_i v = \mathsf{q}^{\langle u_i + 2\rho, u_i \rangle - \langle u_{i-1} + 2\rho, u_{i-1} \rangle - \langle \epsilon_1 + 2\rho, \epsilon_1 \rangle} v. \tag{7.2}$$

$\square$

Proof. This is essentially the content of [19, Proposition 3.6]. We recall the proof for the reader here, in our conventions. By its construction, $\mathsf{Y}_k = \mathsf{T}_{k-1} \cdots \mathsf{T}_1 \mathsf{Y}_1 \mathsf{T}_1 \cdots \mathsf{T}_{k-1}$ is the double-braiding of V around $V^{\otimes k-1} \otimes M$. Hence, applying equation (3.3) we have:

$$\mathsf{Y}_k = \mathrm{Id}_{V^{\otimes n-k}} \otimes \left(\Delta^{(k+1)}(\nu) \cdot (\nu^{-1} \otimes \Delta^{(k)}(\nu^{-1})) \right) \Big|_{V^{\otimes k} \otimes M}.$$

By definition of the line $L_{\underline{u}}$, and equation (3.4), we have:

$$\left(\mathrm{Id}_{V^{\otimes n-k}} \otimes \Delta^{(k+1)}(\nu) \right) \Big|_{L_{\underline{u}}} = \mathsf{q}^{\langle u_i + 2\rho, u_i \rangle},$$

$$\left(\mathrm{Id}_{V^{\otimes n-k}} \otimes \left(\nu^{-1} \otimes \Delta^{(k)}(\nu^{-1}) \right) \right) \Big|_{L_{\underline{u}}} = \mathsf{q}^{-\langle \epsilon_1 + 2\rho, \epsilon_1 \rangle - \langle u_{i-1} + 2\rho, u_{i-1} \rangle}.$$

Hence we compute that Y_i scales the line $L_{\underline{u}}$ as claimed. $\blacksquare$

Remark 7.7. Recall that the bijection $\mathcal{T}ab$ from Section 2.4 identifies looped walks at λ with skew tableaux in the diagram $\mathrm{D}_\lambda^{N,k}$. The diagram $\mathrm{D}_\lambda^{N,k}$ is natural in light of Theorem 7.6: tensoring $L_{\underline{u}}$ with V_{λ^*} corresponds to filling in the lower right corner of the diagram, thus producing an invariant (in the SL case), or a multiple of the determinant (in the GL case), within $V^{\otimes n} \otimes V_{\lambda^*} \otimes V_\lambda$. $\square$

7.4   From skew tableaux to periodic tableaux

While the natural basis of lines coming from Lie theory is indexed by looped walks of length n, and hence skew tableaux of size n, the weight basis for simple and $\mathcal{Y}$-semisimple modules for $\mathbb{H}_{q,t}(GL_n)$-modules is indexed by periodic tableaux on infinite skew diagrams. For simple and $\mathcal{Z}$-semisimple $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$-modules the weight basis is indexed by shift-equivalence classes of periodic tableaux. In this section, we construct a bijection between the two bases.

Recall from Section 2.4 the definition of $\mathrm{D}_\lambda^{N,k}$ and $\mathcal{SK}_\lambda^{N,k}$. We first observe that $\mathrm{D}_\lambda^{N,k}$ is also a fundamental domain of the periodization of $\mu = (k^N)$, i.e. of the $N \times \infty$ strip (see Remark 6.9). Similarly, "periodizing" a standard skew tableau in $\mathcal{T} \in \mathcal{SK}_\lambda^{N,k}$ (i.e., filling in the rest of the entries according to the periodicity constraint) yields a well-defined standard periodic tableau in $\mathrm{P}_n\mathrm{SYT}(k^N)$, as soon as we specify the compatibility with the diagonal labelling. See Figure 14. We formalize this as follows:

Definition 7.8. The *periodization* maps,

$$\mathcal{P}er : \bigsqcup_{\lambda \in \Lambda^+_{\mathfrak{gl}_N}} \mathcal{SK}^{N,k}_\lambda \to \mathrm{P}_n\mathrm{SYT}(k^N), \qquad\qquad \overline{\mathcal{P}er} : \bigsqcup_{\lambda \in \Lambda^+_{\mathfrak{sl}_N}} \mathcal{SK}^{N,k}_\lambda \to \mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n, \qquad (7.3)$$

send $\mathcal{T}$ to the unique periodic tableau in $\mathrm{P}_n\mathrm{SYT}(k^N)$ (resp. $\mathrm{P}_n\mathrm{SYT}(k^N)/\pi^n$) agreeing with $\mathcal{T}$ in the fundamental domain of shape $\mathrm{D}^{N,k}_\lambda$ located along the $N \times \infty$ strip so that diagonal labels coincide. $\square$

Note that as the filling of $\mathcal{T}$ is $\{1, \dots, n\}$ it is easy to see its periodization is standard. While it's clear then that $\mathcal{P}er$ is well-defined, we need the following for $\overline{\mathcal{P}er}$.

Proposition 7.9. The map $\overline{\mathcal{P}er}$ is well-defined. $\square$

Proof. Let $\lambda = \sum_i m_i \varepsilon_i \in \Lambda_{\mathfrak{sl}_N}$ and consider $\mathcal{T} \in \mathcal{SK}^{N,k}_\lambda$. Recall its principal diagonal is labeled $-\frac{|\lambda|}{N}$. We need to check this agrees with the diagonal labels assigned by (6.7) once we extend $\mathcal{T}$ periodically to the $N \times \infty$ strip, see Remark 6.9.

In other words we need to check the $N \times k$ rectangle whose NW diagonal agrees with the principal diagonal of $\mathcal{T}$ has filling sum $\mathfrak{s} - |\lambda| n$. Consider the k entries in the first row of $\mathcal{T}$. They lie in $\{1, \dots, n\}$ and hence contribute to the sum $\mathfrak{s}$; they also sit on diagonals labeled $m_1 - m_N - \frac{|\lambda|}{N}, \dots, m_1 - m_N + k - 1 - \frac{|\lambda|}{N}$. Likewise after periodizing, the k entries on diagonals labeled

$$-kr + m_1 - m_N - \frac{|\lambda|}{N}, \dots, -kr + m_1 - m_N + k - 1 - \frac{|\lambda|}{N}$$

lie in the set $\{-nr + 1, \dots, -nr + n\}$ and so contribute toward sum $\mathfrak{s} - krn$. However the above makes sense not only for $r \in \mathbb{Z}$ but in the case $kr \in \mathbb{Z}$. The entries in the first row that start on principal diagonal labeled $-\frac{|\lambda|}{N}$ correspond to left shift by $r = \frac{m_1 - m_N}{k}$. Considering now all N rows, the filling sum of this rectangle is $\mathfrak{s} - \sum_i k(\frac{m_i - m_N}{k})n = \mathfrak{s} - |\lambda| n$, and so our $p = -|\lambda|$ as in (6.7), as desired. $\blacksquare$

Proposition 7.10. The maps $\mathcal{P}er$ and $\overline{\mathcal{P}er}$ are bijections. $\square$

Proof. The inverse map $\mathcal{P}er^{-1}$ can be described as follows. Given $R \in \mathrm{P}_n\mathrm{SYT}(k^N)$ the boxes filled with $\{1, 2, \dots, n\}$ form a skew diagram $\mathcal{T}$. Since the box k steps to the right of $\boxed{i}$ is $\boxed{i+n}$, $\mathcal{T}$ has shape $(\gamma + (k^N))/\gamma$ for some partition with $\gamma_N = 0$. The principal diagonal of γ has some diagonal label r. Then set $\lambda = (\sum_i \gamma_i \epsilon_i) + r\mathbf{d}$. We may consider this diagonal-labeled skew diagram $\mathcal{T}$ to be in $\mathcal{SK}^{N,k}_\lambda$.

We do similarly for $\overline{\mathcal{P}er}^{-1}$, setting $\lambda = \sum_{i=1}^{N-1} \gamma_i \varepsilon_i$. Having already checked the diagonal labels for $\overline{\mathcal{P}er}$ are well-defined, the principal diagonal will already have inherited label $-\frac{|\lambda|}{N}$.
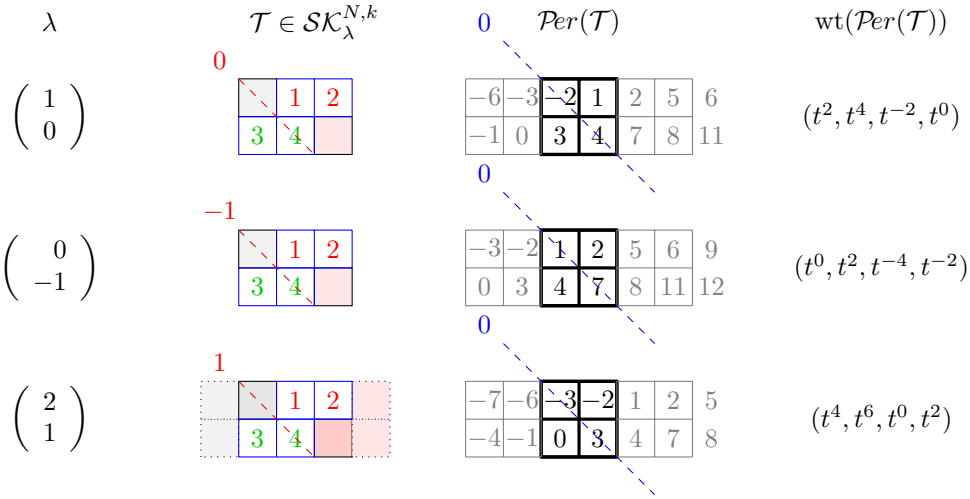
$\blacksquare$

Fig. 14. Here $G = GL_2, k = 2$. The fundamental rectangle of $\mathcal{P}er(\mathcal{T})$ is chosen so that the 0th diagonal matches that of $\mathcal{T} \in \mathcal{SK}_\lambda^{N,k}$.

For an illustration of $\mathcal{P}er$ in the GL case see Figure 14. In that example, note the skew tableaux are only differentiated by their diagonal labels and likewise for the periodic tableaux. For an example of $\overline{\mathcal{P}er}$ in the SL case, compare Figure 2 to Figure 12.

7.5 The isomorphism type of $F_n^G(\mathcal{O}_{\mathsf{q}}(G))$

We are finally ready to prove our main theorem:

Theorem 7.11. Let $\underline{u}$ be a looped walk in $\Lambda_{\mathfrak{gl}_N}^+$, respectively $\Lambda_{\mathfrak{sl}_N}^+$.

1. Each subspace $L_{\underline{u}}$ is moreover a simultaneous $\mathcal{Y}$-$\mathcal{Z}$-weight vector; for any $v \in L_{\underline{u}}$, we have:

$$ Y_i v = t^{2\mathrm{diag}_{\mathcal{T}ab(\underline{u})}(i)} v, \qquad Z_i v = t^{2\mathrm{diag}_{\overline{\mathcal{T}ab}(\underline{u})}(i)} v. $$

2. We have isomorphisms of $\mathbb{H}_{q,t}(GL_n)$-modules, and $\mathbb{H}_{\boldsymbol{q},t}(SL_n)$-modules, respectively:

$$ F_n^{GL}(\mathcal{O}_q(GL_N)) \cong L(k^N), \qquad F_n^{SL}(\mathcal{O}_q(SL_N)) \cong \overline{L}(k^N). $$

$\square$

Proof. For GL_N, all that remains is to simplify the exponent of q appearing in Theorem 7.6. We compute first for GL_N:

$$\langle u_i + 2\rho, u_i \rangle - \langle u_{i-1} + 2\rho, u_{i-1} \rangle - \langle \epsilon_1 + 2\rho, \epsilon_1 \rangle$$

$$= \langle u_{i-1} + \epsilon_{\delta_i(\underline{u})} + 2\rho, u_{i-1} + \epsilon_{\delta_i(\underline{u})} \rangle - \langle u_{i-1} + 2\rho, u_{i-1} \rangle - \langle \epsilon_1 + 2\rho, \epsilon_1 \rangle$$

$$= \langle u_{i-1}, \epsilon_{\delta_i(\underline{u})} \rangle + \langle 2\rho, \epsilon_{\delta_i(\underline{u})} \rangle + \langle \epsilon_{\delta_i(\underline{u})}, u_{i-1} \rangle + \langle \epsilon_{\delta_i(\underline{u})}, \epsilon_{\delta_i(\underline{u})} \rangle - \langle \epsilon_1, \epsilon_1 \rangle - \langle 2\rho, \epsilon_1 \rangle$$

$$= 2 \left(\langle u_{i-1}, \epsilon_{\delta_i(\underline{u})} \rangle + \langle \rho, \epsilon_{\delta_i(\underline{u})} \rangle - \langle \rho, \epsilon_1 \rangle \right)$$

$$= 2(\text{the label of the diagonal on which } \boxed{i} \text{ lies in } \mathcal{T}ab(\underline{u})), \text{ by } (2.3)$$

$$= 2 \operatorname{diag}_{\mathcal{P}er(\mathcal{T}ab(\underline{u}))}(i).$$

Hence we may re-write the scalar (7.2) by which $Y_i = \phi(\mathsf{Y}_i)$ acts, as

$$\mathsf{q}^{2\operatorname{diag}_{\mathcal{T}ab(\underline{u})}(i)} = t^{2\operatorname{diag}_{\mathcal{T}ab(\underline{u})}(i)}.$$

Having matched their support, the isomorphism $F_n^{GL}(\mathcal{O}_{\mathsf{q}}(GL_N)) \cong L(k^N)$ then follows from Theorem 6.10 and Corollary 5.6.

In the case $G = SL_N$, a similar computation gives:

$$\langle u_i + 2\rho, u_i \rangle - \langle u_{i-1} + 2\rho, u_{i-1} \rangle - \langle \varepsilon_1 + 2\rho, \varepsilon_1 \rangle$$

$$= 2 \left(\langle u_{i-1}, \varepsilon_{\delta_i(\underline{u})} \rangle + \langle \rho, \varepsilon_{\delta_i(\underline{u})} \rangle - \langle \rho, \varepsilon_1 \rangle \right)$$

$$= 2 \left(m_{\delta_i(\underline{u})} - m_N - \frac{|u_{i-1}|}{N} - \delta_i(\underline{u}) + 1 \right)$$

$$= 2 \left(-\frac{|u_0| + i - 1}{N} + m_{\delta_i(\underline{u})} - m_N + 1 - \delta_i(\underline{u}) \right)$$

$$= 2\frac{1-i}{N} + 2(\text{the label of the diagonal on which } \boxed{i} \text{ lies in } \overline{\mathcal{T}ab}(\underline{u}))$$

$$= 2\frac{1-i}{N} + 2\operatorname{diag}_{\overline{\mathcal{P}er}(\overline{\mathcal{T}ab}(\underline{u}))}(i).$$

The second equality follows from (2.6), where we write $u_{i-1} = \sum_{j=1}^N m_j \varepsilon_j$. Since $m_j - m_N$ is the length of the jth row of u_{i-1}, $\overline{\mathcal{T}ab}(\underline{u})$ places $\boxed{i}$ in column $m_{\delta_i(\underline{u})} - m_N + 1$ and row $\delta_i(\underline{u})$. Recall from Section 2 that the principal diagonal of $\overline{\mathcal{T}ab}(\underline{u})$ is labeled $-\frac{|u_0|}{N}$.

Hence we may re-write the scalar (7.2) by which $Z_i = \phi(\boldsymbol{q}^{2(i-1)}\mathsf{Y}_i)$ acts, as

$$\boldsymbol{q}^{2(i-1)} \mathsf{q}^{2\frac{1-i}{N}} \mathsf{q}^{2\operatorname{diag}_{\overline{\mathcal{P}er}\overline{\mathcal{T}ab}(u)}(i)} = (\mathsf{q}^{\frac{1}{N}})^{2(i-1)} \mathsf{q}^{2\frac{1-i}{N}} \mathsf{q}^{2\operatorname{diag}_{\overline{\mathcal{P}er}\overline{\mathcal{T}ab}(u)}(i)} = t^{2\operatorname{diag}_{\overline{\mathcal{P}er}\overline{\mathcal{T}ab}(u)}(i)}$$

since we have specialized $t = \mathsf{q}, \boldsymbol{q} = \mathsf{q}^{1/N}$.

This completely determines the support of $F_n^{SL}(\mathcal{O}_q(SL_N))$. Comparing with Theorems 6.13 and 6.14 we see that:

$$F_n^{SL}(\mathcal{O}_q(SL_N)) \cong \overline{L}(k^N)^a,$$

for some $a \in \mathcal{K}^\times$ with $a^n = 1$.

In order to determine the twist parameter a, it suffices to consider a weight vector whose weight $\mathrm{wt}(\overline{R}_0)$ is stabilized by π^N: the constant by which π^N scales such a vector is a^N, and this determines the isomorphism type of the twist as in Theorem 6.14. Hence, in order to conclude that $a = 1$, it suffices to check that $\pi^N v_{\overline{R}_0} = v_{\overline{R}_0}$ with $\overline{R}_0$ as in Figure 13.

We note that $\overline{R}_0 = \overline{\mathcal{P}er}(\overline{\mathcal{T}ab}(\underline{u}_0))$ where $\underline{u}_0$ is the looped walk at $\lambda = 0$ whose steps satisfy $\delta_i(\underline{u}_0) = i \bmod N$. In other words, $\underline{u}_0$ is formed from the unique N-step looped walk $\underline{u}$ at $\lambda = 0$, concatenated with itself k times. We have:

$$v_{\overline{R}_0} \in L_{\underline{u}_0} = \underbrace{W_0^N \otimes \cdots \otimes W_0^N \otimes W_0^N}_{k} \subseteq W_0^{kN} = W_0^n,$$

where we recall that W_0^N denotes the space $\mathrm{Hom}(\mathbf{1}, V^{\otimes N})$ of invariants in V. It was shown in [15] that $\mathsf{X}_1 \cdots \mathsf{X}_N|_{W_0^N} = (\nu^{-1}|_V)^N \mathsf{q}^{N(\frac{1}{N} - N)}$, hence

$$\mathsf{X}_1 \cdots \mathsf{X}_N v_{\overline{R}_0} = \mathsf{q}^{N(\frac{1}{N} - N)} v_{\overline{R}_0}.$$

We observe that $\phi(\boldsymbol{r}^{-N} \mathsf{X}_1 \cdots \mathsf{X}_N \mathsf{T}_w) = \pi^N$, where w is the length $N^2(k-1)$ permutation obtained by raising an n-cycle to the Nth power. This sends

$$\underset{k}{W_0^N} \otimes \underset{k-1}{W_0^N} \otimes \cdots \otimes \underset{1}{W_0^N} \qquad \text{to} \qquad \underset{1}{W_0^N} \otimes \underset{k}{W_0^N} \otimes \cdots \otimes \underset{2}{W_0^N},$$

and in fact T_w acts trivially on $v_{\overline{R}_0}$. This computation is also consistent with $\mathsf{q}^{N(k-1)} \mathsf{q}^{(-\frac{1}{N})\ell(w)} = 1$ in the GL setting where the $(k-1)$ crossings of det contribute the first term.

Hence as we are taking $\boldsymbol{r} = \boldsymbol{q}^{1-N^2} = \mathsf{q}^{1/N - N}$, we have

$$\pi^N v_{\overline{R}_0} = \boldsymbol{r}^{-N} \mathsf{q}^{N(\frac{1}{N} - N)} v_{\overline{R}_0} = \mathsf{q}^{-N(1/N - N)} \mathsf{q}^{N(1/N - N)} v_{\overline{R}_0} = v_{\overline{R}_0}.$$

$\blacksquare$

As a consequence of Theorem 7.11, we can give a combinatorial description of the action of π on the weight basis of $F_n^G(\mathcal{O}_{\mathsf{q}}(G))$, as follows.

Definition 7.12. Given a looped walk $\underline{u}$ of length n, we may construct a new looped walk $\varpi(\underline{u})$, by setting

$$\delta_i(\varpi(\underline{u})) = \delta_{i-1}(\underline{u}) \quad (2 \leq i \leq n), \qquad \delta_1(\varpi(\underline{u})) = \delta_n(\underline{u}),$$

$$\varpi(\underline{u})_i = \underline{u}_{i-1} \quad (1 \leq i \leq n), \qquad \varpi(\underline{u})_0 = \varpi(\underline{u})_1 - \epsilon_{\delta_n(\underline{u})}.$$

$\square$

In other words, $\varpi(\underline{u})$ is the looped walk ending at $\underline{u}_{n-1}$ instead of $\underline{u}_n$.

We note that $\varpi^n(\underline{u}) = \underline{u}$ when $\underline{u}$ is a looped walk of length n in $\Lambda^+_{\mathfrak{sl}_N}$, while in $\Lambda^+_{\mathfrak{gl}_N}$ we have $\varpi^n(\underline{u})_i = \underline{u}_i - k\mathbf{d}$.

Corollary 7.13. We have $\pi(W^n_\lambda) \subset \bigoplus_i W^n_{\lambda+\epsilon_i}$, and moreover

$$\pi L_{\underline{u}} = L_{\varpi(\underline{u})}.$$

$\square$

While the corollary follows immediately from $\mathcal{Y}$-$\mathcal{Z}$-semisimplicity of $F^G_n(\mathcal{O}_\mathsf{q}(G))$ and the results of Section 5, it would not be straightforward to prove in purely Lie theoretic terms. One can also directly deduce the action of the X_i on the module combinatorially, in a similar fashion.

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