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SEMI-UNIFORM DOMAINS AND THE A_∞ PROPERTY FOR HARMONIC MEASURE

JONAS AZZAM

ABSTRACT. We study the properties of harmonic measure in semi-uniform domains. Aikawa and Hirata showed in [AH08] that, for John domains satisfying the capacity density condition (CDC), the doubling property for harmonic measure is equivalent to the domain being semi-uniform. Our first result removes the John condition by showing that any domain satisfying the CDC whose harmonic measure is doubling is semi-uniform. Next, we develop a substitute for some classical estimates on harmonic measure in nontangentially accessible domains that works in semi-uniform domains.

We also show that semi-uniform domains with uniformly rectifiable boundary have big pieces of chord-arc subdomains. We cannot hope for big pieces of Lipschitz subdomains (as was shown for chord-arc domains by David and Jerison [DJ90]) due to an example of Hrycak, which we review in the appendix.

Finally, we combine these tools to study the A_∞ -property of harmonic measure. For a domain with Ahlfors-David regular boundary, it was shown by Hofmann and Martell that the A_∞ property of harmonic measure implies uniform rectifiability of the boundary [HM15, HLMN17]. Since A_∞ -weights are doubling, this also implies the domain is semi-uniform. Our final result shows that these two properties, semi-uniformity and uniformly rectifiable boundary, also imply the A_∞ property for harmonic measure, thus classifying geometrically all domains for which this holds.

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1. INTRODUCTION

In this paper we study a few connections between the geometry of a domain $\Omega \subseteq \mathbb{R}^{d+1}$ and the behavior of its harmonic measure ω_Ω^x with pole $x \in \Omega$. Our motivation is to obtain a characterization of the A_∞ -property for harmonic measure on the boundary, however this first requires understanding how the connectivity properties of a domain relate to the doubling properties of harmonic measure. Below we define a few common notions of connectivity that are studied in this context.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an open set.

- (1) For $x, y \in \overline{\Omega}$, we say a curve $\gamma \subseteq \overline{\Omega}$ is a **C -cigar curve** from x to y if $\min\{\ell(x, z), \ell(y, z)\} \leq C \operatorname{dist}(z, \Omega^c)$ for all $z \in \gamma$, where $\ell(a, b)$ denotes the length of the sub-arc in γ between a and b . We will also say it has **bounded turning** if $\ell(\gamma) \leq C|x - y|$.
- (2) If there is $x \in \Omega$ such that every $y \in \Omega$ is connected to x by a curve γ so that $\ell(y, z) \leq C \operatorname{dist}(z, \Omega^c)$ for all $z \in \Gamma$, we say Ω is **C -John**.
- (3) If every pair $x \in \Omega$ and $\xi \in \partial\Omega$ are connected by a C -cigar with bounded turning, then we say Ω is **C -semi-uniform (SU)**.
- (4) If every $x, y \in \Omega$ are connected by a C -cigar of bounded turning, we say Ω is **uniform**.
- (5) For a ball B of radius r_B centered on $\partial\Omega$, we say $x \in B$ is an **interior/exterior c -corkscrew point** for Ω if $B(x, 2cr_B) \subseteq B \cap \Omega$ (or $B(x, 2cr_B) \subseteq B \setminus \Omega$). We say Ω satisfies the **interior c -Corkscrew condition** if every ball B on $\partial\Omega$ has a interior (or exterior) c -corkscrew point.
- (6) A uniform domain with exterior corkscrews is **nontangentially accessible (NTA)**.

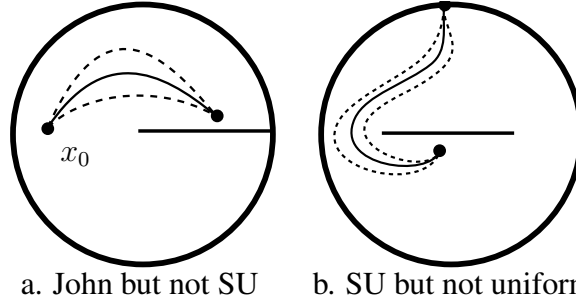
Note that a John domain is necessarily bounded. As mentioned in [AH08], these domains have the following containments:

$$\text{NTA} \subsetneq \text{Uniform} \subsetneq \text{Semi-uniform} \subsetneq \text{John}$$

where the last containment is only true for bounded domains, and each of these containments can be strict. For example, the complement of a 4-corner cantor set in \mathbb{R}^2 is uniform but not NTA. If we set $\Omega = \{(x, y) \in$

$\mathbb{R}^d \times \mathbb{R} : y \neq 0 \text{ or } |x| < 1$, then this is semi-uniform but not uniform. A bounded example of a non-uniform semi-uniform domain is $\Omega = B(0, 1) \setminus [-1/2, 1/2] \subseteq \mathbb{C}$, see Figure 1.b. Note that each point along the segment is easily accessible from any other point in the domain by a curve of bounded turning, but points close to the segment and on opposite sides are not. If $\Omega = \mathbb{B} \setminus [0, 1] \subseteq \mathbb{C}$, then Ω is John but not semi-uniform, as points in the bottom corner do not easily access boundary points above and near the top corner, see Figure 1.a.

Uniform domains were introduced independently by Martio and Sarvas [MS79] and by Jones [Jon80]. To our knowledge, semi-uniform domains were first defined (and only mentioned) by Aikawa and Hirata in [AH08].



a. John but not SU b. SU but not uniform.

FIGURE 1. The domain on the left is John but not SU since points on the boundary just above the slit are not accessible by short curves from points in the domain directly below the slit. The figure on the right is SU since every point on the boundary is accessible from any point in the domain by a cigar curve of bounded turning. Not every pair of points in the domain are connected in this way, since there are no short curves connecting points close to but on opposite sides of the slit, and so it is not uniform.

Jerison and Kenig introduced NTA domains in [JK82, Lemma 4.9] since they were domains which could have very rough or non-smooth boundaries and yet their harmonic measures still had nice properties such as the doubling property: If $\Omega \subseteq \mathbb{C}$ is a John domain, then harmonic measure is doubling, meaning there is $C > 0$ (depending on $x \in \Omega$) so that for all balls B centered on $\partial\Omega$, $\omega_\Omega^x(2B) \leq C\omega_\Omega^x(B)$ (c.f. [GM08, Exercise VII.13(e)]). Because of this and other nice scale invariant properties, these domains have become ubiquitous in the literature on harmonic and elliptic measure (for example, [Wu86, DJ90, KT06, KPT09]). We will discuss more of these properties below.

For some of these properties, the full NTA condition is not needed. For example, Aikawa and Hirata showed that, in the case of John domains satisfying the capacity density condition, doubling is in fact equivalent to semi-uniformity.

Theorem 1.2. [AH08] *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a John domain with the CDC. Then the following are equivalent:*

(1) *There are $A, A_0 > 0$ so that*

$$(1.1) \quad \omega^x(2B) \leq A\omega^x(B) \text{ for all } x \in \Omega \setminus A_0B \text{ and } B \text{ centered on } \partial\Omega.$$

(2) *Ω is semi-uniform*

Another appealing property of NTA domains is the Carleson estimate.

Definition 1.3. A corkscrew domain $\Omega \subseteq \mathbb{R}^{d+1}$ has the **Carleson estimate (CE)** if, for any B centered on $\partial\Omega$ and u is a non-negative harmonic function on $2B \cap \Omega$ vanishing on $2B \cap \partial\Omega$, if $y \in B \cap \Omega$ has $\text{dist}(y, \Omega^c) \geq \varepsilon > 0$, then

$$\sup_{B \cap \Omega} u \lesssim_\varepsilon u(y).$$

We say that Ω has the **lesser Carleson estimate (LCE)** if there is a corkscrew point y (depending on u) for which the above inequality holds.

It was first shown in [JK82] that the CE holds in NTA domains. Later, Aikawa showed in [Aik04, Theorem 1.2] and [Aik08, Corollary 2] that, for CDC John domains, the CE condition is equivalent to Ω being uniform.

Also proved in [JK82] is the fact that for NTA domains Ω , if B is centered on $\partial\Omega$ and $E \subseteq B$ is Borel, then there is $M > 0$ so that

$$(1.2) \quad \frac{\omega_\Omega^x(E)}{\omega_\Omega^x(B)} \sim \omega_\Omega^{x_B}(E) \text{ for all } x \in \Omega \setminus MB.$$

Additionally, (1.2) was extended to uniform domains without exterior corkscrews. In fact, Aikawa showed that a crucial ingredient (the boundary Harnack principle, another nice scale invariant property) holds in any uniform domain [Aik01], and for uniform domains with the capacity density condition, the proof in [JK82] carries over immediately. Later, Mouroglou and Tolsa showed that (1.2) held in any uniform domain [MT15] if we also divide the right hand side by $\omega_\Omega^{x_B}(B)$.

These results show that the exterior corkscrew property was not necessary to achieve the same nice estimates as NTA domains, but only some nice connectivity like uniformity and some nondegeneracy in the boundary like the capacity density condition. For these reasons, uniform domains with ADR boundaries are in some sense the new NTA domains and are often studied, see [HM14],[HMUT14], [ABHM17], [AHM⁺17], [TZ17], and

[HMM⁺17].

Our first objective is to extend some techniques and results that are standard for NTA and uniform domains to semi-uniform domains as Aikawa and Hirata did for doubling measures. Initially, we wanted to prove Theorem 1.2 without the John condition, but this is quite difficult. It seems that to get nice connectivity properties, we need to assume that the doubling condition (1.1) also holds for balls that contain x . However, we can't have (1.1) hold for all $x \in \Omega$, even when Ω is very nice. If B is a ball on the boundary, then as $x \in \Omega$ approaches a point in $2B \cap \Omega \setminus \overline{B}$, $\omega_\Omega^x(2B) \rightarrow 1$ whilst $\omega_\Omega^x(B) \rightarrow 0$.

To avoid this issue, we don't have to require that x remain outside some large ball A_0B , but that it stay away from the boundary of $\partial\Omega$ inside that ball.

Hence, in this paper, we will say harmonic measure is **doubling** if there is a constant $A \geq 2$ and a function $C : (0, \infty) \rightarrow (1, \infty)$ so that, for any ball B centered on $\partial\Omega$ and $\alpha > 0$,

$$(1.3) \quad \omega_\Omega^x(2B) \leq C(\alpha)\omega_\Omega^x(B) \text{ for all } x \text{ such that } \text{dist}(x, AB \cap \partial\Omega) \geq \alpha r_B.$$

The work of Jerison and Kenig actually implies this stronger form of doubling, see [JK82, Lemma 4.9].

Our first result removes the John condition from Theorem 1.2 using this definition of doubling.

Theorem I. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a CDC domain. Then the following are equivalent:

- (1) ω_Ω is doubling.
- (2) Ω is semi-uniform.

Our second main result is a substitute for (1.2) for semi-uniform domains.

Theorem II. Let Ω be a semi-uniform CDC domain, B a ball centered on $\partial\Omega$ with $r_B < \text{diam } \partial\Omega$, and $E \subseteq B \cap \partial\Omega$. Then there is $M > 0$ depending on the CDC and semi-uniformity constants and corkscrew points $x_1, \dots, x_n \in B \cap \Omega$ with n depending on the semi-uniformity constants so that

$$\min_{i=1, \dots, n} \omega_\Omega^{x_i}(E) \lesssim \frac{\omega_\Omega^x(E)}{\omega_\Omega^x(B)} \lesssim \max_{i=1, \dots, n} \omega_\Omega^{x_i}(E) \text{ for all } x \in \Omega \setminus MB.$$

For semi-uniform CDC domains, one can show that the LCE holds, and the proof is more or less the same as that in [JK82] (see Lemma 4.6 below).

We now turn to the second objective of this paper (which was also the motivation for proving the previous two theorems), which concerns the relationship between the absolute continuity properties of harmonic measure and the geometry of the domain. Very recently, it was shown in [AHM⁺16] that for $\Omega \subseteq \mathbb{R}^{d+1}$, if $E \subseteq \partial\Omega$ (with $\mathcal{H}^d(E) < \infty$ if $d > 1$) and $\omega_\Omega|_E \ll \mathcal{H}^d|_E$, then E can be covered by d -dimensional Lipschitz graphs up to harmonic measure zero. This generalizes a result of Pommerenke who showed the same result for simply connected planar domains [Pom86].

Being covered by Lipschitz graphs is not enough to be absolutely continuous with respect to \mathcal{H}^d , however: the complement of the $\frac{2}{3}$ -Cantor set as a subset of \mathbb{R}^2 is one example, so some extra assumptions are needed. Bishop and Jones (generalizing work of McMillan [McM69]) showed that, for simply connected planar domain, absolute continuity occurs in the subset of any rectifiable curve [BJ90].

Higher dimensional versions of their work are false without some extra assumptions due to an example of Wu [Wu86]. We showed recently with Akman and Mourougolou that, under a lower d -regularity assumption on the complement of a domain $\Omega \subseteq \mathbb{R}^{d+1}$ (which is satisfied by many domains, including simply connected planar domains and NTA domains), $\omega_\Omega \ll \mathcal{H}^d$ on any subset of a Lipschitz graph (and in fact more general surfaces) [AAM16] (the techniques of which build off of previous more quantitative results, which we will describe shortly) which generalizes the work of Wu who assumed an exterior corkscrew condition [Wu86]. Combining this with the work of [AHM⁺16] classifies absolute continuity for this kind of domain.

In addition to knowing that harmonic measure and surface measure share the same null sets, one can also ask when this holds in a quantitative sense. Results of this nature typically assume some stronger properties about the surface measure on the boundary. The first example (which forms the foundation of all subsequent results establishing absolute continuity in higher dimensions) is the result of Dahlberg [Dah77] that harmonic measure for a Lipschitz domain is an A_2 -weight.

Recall that a set E is d -**Ahlfors-David regular (ADR)**, or just d -**regular**, if there is $C > 0$ such that

$$C^{-1}r^d \leq \mathcal{H}^d(B(x, r) \cap E) \leq Cr^d \text{ for all } x \in E \text{ and } 0 < r < \text{diam } E.$$

We will say that harmonic measure is A_∞ if, for all $\alpha > 0$, $\omega_\Omega^x \in A_\infty(\partial\Omega \cap B, \mathcal{H}^d)$ for any ball B centered on $\partial\Omega$ and $x \in \Omega$ with $\text{dist}(x, AB) > \alpha r_B$. That is, for all $\delta > 0$ there is $\varepsilon > 0$ (also depending on α) so that if $E \subseteq \partial\Omega \cap B$ and

$$\mathcal{H}^d(E) < \varepsilon \mathcal{H}^d(B \cap \partial\Omega),$$

then

$$\omega_\Omega^x(E) < \delta \omega_\Omega^x(B).$$

Seeking out this form of absolute continuity has applications for PDEs: in [HL16], for example, Hofmann and Le showed that BMO solvability of the Dirichlet problem for the Laplacian is implied by the A_∞ -property (in fact, it is implied by the weak A_∞ property).

We say a domain is a **chord-arc domain (CAD)** if it is NTA with Ahlfors regular boundary. In the plane, this is equivalent to the boundary being a chord-arc (or bi-Lipschitz) curve. Lavrentiev showed in [Lav36] that, for chord-arc domains in the plane, harmonic measure is in A_∞ , and in fact, for Jordan domains with Ahlfors regular boundaries, the converse holds as well (for modern treatments of both these facts, see [GM08, Section VII.4]). Independently, David and Jerison [DJ90] and Semmes [Sem90] proved Lavrentiev's theorem for higher dimensions. The common thread to both proofs is to reduce things to Dahlberg's original result by approximating the domain from within by Lipschitz subdomains. In particular, in [DJ90] the authors first prove that a CAD has **big pieces of Lipschitz subdomains (BPLS)**: for every ball $B = B(x,r)$ $x \in \partial\Omega$ and $0 < r < \text{diam } \Omega$, there is a Lipschitz domain $\Omega_B \subseteq B \cap \Omega$ so that $\mathcal{H}^d(\partial\Omega_B \cap \partial\Omega) \geq cr^d$. Dahlberg's result shows that harmonic measure is an A_∞ -weight, and then using the maximum principle one can show that the A_∞ property for the original domain is inherited from these subdomains.

To date, these are the most general domains for which the A_∞ property (as we have defined it) has been proven to hold, and there is yet no result that says exactly for which domains it holds. The only exception are when assuming the domain is uniform with ADR boundary (see [HM14, HMUT14]), but even in this setting, the A_∞ property is actually equivalent to the domain being a CAD [AHM⁺17]. The most general kind of domain for which the A_∞ property holds that follows immediately from results in the literature (although isn't stated anywhere) are semi-uniform domains with BPLS: Bennewitz and Lewis showed in [BL04] that harmonic measure satisfies a weak-reverse Hölder inequality in corkscrew domains with BPLS; in semi-uniform domains, the corkscrew property is immediate, and because harmonic measure is doubling, harmonic measure actually satisfies the usual reverse Hölder inequality and hence our A_∞ condition by classical results (see [Ste93, Chapter 5]).

There are some necessary conditions our domain must satisfy for harmonic measure to be A_∞ . Firstly, Hofmann and Martell showed that the boundary is **uniformly rectifiable (UR)** [HM15]¹: $\partial\Omega$ is d -regular and

¹They actually show that the so-called weak- A_∞ property implies UR, although we will not discuss this class of measures.

there are $L, c > 0$ so that, for each ball $B(x, r)$ centered on $\partial\Omega$ with $0 < r < \delta\Omega$ there is an L -Lipschitz map $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ so that

$$\mathcal{H}^d(B(x, r) \cap \partial\Omega \cap f(\mathbb{R}^d)) \geq cr^d.$$

This paper is an Arxiv preprint, although later they extended this result to p -harmonic measures in a paper with Le and Nyström [HLMN17]. Mouroglou and Tolsa also developed a local result that works when harmonic measure is not doubling [MT15]. See also [HMT16] and [HMM⁺17] for the elliptic setting. Secondly, since A_∞ -weights are doubling, Theorem I implies that such domain must also be semi-uniform. Our third result confirms that these conditions are also sufficient, thus classifying the A_∞ property for harmonic measure.

Theorem III. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with d -regular boundary. Then the following are equivalent:

- (1) Ω is a semi-uniform domain with ADR and UR boundary.
- (2) Ω is a semi-uniform domain with ADR boundary and very big pieces of chord-arc subdomains (VBPCAS): for every ball B centered on $\partial\Omega$ and $\varepsilon > 0$, there is $\Omega_B \subseteq B \cap \Omega$ a CAD (with constants depending on the semi-uniformity, Ahlfors regularity, and on ε) so that

$$\mathcal{H}^d(\partial\Omega \cap B \setminus \partial\Omega_B) < \varepsilon \mathcal{H}^d(\partial\Omega \cap B).$$

- (3) Harmonic measure is A_∞ .

Recall that, combining the works of [HMUT14] and [AHM⁺17], for uniform domains with ADR boundary, the A_∞ property is equivalent to the boundary being UR and equivalent to the domain being CAD. This equivalence doesn't hold for semi-uniform domains, since the complement of a line segment clearly satisfies the conditions of Theorem III without being CAD.

[Addendum (February 21, 2018): Shortly after posting this paper to Arxiv, Hofmann and Martell posted another paper [HM17] where they showed that the weak A_∞ condition is implied by a so-called ‘‘weak local John condition.’’ This means that, for every $x \in \Omega$ there is a set $F \subseteq \partial\Omega \cap B(x, 2 \operatorname{dist}(x, \Omega^c))$ so that, for all $\xi \in F$, there is a path $\gamma \subseteq \Omega$ from x to ξ so that $\operatorname{dist}(z, \partial\Omega) \geq c|z - \xi|$ for all $z \in \gamma$. Of course, this is weaker than being semi-uniform, and also, if we assume semi-uniformity, then this result combined with Theorem II gives an alternate proof that (1) implies (3) in Theorem III. Indeed, since this implies weak A_∞ , we can obtain the local A_∞ property using doubling and then our global A_∞ condition by using Theorem II as in the proof of Theorem III in Section 7. However, the techniques and approach of Hofmann and Martell in their paper are quite different. They use the method of ‘‘extrapolation’’ of Carleson measures to

estimate harmonic measure directly, whereas we model our argument on that of [DJ90] by building nice chord-arc subdomains that carve out a large piece of the boundary (that is, the implication that (1) implies (2) in Theorem III, which is proven in Lemma 6.4 below) to prove a local A_∞ property, and then we use Theorem II to get the global A_∞ property.]

In light of Bennewitz and Lewis' result mentioned earlier, however, it would be natural to ask if condition (1) was also equivalent to being semi-uniform with ADR boundary and BPLS. This is certainly true for CADs, as shown by David and Jerison in [DJ90] (and this was crucial for their proof), so in the uniform setting, the A_∞ property implies BPLS. However, there are examples of semi-uniform domains with ADR and UR boundary that do not have BPLS. One example is the complement of Hrycak's example, a well-known (and unpublished) set constructed by Hrycak, often cited in the literature on uniform rectifiability to show that not all UR sets have big pieces of Lipschitz graphs. In the appendix, we show the following.

Proposition I. *If $E \subseteq \mathbb{R}^2$ is Hrycak's example, then E^c is a semi-uniform domain with UR boundary and does not have BPLS.*

As mentioned before, the implication (3) implies (1) in Theorem III follows by Theorem I and [HM15], so all we will show in this paper is the implication that (1) implies (2) and (2) implies (3). The proof of (1) implies (2) requires building chord-arc subdomains of Ω that carve out as much of the boundary as we please, and we know that harmonic measure is A_∞ in this domain by [DJ90]. We then use Theorem II and the maximum principle to prove A_∞ for our original measure. The work of Bortz and Hofmann [BH17] comes close to what we need by building a *union* of (possibly disjoint) chord-arc domains, and in essence what we do is show that these chord arc domains can be connected into one single CAD, although our construction in the end is quite different and uses some additional techniques in order to prove semi-uniformity.

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2. PRELIMINARIES

2.1. Notation. We will write $a \lesssim b$ if there is a constant $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant depends on the parameter t . As usual we write $a \sim b$ and $a \sim_t b$ to mean $a \lesssim b \lesssim a$ and $a \lesssim_t b \lesssim_t a$ respectively. We

will assume all implied constants depend on d and hence write \sim instead of \sim_d .

Whenever $A, B \subset \mathbb{R}^{d+1}$ we define

$$\text{dist}(A, B) = \inf\{|x - y|; x \in A, y \in B\}, \text{ and } \text{dist}(x, A) = \text{dist}(\{x\}, A).$$

Let $\text{diam } A$ denote the diameter of A defined as

$$\text{diam } A = \sup\{|x - y|; x, y \in A\}.$$

For a domain Ω and $x \in \Omega$, we will write

$$\delta_\Omega(x) = \text{dist}(x, \partial\Omega).$$

We let $B(x, r)$ denote the open ball centered at x of radius r . For a ball B , we will denote its radius by r_B .

2.2. Harnack Chains.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^{d+1}$. A *Harnack chain* is a (finite or infinite) sequence of balls $\{B_i\}_{i \in [a, b]}$ where $[a, b]$ denote the integers between a and b and a can be $-\infty$ and b can be $+\infty$, such that for all i ,

- (1) $B_i \cap B_{i+1} \neq \emptyset$ if $a \leq i < b$,
- (2) $2B_i \subseteq \Omega$, and
- (3) $r_{B_i} \sim \text{dist}(B_i, \partial\Omega)$.

The *length* of a Harnack chain is just the number of balls in the Harnack chain.

Note that if B_i is a Harnack chain,

$$r_{B_i} \sim r_{B_{i+1}}.$$

To verify that a domain is either uniform or semi-uniform, it will be more convenient to work with equivalent definitions in terms of Harnack chains.

Theorem 2.2. [AHM⁺17, Theorem 2.15] *A domain Ω is uniform if and only if it has interior corkscrews and there is a non-decreasing function $N : [1, \infty) \rightarrow [1, \infty)$ so that for all $x, y \in \Omega$, there is a Harnack chain from x to y in Ω of length $N(|x - y| / \min\{\delta_\Omega(x), \delta_\Omega(y)\})$.*

One can prove a similar Harnack chain version of semi-uniformity.

Theorem 2.3. *A domain $\Omega \subseteq \mathbb{R}^{d+1}$ is semi-uniform if and only if it has interior corkscrews and there is $c > 0$ and a non-decreasing function $N : [1, \infty) \rightarrow [1, \infty)$ so that, for all $x \in \Omega$, $\xi \in \partial\Omega$, and $0 < r < \text{diam } \partial\Omega$, there is a Harnack chain from x to a c -corkscrew point $y \in \Omega \cap B(\xi, r)$ of length $N(|x - y| / \min\{\delta_\Omega(x), r\})$.*

Remark 2.4. The condition that $r < \text{diam } \partial\Omega$ is important. Note that if $\Omega = \mathbb{R}^{d+1} \setminus (\partial\mathbb{B} \cup B(e_{d+1}, \varepsilon))$ where e_{d+1} is the $(d+1)$ st standard basis vector and ε is small, this is an unbounded semi-uniform domain. However, points in \mathbb{B} are not well connected to corkscrew points outside \mathbb{B} because of the small ε -hole, that is, the property stated in the previous theorem does not hold if we allow $r \gg \text{diam } \partial\Omega$. So in particular, the statement implies that a corkscrew ball in a ball B centered on the boundary can be connected down to a smaller corkscrew ball in $M^{-1}B$ with length depending on M for any $M \geq 1$, but it can only be connected up to a larger corkscrew ball in MB with length depending on M so long as $Mr_B < \text{diam } \partial\Omega$.

Proof. We only sketch the details. For the forward direction, one takes a cigar curve γ from ξ to x and then one can show that a Besicovitch subcover of $\{B(z, \delta_\Omega(z)/2) : z \in \gamma\}$ gives the desired Harnack chain. For the reverse direction, the proof of this is similar to the proof of [AHM⁺17, Theorem 2.15], but we will outline the initial steps.

Assume we have a function N satisfying the properties in the theorem. Let $x \in \Omega$ and $\xi \in \partial\Omega$ (and note that the ball $B(\xi, r)$ could be much larger than $B(\xi, |x - \xi|)$).

- (1) If $\delta_\Omega(x) \geq |\xi - x|/8$, let $B' = B(\xi, 2|x - \xi|)$. Then by iterating, we can find for each $i \geq 0$ corkscrew points $x_i \in 2^{-i}B'$ (with $x_0 = x$) and Harnack chain $B_1^i, \dots, B_{n_i}^i$ from x_i to x_{i+1} . If we connect the centers of all the Harnack chains in order by line segments (so we connect the centers of B_j^i to B_{j+1}^i and $B_{n_i}^i$ to B_1^{i+1}), one can show this is a cigar curve of bounded turning as in the proof of [AHM⁺17, Theorem 2.15].
- (2) If $\delta_\Omega(x) < |\xi - x|/8$, let $\zeta \in \partial\Omega$ be closest to x and $B'' = B(\zeta, 2|x - \zeta|)$. Let n be the largest integer for which $2^n r_{B''} < r_{B'}/4$. Since

$$2r_{B''} = 4|\zeta - x| < |\xi - x|/2 = r_{B'}/4,$$

we know $n > 0$. Then

$$r_{2^n B''} = 2^n r_{B''} < \frac{r_{B'}}{4} = \frac{|\xi - x|}{2} \leq \frac{|\xi - \zeta| + |\zeta - x|}{2} \leq \frac{\text{diam } \partial\Omega}{2} + \frac{r_{B''}}{4}$$

hence, since $n > 0$,

$$2^n r_{B''} < \frac{2^n}{2^n - 1/4} \frac{\text{diam } \partial\Omega}{2} < \text{diam } \partial\Omega.$$

Thus, we can apply the condition of the theorem to get that, for $0 \leq i \leq n$, there are corkscrew points $y_i \in 2^i B''$ and a bounded Harnack chain $\tilde{B}_1^i, \dots, \tilde{B}_{m_i}^i$ from y_i to y_{i+1} . If we connect the centers

of these balls in order we obtain a curve γ_1 . Note that

$$\delta_\Omega(y_n) \gtrsim 2^n r_{B''} \sim r_{B'} \sim |x - \xi|$$

and so just as in the previous case, we can find a curve γ_2 connecting the centers of an infinite Harnack chain from y_n to ξ . The union of these two curves γ can be shown as in the proof of [AHM⁺17, Theorem 2.15] to be cigar curves of bounded turning. \square

2.3. Background on Harmonic Measure. For background on harmonic measure and Green's function, we refer the reader to [AG01].

Definition 2.5. For $K \subset \partial\Omega$, we say that Ω has the *capacity density condition (CDC) in K* if $\text{cap}(B(x, r) \cap \Omega^c, B(x, 2r)) \gtrsim r^{d-1}$, for every $x \in K$ and $r < \text{diam } K$, and that Ω has the *capacity density condition* if it has the CDC in $K = \partial\Omega$. Here, $\text{cap}(\cdot, \cdot)$ stands for the variational 2-capacity of the condenser (\cdot, \cdot) (see [HKM06, p. 27] for the definition).

Remark 2.6. This is the traditional definition of CDC, but it also has a geometric formulation. By the main result of [Leh08] (also see [Leh08, Equation (8)]), $\Omega \subseteq \mathbb{R}^{d+1}$ satisfies the CDC (or is *uniformly 2-fat* in that paper's argot) if there are $c > 0$ and $s > d - 1$ so that

$$\mathcal{H}_\infty^s(B(\xi, r) \setminus \Omega) \geq cr^s \text{ for all } x \in \Omega \text{ and } r > 0.$$

Below, when we talk about the *CDC constants*, we will in fact refer to the constants s and c here.

Lemma 2.7 ([HKM06, Lemma 11.21]). *Let $\Omega \subset \mathbb{R}^{d+1}$ be any domain satisfying the CDC condition, B a ball centered on $\partial\Omega$ so that $\Omega \setminus 2B \neq \emptyset$. Then*

$$(2.1) \quad \omega_\Omega^x(2B) \geq c > 0 \text{ for all } x \in \Omega \cap B.$$

where c depends on d and the constant in the CDC.

Using the previous lemma and iterating, it is possible to obtain the following lemma.

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with the CDC, $\xi \in \partial\Omega$ and $0 < r < \text{diam } \partial\Omega/2$. Suppose u is a non-negative function that is harmonic in $B(\xi, r) \cap \Omega$ and vanishes continuously on $\partial\Omega \cap B(\xi, r)$. Then*

$$(2.2) \quad u(x) \lesssim \left(\sup_{y \in B(\xi, r) \cap \Omega} u \right) \left(\frac{|x - \xi|}{r} \right)^\alpha$$

where $\alpha > 0$ depends on the CDC constant and d .

There are two key facts we will use about Green's function.

Lemma 2.9. [Aik08, Lemma 1] For $x \in \Omega \subseteq \mathbb{R}^{d+1}$ and $\phi \in C_c^\infty(\mathbb{R}^{d+1})$,

$$(2.3) \quad \int \phi \omega_\Omega^x = \int_\Omega \Delta \phi(y) G_\Omega(x, y) dy.$$

Lemma 2.10. Let $\Omega \subset \mathbb{R}^{d+1}$ be a CDC domain. Let B be a ball centered on $\partial\Omega$ and $0 < r_B < \text{diam } \partial\Omega$. Then,

$$(2.4) \quad \omega^x(4B) \gtrsim r_B^{d-1} G_\Omega(x, y) \quad \text{for all } x \in \Omega \setminus 2B \text{ and } y \in B \cap \Omega,$$

This follows quickly from the maximum principle, Lemma 2.7, and the fact that, for $x \in \partial 2B \cap \Omega$ and $y \in B$, $r_B^{d-1} G_\Omega(x, y) \lesssim 1$. For proofs, see [AH08, Lemma 3.5] or [AHM⁺16, Lemma 3.3].

Some of the proofs below will use compactness arguments via the following lemma from [AMT17].

Lemma 2.11. [AMT17, Lemma 2.9] Let $\Omega_j \subseteq \mathbb{R}^{d+1}$ be a sequence of CDC domains with the same CDC constants (as in Remark 2.6) such that $0 \in \partial\Omega_j$, $\inf \text{diam } \partial\Omega_j > 0$, and there is a ball $B(x_0, r) \subseteq \Omega_j$ for all j . Then there is a connected open set $\Omega_\infty^{x_0}$ containing $B(x_0, r)$ so that, after passing to a subsequence,

- (1) $G_{\Omega_j}(x_0, \cdot)$ converges uniformly to $G_{\Omega_\infty^{x_0}}(x_0, \cdot)$ on compact subsets of $\{x_0\}^c$,
- (2) $\omega_{\Omega_j}^{x_0} \rightharpoonup \omega_{\Omega_\infty^{x_0}}^{x_0}$, and
- (3) $\Omega_\infty^{x_0}$ has the CDC with the same constants.

We'll need an additional two lemmas building off of this one.

Lemma 2.12. With the assumptions of Lemma 2.11, if $x \in \Omega_\infty^{x_0}$, then we may pass to a further subsequence so that the same conclusions hold with x in place of x_0 and $\Omega_\infty^{x_0} = \Omega_\infty^x$. In particular, $\omega_{\Omega_j}^x \rightharpoonup \omega_{\Omega_\infty^{x_0}}^x$ and $\omega_{\Omega_\infty^{x_0}}^x = \omega_{\Omega_\infty^x}^x$.

Proof. When passing to the subsequence in Lemma 2.11, we can pass to another subsequence so that $G_{\Omega_j}(x, \cdot)$ converges on compact subsets of $\{x\}^c$ to $G_{\Omega_\infty^x}(x, \cdot)$. For $y \in \Omega_\infty^{x_0} \setminus \{x_0, x\}$, a small ball around y is contained in Ω_j for all j large (since $G_{\Omega_\infty^{x_0}}(x_0, \cdot)$ is bounded away from zero on this ball, hence so is $G_{\Omega_j}(x_0, \cdot)$ for large j). Let $\gamma \subseteq \Omega_\infty^{x_0} \setminus \{y\}$ be a curve from x to x_0 . Then $\gamma \subseteq \Omega_j$ for sufficiently large j (this is because $G_{\Omega_\infty^{x_0}}(x_0, \cdot) \geq c > 0$ on γ for some $c > 0$, and thus $G_{\Omega_j}(x_0, \cdot) \geq c/2 > 0$ for j large). Thus, by Harnack's inequality used on a chain of balls along γ whose doubles don't contain y ,

$$G_{\Omega_j}(x_0, y) \sim_y G_{\Omega_j}(x, y)$$

In particular,

$$G_{\Omega_\infty^{x_0}}(x_0, y) \sim_y G_{\Omega_\infty^x}(x, y).$$

and so

$$\begin{aligned}\Omega_\infty^{x_0} \setminus \{x, x_0\} &= \{y \neq x, x_0 : G_{\Omega_\infty}^{x_0}(x_0, y) > 0\} \\ &= \{y \neq x, x_0 : G_{\Omega_\infty}^x(x, y) > 0\} \\ &= \Omega_\infty^x \setminus \{x, x_0\}.\end{aligned}$$

Thus, adding back x and x_0 , we get $\Omega_\infty^{x_0} = \Omega_\infty^x \cup \{x_0\}$. Since the former set is open, so must the latter set, and this can only be if $x_0 \in \Omega_\infty^x$. Indeed, if $x_0 \notin \Omega_\infty^x$, then since $\Omega_\infty^x \cup \{x_0\}$ is open, there is $\varepsilon > 0$ so that $B(x_0, \varepsilon) \subseteq \Omega_\infty^x \cup \{x_0\}$, and so $B(x_0, \varepsilon) \setminus \{x_0\} \subseteq \Omega_\infty^x$. Thus, x_0 is an isolated point of $\partial\Omega_\infty^x$, but this is impossible since domains with the CDC have no isolated points in their boundary. The last sentence follows now because $\Omega_\infty^{x_0} = \Omega_\infty^x$, and so $\omega_{\Omega_j}^{x_0} \rightarrow \omega_{\Omega_\infty}^{x_0} = \omega_{\Omega_\infty}^x$. This proves the lemma. \square

Lemma 2.13. *Under the conditions of Lemma 2.11, if $\omega_{\Omega_j}^{x_0}$ satisfies (1.3), then so does $\omega_{\Omega_\infty}^{x_0}$ with the same constants.*

Proof. Let B be centered on $\partial\Omega_\infty^{x_0}$, $\alpha > 0$, and $x \in \Omega_\infty^{x_0}$ such that $\text{dist}(x, AB \cap \partial\Omega_\infty^{x_0}) \geq \alpha r_B$. Then there is $\xi_j \in \partial\Omega_j$ so that $\xi_j \rightarrow x_B$. For every $0 < r < s < t < 1$, if j large enough and $B_j = B(\xi_j, r_B)$, then $2rB \subseteq 2sB_j$, $sB_j \subseteq tB$, and

$$\text{dist}(x, AsB_j \cap \partial\Omega_j) \geq \alpha r_{sB_j} = \alpha sr_B.$$

Otherwise, if for infinitely many j we could find $\zeta_j \in B(x, \alpha sr_B) \cap AsB_j$, then by passing to a subsequence, they converge to a point $\zeta \in B(x, \alpha r_B)$. Since the $\omega_{\Omega_j}^x$ are uniformly doubling and there is a small ball containing x_0 that is contained in Ω_j for all j large, $\omega_{\Omega_j}^{x_0}(B(\xi, \varepsilon)) \gtrsim_{\varepsilon, x} 1$ for all large j and $\varepsilon > 0$, hence $\omega_{\Omega_\infty}^{x_0}(B(\xi, \varepsilon)) \gtrsim_{\varepsilon, x} 1$ as well, so $\zeta \in \text{supp } \omega_{\Omega_\infty}^{x_0} = \partial\Omega_\infty^{x_0}$, but then $\text{dist}(x, AB \cap \partial\Omega_\infty) < \alpha r_B$, a contradiction.

Thus, recalling the previous lemma,

$$\begin{aligned}\omega_{\Omega_\infty}^{x_0}(2rB) &\leq \liminf_j \omega_{\Omega_j}^x(2rB) \\ &\leq \liminf_j \omega_{\Omega_j}^x(2sB_j) \\ &\stackrel{(1.3)}{\leq} \liminf_j C(\alpha) \omega_{\Omega_j}^x(sB_j) \\ &\leq C(\alpha) \omega_{\Omega_\infty}^x(\overline{tB}).\end{aligned}$$

Letting $t \uparrow 1$ and $r \uparrow 1$ now gives (1.3) when $\Omega = \Omega_\infty^{x_0}$. \square

3. PROOF OF THEOREM I

This section is dedicated to the proof of Theorem I.

The reverse implication follows using Theorem 1.2 and semi-uniformity. Suppose Ω is semi-uniform, then Theorem 1.2 implies (1.1) for some constant constants A and A_0 , which is a priori weaker than (1.3). Let $\alpha > 0$, and let x be such that

$$(3.1) \quad \text{dist}(x, A_0B \cap \partial\Omega) \geq \alpha r_B.$$

We split into three cases:

- (1) If $x \in \Omega \setminus A_0B$, then we have $\omega_\Omega^x(2B) \leq A\omega_\Omega^x(B)$ immediately by (1.1).
- (2) Now suppose $x \in A_0B$ and $\Omega \setminus 2A_0B \neq \emptyset$, then by semi-uniformity and (3.1), we can find a Harnack chain from x to a point $y \in \Omega \setminus A_0B$ (with length depending on α and A_0 , which depends on the semi-uniformity constant), and so

$$\omega_\Omega^x(2B) \sim_{\alpha, A_0} \omega_\Omega^y(2B) \stackrel{(1.1)}{\leq} A\omega_\Omega^y(B) \sim_{\alpha, A_0} A\omega_\Omega^x(B).$$

- (3) If $x \in \Omega \setminus A_0B$ and $\Omega \subseteq 2A_0B$, then by semi-uniformity, there is a Harnack chain from x to a corkscrew point $y \in \frac{1}{2}B$, again with length depending on α and A_0 . Thus,

$$\omega_\Omega^x(B) \sim_\alpha \omega_\Omega^y(B) \stackrel{(2.1)}{\gtrsim} 1 \geq \omega_\Omega^x(2B).$$

For the rest of this section, we will focus on showing that if ω_Ω is doubling in the sense of (1.3), then Ω is semi-uniform.

Lemma 3.1. *If (1.3) holds, then Ω has interior c_1 -corkscrews with $c_1 \in (0, 1/8)$ depending on the CDC and doubling constants.*

The constant c_1 may be larger than $1/8$, but if a domain has c_1 corkscrews, then it has c -corkscrews for $c < c_1$, so the conclusion is still true. It will just be convenient to assume $c_1 < 1/8$ for later on.

Proof. Suppose there is a sequence of domains Ω_j for which (1.3) is satisfied, and a sequence of balls B_j centred on $\partial\Omega$ with $0 < r_{B_j} < \text{diam } \Omega_j/2A$ so that

$$\sup_{x \in B_j \cap \Omega} \text{dist}(x, \partial\Omega) < \frac{r_{B_j}}{j}.$$

Without loss of generality, by scaling and translating our domain, we can assume $0 \in \partial\Omega_j$ and $B_j = \mathbb{B}$, so $r_{B_j} = 1$. Then $\text{diam } \partial\Omega_j > 2A$. By rotating we may assume that there is a fixed point $y \in \Omega_j \setminus 2A\mathbb{B}$ for all j .

By Lemmas 2.7 and 2.8, if we define

$$g_j(x) = \begin{cases} \omega_{\Omega_j}^y(\mathbb{B})^{-1} G_{\Omega_j}(x, y) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

then

$$|g_j(x)| \lesssim j^{-\alpha} \text{ for all } x \in \mathbb{B} \cap \Omega$$

and so $g_j \rightarrow 0$ uniformly in \mathbb{B} . However, if $\phi \in C_c(\mathbb{B})$ is equal to 1 on $\frac{1}{2}\mathbb{B}$, then by (1.3) (since $\text{dist}(y, A\mathbb{B} \cap \partial\Omega) \geq A$) and (2.3),

$$1 \lesssim \frac{\omega_{\Omega_j}^y(\frac{1}{2}\mathbb{B})}{\omega_{\Omega_j}^y(\mathbb{B})} \leq \int \phi \frac{d\omega_{\Omega_j}^y}{\omega_{\Omega_j}^y(\mathbb{B})} = \int g_j \Delta \phi dx \rightarrow 0$$

which is a contradiction. \square

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ is a CDC domain, $c > 0$ and assume (1.3) holds. Let c_1 be as in the previous lemma. For any ball B centered on $\partial\Omega$ and $B' \subseteq B \cap \Omega$ a c -corkscrew ball, there is a Harnack chain of length N (depending on c , the CDC constants, and the doubling constants) whose first ball is B' and whose last ball is a $c_1/2$ -corkscrew ball for $\frac{1}{2}B$.*

Proof. Without loss of generality, we can assume $c < \frac{1}{4}$ and $x_B = 0$.

Suppose $\frac{4}{c} \text{diam } \partial\Omega < r_B$, then $\partial\Omega \subseteq \frac{c}{4}B \subseteq \frac{1}{4}B$, and since B' is a corkscrew ball, $2B' \subseteq B \cap \Omega$, hence $\frac{1}{2}B' \subseteq (\frac{c}{2}B)^c$. We can then connect B' by a short Harnack chain in $B \setminus \frac{c}{4}B$ to the center of a ball B'' of radius $\frac{1}{16}r_B$ with $2B'' \subseteq \frac{1}{2}B \setminus \frac{1}{4}B$, so B'' is a $\frac{1}{8}$ -corkscrew ball for $\frac{1}{2}B$. Since $c_1 \leq \frac{1}{8}$, B'' is also a c_1 -corkscrew for $\frac{1}{2}B$. Now we must prove the lemma in the case that $\frac{4}{c} \text{diam } \partial\Omega \geq r_B$

Suppose there were domains Ω_j and balls B_j centered on $\partial\Omega$ with $0 < r_{B_j} \leq \frac{4}{c} \text{diam } \partial\Omega_j$ whose harmonic measures were doubling in the sense of (1.3) (with the same constants) and c -corkscrew balls $B'_j \subseteq B_j \cap \Omega_j$ for which the shortest Harnack chain whose first ball is B'_j and whose last ball is a $c_1/2$ -corkscrew ball contained in $\frac{1}{2}B$ has length at least j (since Ω_j is connected and has the c_1 -corkscrew property, this is well defined). Without loss of generality, we may assume $B_j = \mathbb{B}$. By passing to a subsequence if necessary, we can find a ball $B(x_0, c - 1/j) \subseteq B'_j$ for all j . Again, we can pass to a subsequence so that the conclusions of Lemma 2.11 hold. In particular, if $\Omega_\infty^{x_0}$ is from the lemma, then it is doubling by Lemma 2.13 with the same constants and so it also has the c_1 -corkscrew property.

Using the Harnack principle, the doubling property, and Lemma 2.1, we have that for all $0 < r < 1$,

$$\omega_\infty^{x_0}(r\mathbb{B}) \geq \limsup_{j \rightarrow \infty} \omega_{\Omega_j}^{x_0}\left(\frac{r}{2}\mathbb{B}\right) \gtrsim \omega_{\Omega_j}^{x_{B'_j}}\left(\frac{r}{2}\mathbb{B}\right) \gtrsim_r \omega_{\Omega_j}^{x_{B'_j}}(\mathbb{B}) \gtrsim 1.$$

Thus, $\omega_\infty^{x_0}(r\mathbb{B}) > 0$ for all $0 < r < 1$, which implies

$$0 \in \text{supp } \omega_\infty^{x_0} = \partial\Omega_\infty^{x_0}.$$

Since $\Omega_\infty^{x_0}$ is a connected c_1 -corkscrew domain, there is a finite Harnack chain from a c_1 -corkscrew ball B' for $\frac{1}{2}\mathbb{B}$ in $\Omega_\infty^{x_0}$ to x_0 contained in $\Omega_\infty^{c_0}$. Hence, there is a ball $B'' \subseteq B'$ that is a $c_1/2$ -corkscrew ball for $\frac{1}{2}\mathbb{B} \cap \Omega_j$ for all j large.

Let N be the length of the chain. Then this Harnack chain is contained in Ω_j for all j sufficiently large (and by replacing them with some smaller balls, we can replace it with a Harnack chain for Ω_j of length no more than a multiple of N). But this is a contradiction for $j \gg N$. \square

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ have the CDC and assume (1.3) holds and $c > 0$. Then for any ball B centered on $\partial\Omega$ and $B' \subseteq B$ be a c -corkscrew ball for B such that $\partial\Omega \setminus 4B \neq \emptyset$, there is a Harnack chain of length N (depending on c , the CDC and doubling constants) whose first ball is B' and whose last ball is a $c_1/2$ -corkscrew ball for $2B$.*

Proof. Suppose there were domains Ω_j and balls B_j so that $\partial\Omega_j \setminus 4B_j \neq \emptyset$ and a c -corkscrew ball $B'_j \subseteq B_j \cap \Omega_j$ so that the length of any Harnack chain from B'_j to a $c_1/2$ -corkscrew ball for $2B_j$ is at least j . Without loss of generality, $B_j = \mathbb{B}$. Pass to a subsequence just as in the previous lemma, so B_j converges to a c -corkscrew ball $B(x_0, c) \subseteq \mathbb{B} \cap \Omega_\infty$. Let $y_j \in \partial\Omega_j \setminus 4\mathbb{B}$. Since we are assuming (1.3) holds

$$\omega_{\Omega_j}^{x_0}(\partial\Omega_j \setminus 3\mathbb{B}) \geq \omega_{\Omega_j}^{x_0}(B(y_j, |y_j|/4)) \stackrel{(1.3)}{\gtrsim} \omega_{\Omega_j}^{x_0}(B(y_j, 4|y_j|)) \geq \omega_{\Omega_j}^{x_0}(2\mathbb{B}) \stackrel{(2.1)}{\sim} 1.$$

Thus, there is $\beta \in (0, 1)$ so that

$$\omega_\infty^{x_0}(\overline{2\mathbb{B}}) \leq \liminf_{j \rightarrow \infty} \omega_{\Omega_j}^{x_0}(3\mathbb{B}) < \beta.$$

This means there is a curve in $\Omega_\infty^{x_0}$ from x_0 to $\Omega \setminus \overline{2\mathbb{B}}$, otherwise $\Omega_\infty^{x_0} \subseteq 2\mathbb{B}$, and harmonic measure for bounded domains is a probability measure, but $\text{supp } \omega_\infty^{x_0} = \partial\Omega_\infty^{x_0} \subseteq \overline{2\mathbb{B}}$, so that

$$1 = \omega_\infty^{x_0}(\mathbb{R}^d) = \omega_\infty^{x_0}(\overline{2\mathbb{B}}) < \beta < 1,$$

a contradiction.

If $\partial\Omega_\infty \subseteq \frac{3}{2}\mathbb{B}$ (and recall $c_1 < 1/8$), then this curve connects x_0 to a ball B' of radius c_1 whose double is contained in $2\mathbb{B} \setminus \frac{3}{2}\mathbb{B}$ (so it is a $\frac{c_1}{2}$ -corkscrew ball for $2\mathbb{B}$). Otherwise, if $\partial\Omega_\infty \not\subseteq \frac{3}{2}\mathbb{B}$, then as $\Omega_\infty^{c_0}$ has the c_1 -interior corkscrew property, there is a c_1 -corkscrew ball $B' \subseteq 2\mathbb{B} \cap \Omega_\infty^{c_0}$, and since $\Omega_\infty^{x_0}$ is connected, the curve can be extended to connect x_0 to this ball. In either case, there is a curve $\gamma \subseteq \Omega_\infty^{x_0}$ from x_0 to the center of a c_1 -corkscrew

ball $B' \subseteq 2\mathbb{B}$ for $\Omega_\infty^{x_0}$. For j large enough, $\frac{1}{2}B'$ is a $c_1/2$ -corkscrew ball for \mathbb{B} in Ω_j . Also for j large enough, this curve is also contained in Ω_j . We can cover γ with boundedly many balls (depending on γ but independent of j) to form a Harnack chain between x_0 and $\frac{1}{2}B'$ in Ω_j , but this is a contradiction for j large enough. \square

We now finish the proof of Theorem I. Assume ω_Ω is doubling. Let $x \in \Omega$, $\xi \in \partial\Omega$, and $0 < r < \text{diam } \partial\Omega$. Let $\zeta \in \partial\Omega$ be closest to x . We will show that there is a function N as in Lemma 2.3.

We will also abuse notation below and write \log for $\log_+ = \max\{0, \log\}$.

There are two cases to consider:

Case 1. Suppose $\delta_\Omega(x) = |x - \zeta| \geq 2^{-4}|x - \xi|$ and $B = B(\xi, 2|x - \xi|)$. Since ω is doubling, x is a $\frac{1}{2^6}$ -corkscrew point for the ball $B(\xi, 2|x - \xi|)$.

Case 1.a. If $r < 4|x - \xi|$, we can iterate Lemma 3.2 to find a Harnack chain from x to a c_1 -corkscrew point $y \in B(\xi, r) \cap \Omega$ of length at most a constant times $\log \frac{|x - \xi|}{r} + 1$. Note that

$$|x - \xi| \leq |x - y| + |y - \xi| \leq |x - y| + r$$

and since \log is increasing, the length of the chain is at most

$$\log \frac{|x - \xi|}{r} + 1 \leq \log \left(\frac{|x - y|}{r} + 1 \right) + 1 \lesssim \log \frac{|x - y|}{r} + 1.$$

Case 1.b. If $r \geq 4|x - \xi|$, then $|x - \xi| \leq r/4 < \text{diam } \partial\Omega/4$ and since x is a corkscrew point for B , we can iterate using Lemma 3.3 instead to find a Harnack chain to a corkscrew point $y \in B(\xi, r)$ with $|y - x| \gtrsim r$ of length at most a constant times

$$\log \frac{r}{|x - \xi|} + 1 \leq \log \frac{r}{\delta_\Omega(x)} + 1 \lesssim \log \frac{|x - y|}{\delta_\Omega(x)} + 1.$$

Case 2. Suppose $|x - \zeta| < 2^{-4}|x - \xi|$. Let $B = B(\zeta, 2|x - \zeta|)$, so x is a $\frac{1}{2}$ -corkscrew point for this ball. Let k be the largest integer for which $\xi \notin 2^{k+2}B$. Since

$$|\xi - \zeta| \geq |\xi - x| - |x - \zeta| > (2^4 - 1)|x - \zeta| \geq 2^3|x - \zeta| = 2^2 r_B,$$

we know $k \geq 0$. Since x is a $\frac{1}{4}$ -corkscrew point for B , by iterating Lemma 3.3, for $0 \leq j \leq k$, we can Harnack chains from a c_1 -corkscrew point in $2^j B$ (that is x if $j = 0$) to a c_1 -corkscrew point in $2^{j+1} B$ of lengths at most some constant N (depending on the CDC and doubling constants). If we combine these balls, we get a Harnack chain from x to a c_1 -corkscrew point

$x' \in 2^{k+1}B$ of total length at most

$$N \cdot (k+1) \lesssim \log \frac{|x-\xi|}{|x-\zeta|} + 1 = \log \frac{|x-\xi|}{\delta_\Omega(x)} + 1.$$

In particular, since x' is a c_1 -corkscrew point in $2^{k+1}B$ and $\xi \in 2^{k+3}B$, we know $|x' - \xi| \leq 2^{k+6}r_B$, and so x' is a corkscrew point in $B(\xi, 2^{k+7}r_B)$. Note $2^{k+7}r_B \sim |x - \xi|$. Indeed,

$$2^{k+3}r_B > |\xi - \zeta| \geq |\xi - x| - |x - \zeta| > (1 - 2^{-4})|\xi - x|$$

and

$$2^{k+2}r_B \leq |\xi - \zeta| \leq |\xi - x| + |x - \zeta| < (1 + 2^{-4})|\xi - x|.$$

Case 2.a. If $r < |\xi - x|$, then the estimates above imply $r < 2^{k+4}r_B$, so we can apply Lemma 3.2 again, and using the fact that $2^k r_B \sim |x - \xi|$, we can make a Harnack chain from x' to a corkscrew point $y \in B(\xi, r)$ with $|y - x| \gtrsim r$ of length at most a constant times

$$\log \frac{2^{k+7}r_B}{r} + 1 \lesssim \log \frac{|x - \xi|}{r} + 1$$

Combing our two chains together gives us a Harnack chain from x to ξ of total length at most a constant times $\log \frac{|x-\xi|}{\min\{\delta_\Omega(x), r\}} + 1$. Moreover,

$$|x - \xi| \leq |x - y| + |y - \xi| \leq |x - y| + r \lesssim |x - y|.$$

This means the chain has length at most $\log \frac{|x-y|}{\min\{\delta_\Omega(x), r\}} + 1$

Case 2.b. If $r \geq |\xi - x|$, then we can apply Lemma 3.3 instead to get a Harnack chain from x' to a corkscrew point $y \in B(\xi, r)$ with $|x - y| \gtrsim r$ of length at most a constant times

$$\log \frac{r}{2^{k+7}r_B} + 1 \lesssim \log \frac{|x - y|}{|\xi - x|} + 1$$

and connecting these two Harnack chains gives a chain of length at most

$$\log \frac{|x - \xi|}{\delta_\Omega(x)} + \log \frac{|x - y|}{|\xi - x|} + 1 \sim \log \frac{|x - y|}{\delta_\Omega(x)} + 1.$$

Taking the maximum of in Case 2.a. and 2.b. gives a chain of length at most $\log \frac{|x-y|}{\min\{\delta_\Omega(x), r\}} + 1$.

Taking the minimum of all the possible estimates we have for the possible length of a Harnack chain gives us our desired function N and semi-uniformity now follows from Lemma 2.3.

Remark 3.4. Note that as a corollary of the proof, we have that, for a semi-uniform domain, the function N given in Theorem 2.3 is $N(x) =$

$C \min\{1, \log x+1\}$, that is, the shortest Harnack chain from x to a corkscrew point $y \in B(\xi, r)$ is at most a constant times

$$\log \frac{|x-y|}{\min\{\delta_\Omega(x), r\}} + 1.$$

4. PROOF OF THEOREM II: PART I

Definition 4.1. For a domain $\Omega \subseteq \mathbb{R}^{d+1}$, we say that points y_1, \dots, y_n are reference points for a ball B centered on $\partial\Omega$ if

$$(4.1) \quad \min_{i=1, \dots, n} k_\Omega(x, y_i) \lesssim \log \frac{r_B}{\delta_\Omega(x)} + 1 \quad \text{for all } x \in B.$$

where $k_\Omega(x, y)$ denotes the quasihyperbolic distance between x and y . As observed in [AH08, p. 434], if $N_\Omega(x, y)$ denotes the length of the shortest Harnack chain between x and y , then

$$N_\Omega(x, y) \sim k_\Omega(x, y) + 1.$$

Remark 4.2. We first make some observations about reference points.

- (1) If y_i is a reference point for B , then $\delta_\Omega(y_i) \gtrsim r_B$.
- (2) By semi-uniformity, if y_1, \dots, y_n are reference points for B that aren't necessarily in B , then using Harnack chains we can find new reference points z_1, \dots, z_n that are corkscrew points in B (with different corkscrew and reference point constants). This is because semi-uniformity implies we may find Harnack chains from the y_i to corkscrew points $z_i \in B$, so for $x \in B$,

$$k_\Omega(x, z_i) \leq k_\Omega(x, y_i) + k_\Omega(y_i, z_i) \lesssim k_\Omega(x, y_i) + 1.$$

- (3) Similarly, if y_1, \dots, y_n are in B already, $M \geq 1$, and $2Mr_B < \text{diam } \partial\Omega$, we can also find reference points z_1, \dots, z_n for B outside MB , say, though with constants depending also on M . Indeed, if $z \in \partial\Omega \setminus 2MB$, by Theorem 2.3, we can find bounded Harnack chains from each y_i to a corkscrew ball z_i for $B(z, Mr_B) \subseteq MB^c$, so now (4.1) holds with the z_i in place of the y_i with constant depending on M .
- (4) If B is a ball centered on $\partial\Omega$ with $r < 2 \text{diam } \Omega$, we can always find n reference points with n at most a constant depending on the semi-uniformity. Aikawa, Hirata, and Lundh showed this held for any John domain [AHL06, Proposition 2.1]. For a general semi-uniform domain (which won't be John if it is unbounded), we prove this as follows. First, we can assume that $r_B < \text{diam } \Omega/8$. Semi-uniformity and Theorem 2.3 imply that for any $x_0 \in B \cap \Omega$, $\zeta \in \partial\Omega$ the closest point to x_0 and $B' = B(\zeta, 2\delta_\Omega(x_0))$, and k the maximal integer so that $2^k B' \subseteq 2B$, there are corkscrew points $x_i \in 2^i B'$

and Harnack chains from x_i to x_{i+1} of length at most some number N . Note that

$$k \sim \left\lceil \log \frac{r_B}{r_{B'}} \right\rceil + 1 \lesssim \log \frac{r_B}{\delta_\Omega(x_0)} + 1$$

and $\delta_\Omega(x_k) \gtrsim 2^k r_{B'} \gtrsim r_B$. Let Q_{x_0} be the Whitney cube in Ω to which x_k belongs. Then $\ell(Q_{x_0}) \sim r_B$, $Q_{x_0} \cap 2B \neq \emptyset$, and we have shown that there is a Harnack chain from the center of Q_{x_0} to x_0 of length at most a constant times $\log \frac{r_B}{\delta_\Omega(x_0)} + 1$. Since the number of Whitney cubes Q satisfying $\ell(Q) \sim r_B$ and $Q \cap B \neq \emptyset$ is uniformly bounded, we can take their centers as our reference points for B .

If $2 \operatorname{diam} \partial\Omega > r_B \geq \operatorname{diam} \partial\Omega/8$, we can cover $\partial\Omega$ with a bounded number of balls of radius $\operatorname{diam} \partial\Omega/16$ and then the union of their respective reference points are a set of reference points for B .

Note that this is not always possible for $r \gg \operatorname{diam} \partial\Omega$, and the example is the same as Remark 2.4.

The objective of this section is to prove the following:

Lemma 4.3. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a CDC semi-uniform domain, $B' \subseteq B$ two balls centered on $\partial\Omega$ with $r_B < \operatorname{diam} \partial\Omega$ and $y_1, \dots, y_n \in \Omega$ a set of reference points for $2B$. There is $M > 0$ depending on the CDC and semi-uniformity constants so that*

$$\frac{\omega_\Omega^x(B')}{\omega_\Omega^x(B)} \lesssim \sum_{i=1}^n \omega_\Omega^{y_i}(B') \quad \text{for all } x \in \Omega \setminus MB.$$

In particular, if $E \subseteq B$ is a Borel set, then

$$\frac{\omega_\Omega^x(E)}{\omega_\Omega^x(B)} \lesssim \sum_{i=1}^n \omega_\Omega^{y_i}(E) \quad \text{for all } x \in \Omega \setminus MB.$$

We recall the following lemma from [AH08, Lemma 3.6]. The statement there is slightly different, but the proof is exactly the same.

Lemma 4.4. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a CDC domain, B a ball centered on $\partial\Omega$ with $r_B < \operatorname{diam} \partial\Omega$, and $y_1, \dots, y_n \in \Omega$ be reference points for $2B$. Then*

$$(4.2) \quad \omega_\Omega^x(B) \lesssim r_B^{d-1} \sum_{i=1}^n G_\Omega(x, y_i) \quad \text{for } x \in \Omega \setminus 2B.$$

The implied constant depends on the CDC constant and reference point constants.

Remark 4.5. Note that by Remark 4.2, the reference points can also be taken to be corkscrew points in B .

Lemma 4.6. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a CDC domain and B a ball centered on $\partial\Omega$ with $r_B < \text{diam } \partial\Omega$. Let y_1, \dots, y_n be reference points for $2B$, and M large enough (depending on the reference point constants) so that each $x \in 2B \cap \Omega$ can be connected to one of the y_i by a Harnack chain of length $N_\Omega(x, y_i)$ in MB . Let u be a non-negative harmonic function on $\Omega \cap MB$ vanishing continuously on $2B \cap \partial\Omega$. then*

$$\sup_{B \cap \Omega} u \lesssim \sum_{j=1}^n u(y_j).$$

Proof. The proof of this is almost exactly like that of [JK82, Lemma 4.4], we just point out the required modifications in its proof. First, [JK82, Lemma 4.1] still holds in CDC domains, as it is just Lemma 2.8. In particular, there is M_1 so that for any $\xi \in \partial\Omega$ and $s < r_B$,

$$\sup\{u(x) : x \in B(\xi, M_1^{-1}s) \cap \Omega\} < \frac{1}{2} \sup\{u(x) : x \in B(\xi, s) \cap \Omega\}.$$

Next, if we assume $\max u(y_i) = 1$, then using Harnack chains and the fact that the y_1, \dots, y_n are reference points for $2B$, one can show that there is $M_2 > 1$ depending on M_1 and the reference point constants so that if $u(y) > M_2^h$ for an integer h and $y \in 2B \cap \Omega$, then $\delta_\Omega(y) < M_1^{-h}$. The proof now follows that of [JK82, Lemma 4.4] word by word. \square

A domain Ω satisfies the **boundary Harnack principle (BHP)** if there is $M \geq 1$ so that, if u, v are non-negative harmonic functions vanishing continuously on $MB \cap \partial\Omega$ and x_0 is a corkscrew point in B , then

$$\frac{u(x)}{v(x)} \sim \frac{u(x_0)}{v(x_0)} \text{ for all } x \in B \cap \Omega.$$

This is shown for NTA domains in [JK82, Lemma 4.10] and was a key ingredient in Jerison and Kenig's proof of (1.2), see [JK82, Lemma 4.11]. However, Aikawa has shown that, if Ω is a CDC John domain, the BHP is equivalent to Ω being a uniform domain [Aik06], so we can't expect such an estimate to hold in our setting. The following lemma serves as a weak substitute for the BHP in semi-uniform domains, and its proof is based on that of [JK82, Lemma 4.10].

Lemma 4.7. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a semi-uniform CDC domain. Then there are constants $2M_0 < M_1$ depending on the semi-uniformity constants so that the following holds. Let B a ball centered on $\partial\Omega$ with $r_B < \text{diam } \partial\Omega$, and $y_1, \dots, y_n \in B$ be reference points for B . Let u be a non-negative harmonic function on $M_1B \cap \Omega$ that vanishes continuously on $2M_0B \cap \partial\Omega$. Then*

$$(4.3) \quad u(x) \lesssim \left(\sum_{i=1}^n u(y_i) \right) \sum_{i=1}^n r_B^{d-1} G_\Omega(x, y_i) \text{ for all } x \in B.$$

Proof. Let \mathscr{W} denote the Whitney cubes in Ω . Pick M_0 large enough (depending on the semi-uniformity constants) so that each $x \in 2B \cap \Omega$ can be connected to one of the y_i by a Harnack chain of length $N_\Omega(x, y_i)$ so that $2Q \subseteq M_0B$ for each Whitney cube Q for Ω that intersects the Harnack chain. Note that by semi-uniformity, we also know that the y_1, \dots, y_n are also reference points for $2M_0B$ with different reference point constants: each point $x \in 2M_0B \cap \Omega$ can be connected by a Harnack chain of length $\lesssim \log \frac{2M_0r_B}{\delta_\Omega(x)} + 1$ to a corkscrew point in B , and this can be extended by a bounded number of balls to one of the y_i of total length $N'_\Omega(x, y_i) \sim N_\Omega(x, y_i)$, say. Now pick M_1 large enough (depending on the new reference point constants) so that each $x \in 2M_0B \cap \Omega$ can be connected to a y_i by a Harnack chain of length $N'_\Omega(x, y_i)$ contained in M_1B .

Without loss of generality, $\sum_{i=1}^n u(y_i) = 1$. For each $Q \in \mathscr{W}$ intersecting $2B$, there is $\{R_i^Q\}_{i=1}^{N_Q}$ a chain of cubes, the first containing one of the y_j , the last equalling Q , where

$$N_Q \lesssim 1 + \log \frac{r_B}{\ell(Q)}.$$

Let $\lambda > 1$ be small and

$$\Omega' := \bigcup_{\substack{Q \in \mathscr{W} \\ Q \cap 2B \neq \emptyset}} \bigcup_{i=1}^{N_Q} \lambda R_i^Q \supseteq 2B.$$

Note that by construction and our choice of M_0 that

$$2B \cap \Omega \subseteq \Omega' \subseteq M_0B \cap \Omega.$$

By Lemma 4.6 and our choice of M_1 ,

$$\sup_{M_0B \cap \Omega} u \lesssim \sum u(y_i) \leq 1.$$

In particular, by the maximum principle we have

$$(4.4) \quad u(x) \leq \omega_{\Omega'}^x(\partial\Omega' \cap \Omega) \quad \text{for all } x \in \Omega'.$$

Let $\{B_i\}_{i=1}^m$ be a finite collection of balls (with m depending only on d) of radius ηr_B (where $\eta \in (0, \eta)$ will be chosen later) centered along $\partial\Omega'$ that cover

$$L = \{x \in \partial\Omega' : \delta(x) < \eta^2 r_B\}.$$

See Figure 2.

Claim: for each i , the points y_1, \dots, y_n are reference points for $2B_i$ with respect to Ω' .

Let $x \in B_i \cap \Omega'$, so $x \in \lambda Q$ for some $Q \in \mathscr{W}$. Then $Q = R_i^{Q''}$ for i and some cube $Q'' \in \mathscr{W}$ such that $Q'' \cap B \neq \emptyset$ and $R_1^{Q''}$ contains y_j for some $j \in \{1, 2, \dots, n\}$. Moreover, $i \lesssim 1 + \log \frac{r_B}{\ell(Q)}$. Thus, we can find

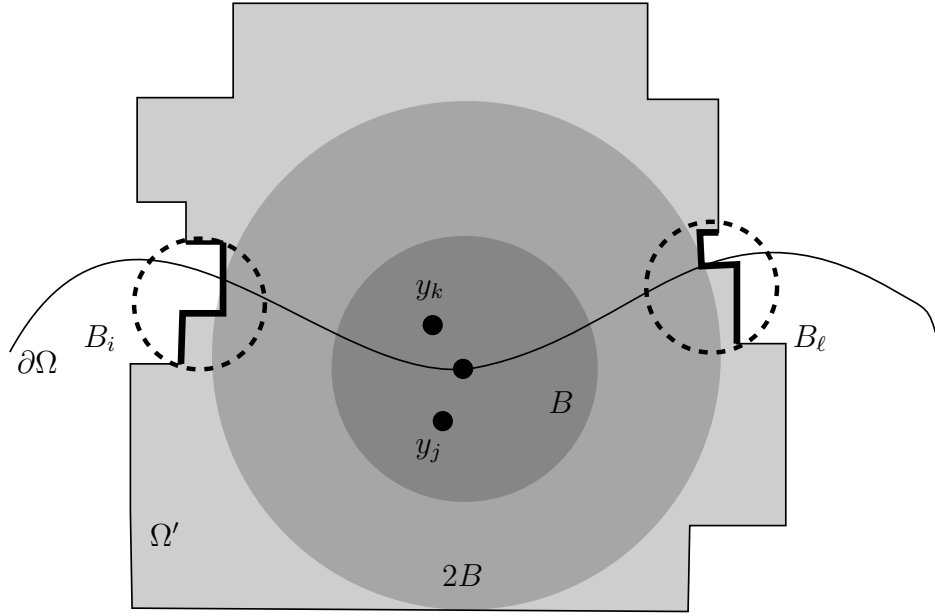


FIGURE 2. The dark lines denote the set $L \subseteq \partial\Omega'$.

a Harnack chain in $\bigcup_{k=1}^i \lambda R_k^{Q''} \subseteq \Omega'$ from y_j to x_Q (the center of Q) of length at most a constant times $1 + \log \frac{r_B}{\ell(Q)}$. If $\delta_{\Omega'}(x) \geq \varepsilon \ell(Q)$, then we can add on a finite number (depending on ε) of balls to complete this chain into a Harnack chain from y_j to x . If $\delta_{\Omega'}(x) < \varepsilon \ell(Q)$, then for $\varepsilon > 0$ small enough (depending on λ), $\delta_{\lambda Q}(x) \sim \delta_{\Omega'}(x)$, and since λQ is chord-arc, we can connect x to x_Q by a Harnack chain of length at most a constant times

$$\log \frac{\ell(Q)}{\delta_{\lambda Q}(x)} + 1 \sim \log \frac{\ell(Q)}{\delta_{\Omega'}(x)} + 1$$

and then we can connect y to x_Q by a Harnack chain of length at most a constant times $\log \frac{r_B}{\delta_{\Omega'}(x)} + 1$.

In either case, by summing up the estimates for the lengths, we obtain a Harnack chain of length at most $\log \frac{r_B}{\delta_{\Omega'}(x)} + 1$ from x to some y_i . This completes the claim.

For η small enough, we can ensure that, for each j , $B \cap 2B_j = \emptyset$, and so Lemma 4.4, the maximum principle implies that for all $x \in B$,

$$(4.5) \quad \omega_{\Omega'}^x(L) \leq \sum_{j=1}^m \omega_{\Omega'}^x(B_j) \lesssim \sum_{j=1}^m \sum_{i=1}^n r_{B_j}^{d-1} G_{\Omega'}(x, y_i) \lesssim \sum_{i=1}^n r_B^{d-1} G_{\Omega}(x, y_i).$$

If $x \in \partial\Omega' \setminus L$, then by construction, there is a short Harnack chain from x to one of the y_i , and thus to a point $z_i \in \partial B(y_i, \varepsilon c r_B)$, where c is the

corkscrew constant and $\varepsilon > 0$ is small enough so that $B(y_i, 2\varepsilon cr_B)$ are disjoint (we can replace them with other reference points so that this happens by Remark 4.2). Since

$$1 \lesssim r_B^{d-1} G_\Omega(z_i, y_i) \lesssim r_B^{d-1} G_\Omega(x, y_i),$$

we have $\sum_j r_B^{d-1} G_\Omega(x, y_j) \gtrsim 1$ for $x \in \partial\Omega' \setminus L \cup \bigcup \partial B(y_j, \varepsilon cr_B)$, and so by the maximum principle on $\Omega' \setminus \bigcup \partial B(y_j, \varepsilon cr_B)$ that

$$\omega_{\Omega'}^x(\partial\Omega' \setminus L) \lesssim \sum_{i=1}^n r_B^{d-1} G_\Omega(x, y_i) \quad \text{for all } x \in \Omega'.$$

Combining the above estimates and using the bounded overlap of the B_j , we obtain that for $x \in B \cap \Omega$,

$$u(x) \stackrel{(4.4)}{\leq} \omega_{\Omega'}^x(\partial\Omega' \cap \Omega) \leq \omega_{\Omega'}^x(L) + \omega_{\Omega'}^x(\partial\Omega' \setminus L) \lesssim \sum_{i=1}^n r_B^{d-1} G_\Omega(x, y_i)$$

and this gives (4.3). \square

Proof of Lemma 4.3. Let $y_1, \dots, y_n \in \frac{1}{4}B$ be reference points for B and $z_1, \dots, z_n \in B'$ be reference points for B' . Then for $x \in \Omega \setminus MB$ and M large enough, Lemma 4.4 implies

$$\omega_\Omega^x(B') \lesssim r_{B'}^{d-1} \sum_{i=1}^n \underbrace{G_\Omega(x, z_i)}_{=: u(z_i)}.$$

Note that by (4.3),

$$u(z_i) \lesssim \left(\sum_{j=1}^n u(y_j) \right) \left(\sum_{j=1}^n r_B^{d-1} G_\Omega(z_i, y_j) \right)$$

By (2.4) and the doubling property,

$$u(y_i) = G_\Omega(x, y_i) \lesssim r_B^{1-d} \omega_\Omega^x(B).$$

And similarly,

$$G_\Omega(z_i, y_j) \lesssim r_{B'}^{1-d} \omega_\Omega^{y_j}(B').$$

So the above estimates combined give

$$\omega_\Omega^x(B') \lesssim \omega_\Omega^x(B) \sum_{i=1}^n \omega_\Omega^{y_i}(B').$$

\square

Remark 4.8. It's natural to ask whether we can get away with just one y_j in the above estimate instead of having to sum over all reference points, or in other words, whether $\omega_{\Omega}^{x_1}(B) \sim \omega_{\Omega}^{x_2}(B)$ for any two corkscrew points $x_1, x_2 \in B$, but this is not the case. If we consider the von Koch snowflake, or any NTA domain Ω_1 whose exterior domain $\Omega_2 = (\Omega_1^c)^\circ$ is also NTA and whose common boundary is purely unrectifiable boundary, then $\omega_{\Omega_1}^{x_1}$ and $\omega_{\Omega_2}^{x_2}$ must be mutually singular. If we remove a ball B_0 from the boundary of this domain, we now have a connected domain $\Omega = \Omega_1 \cup \Omega_2 \cup B_0$, and if x_1 and x_2 are two corkscrew points for some fixed ball B centered on the boundary away from B_0 (for Ω_1 and Ω_2 respectively), then as the ball shrinks, $\omega_{\Omega}^{x_i} \rightarrow \omega_{\Omega_i}^{x_i}$. In particular, we can find a ball $B' \subseteq B$ so that $\omega_{\Omega_1}^{x_1}(B') \ll \omega_{\Omega_2}^{x_2}(B')$, say, and then this will imply, for B_0 small enough, $\omega_{\Omega}^{x_1}(B') \ll \omega_{\Omega}^{x_2}(B')$.

5. PROOF OF THEOREM II: PART II

The objective of this section is to prove the counterpart to Lemma 4.3.

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a semi-uniform CDC domain, B a centered on $\partial\Omega$ with $r_B < \text{diam } \partial\Omega/4$, $E \subseteq \partial\Omega \cap B$ Borel, and $x_1, \dots, x_n \in \Omega$ a set of reference points for B . There is M depending on the CDC and doubling constants and integer i so that*

$$\frac{\omega_{\Omega}^x(E)}{\omega_{\Omega}^x(B)} \gtrsim \min_{i=1, \dots, n} \omega_{\Omega}^{x_i}(E) \text{ for all } x \in \Omega \setminus MB.$$

For this, we will need different estimates on Green's function.

Lemma 5.2. *Let $\partial\Omega \subseteq \mathbb{R}^{d+1}$ be a semi-uniform domain and B a ball centered on $\partial\Omega$. Let $x_1, \dots, x_n \in \Omega \setminus 2B$ be reference points for B , and for each $x \in \Omega$, let $\{Q_i(x)\}_{i=1}^{N(x)}$ be a Harnack chain of cubes of minimal length from one of the x_1, \dots, x_n to x (which exists by semi-uniformity). Let*

$$\Omega' = \bigcup_{x \in \Omega \setminus B} \bigcup_{i=1}^{N(x)} \lambda Q_i(x).$$

Then x_1, \dots, x_n are reference points for $\Omega' \cap 2B$.

Proof. Let $x \in \Omega' \cap 2B$. Then $x \in \lambda Q_i(y) \cap 2B$ for some $y \in \Omega \setminus B$, where Q_1 contains some reference point x_j . Let $Q = Q_i(y)$. Then there is a Harnack chain from x_Q to x_j of length $\lesssim \log \frac{r_B}{\ell(Q)} + 1$ (one can find this chain by considering the union of corkscrew balls with respect to Ω' covering $\frac{1+\lambda}{2}Q_k$ but whose doubles are contained in the λQ_k and follow this back up to x_j). If $\delta_{\Omega'}(x) \geq \varepsilon \ell(Q)$ for some $\varepsilon > 0$, then it is easy to find a Harnack chain of bounded length from x to x_Q (of length depending on

ε). If $\delta_{\Omega'}(x) < \varepsilon \ell(Q)$, then for $\varepsilon > 0$ small enough, $\delta_{\Omega'}(x) \sim \delta_{\lambda Q}(x)$ and there is a Harnack chain in λQ from x to x_Q of length at most a constant times

$$\log \frac{\ell(Q)}{\delta_{\lambda Q}(x)} + 1 \lesssim \log \frac{\ell(Q)}{\delta_{\Omega'}(x)} + 1.$$

Connecting these chains give a Harnack chain from x to x_i of length at most $C \log \frac{r_B}{\delta_{\Omega'}(x)} + 1$, and this finishes the proof. \square

Proof of Lemma 5.1. Let $x \in \Omega \setminus 4B$. By Remark 4.2, since $4r_B < \text{diam } \partial\Omega$, we can assume that our reference points are outside $2B$. Let $y_i \in \partial\Omega'$ be a corkscrew point in Ω accessible from x_i , so that $\delta_\Omega(y_i) \sim r_B$ and there is a Harnack chain of bounded length from y_i to x_i . To see the existence of such y_i , first note that there is $c > 0$ depending on the semi-uniformity constants so that $cB \subseteq (\Omega')^c$. Indeed, suppose there was some $x \in \Omega \setminus B$ so that $Q_j(x) \cap cB \neq \emptyset$ for some j . Let k be the first integer so that $Q_k(x) \cap B \neq \emptyset$. Note that $Q_k(x), Q_{k+1}(x), \dots$ is a minimal Harnack chain of Whitney cubes from the center of $Q_k(x)$ to a reference point (otherwise we could have shortened our original minimal chain). By semi-uniformity, there should be a Harnack chain of length depending only on d and the uniformity constants from the center of $Q_k(x)$ to one of the reference points, so in particular, it cannot reach cB for c large enough, which is a contradiction. Thus, $cB \subseteq (\Omega')^c$ for some c . Now semi-uniformity implies the existence of a Harnack chain from x_i to a corkscrew point in cB of bounded length, and so one of the balls in the Harnack chain must cross $\partial\Omega'$, and we pick a point y_i in the intersection. See Figure 3.

In particular, we may find a Harnack chain from x_i to a corkscrew point $z_i \in B(y_i, \frac{\delta_\Omega(y_i)}{10}) \cap \Omega'$.

By the previous lemma, x_1, \dots, x_n are all reference points in Ω' for $2B$. By Lemma 4.4 (using the fact that $x \notin 2(2B)$) and using Harnack chains from x_i to z_i and the fact that $r_B \sim \delta_\Omega(y_i)$,

$$\begin{aligned} \omega_{\Omega'}^x(2B) &\lesssim r_B^{d-1} \sum_{i=1}^n G_{\Omega'}(x, x_i) \sim r_B^{d-1} \sum_{i=1}^n G_{\Omega'}(x, z_i) \\ &\stackrel{(2.4)}{\lesssim} \sum_{i=1}^n \omega_{\Omega'}^x(B(y_i, \delta_\Omega(y_i)/2)). \end{aligned}$$

In particular, there is j so that

$$(5.1) \quad \omega_{\Omega'}^x(B(y_j, \delta_\Omega(y_j)/2)) \gtrsim \omega_{\Omega'}^x(2B).$$

Using Harnack chains, we know $\omega_\Omega^y(E) \sim \omega_\Omega^{y_i}(E)$ for all $y \in B(y_i, \delta_\Omega(y_i)/2)$. Also, since $\partial\Omega \cap B \subseteq (\Omega')^c$, we know $\omega_\Omega(B)$ vanishes continuously along

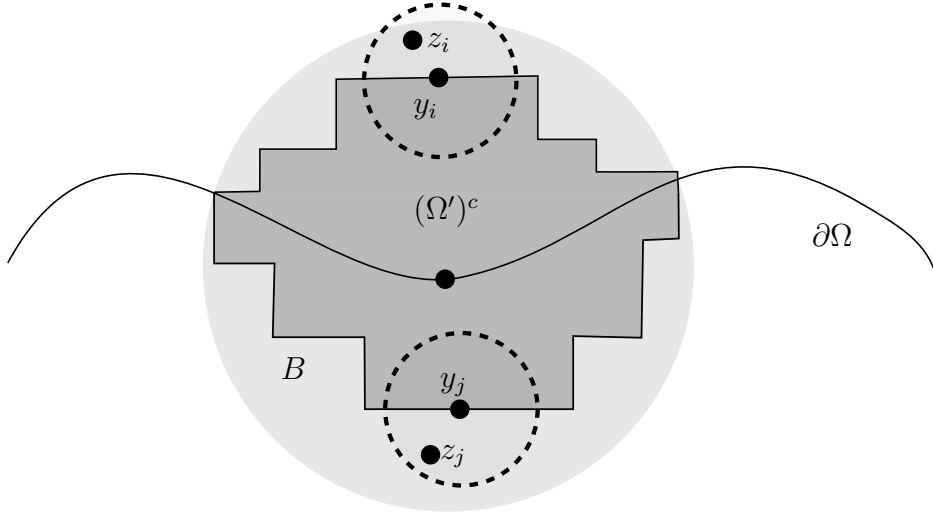


FIGURE 3. Observe that $\partial\Omega \cap B \subseteq (\Omega')^c$, so by the maximum principle, $\omega_{\Omega'}^x(B) \geq \omega_{\Omega}^x(B)$ for $x \in \Omega'$.

$\partial\Omega' \setminus \bar{B}$ and is at most 1 on $\bar{B} \cap \partial\Omega'$. Thus, by the maximum principle,

$$(5.2) \quad \omega_{\Omega}^z(B) \leq \omega_{\Omega'}^z(\bar{B}) \leq \omega_{\Omega'}^z(2B) \text{ for all } z \in \Omega'.$$

Hence, by the strong Markov Property of harmonic measure and the maximum principle

$$\begin{aligned} \omega_{\Omega}^x(E) &= \int_{\partial\Omega' \cap \Omega} \omega_{\Omega}^y(E) d\omega_{\Omega'}^x(y) \geq \int_{B(y_j, \delta_{\Omega}(y_j)/2)} \omega_{\Omega}^y(E) d\omega_{\Omega'}^x(y) \\ &\gtrsim \int_{B(y_j, \delta_{\Omega}(y_j)/2)} \omega_{\Omega}^{y_j}(E) d\omega_{\Omega'}^x(y) = \omega_{\Omega}^{y_j}(E) \omega_{\Omega'}^x(B(y_j, \delta_{\Omega}(y_j)/2)) \\ &\stackrel{(5.1)}{\gtrsim} \omega_{\Omega}^{y_j}(E) \omega_{\Omega'}^x(2B) \stackrel{(5.2)}{\geq} \omega_{\Omega}^{y_j}(E) \omega_{\Omega}^x(B) \sim \omega_{\Omega}^{x_j}(E) \omega_{\Omega}^x(B). \end{aligned}$$

□

Proof of Theorem II. The first inequality in Theorem II follows from Lemma 4.3 by choosing some reference points y_1, \dots, y_n that are also corkscrew points for B (recall Remark 4.2). The second inequality follows from 5.1 when $r_B < \frac{1}{4} \text{diam } \partial\Omega$. If $\text{diam } \partial\Omega > r_B \geq \text{diam } \partial\Omega$, things are a little easier: by Remark 4.2, and assuming $0 \in \partial\Omega$ and $\text{diam } \partial\Omega = 1$, we can find Reference points y_i for $2\mathbb{B}$ that are contained inside \mathbb{B} . In particular, every $x \in \partial 2B$ can be connected to a reference point y_i by a short Harnack chain. Also, for each such x , $\omega_{\Omega}^x(B) \sim \omega_{\Omega}^x(2\mathbb{B}) \gtrsim 1$ by (2.1) and the doubling condition. These two observations imply

$$\omega_{\Omega}^x(E) \gtrsim \min_i \omega_{\Omega}^{y_i}(E) \gtrsim \omega_{\Omega}^x(B) \min_i \omega_{\Omega}^{y_i}(E) \text{ for } x \in \partial 2\mathbb{B}$$

Thus, the second inequality Theorem II holds in this case as well by the maximum principle. \square

6. CHORD-ARC SUBDOMAINS OF SEMI-UNIFORM DOMAINS WITH UR BOUNDARY

We recall the construction of cubes on a metric space, originally due to David and Christ ([Dav88], [Chr90]), but the current formulation is from Hytönen and Martikainen [HM12]. This construction works for any doubling metric space X , but we state it for the case $X = \partial\Omega$, where $\partial\Omega$ is the boundary of some domain.

Definition 6.1. We say that a set X is δ -separated or a δ -net if for all $x, y \in X$ we have $|x - y| \geq \delta$.

Theorem 6.2. Let X be a metric space and let X_k be a nested sequence of maximal ρ^k -nets for X where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection \mathcal{D}_k of “cubes,” which are Borel subsets of E such that the following hold.

- (1) For every integer k , $X = \bigcup_{Q \in \mathcal{D}_k} Q$.
- (2) If $Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
- (3) For $Q \in \mathcal{D}$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_k$ and set $\ell(Q) = 5\rho^{k(Q)}$. Then there is $\zeta_Q \in X_k$ so that

$$(6.1) \quad B(\zeta_Q, c_0\ell(Q)) \subseteq Q \subseteq B(\zeta_Q, \ell(Q)) =: B_Q$$

and

$$X_k = \{\zeta_Q : Q \in \mathcal{D}_k\}.$$

If μ is a doubling measure on E , $\tau \in (0, 1)$, $Q \in \mathcal{D}_k$ for some k , and

$$(1 - \tau)Q := \{x \in Q : \text{dist}(x, E \setminus Q) \geq \tau\ell(Q)\},$$

then $\mu(Q \setminus (1 - \tau)Q) \lesssim \tau^\alpha$ for α and implied constant depending on the doubling constant for μ .

We next recall a theorem from [HMM14] (which is a bilateral version of the Coronization theorem of David and Semmes [DS91]).

Lemma 6.3. Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain with UR boundary, $K \gg 1$, $0 < \eta \ll 1$, and let $\sigma = \mathcal{H}^d|_{\partial\Omega}$. Let \mathcal{D} denote the cubes from Theorem 6.2 for $E = \partial\Omega$. Then we may partition $\mathcal{D} = \mathcal{G} \cup \mathcal{B}$ such that the following hold:

- (1) The cubes in \mathcal{B} satisfy a Carleson packing condition:

$$\sum_{\substack{Q \subseteq R \\ Q \in \mathcal{B}}} \sigma(Q) \lesssim \sigma(R) \quad \text{for all } R \in \mathcal{D}.$$

- (2) $\mathcal{G} = \bigcup_{S \in \mathcal{F}} S$ where each $S \in \mathcal{F}$ is a stopping-time region, meaning
- (a) S contains a unique maximal element $Q(S)$ so that $Q \subseteq Q(S)$ for all $Q \in S$.
 - (b) If $Q \in S$ and if $Q \subseteq R \subseteq Q(S)$, then $R \in S$,
 - (c) If $Q \in S$, either all of its children are in S or none of them are. This last property ensures that every $x \in Q(S)$ is either contained in infinitely many cubes from S or it is contained in a minimal cube that does not properly contain other cubes from S . We denote the set of minimal cubes $m(S)$.
- (3) The cubes $\{Q(S) : S \in \mathcal{F}\}$ have a Carleson packing condition:

$$\sum_{\substack{Q(S) \subseteq R \\ S \in \mathcal{F}}} \sigma(Q(S)) \lesssim \sigma(R) \quad \text{for all } R \in \mathcal{D}.$$

- (4) For each $S \in \mathcal{F}$, there is a d -dimensional η -Lipschitz graph Γ_S so that, for all $Q \in S$,

$$(6.2) \quad \sup_{x \in K^2 B_Q \cap E} \text{dist}(x, \Gamma_S) + \sup_{X \in K^2 B_Q \cap \Gamma_S} \text{dist}(x, E) < \eta \ell(Q).$$

The main objective of this section is to prove the following lemma.

Lemma 6.4. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be semi-uniform with UR boundary and $\mathcal{D} = \mathcal{G} \cup \mathcal{B}$ the decomposition as in Lemma 6.3. For each $\delta > 0$, $Q_0 \in \{Q(S) : S \in \mathcal{F}\}$, and x_0 a corkscrew point for B_{Q_0} , there is a CAD $\Omega_0 \subseteq MB_0$ (with constants depending on δ and the UR constants of $\partial\Omega$) so that*

$$\sigma(Q_0 \setminus \partial\Omega_0) < \delta \sigma(Q_0)$$

and

$$\delta_{\Omega_0}(x_0) \sim \text{diam } \Omega_0 \sim \ell(Q_0).$$

The rest of this section is dedicated to the proof of 6.4. The arguments below take inspiration not just from [BH17] and [HM14], but also from [DS91, Chapter 16]. We understand that there are many constructions of chord-arc subdomains for uniform domains and NTA domains, and though we are working in semi-uniform domains, the details are similar and in some cases identical to steps in these other cases (see for example [HM14]). However, to spare the reader the task of checking these references to adapt the steps themselves, we present all of them here so there is no ambiguity.

First we give a vague sketch of the proof. According to the corona decomposition, we can decompose the cubes in Q_0 into bad cubes \mathcal{B} and stopping time regions $S \in \mathcal{F}$. For each $S \in \mathcal{F}$, $\partial\Omega$ is well approximated by a graph Γ_S near cubes in S . Thus, we can easily construct two Lipschitz domains Ω_S^+ and Ω_S^- above and below the graph whose boundaries are

close to $\partial\Omega$ near cubes in S . In particular, if $x \in Q_0$ is contained in infinitely many $Q \in S$, then $x \in \partial\Omega_S^\pm$. We then connect some of these into one CAD as follows: first, $Q_0 = Q(S_0)$ for some stopping-time region S_0 , so we add either of $\Omega_{S_0}^\pm$ to our CAD (whichever contains the corkscrew point x_0 , and one of them should since $\partial\Omega$ is very flat inside KB_{Q_0} because of 6.2). For the stopping-time regions S just below S_0 , semi-uniformity implies we may connect at least one of the Ω_S^\pm (say it is Ω_S^+) up to Ω_{S_0} by a Harnack chain, and we declare Ω_S to be Ω_S^+ plus this chain. We continue so forth, adding CADs Ω_S for N levels of stopping-time regions S to construct one large CAD Ω_0 . We pick N large enough so that, by the Carleson packing condition, most of Q_0 will be contained in at least one of the $\partial\Omega_S$.

We have to be more careful than this, however, since we could add two CADs Ω_S and $\Omega_{S'}$ corresponding to stopping-time regions S and S' for which $Q(S)$ and $Q(S')$ are adjacent, and if $\partial\Omega_S$ and $\partial\Omega_{S'}$ contain all of $Q(S)$ and $Q(S')$, this could cause a pinch in the domain Ω_0 . To remedy this, we also remove a small neighborhood of the boundaries of the $Q(S)$ and remove cubes from our stopping-times that fall into these gaps. In this way, the points in $\partial\Omega_S$ and $\partial\Omega_{S'}$ that are close to $\partial\Omega$ will be far enough away from each other.

We proceed with the proof. Let $\delta > 0$. Fix $S_0 \in \mathcal{F}$ and set $Q_0 = Q(S_0)$. Let N be large (we will choose it shortly). Let

$$\mathcal{T} = \mathcal{B} \cup \{Q(S) : S \in \mathcal{F}\} \cup \bigcup_{S \in \mathcal{F}} m(S).$$

and

$$E_0 = \{x \in Q_0 : \sum_{\substack{Q \in \mathcal{T} \\ Q \subseteq Q_0}} \mathbb{1}_Q \geq N\}.$$

Then

$$\sigma(E_0) \leq \frac{1}{N} \int \sum_{\substack{Q \in \mathcal{T} \\ Q \subseteq Q_0}} \mathbb{1}_Q d\sigma \leq \frac{1}{N} \sum_{\substack{Q \in \mathcal{T} \\ Q \subseteq Q_0}} \sigma(Q) \lesssim \frac{\sigma(Q_0)}{N}.$$

We fix N so that

$$\sigma(E_0) < \frac{\delta}{2} \sigma(Q_0).$$

Recall from Theorem 6.2 that, for $\tau > 0$,

$$\sigma(Q \setminus (1 - \tau)Q) \lesssim \tau^\alpha \sigma(Q).$$

Let

$$E_1 = \bigcup_{\substack{Q \in \mathcal{F} \\ Q \subseteq Q_0}} Q \setminus (1-\tau)Q.$$

Then

$$\sigma(E_1) \leq \sum_{\substack{S \in \mathcal{F} \\ Q(S) \subseteq Q_0}} \sigma(Q(S) \setminus (1-\tau)Q(S)) \lesssim \sum_{\substack{S \in \mathcal{F} \\ Q(S) \subseteq Q_0}} \tau^\alpha \sigma(Q(S)) \lesssim \tau^\alpha \sigma(Q_0).$$

So for τ small enough

$$\sigma(E_1) < \frac{\delta}{2} \sigma(Q_0).$$

Let

$$G = Q_0 \setminus (E_0 \cup E_1).$$

Then

$$(6.3) \quad \sigma(Q_0 \setminus G) < \delta \sigma(Q_0).$$

For $S \in \mathcal{F}$ with $Q(S) \subseteq Q_0$, let Q_j be the cubes that have a sibling not intersecting G and set

$$\tilde{S} = \{Q \in S : Q \not\subseteq Q_j \text{ for some } j\}$$

and

$$\tilde{\mathcal{F}} = \{\tilde{S} : S \in \mathcal{F}, Q(S) \subseteq Q_0\}.$$

Remark 6.5. We make a few remarks:

- (1) Note that for δ small, there is $c(\delta) \in (0, 1)$ so that any $Q \subseteq Q_0$ with $\ell(Q) \geq c(\delta) \ell(Q_0)$ intersects G .
- (2) Each $S \in \tilde{\mathcal{F}}$ are stopping-time regions.
- (3) Each $x \in G$ is contained in infinitely many cubes from one $S \in \tilde{\mathcal{F}}$, and is only contained in N many $Q(S)$ with $S \in \tilde{\mathcal{F}}$.
- (4) Finally,

(6.4) if $Q, R \in \mathcal{F}$ and $Q \cap R = \emptyset$, then

$$\text{dist}(Q \cap G, R \cap G) > \tau \max\{\ell(Q), \ell(R)\}.$$

For $n \geq 0$, let $\mathcal{F}_0 = \{S_0\}$ (recall S_0 is the stopping time region with $Q(S_0) = Q_0$) and \mathcal{F}_n denote those $\tilde{S} \in \tilde{\mathcal{F}}$ that are properly contained in n many cubes of the form $Q(\tilde{S}') \subseteq Q_0$ where $\tilde{S}' \in \mathcal{F}$ and $Q(\tilde{S}') \cap G \neq \emptyset$. So in particular, $\mathcal{F}_0 = \{\tilde{S}_0\}$ and \mathcal{F}_n are those stopping-times $\tilde{S} \in \tilde{\mathcal{F}}$ that are properly contained in n other $Q(\tilde{S}) \subseteq Q_0$ with $\tilde{S} \in \tilde{\mathcal{F}}$.

Let $S \in \tilde{\mathcal{F}}$, and let $\hat{S} \in \mathcal{F}$ be so that $\tilde{S} = S$. For $x \in \mathbb{R}^{d+1}$, define

$$d_S(x) = \inf_{Q \in S} (\text{dist}(x, Q) + \ell(Q)).$$

Note that d_S is 1-Lipschitz: indeed, if $x, y \in \mathbb{R}^{d+1}$ and $Q \in S$, then

$$d_S(x) \leq \text{dist}(x, Q) + \ell(Q) \leq |x - y| + \text{dist}(y, Q) + \ell(Q),$$

and infimizing over all $Q \in S$ gives $d_S(x) \leq |x - y| + d_S(y)$, which proves the claim.

Let $\Gamma_S := \Gamma_{\hat{S}}$, where $\Gamma_{\hat{S}}$ is from Lemma 6.3. Assume Γ_S is a graph over a d -plane P_S of a function f_S . Let e_S be a normal unit vector to P_S , π_S the projection into P_S , and for $x \in \mathbb{R}^{d+1}$, define

$$F_S(x) = \pi_S(x) + f_S(\pi_S(x))e_S.$$

Then each $x \in \Gamma_S$ can be written as

$$x = F_S(x).$$

Set

$$U_S^+ = \{x + te_S \in KB_{Q(S)} : x \in P_S, t > f_S(x) + d_S(F_S(x))\}$$

and

$$U_S^- = \{x + te_S \in KB_{Q(S)} : x \in P_S, t < f_S(x) - d_S(F_S(x))\}$$

Note that these two domains are Lipschitz domains. See Figure 4.

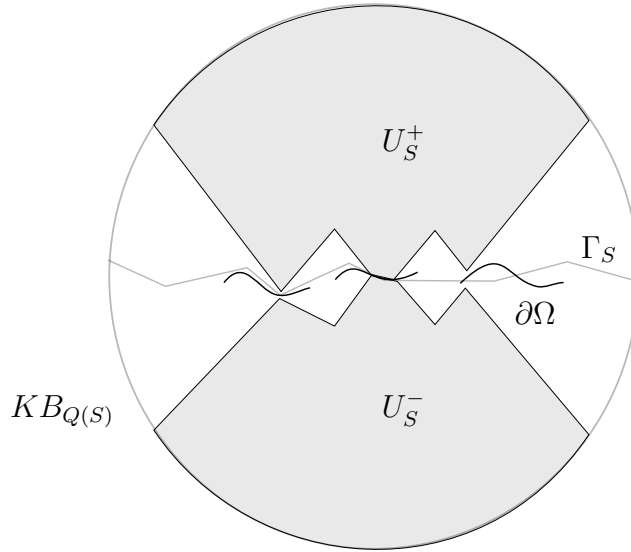


FIGURE 4. The domains U_S^\pm above and below $\partial\Omega$ in $KB_{Q(S)}$.

Let \mathscr{W} denote the Whitney cubes in Ω and define

$$\mathscr{W}_S^\pm = \{Q \in \mathscr{W} : Q \cap \overline{U_S^\pm} \neq \emptyset\}$$

and for $\lambda > 1$ close to one, let

$$\Omega_S^\pm = \bigcup_{Q \in \mathcal{W}_S^\pm} \lambda Q.$$

See Figure 5. It is not hard to show that Ω_S^\pm are CADs.

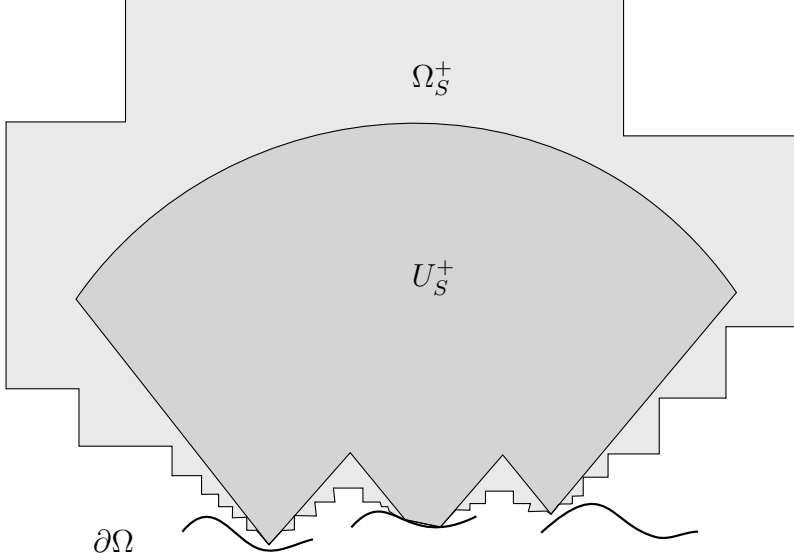


FIGURE 5. The domain Ω_S^+ is obtained by fattening U_S^+ with fattened Whitney cubes.

For $Q \in m(S)$, recall from Theorem 6.2 that ζ_Q is the center of Q . Let γ_Q^\pm be the center of the cube $I_Q^\pm \in \mathcal{W}_S^\pm$ for which

$$F_S(\pi_S(\zeta_Q)) \pm d_S(F_S(\zeta_Q))e_S \in I_Q^\pm.$$

Note that $F_S(\pi_S(\zeta_Q)) \in \Gamma_S \cap \frac{K}{2}B_{Q(S)}$ by definition and for $K \gg 1$, so the vector above is in $U_S^\pm \cap \frac{K}{2}B_{Q(S)}$, hence I_Q^\pm is well defined.

Note that by the interior corkscrew property, at least one of the domains Ω_S must be in Ω , and both could be. We will pick exactly one of them for each $S \in \mathcal{F}$ as follows.

Pick a corkscrew point for B_{Q_0} . For $\eta > 0$ small enough, we can guarantee it is also a corkscrew point for either $\Omega_{S_0}^\pm$, say it is $\Omega_{S_0}^+$ and set

$$\Omega_{S_0} = \Omega_{S_0}^+ \quad \text{and} \quad U_{S_0} = U_{S_0}^+.$$

Let $n \geq 0$ and suppose we have defined Ω_S for all $S \in \mathcal{F}_n$. Let $S \in \mathcal{F}_{n+1}$. Note that by assumption, at most N ancestors of $Q(S)$ are not in some other stopping time $S' \in \tilde{S}$. In particular, if $S' \in \mathcal{F}_n$ is such that $Q(S) \subseteq Q(S')$, there is $\hat{Q}(S) \in m(S')$ so that $\hat{Q}(S) \supseteq Q(S)$ and $\ell(\hat{Q}(S))/\ell(Q(S)) \lesssim_N 1$.

By semi-uniformity, there is a Harnack chain from x_S (which we define to be whichever of $\gamma_{Q(S)}^\pm$ is in $\Omega_{S'}$) to a corkscrew in $B_{Q(S)}$. See Figure 6 below. However, by (6.2) and since Γ_S is η -Lipschitz, for $\eta > 0$ small enough, we know this corkscrew point is also a corkscrew point for either Ω_S^\pm , we suppose it is Ω_S^+ and call the corkscrew point $x(S)$.

Let T'_S be the union of the balls in this Harnack chain and let

$$T_S = \bigcup_{\substack{Q \in \mathcal{W} \\ Q \cap T'_S \neq \emptyset}} \lambda Q.$$

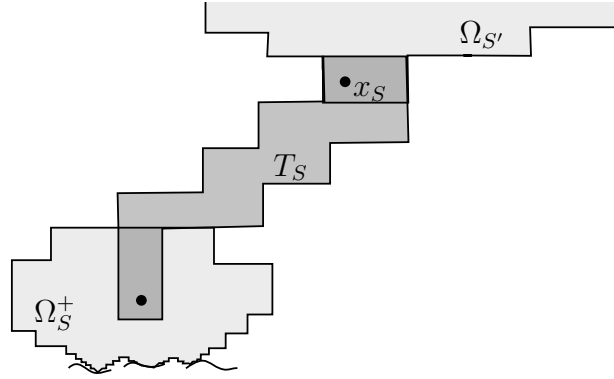


FIGURE 6. This shows how Ω_S constructed by adding to Ω_S^+ a path of cubes T_S from Ω_S^+ to the domain $\Omega_{S'}$ above it.

We now set

$$\Omega_S = \Omega_S^+ \cup T_S.$$

Note that

$$x_S \in \Omega_{S'} \cap \Omega_S$$

and is a corkscrew point for each of these domains in the sense that

$$\delta_{\Omega_S}(x_S) \sim \delta_{\Omega_{S'}}(x_S) \sim \delta_{\Omega}(x_S) \sim \ell(Q(S)).$$

It is also not hard to show that each Ω_S is a CAD.

Now we define

$$\Omega_0 = \bigcup_{S \in \mathcal{F}} \Omega_S.$$

Note that $\text{diam } \Omega_0 \sim \ell(Q_0)$, and it is not hard to see that

$$\partial\Omega_0 \cap \partial\Omega = \overline{G}.$$

Also, let \mathscr{W}_0 be all cubes $Q \in \mathscr{W}$ that intersect $U_S \cup T_S$ for some $S \in \tilde{\mathscr{F}}$, then we have by definition that

$$\Omega_0 = \bigcup_{Q \in \mathscr{W}_0} \lambda Q.$$

Lemma 6.6. *For $S \in \tilde{\mathscr{F}}$,*

$$(6.5) \quad \text{dist}(x, \partial\Omega) \lesssim \eta d_S(x) \text{ for all } x \in \Gamma_S \cap K^2 B_{Q(S)}$$

and

$$(6.6) \quad \text{dist}(x, \Gamma_S) \lesssim \eta d_S(x) \text{ for all } x \in \partial\Omega \cap K^2 B_{Q(S)}$$

Proof. First we'll prove (6.5), the proof of (6.6) is identical. Let $x \in \Gamma_S \cap K^2 B_{Q(S)}$ and $Q' \in S$ be such that

$$\ell(Q') + \text{dist}(x, Q') \leq 2d_S(x).$$

We divide into a few cases.

- (1) If $\ell(Q') > \text{dist}(x, Q')$, then $x \in K^2 B_Q$ for $K \gg 1$, and so (6.2) implies

$$\text{dist}(x, \partial\Omega) < \eta \ell(Q') \leq 2\eta d_S(x).$$

- (2) Now suppose $\ell(Q') \leq \text{dist}(x, Q')$. Replace Q' with its largest ancestor $Q \in S$ for which $\ell(Q) \leq \text{dist}(x, Q)$, so we still have

$$\ell(Q) + \text{dist}(x, Q) \leq 2 \text{dist}(x, Q) \leq 2 \text{dist}(x, Q') \leq 4d_S(x).$$

- (a) If $Q \neq Q(S)$, then $\ell(Q) \sim \text{dist}(x, Q)$, so for K large enough, $x \in K^2 B_Q$, and so (6.2) implies

$$\text{dist}(x, \partial\Omega) < \eta \ell(Q) \lesssim \eta d_S(x).$$

- (b) If $Q = Q(S)$, then

$$\ell(Q) \leq \text{dist}(x, Q) \leq \text{diam } K^2 B_{Q(S)} = 2K^2 \ell(Q(S)) = 2K^2 \ell(Q).$$

Thus, since $x \in K^2 B_{Q(S)}$, (6.2) again implies

$$\text{dist}(x, \partial\Omega) < \eta \ell(Q(S)) = \eta \ell(Q) \leq 2\eta d_S(x).$$

□

Lemma 6.7.

$$\text{dist}(x, G) \lesssim \text{dist}(x, \partial\Omega) \text{ for all } x \in \Omega_0.$$

In particular, this follows from

$$(6.7) \quad d_S(x) \sim \text{dist}(x, G \cap Q(S)) \sim \text{dist}(x, \partial\Omega) \text{ for all } x \in \Omega_S.$$

Proof. Suppose $x \in \Omega_S$, then $x \in \lambda I$ for some $I \in \mathscr{W}_0$ so that $I \cap U_S \neq \emptyset$. Let $y \in I \cap U_S$ and $y' = F_S(\pi_S(y)) \in \Gamma_S$.

We first claim that

$$(6.8) \quad \text{dist}(y, \partial\Omega) \sim |y - y'|.$$

Note that since $y \in U_S^+$,

$$(6.9) \quad \begin{aligned} |y - y'| &\geq (y - y') \cdot e_S = y \cdot e_S - F_S(\pi_S(y)) \cdot e_S \\ &> f_S(\pi_S(y)) + d_S(F_S(y)) - f_S(\pi_S(y)) = d_S(F_S(y)). \end{aligned}$$

This and the fact that Γ_S is a Lipschitz graph imply

$$(6.10) \quad d_S(y') \leq |y - y'| \sim \text{dist}(y, \Gamma_S).$$

By (6.2) we have $\text{dist}(\zeta_{Q(S)}, \Gamma_S) < \eta\ell(Q(S))$, and since $y \in KB_{Q(S)}$,

$$(6.11) \quad |y - y'| \sim \text{dist}(y, \Gamma_S) \leq |y - \zeta_{Q(S)}| + \eta\ell(Q(S)) < (K+1)\ell(Q(S)),$$

and so

$$(6.12) \quad |y' - \zeta_{Q(S)}| \leq |y' - y| + |y - \zeta_{Q(S)}| \stackrel{(6.11)}{\leq} \lesssim K\ell(Q(S)).$$

Thus, for some constant C and for K large enough (depending on C),

$$y' \in \Gamma_S \cap CKB_{Q(S)} \subseteq K^2B_{Q(S)}.$$

Hence, for $\eta \ll 1$, (6.5) implies

$$\text{dist}(y, \partial\Omega) \leq |y - y'| + \text{dist}(y', \partial\Omega) < |y - y'| + C\eta d_S(y') \leq 2|y - y'|.$$

So it suffices to show $|y - y'| \lesssim \text{dist}(y, \partial\Omega)$.

If $B(y, |y - y'|) \cap \partial\Omega = \emptyset$, then the inequality follows immediately, so assume instead that there is

$$(6.13) \quad \zeta \in B(y, |y - y'|) \cap \partial\Omega.$$

Since $y, KB_{Q(S)}$ and $y' \in CKB_{Q(S)}$, we have

$$\zeta \in (C+1)KB_{Q(S)} \subseteq K^2B_{Q(S)}$$

so by (6.6),

$$\text{dist}(\zeta, \Gamma_S) < \eta d_S(\zeta) \leq \eta(d_S(y') + |y' - \zeta|) \stackrel{(6.13)}{\leq} \eta(d_S(y') + 2|y' - y|) \stackrel{(6.9)}{\lesssim} \eta|y - y'|.$$

Let $z = F_S(\pi_S(\zeta))$. Since Γ_S is a η -Lipschitz graph, the previous inequality implies $|\zeta - z| \lesssim \eta|y - y'|$ and so

$$\begin{aligned} e_S(y' - \zeta) &\leq e_S \cdot (y' - z) + C\eta|y - y'| \leq \eta|\pi_S(y') - \pi_S(z)| + C\eta|y - y'| \\ &\leq \eta|y' - \zeta| + C\eta|y - y'| \lesssim \eta|y - y'|. \end{aligned}$$

Thus, for all $\zeta \in B(y, |y - y'|) \cap \partial\Omega$,

$$|y - \zeta| \geq e_S \cdot (y - \zeta) \geq e_S \cdot (y - y') + e_S \cdot (y' - \zeta) \geq |y - y'| - C\eta|y - y'| \gtrsim |y - y'|.$$

Infimizing over all $\zeta \in B(y, |y - y'|) \cap \partial\Omega$, we get

$$\text{dist}(y, \partial\Omega) = \text{dist}(y, B(y, |y - y'|) \cap \partial\Omega) \gtrsim |y - y'|.$$

And this proves (6.8).

Since I is a Whitney cube, $\text{dist}(z, \partial\Omega) \sim \text{diam}(I)$ for all $z \in \lambda I$, and so

$$d_S(x) \leq d_S(y') + |y' - y| + |y - x| \stackrel{(6.9)}{\leq} 2|y' - y| + \text{diam } I \stackrel{(6.8)}{\lesssim} \text{dist}(y, \partial\Omega)$$

and so

$$d_S(x) \lesssim \text{dist}(y, \partial\Omega) \sim \text{dist}(x, \partial\Omega) \leq \text{dist}(x, G \cap Q(S)).$$

Moreover, if $T \in S$ is such that $\text{dist}(x, T) + \ell(T) < 2d_S(x)$, then $T \cap G \neq \emptyset$, so we have that

$$\text{dist}(x, G \cap Q(S)) \leq \text{dist}(x, T) + \text{diam } T \lesssim d_S(x).$$

□

Lemma 6.8. *The domain Ω_0 has the interior corkscrew property.*

Proof. Let $z \in \partial\Omega_0$ and $0 < r < \text{diam } \Omega_0$. Then

$$d_{S_0}(x) \leq \ell(Q_0) + \text{diam } \Omega_0 \lesssim \ell(Q_0),$$

and so $d_{S_0}(x) < M\ell(Q_0)$ for some number M depending on the semi-uniformity of Ω .

Let $x \in \partial\Omega_0$ and $0 < r < \text{diam } \partial\Omega_0$. We split into several cases.

- (1) If $x \in \partial\lambda I$ for some $I \in \mathscr{W}_0$ and $r \leq M\ell(I)$, then we can pick any ball inside $B(x, r) \cap \lambda I$ of radius comparable to r .
- (2) Suppose $x \in \partial\lambda I$ for some $I \in \mathscr{W}_0$ and $M\ell(I) < r < \text{diam } \partial\Omega_0$. Note that by (6.7),

$$\text{dist}(x, G) \lesssim \text{dist}(x, \partial\Omega) \lesssim \ell(I) \leq M^{-1}r,$$

hence for M large enough, we can find $y \in G \cap B(x, \frac{r}{2})$. Thus, we just need to find an interior corkscrew for $B(y, r/2)$, since it will then also be one for $B(x, r)$.

Let $S \in \tilde{\mathcal{F}}$ be so that $y \in Q(S)$ and $\ell(Q(S))$ is minimal such that $\ell(Q(S)) > r/M^2$.

- (2.a) If $\ell(Q(S)) < r/M$, then for M large enough, $\Omega_S \subseteq B(x, r)$ and x_S is a corkscrew point for $B(x, r)$.
- (2.b) If $\ell(Q(S)) \geq r/M$, then recall from Remark 6.5 that since $y \in G$, y is contained in infinitely many cubes from some stopping time in \tilde{F} .

- (2.b.i) If this stopping time is not S , then there is $S'\tilde{F}$ a maximal stopping time for which $Q(S') \subsetneq Q(S)$ and $y \in Q(S')$. Then $\ell(Q(S')) \leq r/M^2$ by the minimality of S' , and so $\Omega_{S'} \subseteq B(y, \frac{r}{4M})$ for M large enough. Since Ω_S is chord-arc, it is uniform and there is a cigar curve γ of bounded turning between $x_{S'}$ and x_S . Let $z \in \partial B(y, \frac{r}{2M}) \cap \gamma$. Then

$$\begin{aligned} \delta_{\Omega_0}(z) &\geq \delta_{\Omega_S}(z) \gtrsim \min\{|z - x_{S'}|, |z - x_S|\} \\ &\geq \min\left\{\frac{r}{2N} - \frac{r}{4M}, \frac{r}{M} - \frac{r}{2M}\right\} \gtrsim \frac{r}{M}. \end{aligned}$$

Thus, z is a corkscrew point for $B(y, r/2)$ with constant depending on M .

- (2.b.ii) If it is S , then $y \in \partial\Omega_S$, and we can connect y to x_S directly with a cigar curve of bounded turning in Ω_S and then the proof is just as in the previous case.

- (3) If $x \in \overline{G} = \partial\Omega \cap \partial\Omega_0$, then we repeat the argument in Case 2 with x in place of y .

□

Lemma 6.9. *The domain Ω_0 is uniform.*

Proof. Let $x, y \in \Omega_0$, $\varepsilon = \min\{\delta_{\Omega_0}(x), \delta_{\Omega_0}(y)\}$, and $\Lambda = |x - y|$. By Lemmas 2.2 and 6.10, it suffices to show that there is a Harnack chain between every such x and y of length at most $C \log \frac{\Lambda}{\varepsilon} + 1$. As in [HM14], we will first make some reductions.

Case 1. First, we may assume that any cubes $Q, R \in \mathscr{W}_0$ containing x and y respectively satisfy

$$(6.14) \quad \text{dist}(\lambda Q, \lambda R) \gtrsim \max\{\ell(Q), \ell(R)\}.$$

Since otherwise Q and R must be adjacent and $\lambda Q \cup \lambda R$ forms an NTA domain and it is easy to find a Harnack chain in Ω_0 between x and y .

Case 2. Next, we claim it suffices to assume x and y are centers of cubes in \mathscr{W}_0 .

Indeed, if $x \in \lambda Q$ and $\delta_{\Omega_0}(x) \gtrsim \ell(Q)$, then it is not hard to find a short Harnack chain from x to x_Q . If $\delta_{\Omega_0}(x) < \ell(Q)$, then we may connect x to a point $x' \in B(x, \delta_{\Omega_0}(x)) \cap \lambda Q$ of distance at least a constant times $\delta_{\Omega_0}(x)$ from $\partial\lambda Q$, and thus there is a Harnack chain from that point to x_Q of length depending on $\log \text{diam } \lambda Q / \delta_{\Omega_0}(x) \lesssim \log \Lambda / \varepsilon$. Similarly, if $y \in \lambda R$ for some $R \in \mathscr{W}_0$, we can connect y to x_R by a chain of length depending on R . Then, by assumption, we can connect x_Q and x_R by a chain of length at

most a constant times

$$\log \frac{|x_Q - x_R|}{\min\{\delta_{\Omega_0}(x_Q), \delta_{\Omega_0}(x_R)\}} + 1 \lesssim \log \frac{\Lambda}{\varepsilon} + 1$$

where in the last inequality we used the fact that (6.14) implies $|x_Q - x_R| \sim |x - y|$. Combining these two chains together gives the desired chain between x and y .

Case 3. We now assume $x = x_Q$ and $y = x_R$ for some cubes $Q, R \in \mathcal{W}_0$ satisfying (6.14).

Case 3.a. Suppose $x_Q, x_R \in \Omega_S$. Then since Ω_S is a CAD, it is not hard to find the desired Harnack chain between x_Q and x_R .

Case 3.b. Suppose $x_Q \in \Omega_{S_1}$ and $x_R \in \Omega_{S_n} \setminus \Omega_{S_{n-1}}$ for some $S_1, S_n \in \tilde{\mathcal{F}}$ where $Q(S_1) \subseteq Q(S_n)$ and there are S_2, \dots, S_{n-1} so that $Q(S_i) \subseteq Q(S_{i+1})$.

Note that since $Q \in \mathcal{W}_S$ and $\lambda Q \subseteq \Omega_S \subseteq \Omega_0$,

$$\ell(Q) \lesssim \delta_{\Omega_S}(x_Q) \leq \delta_{\Omega_0}(x_Q) \leq \delta_{\Omega}(x_Q) \sim \ell(Q).$$

Since Ω_{S_1} is a CAD, there is a Harnack chain from x_Q to x_{S_1} of length

$$\lesssim \log \frac{\ell(Q(S_1))}{\delta_{\Omega_{S_1}}(x_Q)} + 1 \lesssim \log \frac{\ell(Q(S_1))}{\min\{\ell(Q), \ell(R)\}} + 1.$$

For each i , there is a Harnack chain in $\Omega_{S_{i+1}}$ from x_{S_i} to $x_{S_{i+1}}$ of length

$$\lesssim \log \frac{|x_{S_i} - x_{S_{i+1}}|}{\min\{\delta_{\Omega_{S_{i+1}}}(x_{S_i}), \delta_{\Omega_{S_{i+1}}}(x_{S_{i+1}})\}} + 1 \lesssim \log \frac{\ell(Q(S_{i+1}))}{\ell(Q(S_i))} + 1.$$

Finally, we can also show there is a Harnack chain in Ω_{S_n} between $x_{S_{n-1}}$ to x_R of length

$$\lesssim \log \frac{|x_R - x_{S_{n-1}}|}{\min\{\delta_{\Omega_{S_n}}(x_{S_{n-1}}), \delta_{\Omega_{S_n}}(x_R)\}} + 1 \leq \log \frac{|x_R - x_Q| + \ell(Q(S_{n-1}))}{\min\{\ell(R), \ell(Q(S_{n-1}))\}} + 1.$$

Here, we used the fact that $|x_Q - x_{S_{n-1}}| \lesssim \ell(Q(S_{n-1}))$. Adding up these inequalities, we get that there is a Harnack chain from x_R to x_Q of length no more than

$$\begin{aligned} &\lesssim \log \frac{\ell(Q(S_1))}{\min\{\ell(Q), \ell(R)\}} + \sum_{i=1}^{n-2} \left(\log \frac{\ell(Q(S_{i+1}))}{\ell(Q(S_i))} + 1 \right) \\ &\quad + \log \frac{|x_R - x_Q| + \ell(Q(S_{n-1}))}{\min\{\ell(R), \ell(Q(S_{n-1}))\}} \\ &\lesssim \log \frac{\ell(Q(S_{n-1}))}{\min\{\ell(Q), \ell(R)\}} + \log \frac{|x_R - x_Q| + \ell(Q(S_{n-1}))}{\min\{\ell(R), \ell(Q(S_{n-1}))\}} + N \end{aligned}$$

Our goal is to show

$$(6.15) \quad \ell(Q(S_{n-1})) \lesssim |x_Q - x_R|$$

in which case the total length of our combined chain will be at most

$$(6.16) \quad \lesssim \log \frac{|x_R - x_Q|}{\min\{\ell(Q), \ell(R)\}} + N \lesssim \log \frac{\Lambda}{\varepsilon} + 1$$

Case 3.b.i. If $\max\{\ell(Q), \ell(R)\} \geq \tau^2 \ell(Q(S_{n-1}))$, then by (6.14), we also have that $|x_Q - x_R| \gtrsim \ell(Q(S_{n-1}))$, which proves (6.15).

Case 3.b.ii. Now suppose $\max\{\ell(Q), \ell(R)\} < \tau^2 \ell(Q(S_{n-1})) < \tau^2 \ell(Q(S_n))$. By (6.7), we may find $\xi_R \in Q(S_n) \cap G$ so that

$$|\xi_R - x_R| \lesssim \text{dist}(x_R, \partial\Omega) \sim \ell(R) < \tau^2 \ell(Q(S_{n-1})).$$

Similarly, we may find $\xi_Q \in Q(S_1) \cap G \subseteq Q(S_{n-1}) \cap G$ so that

$$|\xi_Q - x_Q| \lesssim \tau^2 \ell(Q(S_{n-1})).$$

Suppose $\xi_R \in Q(S_{n-1})$. Let $T_0 \in m(S_n)$ contain $Q(S_{n-1})$. If $T \in S_n$ is disjoint from T_0 , then by (6.4)

$$\begin{aligned} \text{dist}(x_R, T) + \ell(T) &\geq \text{dist}(\xi_R, T) - |\xi_R - x_R| \geq \tau \ell(T_0) - C\tau^2 \ell(Q(S_{n-1})) \\ &\gtrsim \tau \ell(Q(S_{n-1})) \end{aligned}$$

and if $T \in S_n$ contains T_0 , clearly

$$\text{dist}(x_R, T) + \ell(T) \geq \ell(T) \geq \ell(Q(S_{n-1})).$$

Thus, infimizing over all $T \in S_n$, we get

$$\tau^2 \ell(Q(S_{n-1})) > \ell(R) \stackrel{(6.7)}{\gtrsim} d_{S_n}(x_R) \gtrsim \tau \ell(Q(S_{n-1})),$$

which is a contradiction for τ small enough. Thus, $\xi_R \in Q(S_n) \setminus Q(S_{n-1})$, so $|\xi_Q - \xi_R| \geq \tau \ell(Q(S_{n-1}))$ by (6.4) since both points are in G , thus

$$\begin{aligned} |x_Q - x_R| &\geq |\xi_Q - \xi_R| - |\xi_Q - x_Q| - |\xi_R - x_R| \gtrsim \tau \ell(Q(S_{n-1})) - \tau^2 \ell(Q(S_{n-1})) \\ &\gtrsim \tau \ell(Q(S_{n-1})) \end{aligned}$$

and thus (6.15) holds in this case as well.

Case 3.c. Finally, suppose $x_Q \in \Omega_S$ and $x_R \in \Omega_{S'}$ where $Q(S) \cap Q(S') = \emptyset$. Let \tilde{S} be the minimal $\tilde{S} \in \tilde{\mathcal{F}}$ for which $Q(S), Q(S') \subseteq Q(\tilde{S})$. Let Q_S and $Q_{S'}$ be the minimal cubes in \tilde{S} containing $Q(S)$ and $Q(S')$ respectively.

We first claim that

$$(6.17) \quad |x_Q - x_R| \gtrsim \tau^2 \max\{\ell(Q_S), \ell(Q_{S'})\}.$$

If

$$\max\{\ell(Q), \ell(R)\} \geq \tau^2 \max\{\ell(Q_S), \ell(Q_{S'})\},$$

then this holds by (6.14), so assume

$$\max\{\ell(Q), \ell(R)\} < \tau^2 \max\{\ell(Q_S), \ell(Q_{S'})\}.$$

Let $\xi_Q \in G \cap Q(S)$ and $\xi_R \in G \cap Q(S')$ be closest to x_Q and x_R respectively. Then by (6.4),

$$\begin{aligned} |x_Q - x_R| &\geq |\xi_Q - \xi_R| - |\xi_Q - x_Q| - |\xi_R - x_R| \\ &\gtrsim \tau(\ell(Q_S) + \ell(Q_{S'}) - \tau^2 \max\{\ell(Q_S), \ell(Q_{S'})\}) \\ &\gtrsim \tau \max\{\ell(Q_S), \ell(Q_{S'})\} \end{aligned}$$

and this finishes the claim.

We'll first build a chain from x_S to $x_{S'}$. If $\check{S} \in \mathcal{F}_n$, let $\check{S}, \check{S}' \in \mathcal{F}_{n+1}$ be so that $Q(\check{S}) \subseteq Q_S$ and $Q(\check{S}') \subseteq Q_{S'}$ contain $Q(S)$ and $Q(S')$ respectively. Observe that $\ell(Q_S) \sim_N \ell(Q(\check{S}))$ and $\ell(Q_{S'}) \sim_N \ell(Q(\check{S}'))$. By case 3.b, we know there are chains from x_S to $x_{\check{S}} \in \Omega_{\check{S}} \cap \Omega_{\check{S}}$ and from $x_{S'}$ to $x_{\check{S}'} \in \Omega_{\check{S}'} \cap \Omega_{\check{S}}$ of total length at most

$$(6.18) \quad \log \frac{\ell(Q(\check{S}))}{\ell(Q(S))} + \log \frac{\ell(Q(\check{S}'))}{\ell(Q(S'))} + 1 \lesssim \log \frac{\max\{\ell(Q(\check{S})), \ell(Q(\check{S}'))\}}{\min\{\ell(Q), \ell(R)\}} + 1$$

Next, there is a Harnack chain from $x_{\check{S}}$ to $x_{\check{S}'}$ in $\Omega_{\check{S}}$ of length at most

$$\begin{aligned} \log \frac{|x_{\check{S}} - x_{\check{S}'}|}{\min\{\delta_{\Omega_{\check{S}}}(x_{\check{S}}), \delta_{\Omega_{\check{S}}}(x_{\check{S}'})\}} + 1 \\ \lesssim \log \frac{|x_{\check{S}} - x_{\check{S}'}|}{\min\{\ell(Q_S), \ell(Q_{S'})\}} + 1 \\ \lesssim \log \frac{|x_{\check{S}} - x_{\check{S}'}|}{\min\{\ell(Q), \ell(R)\}} + 1 \end{aligned}$$

It is not hard to show that $|x_{\check{S}} - x_Q| \lesssim \ell(Q(\check{S})) \lesssim \ell(Q_S)$ and similarly $|x_{\check{S}'} - x_R| \lesssim \ell(Q_{S'})$, and so

$$|x_{\check{S}} - x_{\check{S}'}| \lesssim |x_Q - x_R| + \ell(Q_S) + \ell(Q_{S'}) \stackrel{(6.17)}{\lesssim} \tau^{-2} |x_Q - x_R|.$$

Thus, combining the estimate (6.18) for the length of the chain from x_S to $x_{\check{S}}$ and from $x_{S'}$ to $x_{\check{S}'}$, and the estimate for the length of the chain from $x_{\check{S}}$ to $x_{\check{S}'}$, we obtain a chain from x_S to $x_{S'}$ of length at most

$$\log \frac{|x_Q - x_R|}{\min\{\ell(Q), \ell(R)\}} + 1.$$

Finally, we can connect x_Q to x_S and x_R to $x_{S'}$ by chains of total length at most

$$\lesssim \log \frac{\ell(Q(S))}{\ell(Q)} + \log \frac{\ell(Q(S'))}{\ell(R)} + 1 \lesssim \log \frac{\max\{\ell(Q(\check{S})), \ell(Q(\check{S}'))\}}{\min\{\ell(Q), \ell(R)\}} + 1$$

but this is the term in 6.18 which we have already bounded. \square

Lemma 6.10. *The domain Ω_0 has the exterior corkscrew property.*

The proof is similar to [AHM⁺17, Lemma 4.1]. We first recall some facts about UR sets. For a set E , $x \in E$, a hyperplane P , and $r > 0$ define

$$b\beta_E(x, r, P) = r^{-1} \left(\sup_{y \in E \cap B(x, r)} \text{dist}(y, P) + \sup_{y \in P \cap B(x, r)} \text{dist}(y, E) \right)$$

and then define

$$b\beta_E(x, r) = \inf_P b\beta_E(x, r, P)$$

where the infimum is over all d -dimensional hyperplanes $P \subseteq \mathbb{R}^{d+1}$.

Definition 6.11. We say that an d -dimensional ADR set E satisfies the *bilateral weak geometric lemma* or *BWGL* if, for each $\varepsilon > 0$, the set

$$B_\varepsilon := \{(x, r) : x \in E, r > 0, b\beta_E(x, r) \geq \varepsilon\}$$

is a *Carleson set*, i.e., if we define

$$\hat{\sigma}(A) = \iint_A d\mathcal{H}^d \frac{dt}{t}, \quad A \subset E \times (0, \infty),$$

then

$$(6.19) \quad \hat{\sigma}(B_\varepsilon \cap (B(x, r) \times (0, r))) \lesssim r^d$$

for all $x \in E$ and $0 < r < \text{diam } E$.

Theorem 6.12 ([DS93, Theorem 2.4, Part I]). *A d -dimensional ADR set E is uniformly rectifiable if and only if it satisfies the BWGL.*

Proof of Lemma 6.10. Let $x \in \partial\Omega_0$ and $r > 0$. If $r \leq 2\delta_\Omega(x)$, then $x \in \partial\lambda Q$ for some Whitney cube $Q \in \mathcal{W}_S$ for some $S \in \tilde{\mathcal{F}}$. Then it is not hard to see that, since Ω_0 is a union of dilated Whitney cubes that we may find a large corkscrew point in $B(x, r) \setminus \Omega_0$.

If $r > 2\delta_\Omega(x)$, then there is $y \in \partial\Omega \cap B(x, r/2)$. Since $\partial\Omega$ is uniformly rectifiable, by Theorem 6.12, (6.19) holds, and so for any $\varepsilon > 0$ and $M > 0$, we may find a ball $B(z, r')$ with $r/4 > r' \gtrsim_{\varepsilon, M} r$ and $z \in B(y, r/4) \cap \partial\Omega$ for which

$$(6.20) \quad b\beta_{\partial\Omega}(z, Mr', P) < \varepsilon.$$

for some d -plane P . By replacing ε with 2ε if need be, we can assume $z \in P$. Let $x^\pm = z \pm \frac{r'}{2}e_P$, where e_P is the unit normal vector to P .

Claim: At least one of x^+ and x^- is in Ω_0^c . Suppose instead that both were contained in Ω_0 . Then by uniformity, there is a C -cigar curve $\gamma \subseteq \Omega_0$

from x^+ to x^- . If $\zeta \in P \cap \gamma$, we must have $\zeta \in B(z, C|x^+ - x^-|) \subseteq B(z, 2Cr')$ and, since $\Omega_0 \subseteq \Omega$,

$$\begin{aligned} \text{dist}(\zeta, \partial\Omega) &\geq \text{dist}(\zeta, \partial\Omega_0) \geq C^{-1} \min\{\ell(\zeta, x^+), \ell(\zeta, x^-)\} \\ &\geq C^{-1} \min\{|\zeta - x^+|, |\zeta - x^-|\} \geq \frac{r'}{2C}. \end{aligned}$$

If we pick $M > 3C$ and $\varepsilon < \frac{1}{4CM}$, then this is a contradiction since (6.20) implies

$$\sup_{w \in P \cap B(z, Mr')} \text{dist}(w, \partial\Omega) < M\varepsilon r' < \frac{r'}{4C}.$$

This proves the claim.

Without loss of generality, we may assume $x^+ \in \Omega_0^c$. Note that by (6.20),

$$\text{dist}(x^+, \partial\Omega) \geq \text{dist}(x^+, P) - M\varepsilon r' \geq \frac{r'}{4}$$

so $B(x^+, r'/4) \subseteq B(z, r') \setminus \partial\Omega \subseteq B(x, r) \setminus \partial\Omega$. If $B(x^+, r'/8) \cap \partial\Omega_0 = \emptyset$, then $B(x^+, r'/8) \subseteq B(x, r) \setminus \Omega_0$ is our desired exterior corkscrew ball. Otherwise, there is $z' \in B(x^+, r'/8) \cap \partial\Omega_0$, and by the above displayed inequality, $z' \notin \partial\Omega$, so $z' \in \partial\lambda Q$ for some $Q \in \mathscr{W}_0$. By case 1, we may find an exterior corkscrew point in $B(z', r'/8) \setminus \Omega_0$, and then use this as our exterior corkscrew ball for $B(x, r)$. □

Lemma 6.13. *The domain Ω_0 has ADR boundary.*

Proof. Upper regularity follows from the Carleson packing condition and the fact that each Ω_S is ADR, and lower regularity follows from the interior and exterior corkscrew conditions. See for example [HM14, Appendix A.3]. □

We have now established that Ω_0 is uniform with exterior corkscrews and ADR boundary, that is, Ω_0 is a CAD, and this finishes the proof of Lemma 6.4.

7. THE PROOF OF THEOREM III

We now prove Theorem III. The implication (1) implies (2) is Theorem 6.4, and as mentioned before, (2) implies (1) is immediate, so (1) and (2) are equivalent, and (as mentioned in the introduction) (3) implies (1) follows by earlier work, so we will just show that (2) implies (3). We will first prove the A_∞ -property in cubes that are tops of stopping-time regions. Let $\delta > 0$, $Q_0 \in \{Q(S) : S \in \mathscr{F}\}$, $x \in \Omega \setminus MB_{Q_0}$ (where M is as in Lemma 5.1), and $E \subseteq Q_0$ be so that

$$\frac{\omega_\Omega^x(E)}{\omega_\Omega^x(Q_0)} < \varepsilon$$

where $\varepsilon > 0$ is a constant we will choose to be small later depending on δ . By Lemma 5.1, there is a corkscrew point x_0 for B_{Q_0} so that

$$\omega_{\Omega}^{x_0}(E) \lesssim \varepsilon.$$

Let Ω_0 be from Lemma 6.4 contain x_0 as a corkscrew point and such that

$$\mathcal{H}^d(Q_0 \setminus \partial\Omega_0) < \frac{\delta}{2} \mathcal{H}^d(Q_0).$$

Then by the maximum principle,

$$\omega_{\Omega_0}^{x_0}(E) \leq \omega_{\Omega}^{x_0}(E) \lesssim \varepsilon.$$

By the main result in [DJ90], $\omega_{\Omega_0}^{x_0}$ is an A_∞ -weight, and so for $\varepsilon > 0$ small enough, we can guarantee that

$$\mathcal{H}^d(\partial\Omega_0 \cap E) < \frac{\delta}{2} \mathcal{H}^d(Q_0)$$

Thus,

$$\mathcal{H}^d(E) < \delta \mathcal{H}^d(Q_0).$$

This proves $\omega_{\Omega}^x \in A_\infty(\mathcal{H}^d|_{\partial\Omega}, Q_0)$.

Now we prove the general result. Let $\alpha > 0$, B be an arbitrary ball centered on $\partial\Omega$. By rescaling, we can assume without loss of generality that $r_B = c_0$ where c_0 is as in Theorem 6.2, and then pick a nested sequence of maximal ρ^k -nets for $\partial\Omega$ X_k so that $x_B \in X_0$. In this way, there is a cube $Q_0 \in \mathcal{D}_0$ with center $\zeta_Q = x_B$, so $c_0 B_{Q_0} = B$, and hence $B \cap \partial\Omega \subseteq Q_0$. Let $x \in \Omega$ be so that $\text{dist}(x, MB) \geq \alpha > 0$.

Let Q_1, \dots, Q_L be the k th generation descendants of Q_0 (so there are only boundedly many depending on k) with k large enough depending on α and M , so that $x \in \Omega \setminus MB_{Q_i}$ for all i .

Let \mathcal{F}_i be stopping-time regions as in Lemma 6.3 but with Q_i in place of Q_0 and $\mathcal{F} = \bigcup \mathcal{F}_i$. Since the $\{Q(S) : S \in \mathcal{F}\}$ satisfy a Carleson packing condition, it is not hard to show that we may find finitely many S_1, \dots, S_N so that

- (1) $Q(S_i) \subseteq Q_j$ for some j
- (2) $Q(S_i) \cap Q(S_j) = \emptyset$ when $i \neq j$,
- (3) $N \sim_{\delta'} 1$,
- (4) $\mathcal{H}^d(Q_0 \setminus \bigcup_{i=1}^N Q(S_i)) < \delta' \mathcal{H}^d(Q_0)$

where δ' is a small number we will fix shortly.

Let $E \subseteq B \cap \partial\Omega$ and suppose $\omega_{\Omega}^x(E) < \varepsilon$. By the doubling property for ω_{Ω} ,

$$\omega_{\Omega}^x(E \cap Q(S_i)) \leq \omega_{\Omega}^x(E) < \varepsilon \omega_{\Omega}^x(B) \lesssim_N \varepsilon \omega_{\Omega}^x(Q(S_i)).$$

and since $\omega_\Omega^x \in A_\infty(Q(S_i))$ for each i , and $\varepsilon > 0$ small enough (depending on N and hence only on δ), we then have

$$\begin{aligned} \mathcal{H}^d(E) &\leq \mathcal{H}^d(E \setminus \bigcup_{i=1}^N Q(S_i)) + \sum_{i=1}^N \mathcal{H}^d(E \cap Q(S_i)) \\ &\leq \delta' \mathcal{H}^d(Q_0) + \delta' \sum_{i=1}^N \mathcal{H}^d(Q(S_i)) < \delta' \mathcal{H}^d(Q_0) \\ &\lesssim \delta' \mathcal{H}^d(B \cap \partial\Omega) \end{aligned}$$

and so for δ' small enough, we have $\mathcal{H}^d(E) < \delta \mathcal{H}^d(B \cap \partial\Omega)$, as desired.

APPENDIX A. HRYCÁK'S EXAMPLE

Here we sketch the construction of Hrycak's example and how to use it to give a semi-uniform domain with UR boundary but without BPLS. We fix an integer n and define a set $E \subseteq \mathbb{R}^2$ using the method of Venician blinds. Let $E_0 = I_0$ be the unit line segment. Now divide it into n half-open sub-intervals and rotate each interval $\theta = \frac{2\pi}{n}$ radians counterclockwise around its left endpoint, call this new set E_1 , see Figure 7. Let the new intervals be called I_1, \dots, I_n , ordered by where their left endpoint lies along the real line.

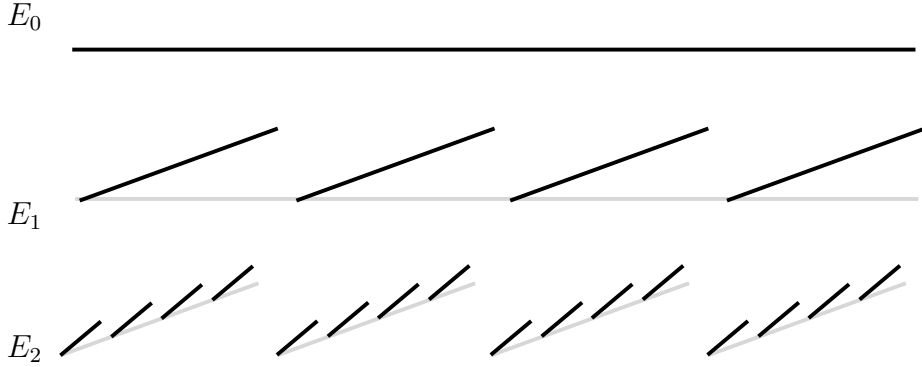


FIGURE 7. Hrycak's example.

Repeat this on each new interval and so on for a total of n times, defining sets E_2, \dots, E_n along the way, so that E_j is a union of a set of n^j half-open intervals I_α where α is a multi-index $\alpha = k_1 \dots k_j$ and $k_i \in \{1, 2, \dots, n\}$, and for $|\alpha| = j$ and each $i \in \{1, 2, \dots, n\}$, $I_{\alpha i}$ is the i th subinterval of I_α rotated about its left endpoint.

The resulting set $E = E_n$ is an ADR set whose orthogonal projections in the directions $0, \theta, 2\theta, \dots, (n-1)\theta$ are at most Cn^{-1} for some constant C . In particular, in order for a Lipschitz graph to intersect at least $c\%$ of it, say, the Lipschitz constant must be at least a constant times n .

The remainder of the proof focuses on showing that E^c is semi-uniform. We will do this by showing that E bi-Lipschitz embeds into the real line. By a theorem of MacManus, we can extend the inverse of this map to a global bi-Lipschitz map of \mathbb{R}^2 , and hence E^c is the bi-Lipschitz image of a domain of the form $\mathbb{R}^2 \setminus A$ where $A \subseteq \mathbb{R}$, and it is immediate that such domains are semi-uniform. Now we begin with the details.

We now construct a bi-Lipschitz embedding of E into the real line. Let K_1, \dots, K_n be the intervals in the unit interval (ordered by their position on the real line) obtained by deleting $n - 1$ equally spaced intervals of length n^{-2} . Then

$$|K_j| = (1 - (n - 1) \cdot \frac{1}{n^2})/n =: \frac{c_n}{n}.$$

If $|\alpha| = j - 1$ and we have defined K_α , let $K_{\alpha 1}, \dots, K_{\alpha n}$ be the intervals in K_α (ordered by their position on the real line) obtained by removing $n - 1$ intervals of length $|K_\alpha|n^{-2}$. Note that since $c_n \geq \frac{1}{2}$,

$$|K_{\alpha i}| = |K_\alpha| \frac{c_n}{n} = \frac{c_n^j}{n^j} \in [n^{-j}/2, n^{-j}]$$

and the gaps between the $K_{\alpha i}$ are

$$|K_\alpha|n^{-2} \in [n^{-j-2}/2, n^{-j-2}].$$

It is not hard to show that, for $|\alpha| < n$, $\bigcup_{i=1}^n I_{\alpha i}$ bi-Lipschitz embeds into I_α by some L -bi-Lipschitz map f_α (with L just some constant independent of α) so that its image is I_α with $n - 1$ equally spaced intervals of length $|I_\alpha|c_n n^{-1}$.

In particular, if $J_{\alpha 1}, \dots, J_{\alpha n}$ are these intervals in order along I_α , then $f_\alpha : I_{\alpha i} \rightarrow J_{\alpha i}$ is affine and surjective, and

$$\frac{|f_\alpha(I_{\alpha i})|}{|I_{\alpha i}|} = c_n$$

We let f_0 be the map that takes $\bigcup_{i=1}^n I_i$ into the unit interval I_0 . Hence, if $\alpha = \alpha_1 \cdots \alpha_k$, define

$$F_\alpha := f_0 \circ f_{\alpha_1} \circ f_{\alpha_1 \alpha_2} \circ \cdots \circ f_{\alpha_1 \alpha_2 \cdots \alpha_{k-1}},$$

then F_α is an affine c_n^k -bi-Lipschitz map on I_α , but since $c_n^n \sim 1$ for all n , we get that F_α is C -bi-Lipschitz on I_α for some universal constant C .

For $x \in I_\alpha \subseteq E$ with $|\alpha| = n$, we define $F(x) = F_\alpha(x)$. Moreover, we can do this in such a way that $F(I_\alpha) = K_\alpha$ for all $|\alpha| = n$.

We will now show this map is bi-Lipschitz.

For $x \in I_\beta$ with $|\beta| = n$ and $\alpha = \beta_1 \cdots \beta_j$ with $j < n$, define

$$f^\alpha(x) := f_{\beta_1 \cdots \beta_{j-1}} \circ \cdots \circ f_{\beta_1 \cdots \beta_{n-1}}(x) \in I_\alpha$$

This defines a map $f^\alpha : E \rightarrow E_j$ so that $F_\alpha(f^\alpha(x)) = F(x)$.

If $x \in I_\alpha$ with $|\alpha| = n$, then if $\beta = \alpha_1 \dots \alpha_j$

$$(A.1) \quad |f^\beta(x) - x| \lesssim \sum_{i=j+1}^n \frac{1}{n^{i+1}} \lesssim \frac{1}{n^{j+2}}.$$

Now let $x \in I_\alpha$ and $y \in I_{\alpha'}$ with $|\alpha| = |\alpha'| = n$, we can assume $\alpha \neq \alpha'$. Let α_0 be the largest common truncation of α and α' , or equivalently, so that K_{α_0} is the smallest common ancestor to K_α and $K_{\alpha'}$, set $j = |\alpha_0|$ and $x_0 = f^{\alpha_0}(x)$ and $y_0 = f^{\alpha_0}(y)$. Observe that $F(x) = F_{\alpha_0}(x_0) \in K_\beta$ and $F(y) = F_{\alpha_0}(y_0) \in K_{\beta'}$ where K_β and $K_{\beta'}$ are the intervals created from K_{α_0} so that β is a truncation of α and β' is a truncation of α' . Hence

$$|x_0 - y_0| \gtrsim |F_{\alpha_0}(x_0) - F_{\alpha_0}(y_0)| \geq \text{dist}(K_\beta, K_{\beta'}) \gtrsim n^{-2} |K_{\alpha_0}| \gtrsim n^{-j-2}.$$

If $|x - y| \geq Mn^{-j-2}$, then

$$\left| |x - y| - |x_0 - y_0| \right| \leq |x - x_0| + |y - y_0| \stackrel{(A.1)}{\lesssim} n^{-j-2} \lesssim M^{-1} |x - y|$$

and so for M large enough, $|x - y| \sim |x_0 - y_0|$. Hence,

$$|x - y| \sim |x_0 - y_0| \sim |F_{\alpha_0}(x_0) - F_{\alpha_0}(y_0)| = |F(x) - F(y)|.$$

If $|x - y| < Mn^{-j-2}$, then, because $x \in I_\alpha$ and $y \in I_{\alpha'}$ and $|\alpha| = |\alpha'| \geq j + 1$, for n large enough,

$$|x - y| \geq \text{dist}(I_\alpha, I_{\alpha'}) \gtrsim n^{-j-2}$$

hence

$$\begin{aligned} |F(x) - F(y)| &= |F_{\alpha_0}(x_0) - F_{\alpha_0}(y_0)| \lesssim |x_0 - y_0| \leq |x_0 - x| + |y_0 - y| \\ &\lesssim n^{-j-2} \lesssim |x - y| \end{aligned}$$

and if I_β and $I_{\beta'}$ are the intervals created from I_{α_0} so that β is a truncation of α and β' is a truncation of α' , then

$$|x - y| \leq Mn^{-j-2} \lesssim M \text{dist}(K_\beta, K_{\beta'}) \leq M |F(x) - F(y)|.$$

Thus, $F : E \rightarrow \mathbb{R}$ is bi-Lipschitz with constant independent of n . In particular, E is UR. By [Mac95, Theorem 1], this bi-Lipschitz map can be extended to a bi-Lipschitz map f of the plane onto itself. In particular, since $f(E)$ is a subset of the real line, hence if we define $\Omega = \mathbb{R}^2 \setminus E$, then $f(\Omega)$ is a domain whose boundary is contained in the real line. It is thus easy to prove that $f(\Omega)$ is a semi-uniform domain, and hence so is Ω since f is bi-Lipschitz (with a different semi-uniformity constant, but ultimately one that is independent of n). Thus, since we can pick n as large as we wish, for any L and $\varepsilon > 0$, we can now construct a semi-uniform domain $\Omega \subseteq \mathbb{R}^2$ with ADR and UR boundary (both independent of L) so that the intersection of $\partial\Omega$ with any Lipschitz graph of constant L has measure at most ε .

REFERENCES

- [Aik01] H. Aikawa. Boundary Harnack principle and Martin boundary for a uniform domain. *J. Math. Soc. Japan*, 53(1):119–145, 2001. 4
- [Aik04] H. Aikawa. Potential-theoretic characterizations of nonsmooth domains. *Bull. London Math. Soc.*, 36(4):469–482, 2004. 4
- [Aik06] H. Aikawa. Characterization of a uniform domain by the boundary Harnack principle. In *Harmonic analysis and its applications*, pages 1–17. Yokohama Publ., Yokohama, 2006. 22
- [Aik08] H. Aikawa. Equivalence between the boundary Harnack principle and the Carleson estimate. *Math. Scand.*, 103(1):61–76, 2008. 4, 13
- [AH08] H. Aikawa and K. Hirata. Doubling conditions for harmonic measure in John domains. *Ann. Inst. Fourier (Grenoble)*, 58(2):429–445, 2008. 1, 2, 3, 4, 13, 20, 21
- [AHL06] H. Aikawa, K. Hirata, and T. Lundh. Martin boundary points of a John domain and unions of convex sets. *J. Math. Soc. Japan*, 58(1):247–274, 2006. 20
- [AAM16] M. Akman, J. Azzam, and M. Mourgoglou. Absolute continuity of harmonic measure for domains with lower regular boundaries. *arXiv preprint arXiv:1605.07291*, 2016. 6
- [ABHM17] M. Akman, M. Badger, S. Hofmann, and J.M. Martell. Rectifiability and elliptic measures on 1-sided NTA domains with Ahlfors-David regular boundaries. *Trans. Amer. Math. Soc.*, 369(8):5711–5745, 2017. 4
- [AG01] D. H. Armitage and S. J. Gardiner. *Classical potential theory*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001. 12
- [AHM⁺16] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg. Rectifiability of harmonic measure. *Geom. Funct. Anal.*, 26(3):703–728, 2016. 6, 13
- [AHM⁺17] J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, and T. Toro. A new characterization of chord-arc domains. *J. Eur. Math. Soc. (JEMS)*, 19(4):967–981, 2017. 4, 7, 8, 10, 11, 12, 43
- [AMT17] J. Azzam, M. Mourgoglou, and X. Tolsa. A two-phase free boundary problem for harmonic measure and uniform rectifiability. *arXiv preprint arXiv:1710.10111*, 2017. 13
- [BL04] B. Bennewitz and J. L. Lewis, *On weak reverse Hölder inequalities for non-doubling harmonic measures*. *Complex Var. Theory Appl.* 49 (2004), no. 7-9, 571–582. 7
- [BH17] S. Bortz and S. Hofmann. Harmonic measure and approximation of uniformly rectifiable sets. *Revista Mat.*, 33(1):pp. 351–373, 2017. 9, 30
- [BJ90] C. J. Bishop and P. W. Jones. Harmonic measure and arclength. *Ann. of Math. (2)*, 132(3):511–547, 1990. 6
- [Chr90] M. Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990. 29
- [Dah77] B. E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977. 6
- [Dav88] G. David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoamericana*, 4(1):73–114, 1988. 29

- [DJ90] G. David and D. S. Jerison. Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. *Indiana Univ. Math. J.*, 39(3):831–845, 1990. [1](#), [3](#), [7](#), [9](#), [45](#)
- [DS91] G. David and S. W. Semmes. Singular integrals and rectifiable sets in \mathbf{R}^n : Beyond Lipschitz graphs. *Astérisque*, (193):152, 1991. [29](#), [30](#)
- [DS93] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Monographs and Surveys **38**, AMS 1993 [43](#)
- [GM08] J. B. Garnett and D. E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original. [3](#), [7](#)
- [HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original. [12](#)
- [HL16] S. Hofmann and P. Le. BMO solvability and absolute continuity of harmonic measure. *arXiv preprint arXiv:1607.00418*, 2016. [7](#)
- [HLMN17] S. Hofmann, P. Le, J.M. Martell, and K. Nyström. The weak- A_∞ property of harmonic and p -harmonic measures implies uniform rectifiability. *Anal. PDE*, 10(3):513–558, 2017. [1](#), [8](#)
- [HM12] T. Hytönen and H. Martikainen. Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. *J. Geom. Anal.*, 22(4):1071–1107, 2012. [29](#)
- [HM14] S. Hofmann and J. M. Martell. Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p . *Ann. Sci. Éc. Norm. Supér. (4)*, 47(3):577–654, 2014. [4](#), [7](#), [30](#), [39](#), [44](#)
- [HM15] S. Hofmann, J.M. Martell. Uniform rectifiability and harmonic measure, IV: Ahlfors regularity plus Poisson kernels in L^p implies uniform rectifiability. Preprint 2015. arXiv:1505.06499. [1](#), [7](#), [9](#)
- [HM17] S. Hofmann and J. M. Martell. A sufficient geometric criterion for quantitative absolute continuity of harmonic measure. *Arxiv Preprint arXiv:1712.03696*. [8](#)
- [HMM14] S. Hofmann, J. M. Martell, and S. Mayboroda. Uniform rectifiability and harmonic measure III: Riesz transform bounds imply uniform rectifiability of boundaries of 1-sided NTA domains. *Int. Math. Res. Not. IMRN*, (10):2702–2729, 2014. [29](#)
- [HMM⁺17] S. Hofmann, J.M., S. Mayboroda, T. Toro, and Z. Zhao. Uniform rectifiability and elliptic operators with small carleson norm. *arXiv preprint arXiv:1710.06157*, 2017. [5](#), [8](#)
- [HMT16] S. Hofmann, J.M. Martell, and T. Toro. a_∞ implies nta for a class of variable coefficient elliptic operators. *arXiv preprint*. [8](#)
- [HMUT14] S. Hofmann, J. M. Martell, and I. Uriarte-Tuero. Uniform rectifiability and harmonic measure, II: Poisson kernels in L^p imply uniform rectifiability. *Duke Math. J.*, 163(8):1601–1654, 2014. [4](#), [7](#), [8](#)
- [JK82] D. S. Jerison and C. E. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains. *Adv. in Math.*, 46(1):80–147, 1982. [3](#), [4](#), [5](#), [22](#)
- [Jon80] P. W. Jones. Extension theorems for BMO. *Indiana Univ. Math. J.*, 29(1):41–66, 1980. [3](#)

- [KPT09] C. E. Kenig, D. Preiss, and T. Toro. Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions. *J. Amer. Math. Soc.*, 22(3):771–796, 2009. [3](#)
- [KT06] C. E. Kenig and T. Toro. Free boundary regularity below the continuous threshold: 2-phase problems. *J. Reine Angew. Math.*, 596:1–44, 2006. [3](#)
- [Lav36] M Lavrentiev. Boundary problems in the theory of univalent functions. *Mat. Sb.(NS)*, 1(43):815–844, 1936. [7](#)
- [HK98] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.
- [Leh08] J. Lehrbäck. Pointwise Hardy inequalities and uniformly fat sets. *Proc. Amer. Math. Soc.*, 136(6):2193–2200, 2008. [12](#)
- [Lew88] John L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308(1):177–196, 1988.
- [Mac95] P. MacManus, *Bi-Lipschitz extensions in the plane*, *J. Anal. Math.* **66** (1995), 85–115. MR 1370347 (97b:30028) [48](#)
- [McM69] J. E. McMillan. Boundary behavior of a conformal mapping. *Acta Math.*, 123:43–67, 1969. [6](#)
- [MS79] O. Martio and J. Sarvas. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 4(2):383–401, 1979. [3](#)
- [MT15] M. Mourgoglou and X. Tolsa. Harmonic measure and riesz transform in uniform and general domains. *arXiv preprint arXiv:1509.08386*, 2015. [4](#), [8](#)
- [Pom86] Ch. Pommerenke. On conformal mapping and linear measure. *J. Analyse Math.*, 46:231–238, 1986. [6](#)
- [Sem90] S. Semmes. Analysis vs. geometry on a class of rectifiable hypersurfaces in \mathbf{R}^n . *Indiana Univ. Math. J.*, 39(4):1005–1035, 1990. [7](#)
- [Ste93] E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192 (95c:42002) [7](#)
- [TZ17] T. Toro and Z. Zhao. Boundary rectifiability and elliptic operators with w 1, 1 coefficients. *To appear in Adv. in Calc. of Var.*, 2017. [4](#)
- [Wu86] J-M. Wu. On singularity of harmonic measure in space. *Pacific J. Math.*, 121(2):485–496, 1986. [3](#), [6](#)

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