MODELS AND INTEGRAL DIFFERENTIALS OF HYPERELLIPTIC CURVES

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Abstract. Let $C : y^2 = f(x)$ be a hyperelliptic curve of genus $g \geq 2$, defined over a discretely valued complete field K, with ring of integers O_K. Under certain conditions on C, mild when residue characteristic is not 2, we explicitly construct the minimal regular model with normal crossings $\mathcal{C}/O_K$ of C. In the same setting we determine a basis of integral differentials of C, that is an O_K-basis for the global sections of the relative dualising sheaf $\omega_{\mathcal{C}/O_K}$.

1. Introduction

The purpose of this paper is to construct regular models of hyperelliptic curves and to describe a basis of integral differentials attached to them. Moreover, we want these constructions explicit and easy to compute.

1.1. **Main results.** Let K be a discretely valued field of residue characteristic p, with discrete valuation v and ring of integers O_K. Since regular models do not change under completion of the base field, we also assume K to be complete. Fix a separable closure K^{s} of K and let k^{s}/k be the corresponding extension of residue fields. Suppose C/K is a hyperelliptic curve of genus $g \geq 2$ given by a Weierstrass equation $y^2 = f(x)$ and write

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).$$

Definition 1.1 A *cluster* (for C) is a non-empty subset $\mathfrak{s} \subset \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in K^{\mathrm{s}} \mid v(x - z) \geq d\}$ for some $z \in K^{\mathrm{s}}$ and $d \in \mathbb{Q}$. We denote by Σ_C the set of clusters for C.

Let K^{nr} be the maximal unramified extension of K in K^{s} and let $I_K = \mathrm{Gal}(K^{\mathrm{s}}/K^{nr})$ be the inertia subgroup. To construct the minimal regular model with normal crossings of C, we assume C is y-regular (Definition 4.5) and Σ_C is almost rational (Definition 3.25) over K^{nr}. Before stating the main result, we want to discuss some special cases in which these two conditions are satisfied.

First, if either $p \neq 2$ or Σ_C only contains clusters of odd size, then C is y-regular over K^{nr}. Second, if $K(\mathfrak{R})/K$ is tamely ramified and every cluster $\mathfrak{s} \in \Sigma_C$ is I_K-invariant, then Σ_C is almost rational over K^{nr}. On the other hand, there are examples of curves with $K(\mathfrak{R})/K$ wildly ramified, but where the cluster picture is almost rational, e.g., $f(x) = x^p - p$, with $p > 3$. Finally, if $g = 2$, then 107/120 Namikawa-Ueno types ([NU]) arise from hyperelliptic curves satisfying the conditions above.

Theorem 1.2 (Theorems 4.16 and 6.4) *Let C/K be a hyperelliptic curve as above, y-regular over K^{nr}. Suppose Σ_C is almost rational over K^{nr}. Then the (rational) cluster picture of C uniquely determines:*

(i) the minimal regular model with normal crossings $\mathcal{C}^{\min}$,
(ii) a basis of integral differentials of C (see §1.2).

Note that the model $\mathcal{C}^{\min}$ in (i) is defined by giving an explicit open affine cover. Moreover, if Σ_C is not almost rational, a stronger version of Theorem 1.2 still gives a proper flat model of C (see Theorem 4.12).

Finally, the author believes that an approach similar to that used in this paper could eventually give a full characterisation of minimal models with normal crossings of hyperelliptic curves (over any discretely valued field).

1.2. **Motivation.** In this subsection we want to present two important applications of Theorem 1.2.

Let C be a hyperelliptic curve of genus $g \geq 2$ defined over a number field F, and let $J = \mathrm{Jac}(C)$ be the Jacobian of C. The Birch and Swinnerton-Dyer conjecture for the g-dimensional abelian variety J is

Conjecture 1.3 (Birch and Swinnerton-Dyer conjecture)

$$\lim_{s\to 1}(s-1)^{-r}L(J,s) = \frac{\Omega_J \cdot R_J \cdot |\text{Ш}(J)| \cdot \prod_{v\nmid\infty} c_v}{\sqrt{|D_F|^d} \cdot |J(F)_{\mathrm{tors}}|^2}$$

where r is the rank, $L(J,s)$ is the L-function, Ω_J is the period, R_J is the regulator, $\text{Ш}(J)$ is the Shafarevic-Tate group (conjecturally finite), $c_{\mathfrak{p}}$ is the Tamagawa number for v place of F, D_F is the discriminant of $F/\mathbb{Q}$, and $J(F)_{\mathrm{tors}}$ are the torsion points in $J(F)$.

For any place $v \nmid \infty$ of F, let $K := F_v$ be the completion of F at v. First note that the Tamagawa number at v can be found from the minimal regular model of C_K, so Theorem 1.2(i) can be applied to compute it in concrete examples.

We now focus on the period Ω_J. Fix a regular model $\mathcal{C}/O_K$ of C_K/K. It is well-known that the K-vector space of regular differentials $\Omega^1_{C_K/K}(C_K)$ is spanned by the basis

$$\underline{\omega} = \left(\frac{dx}{2y}, x\frac{dx}{2y}, \ldots, x^{g-1}\frac{dx}{2y}\right).$$

Consider the global sections of the relative dualising sheaf $\omega_{\mathcal{C}/O_K}$. It is an O_K-free module of rank g that can be viewed as an O_K lattice

$$\omega_{\mathcal{C}/O_K}(\mathcal{C}) \subset \Omega^1_{C_K/K}(C_K), \tag{1}$$

since $\omega_{\mathcal{C}/O_K}|_C = \Omega^1_{C_K/K}$. Via (1), we will call the elements of $\omega_{\mathcal{C}/O_K}(\mathcal{C})$ *integral differentials* of C (at v). In particular, there exists a matrix $A_v \in M_{g\times g}(K)$ such that $A_v \cdot \underline{\omega}$ is a basis of integral differentials of C, i.e., an O_K-basis of $\omega_{\mathcal{C}/O_K}(\mathcal{C})$. Then

$$\Omega_J = \Omega_{\infty,\underline{\omega}} \cdot \prod_v |\det A_v|_v^{-1}.$$

The quantity $\Omega_{\infty,\underline{\omega}}$ is relatively easy to find in actual computations (see [vB] for more details). Theorem 1.2(ii) gives an explicit formula to compute the local part $|\det A_v|_v$ for any v.

A second important application concerns the so-called *conductor-discriminant inequalities*. Let K be a complete discretely valued field of residue characteristic p, with discrete valuation v. Let C/K be a hyperelliptic curve of genus $g \geq 2$ and let Δ be the (valuation of the) discriminant of C associated with this equation. Moreover, let $\mathcal{C}/O_K$ be the minimal regular model of C and let $\mathrm{Art}(\mathcal{C}/O_K)$ denote the Artin conductor of $\mathcal{C}$. If $p \neq 2$, Obus and Srinivasan show in [OS] that

$$\mathrm{Art}(\mathcal{C}/O_K) \leq \Delta_{C/O_K},$$

where Δ_{C/O_K} is the (valuation of the) minimal discriminant of C. However, this inequality is often non-sharp. Let A be the matrix above, so that $A \cdot \underline{\omega}$ is an

O_K-basis of $\omega_{\mathcal{C}/O_K}(\mathcal{C})$. Define

$$\Delta_{\min} := g \cdot \Delta - (8g+4) \cdot v(\det A).$$

It follows from [Kau, Proposition 2.2(1)] that $\Delta_{\min}$ is independent of the choice of the Weierstrass equation, and easily $\Delta_{\min} \leq \Delta_{C/O_K}$.

Conjecture 1.4 *With the notation above*

$$\mathrm{Art}(\mathcal{C}/O_K) \leq \Delta_{\min}.$$

This conjecture is proved in the genus 2 case due to the work of Liu [LiC]. Furthermore, in the semistable case, Maugeais shows it in [Mau], and it also easily follows from [$\mathrm{M}^2\mathrm{D}^2$] and [Kun]. Although Theorem 1.2 does not prove the inequality, it allows us to compute $\Delta_{\min}$ explicitly in many more cases.

1.3. **Related works of other authors.** Let us keep the notations of §1.1. In genus 1, thanks to Tate's algorithm, we have a full understanding of the minimal regular model of an elliptic curve C (see for example [Sil2, IV.8.2]) . Furthermore, $\underline{\omega}$ is always a basis of integral differentials ([LiA, Theorem 9.4.35]).

If C has genus 2, then Namikawa and Ueno [NU] and Liu [LiQ] give a full classification of the possible minimal regular models of C. In [LiC, 1.3], Liu shows that there exists a Moebius transformation so that $\underline{\omega}$ is basis of integral differentials of the transformed curve. However, note this is a theoretical result, that is to find such a change of variable we need to know a basis of integral differentials of C.

The results presented so far work also in residue characteristic equal to 2. If p is odd, then Liu and Lorenzini show in [LL] that regular models of C can be seen as double cover of well-chosen regular models of $\mathbb{P}^1_K$. Since the latter can be found by using the MacLane valuations ([Mac]) approach in [OW], this argument gives a way to describe any regular model of a hyperelliptic curve. However, this construction is only qualitative (it does not give explicit equations) and it has not been generalised to the $p = 2$ case.

If $p \neq 2$ and C is semistable, then in [$\mathrm{M}^2\mathrm{D}^2$] the authors construct a minimal regular model in terms of the cluster picture of C. Note that the components of the stable model used in this paper are given explicitly. Under the same assumptions, Kausz [Kau, Proposition 5.5] gives a basis of integral differentials which Kunzweiler in [Kun] rephrases in terms of the cluster invariants introduced in [$\mathrm{M}^2\mathrm{D}^2$]. These results can be recovered from Theorem 1.2 (see Corollary 4.19).

If $p \neq 2$ and C is semistable over some tamely ramified extension of L/K, then Faraggi and Nowell [FN] find the minimal regular model of C with strict normal crossings taking the quotient of the stable model of C_L and resolving the (tame) singularities. However, since they do not give equations for the model they find, it does not immediately yield all arithmetic invariants, such as a basis of integral differentials.

The last work we want to recall represents a very important ingredient of the strategy we will use in this paper (described more precisely in the next subsection). T. Dokchitser in [Dok] shows that the toric resolution of C gives a regular model in case of Δ_v-regularity ([Dok, Definition 3.9]). This result, used also in [FN], holds for general curves and in any residue characteristic. In his paper, Dokchitser also describes a basis of integral differentials since his model is explicit, i.e., given as open cover of affine schemes. In Corollary 3.24 and Theorem 6.1, we will rephrase his results for hyperelliptic curves by using cluster picture invariants from §3.

1.4. **Strategy and outline of the paper.** In [Dok], Dokchitser not only describes a regular model of C in case of Δ_v-regularity, but also constructs a proper flat model $\mathcal{C}_\Delta$ without any assumptions on C. Assume C is y-regular and Σ_C almost rational over K^{nr} with rational centres $w_1, \dots, w_m \in K^{nr}$ (see Definitions 3.25 and 3.8). Our approach to construct the minimal regular model with normal crossings of C is composed by the following steps:

- Consider the x-translated hyperelliptic curves $C^h/K^{nr} : y^2 = f(x + w_h)$, for $h = 1, \dots, m$. For each h, [Dok, Theorem 3.14] constructs a proper flat model $\mathcal{C}^h_\Delta$, possibly singular.
- We glue the regular parts of these schemes along common opens, and show that the result is a proper flat regular model $\mathcal{C}$ of $C_{K^{nr}}$.
- We give a complete description of what components of the special fibre of $\mathcal{C}$ have to be blown down to obtain the minimal model with normal crossings $\mathcal{C}^{\min}$ of $C_{K^{nr}}$.
- Finally, we describe the action of the Galois group $G_k = \mathrm{Gal}(k^{\mathrm{s}}/k)$ on the special fibre of $\mathcal{C}^{\min}$.

Since all models $\mathcal{C}^h_\Delta$ are explicitly described, the model $\mathcal{C}$ is explicit as well. This allows us to study the global sections of its relative dualising sheaf $\omega_{\mathcal{C}/O_K}(\mathcal{C})$.

In §2, we present some basic results on Newton polygons related to what is used in [Dok]. We use them in the following sections. In §3, we recall the most important notions of [M^{2}D^2] and introduce the new definition of rational cluster picture. Moreover, we compare it with the definitions and results given §2. This comparison allows us to rephrase the objects in [Dok] in terms of cluster invariants in §4. In the same section we also state the main theorems which describe a proper flat model (Theorem 4.12) and the minimal regular model with normal crossings (Theorem 4.16) of C. Although the description of these models is clear from the statement of the theorems, its detailed construction is presented only in §5. Here we use the toric resolution of [Dok], making concrete the strategy sketched above. Finally, in §6, Theorem 6.4 describes a basis of integral differentials of C, in terms of the cluster invariants defined in §3.

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2. Newton polygon

Let K be a complete field with a discrete valuation v, ring of integers O_K, uniformiser π, and residue field k of characteristic p. We fix $\overline{K}$, an algebraic closure of K, of residue field $\bar{k}$, and we denote by K^{s} the separable closure of K in $\overline{K}$, and by k^{s} its residue field. Note that k^{s} is the separable closure of k in $\bar{k}$. We write G_K, G_k for the Galois groups $\mathrm{Gal}(K^{\mathrm{s}}/K)$, $\mathrm{Gal}(k^{\mathrm{s}}/k)$, respectively. Finally, write K^{nr} for the maximal unramified extension of K.

Let $f \in K[x]$ be a polynomial of degree d, say

$$f(x) = \sum_{i=0}^{d} a_i x^i.$$

The *Newton polygon* of f, denoted $\mathtt{NP}(f)$, is

$$\mathtt{NP}(f) = \text{lower convex hull } \{(i, v(a_i)),\ i = 0, \dots, d\} \subset \mathbb{R} \times (\mathbb{R} \cup \{\infty\}).$$

We recall the following well-known result.

Theorem 2.1 *Let $i_0 < \ldots < i_s$ be the set of indices in $\{0, \dots, d\}$ such that the points $(i_0, v(a_{i_0})), \dots, (i_s, v(a_{i_s}))$ are the vertices of $\mathtt{NP}(f)$. For any $j = 1, \dots, s$,* *denote by ρ_j the slope of the edge of* $\mathtt{NP}(f)$ *which links the points* $(i_{j-1}, v(a_{i_{j-1}}))$ *and* $(i_j, v(a_{i_j}))$*. Then f factors over K as a product*

$$f = g_1 \cdots g_s,$$

where, for all $j = 1, \ldots, s$,

- *the degree of g_j is $d_j = i_j - i_{j-1}$,*
- *all the roots of g_j have valuation $-\rho_j$ in $\overline{K}$.*

Remark 2.2. If $x \mid f$, then g_1 is equal to the maximal power of x which divides f, $\mathtt{NP}(g_1)$ can be viewed as a vertical line, and $\rho_1 = \infty$.

Corollary 2.3 *With the notation of Theorem 2.1, the polynomial f has exactly d_j roots of valuation $-\rho_j$ for all $j = 1, \ldots, s$.*

Corollary 2.4 *If $f = \sum a_i x^i$ is irreducible of degree d, then* $\mathtt{NP}(f)$ *is a line linking the points* $(0, v(a_0))$ *and* $(d, v(a_d))$*.*

Definition 2.5 (Restriction and reduction) Let $f = \sum_{i=0}^{d} a_i x^i \in K[x]$ and consider an edge L of its Newton polygon $\mathtt{NP}(f)$. Then $L = \mathtt{NP}(g_i)$ for some g_i in the factorisation of f of Theorem 2.1. Consider the two endpoints of L $(i_1, v(a_{i_1})), (i_2, v(a_{i_2}))$, $i_1 < i_2$. Denote by ρ the slope of L and by n the denominator of ρ. Define the *restriction* of f to L to be

$$f|_L = \sum_{i \geq 0} a_{ni+i_1} x^i.$$

Moreover we define the *reduction* of f (with respect to L) to be the polynomial

$$\overline{f|_L} = \pi^{-c} f|_L(\pi^{-n\rho} x) \bmod \pi \in k[x],$$

where $c = v(a_{i_1}) = v(a_{i_2}) + (i_1 - i_2)\rho$.

Remark 2.6. These definitions coincide with the ones given in [Dok, Definitions 3.4, 3.5] when the number of variables n is 1.

Until the end of the section let $f \in K[x]$, consider a factorisation $f = g_1 \cdots g_s$ as in Theorem 2.1 and denote by L_j the Newton polygon of g_j.

Remark 2.7. By the lower convexity of $\mathtt{NP}(f)$, for all $j = 1, \ldots, s$, the two polynomials $\overline{f|_{L_j}}$ and $\overline{g_j|_{L_j}}$ are equal up to units. In particular they define the same k-scheme in $\mathbb{G}_{m,k}$.

Furthermore, if either $x \nmid f$ or $j \neq 1$, then $x \nmid g_j$ and so $x \nmid \overline{g_j|_{L_j}}$ and $x \nmid \overline{f|_{L_j}}$.

Definition 2.8 We say that f is $\mathtt{NP}$-*regular* if the k-scheme

$$X_{L_j} : \{\overline{f|_{L_j}} = 0\} \subset \mathbb{G}_{m,k}$$

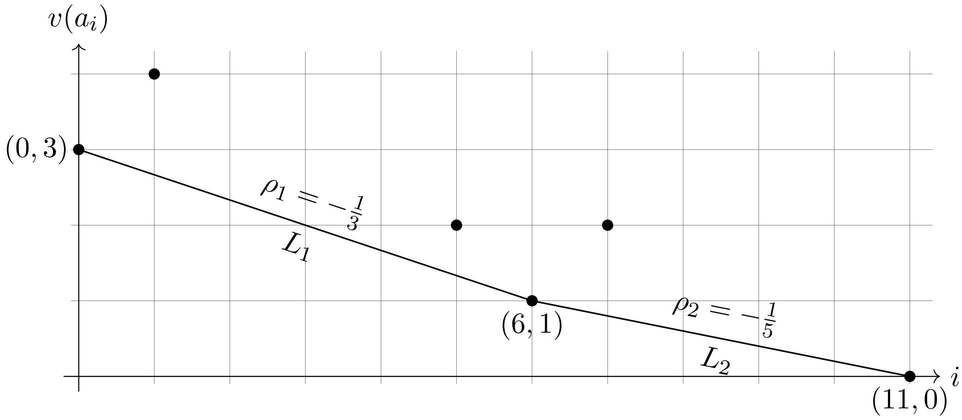
is smooth for all $j = 1, \ldots, s$.

Lemma 2.9 *The polynomial $f = g_1 \cdots g_s$ is* $\mathtt{NP}$*-regular if and only if g_j is* $\mathtt{NP}$*-regular for every j.*

Proof. The lemma follows by Remark 2.7. □

We conclude this section with two examples.

Example 2.10. Let $f = x^{11} + 9x^7 - 3x^6 + 9x^5 + 81x - 27 \in \mathbb{Q}_3[x]$. Then the Newton polygon of f is



Corollary 2.3 implies that f has 6 roots of valuation $\frac{1}{3}$ and 5 roots of valuation $\frac{1}{5}$. Furthermore, the two polynomials g_1 and g_2 in the factorisation $f = g_1 \cdot g_2$ of Theorem 2.1 turn out to be

$$g_1 = x^6 + 9, \qquad g_2 = x^5 + 9x - 3.$$

Finally,

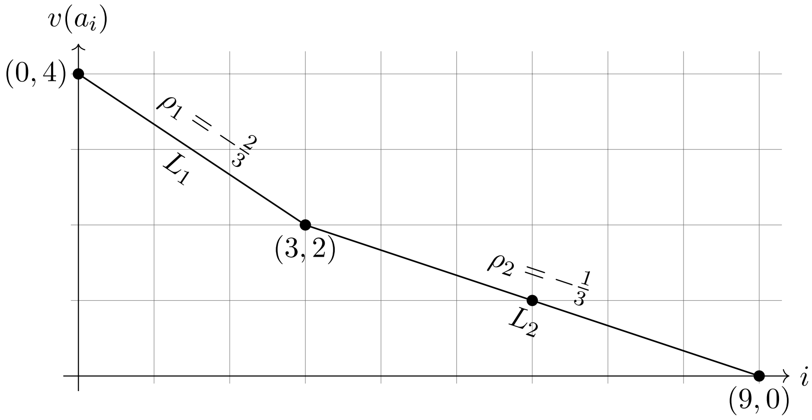
$$f|_{L_1} = -3x^2 - 27 = -3 \cdot g_1|_{L_1}, \qquad f|_{L_2} = x - 3 = g_2|_{L_2};$$

and

$$\overline{f|_{L_1}} = -x^2 - 1 = -(x^2+1) = -\overline{g_1|_{L_1}}, \qquad \overline{f|_{L_2}} = x - 1 = \overline{g_2|_{L_2}} \qquad \text{in } \mathbb{F}_3[x].$$

Thus f is `NP`-regular.

Example 2.11. Let's do now an example of a polynomial that is not `NP`-regular. Let $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$. Then the Newton polygon of f is



Corollary 2.3 implies that f has 3 roots of valuation $\frac{2}{3}$ and 6 roots of valuation $\frac{1}{3}$. Furthermore, the two polynomials g_1 and g_2 in the factorisation $f = g_1 \cdot g_2$ of Theorem 2.1 are (up to units)

$$g_1 = x^3 + 9, \qquad g_2 = x^6 + 3x^3 + 9.$$

Finally,

$$f|_{L_1} = x^3 + 12x^2 + 36x + 81 \qquad f|_{L_2} = x^2 + 12x + 36,$$
$$g_1|_{L_1} = x + 9, \qquad g_2|_{L_2} = x^2 + 3x + 9;$$

and

$$\overline{f|_{L_1}} = x + 1 = \overline{g_1|_{L_1}}, \qquad \overline{f|_{L_2}} = (x+2)^2 = \overline{g_2|_{L_2}} \qquad \text{in } \mathbb{F}_3[x].$$

Then f is not `NP`-regular. In fact, according to Lemma 2.9, g_2 is not `NP`-regular.

3. Clusters

Throughout this section let $f \in K[x]$ be a separable polynomial and denote by $\mathfrak{R}$ the set of its roots and by c_f its leading coefficient. Then

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).$$

Definition 3.1 ([M²D², Definition 1.1]) A *cluster* (for f) is a non-empty subset $\mathfrak{s} \subset \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in \overline{K} \mid v(x - z) \geq d\}$ for some $z \in \overline{K}$ and $d \in \mathbb{Q}$. If $|\mathfrak{s}| > 1$ we say that $\mathfrak{s}$ is *proper* and define its *depth* $d_{\mathfrak{s}}$ to be

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r').$$

Note that every proper cluster is cut out by a disc of the form

$$\mathcal{D} = \{x \in \overline{K} \mid v(x - r) \geq d_{\mathfrak{s}}\}$$

for any $r \in \mathfrak{s}$.

Definition 3.2 ([M²D², Definition 1.3]) If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is maximal subcluster, then we say that $\mathfrak{s}'$ is a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ is the *parent* of $\mathfrak{s}'$, and we write $\mathfrak{s}' < \mathfrak{s}$. Since every cluster $\mathfrak{s} \neq \mathfrak{R}$ has one and only one parent we write $P(\mathfrak{s})$ to refer to the unique parent of $\mathfrak{s}$.

We say that a proper cluster $\mathfrak{s}$ is *minimal* if it does not have any proper child.

For two clusters (or roots) $\mathfrak{s}_1, \mathfrak{s}_2$, we write $\mathfrak{s}_1 \wedge \mathfrak{s}_2$ for the smallest cluster that contains them.

Definition 3.3 ([M²D², Definition 1.4]) A cluster $\mathfrak{s}$ is *odd/even* if its size is odd/even. If $|\mathfrak{s}| = 2$, then we say $\mathfrak{s}$ is a *twin*. A cluster $\mathfrak{s}$ is *übereven* if it has only even children.

Definition 3.4 ([M²D², Definition 1.9]) A *centre* $z_{\mathfrak{s}}$ of a proper cluster $\mathfrak{s}$ is any element $z_{\mathfrak{s}} \in K^{\mathrm{s}}$ such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$, where

$$\mathcal{D} = \{x \in \overline{K} \mid v(x - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}\}.$$

Equivalently, $v(r - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}$ for all $r \in \mathfrak{s}$. The *centre* of a non-proper cluster $\mathfrak{s} = \{r\}$ is r.

Definition 3.5 ([M²D², Definition 1.6]) For a cluster $\mathfrak{s}$ set

$$\nu_{\mathfrak{s}} := v(c_f) + \sum_{r \in \mathfrak{R}} d_{r \wedge \mathfrak{s}}.$$

Definition 3.6 ([M²D², Definition 1.26]) The *cluster picture* Σ_f of f is the set of all its clusters. We denote by $\mathring{\Sigma}_f$ the subset of Σ_f of proper clusters.

Definition 3.7 We say that Σ_f is *nested* if one of the following equivalent conditions is satisfied:

(i) there exists $z \in K^{\mathrm{s}}$ such that z is a centre for all proper clusters $\mathfrak{s} \in \Sigma_f$;
(ii) there is only one minimal cluster in Σ_C;
(iii) every non-minimal proper cluster has exactly one proper child.

Definition 3.8 A *rational centre* of a cluster $\mathfrak{s}$ is any element $w_{\mathfrak{s}} \in K$ such that

$$\min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}) = \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w).$$

If $\mathfrak{s} = \{r\}$, with $r \in K$, then $w_{\mathfrak{s}} = r$.

If $w_{\mathfrak{s}}$ is a rational centre of a proper cluster $\mathfrak{s}$, we define the *radius* of $\mathfrak{s}$ to be

$$\rho_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}).$$

Definition 3.9 A *rational cluster* is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \overline{K} \mid v(x-w) \geq d\}$ with $w \in K$ and $d \in \mathbb{Q}$.

We define the *rational cluster picture* $\Sigma_f^{\mathrm{rat}} \subseteq \Sigma_f$ to be the set of rational clusters. We denote by $\mathring{\Sigma}_f^{\mathrm{rat}}$ the subset of Σ_f^{rat} of proper rational clusters.

Definition 3.10 Given a proper cluster $\mathfrak{s} \in \Sigma_f$, we define the rationalisation $\mathfrak{s}^{\mathrm{rat}}$ of $\mathfrak{s}$ to be the rational cluster

$$\mathfrak{s}^{\mathrm{rat}} = \mathfrak{R} \cap \{x \in \overline{K} \mid v(x - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}\},$$

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ and $\rho_{\mathfrak{s}}$ is its radius.

Lemma 3.11 *Let $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ be a proper cluster of rational centre $w_{\mathfrak{s}}$. Let $\mathfrak{s}'$ be a child of $\mathfrak{s}$ of rational centre $w_{\mathfrak{s}}$ (let $\mathfrak{s}' = \varnothing$ if it does not exist). Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_{\mathfrak{s}} \in \mathbb{Z}$.*

Proof. As $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$, one has $\mathfrak{s} = \mathfrak{s}^{\mathrm{rat}}$. Then $b_{\mathfrak{s}}$ divides the degree of the minimal polynomial of r, for any $r \in \mathfrak{s}$, with $v(w_{\mathfrak{s}} - r) = \rho_{\mathfrak{s}}$. Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_{\mathfrak{s}} \in \mathbb{Z}$, where

$$\mathfrak{s}' = \mathfrak{R} \cap \{x \in \overline{K} \mid v(x - w_{\mathfrak{s}}) > \rho_{\mathfrak{s}}\},$$

as required. □

Remark 3.12. Let $\mathfrak{s} \in \Sigma_f$ be a proper G_K-invariant cluster and assume $K(\mathfrak{s})/K$ is tame. Then by [M²D², Lemma B.1] the cluster $\mathfrak{s}$ has a centre $z_{\mathfrak{s}} \in K$ and so $\rho_{\mathfrak{s}} = d_{\mathfrak{s}}$ and $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$. On the other hand, if a proper cluster $\mathfrak{s} \in \Sigma_f$ satisfies $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$, then a rational centre $w_{\mathfrak{s}} \in K$ of its is also a centre. Then $\mathfrak{s}$ is G_K-invariant.

Lemma 3.13 *Let $\mathfrak{s}$ be a proper cluster. Then $d_{\mathfrak{s}} \geq \rho_{\mathfrak{s}}$.*

Proof. First we want to show that

$$\min_{r,r' \in \mathfrak{s}} v(r - r') = \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r - z).$$

Clearly $\min_{r,r' \in \mathfrak{s}} v(r - r') \leq \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r - z)$. Let $z_{\mathfrak{s}} \in K^{\mathrm{s}}$ such that

$$\max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r - z) = \min_{r \in \mathfrak{s}} v(r - z_{\mathfrak{s}}).$$

Then, for any $r, r' \in \mathfrak{s}$, one has

$$v(r - r') \geq \min\{v(r - z_{\mathfrak{s}}), v(r' - z_{\mathfrak{s}})\} \geq \min_{r \in \mathfrak{s}} v(r - z_{\mathfrak{s}}),$$

and so

$$\min_{r,r' \in \mathfrak{s}} v(r - r') \geq \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r - z),$$

as wanted.

From

$$d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r - r') = \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r - z) \geq \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w) = \rho_{\mathfrak{s}},$$

the lemma follows. □

Lemma 3.14 *Let $\mathfrak{s}$ be a proper cluster with rational centre $w_{\mathfrak{s}}$ and let $\mathfrak{t} \in \Sigma_f$ satisfying $\mathfrak{t} \supseteq \mathfrak{s}$. Then $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{t}$ and $\rho_{\mathfrak{t}} \leq \rho_{\mathfrak{s}}$. Furthermore, if $\mathfrak{s}$ is a rational cluster and $\mathfrak{t} \supsetneq \mathfrak{s}$, then $\rho_{\mathfrak{t}} < \rho_{\mathfrak{s}}$.*

Proof. It suffices to prove the lemma for $\mathfrak{t} = P(\mathfrak{s})$. Hence we want to show that $\min_{r \in P(\mathfrak{s})} v(r - w_{\mathfrak{s}}) = \rho_{P(\mathfrak{s})}$ and $\rho_{P(\mathfrak{s})} \leq \rho_{\mathfrak{s}}$. First of all,

$$\min_{r \in P(\mathfrak{s})} v(r - w_{\mathfrak{s}}) \leq \max_{w \in K} \min_{r \in P(\mathfrak{s})} v(r - w) = \rho_{P(\mathfrak{s})}.$$

Moreover

$$\rho_{P(\mathfrak{s})} = \max_{w\in K} \min_{r\in P(\mathfrak{s})} v(r-w) \le \max_{w\in K} \min_{r\in \mathfrak{s}} v(r-w) = \rho_{\mathfrak{s}}.$$

If $r\in\mathfrak{s}$ then $v(w_{\mathfrak{s}}-r)\ge\rho_{\mathfrak{s}}$, by definition of $\rho_{\mathfrak{s}}$. On the other hand, if $r\in P(\mathfrak{s})\smallsetminus\mathfrak{s}$ then fixing $r'\in\mathfrak{s}$ we have

$$v(r-w_{\mathfrak{s}}) = v(r-r'+r'-w_{\mathfrak{s}}) \ge \min\{v(r-r'), v(r'-w_{\mathfrak{s}})\} \ge \min\{d_{P(\mathfrak{s})}, \rho_{\mathfrak{s}}\} \ge \rho_{P(\mathfrak{s})},$$

by the previous lemma. Thus $\min_{r\in P(\mathfrak{s})} v(r-w_{\mathfrak{s}}) = \rho_{P(\mathfrak{s})}$, as required.

Now suppose $\mathfrak{s}\in\Sigma_f^{\mathrm{rat}}$ with $\mathfrak{t}\supsetneq\mathfrak{s}$. From Definition 3.8, it follows that

$$\{x\in\overline{K} \mid v(x-w_{\mathfrak{s}})\ge\rho_{\mathfrak{s}}\}\cap\mathfrak{R} = \mathfrak{s}\subsetneq\mathfrak{t}\subseteq\{x\in\overline{K}\mid v(x-w_{\mathfrak{s}})\ge\rho_{\mathfrak{t}}\}\cap\mathfrak{R},$$

as $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{t}$. Thus $\rho_{\mathfrak{t}}<\rho_{\mathfrak{s}}$. □

Lemma 3.15 *Every cluster $\mathfrak{s}$ with $\rho_{\mathfrak{s}}<d_{\mathfrak{s}}$ has no rational subcluster $\mathfrak{s}'\subsetneq\mathfrak{s}$.*

Proof. Suppose by contradiction there exists $\mathfrak{s}'\in\Sigma_C^{\mathrm{rat}}$, $\mathfrak{s}'\subsetneq\mathfrak{s}$, and fix a rational centre $w_{\mathfrak{s}'}$ of $\mathfrak{s}'$. Then $w_{\mathfrak{s}'}$ is a rational centre of $\mathfrak{s}$ by the previous lemma. If $|\mathfrak{s}'|=1$, then $w_{\mathfrak{s}'}$ is also a centre of $\mathfrak{s}$ and this contradicts $\rho_{\mathfrak{s}}<d_{\mathfrak{s}}$; so assume $\mathfrak{s}'$ proper. Let $r'\in\mathfrak{s}'$ such that $v(r'-w_{\mathfrak{s}'})=\rho_{\mathfrak{s}'}$ and $r\in\mathfrak{s}$ such that $v(r-w_{\mathfrak{s}'})=\rho_{\mathfrak{s}}$. But then $d_{\mathfrak{s}}\le v(r-w_{\mathfrak{s}'}+w_{\mathfrak{s}'}-r')=\rho_{\mathfrak{s}}$ again by Lemma 3.14.

In particular, the lemma above shows that if $\mathfrak{s}\in\Sigma_f$ and $\mathfrak{s}'\in\Sigma_f^{\mathrm{rat}}$ is a maximal rational subcluster of $\mathfrak{s}$, then $\mathfrak{s}'$ is a child of $\mathfrak{s}$. Moreover, the parent of a rational cluster is rational.

Definition 3.16 We say that a proper rational cluster $\mathfrak{s}\in\Sigma_f^{\mathrm{rat}}$ is *(rationally) minimal* if it does not have any proper rational subcluster.

Lemma 3.17 *Let $\mathfrak{s},\mathfrak{s}'\in\Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s}'\not\subseteq\mathfrak{s}$. If $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ then*

$$\min_{r\in\mathfrak{s}'} v(r-w_{\mathfrak{s}}) = \rho_{\mathfrak{s}\wedge\mathfrak{s}'}.$$

Proof. By Lemma 3.14 we have

$$\min_{r\in\mathfrak{s}\wedge\mathfrak{s}'} v(r-w_{\mathfrak{s}}) = \rho_{\mathfrak{s}\wedge\mathfrak{s}'}.$$

Therefore $\min_{r\in\mathfrak{s}'} v(w_{\mathfrak{s}}-r)\ge\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Suppose by contradiction that

$$\min_{r\in\mathfrak{s}'} v(r-w_{\mathfrak{s}}) =: \rho > \rho_{\mathfrak{s}\wedge\mathfrak{s}'}.$$

It follows from Lemma 3.14 that

$$\min_{r\in\mathfrak{s}} v(r-w_{\mathfrak{s}}) = \rho_{\mathfrak{s}} > \rho_{\mathfrak{s}\wedge\mathfrak{s}'}$$

as $\mathfrak{s}'\not\subseteq\mathfrak{s}$. But then there exists $\tilde{r}\in(\mathfrak{s}\wedge\mathfrak{s}')\smallsetminus(\mathfrak{s}\cup\mathfrak{s}')$ such that $v(\tilde{r}-w_{\mathfrak{s}})=\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Consider the rational cluster

$$\mathfrak{t} := \mathfrak{R}\cap\left\{x\in\overline{K}\mid v(x-w_{\mathfrak{s}})\ge\min\{\rho,\rho_{\mathfrak{s}}\}\right\}\in\Sigma_f^{\mathrm{rat}}.$$

Then $\mathfrak{s},\mathfrak{s}'\subseteq\mathfrak{t}$, but since $\overline{r}\notin\mathfrak{t}$ we have $\mathfrak{s}\wedge\mathfrak{s}'\not\subseteq\mathfrak{t}$ that contradicts the minimality of $\mathfrak{s}\wedge\mathfrak{s}'$. □

Lemma 3.18 *Let $\mathfrak{t}\in\Sigma_f$ with at least two children in Σ_f^{rat}. Then $d_{\mathfrak{t}}=\rho_{\mathfrak{t}}\in\mathbb{Z}$ and $\mathfrak{t}\in\Sigma_f^{\mathrm{rat}}$. More precisely, if $\mathfrak{s},\mathfrak{s}'\in\Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s}\subsetneq\mathfrak{s}\wedge\mathfrak{s}'\supsetneq\mathfrak{s}'$, then*

$$\rho_{\mathfrak{s}\wedge\mathfrak{s}'} = v(w_{\mathfrak{s}}-w_{\mathfrak{s}'}) = d_{\mathfrak{s}\wedge\mathfrak{s}'},$$

where $w_{\mathfrak{s}}$ and $w_{\mathfrak{s}'}$ are rational centres of $\mathfrak{s}$ and $\mathfrak{s}'$ respectively.

Proof. Clearly it suffices to prove the second statement as $v(w_{\mathfrak{s}}-w_{\mathfrak{s}'}) \in \mathbb{Z}$. For our assumptions $\mathfrak{s}' \not\subseteq \mathfrak{s}$. Then by Lemma 3.17 there exists $r \in \mathfrak{s}'$ so that $v(r-w_{\mathfrak{s}}) = \rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Thus,

$$v(w_{\mathfrak{s}}-w_{\mathfrak{s}'}) = \min\{v(w_{\mathfrak{s}}-r), v(r-w_{\mathfrak{s}'})\} = \rho_{\mathfrak{s}\wedge\mathfrak{s}'},$$

as

$$v(r-w_{\mathfrak{s}'}) \geq \rho_{\mathfrak{s}'} > \rho_{\mathfrak{s}\wedge\mathfrak{s}'}$$

by Lemma 3.14. Finally, $d_{\mathfrak{s}\wedge\mathfrak{s}'} = \rho_{\mathfrak{s}\wedge\mathfrak{s}'}$ by Lemma 3.15. □

Definition 3.19 For a proper cluster $\mathfrak{s}$ set

$$\epsilon_{\mathfrak{s}} := v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}}.$$

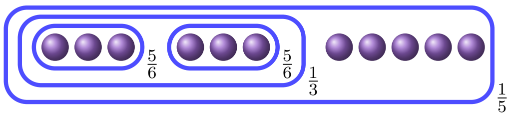
Example 3.20. Let $f = x^{11} - 3x^6 + 9x^5 - 27 \in \mathbb{Q}_3[x]$. The set of roots of f is

$$\mathfrak{R} = \{\sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}, -\sqrt[3]{3}, -\zeta_3\sqrt[3]{3}, -\zeta_3^2\sqrt[3]{3}, \sqrt[5]{3}, \zeta_5\sqrt[5]{3}, \zeta_5^2\sqrt[5]{3}, \zeta_5^3\sqrt[5]{3}, \zeta_5^4\sqrt[5]{3}\},$$

where ζ_q is a primitive q-th root of unity for $q = 3, 5$. Then the proper clusters of f are

$$\mathfrak{s}_1 = \{\sqrt[3]{3}, \zeta_3\sqrt[3]{3}, \zeta_3^2\sqrt[3]{3}\}, \quad \mathfrak{s}_2 = \{-\sqrt[3]{3}, -\zeta_3\sqrt[3]{3}, -\zeta_3^2\sqrt[3]{3}\}, \quad \mathfrak{s}_3 = \mathfrak{s}_1 \cup \mathfrak{s}_2, \quad \mathfrak{R}$$

with $d_{\mathfrak{s}_1} = d_{\mathfrak{s}_2} = \frac{5}{6}$, $d_{\mathfrak{s}_3} = \frac{1}{3}$ and $d_{\mathfrak{R}} = \frac{1}{5}$. The graphic representation of the cluster picture of f is then



where the subscripts of clusters (represented as circles) are their depths.

Furthermore, note that 0 is a rational centre for all (proper) clusters and we have $\rho_{\mathfrak{s}_1} = \rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \frac{1}{3}$ and $\rho_{\mathfrak{R}} = \frac{1}{5}$.

Finally, for every cluster $\mathfrak{s}$ we can also compute $\nu_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}}$, that are

$$\nu_{\mathfrak{s}_1} = \nu_{\mathfrak{s}_2} = \frac{9}{2}, \quad \nu_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_1} = \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = 3, \quad \nu_{\mathfrak{R}} = \epsilon_{\mathfrak{R}} = \frac{11}{5}.$$

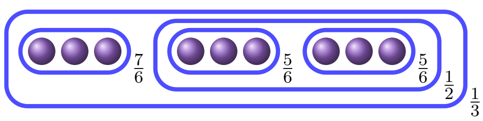
Example 3.21. Let $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$ and fix an isomorphism $\overline{\mathbb{Q}}_3 \simeq \mathbb{C}$. Then the set of roots of f is

$$\mathfrak{R} = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}, \zeta_9\sqrt[3]{3}, \zeta_9^2\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\},$$

where $\zeta_q = e^{2\pi i/q}$ is a primitive q-th root of unity for $q = 3, 9$. Then the proper clusters of f are

$$\mathfrak{s}_1 = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}\}, \quad \mathfrak{s}_2 = \{\zeta_9\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}\},$$
$$\mathfrak{s}_3 = \{\zeta_9^2\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\}, \quad \mathfrak{s}_4 = \mathfrak{s}_2 \cup \mathfrak{s}_3, \quad \mathfrak{R}$$

with $d_{\mathfrak{s}_1} = \frac{7}{6}$, $d_{\mathfrak{s}_2} = d_{\mathfrak{s}_3} = \frac{5}{6}$, $d_{\mathfrak{s}_4} = \frac{1}{2}$, and $d_{\mathfrak{R}} = \frac{1}{3}$. The cluster picture of f is then



It is easy to see that 0 is a rational centre for all (proper) clusters and that $\rho_{\mathfrak{s}_1} = \frac{2}{3}$, $\rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \rho_{\mathfrak{s}_4} = \rho_{\mathfrak{R}} = \frac{1}{3}$. Finally,

$$\nu_{\mathfrak{s}_1} = \frac{11}{2}, \quad \nu_{\mathfrak{s}_2} = \nu_{\mathfrak{s}_3} = 5, \quad \nu_{\mathfrak{s}_4} = 4, \quad \nu_{\mathfrak{R}} = 3; \qquad \epsilon_{\mathfrak{s}_1} = 4, \quad \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_4} = \epsilon_{\mathfrak{R}} = 3.$$

The goal of this section is to describe the $\mathtt{NP}$-regularity of $f \in K[x]$ in terms of conditions on its cluster picture Σ_f.

Lemma 3.22 *Suppose that* $\mathtt{NP}(f) =: L$ *has slope* $-\rho$ *and let* n *be the denominator of* ρ. *Then* f *is* $\mathtt{NP}$*-regular if and only if all proper clusters* $\mathfrak{s} \in \Sigma_f$ *with* $|\mathfrak{s}| > |\rho|_p$ [1] *satisfy* $d_{\mathfrak{s}} = \rho$.

More precisely, if $\mathfrak{s} \in \Sigma_f$ *with* $|\mathfrak{s}| > |\rho|_p$ *but* $d_{\mathfrak{s}} > \rho$, *then* $\overline{f|_L}$ *has a multiple root* $\overline{u} = \left(\frac{r}{\pi^\rho}\right) \in \overline{k}$, *for some (any)* $r \in \mathfrak{s}$. *The multiplicity of* $\overline{u}$ *equals* $|\mathfrak{s}|/p^{v_p(n)}$, *where* $\mathfrak{s} = \{r \in \mathfrak{R} \mid \overline{u} = \left(\frac{r}{\pi^\rho}\right)\}$. *Furthermore, all multiple roots of* $\overline{f|_L}$ *come from a cluster* $\mathfrak{s}$ *as described above.*

Proof. Write $n = m \cdot p^k$ where $p \nmid m$. Let $\mathfrak{R} = \{r_i \mid i = 1, \ldots, D\}$ be the set of roots of f, where $D := \deg f$, and let

$$r_i = u_i \pi^\rho + \ldots$$

be the p-adic expansion of r_i (we fix here a choice of $\sqrt[\ell]{\pi}$, for ℓ large enough), where $u_i \in K^{nr}$ root of unity of order prime to p. Firstly, note that there exists a proper cluster $\mathfrak{s}$ with $|\mathfrak{s}| > |\rho|_p$ and $d_{\mathfrak{s}} > \rho$ if and only if there exists a subset $I \subseteq \{1, \ldots, D\}$ of size $|I| > p^k$ such that $v(u_{i_1} - u_{i_2}) > 0$ for all $i_1, i_2 \in I$. Secondly, recall that f is not $\mathtt{NP}$-regular if and only if $\overline{f|_L}$ has a multiple root in $\overline{k}$. Therefore we will prove that $\overline{f|_L}$ has a multiple root if and only if there exists a subset $I \subseteq \{1, \ldots, D\}$ with size $|I| > p^k$ and such that $v(u_{i_1} - u_{i_2}) > 0$ for all $i_1, i_2 \in I$.

Let $f' \in K[x]$ defined by $f'(x) := f|_L(x^n)$. By definition of $f|_L$, we have that f is $\mathtt{NP}$-regular if and only if f' is $\mathtt{NP}$-regular. Moreover, since

$$\overline{f} := \pi^{-(v(c_f)+D\rho)} f(\pi^\rho x) = \pi^{-(v(c_{f'})+D\rho)} f'(\pi^\rho x) =: \overline{f'} \quad \text{in } \overline{k}[x],$$

and $\{\overline{u_i} \mid i = 1, \ldots, D\}$ is the multiset of roots of $\overline{f}$, we can assume without loss of generality that $f = f'$.

Then we prove the lemma with the additional assumption $f(x) = f|_L(x^n)$. Let $\{t_j \mid j = 1, \ldots, D/n\}$ be the multiset of roots of $f|_L$. Hence there exists an n-to-1 map

$$\phi : \{r_i\} \longrightarrow \{t_j\}, \qquad r_i \longmapsto r_i^n$$

which induces an n-to-1 map

$$\overline{\phi} : \{\overline{u}_i\} \longrightarrow \{\overline{w}_j\}, \qquad \overline{u}_i \longmapsto \overline{u}_i^m$$

where $t_j = w_j \pi^{n\rho} + \ldots$ is the p-adic expansion of t_j and $\overline{u}_i, \overline{w}_j$ denote the reductions to $\overline{k}$ of u_i, w_j, respectively. Note that $\overline{w_j} \in \overline{k}$ are the roots of $\overline{f|_L}$.

Now, suppose that f is not $\mathtt{NP}$-regular. We want to show that there exists a subset $I \subset \{1, \ldots, D\}$ with $|I| > p^k$ such that $v(u_{i_1} - u_{i_2}) > 0$ for all $i_1, i_2 \in I$. Since f is not $\mathtt{NP}$-regular, its reduction $\overline{f|_L}$ has a multiple root. Then there exist $j_1, j_2 \in \{1, \ldots, D/n\}$ so that $\overline{w}_{j_1} = \overline{w}_{j_2} =: \overline{w}$. Hence, by the definition of $\overline{\phi}$, we

[1] Here $|\cdot|_p$ denotes the standard p-adic absolute value attached to $\mathbb{Q}$, i.e., $|x|_p = p^{-v_p(x)}$ for all $x \in \mathbb{Q}$

have at least $2p^k$ u_i's with same reduction in $\overline{k}$ (and such that $\overline{\phi}(\overline{u}_i) = \overline{w}$). Let I denote the set of their indices. Then $|I| \geq 2p^k > p^k$ and $v(u_{i_1} - u_{i_2}) > 0$ for all $i_1, i_2 \in I$, as required.

Finally, suppose that there exists a subset $I \subset \{1, \ldots, D\}$ with $|I| > p^k$ and such that $v(u_{i_1} - u_{i_2}) > 0$ for all $i_1, i_2 \in I$. This means $\overline{u}_{i_1} = \overline{u}_{i_2}$ for all $i_1, i_2 \in I$. We want to show that $\overline{f|_L}$ has a multiple root, that is there exist two indices $j_1, j_2 \in \{1, \ldots, D/n\}$ such that $\overline{w}_{j_1} = \overline{w}_{j_2}$. Suppose not and let $j \in \{1, \ldots, D/n\}$ such that $\overline{w}_j = \overline{u}_i^m = \overline{\phi}(\overline{u}_i)$ for some (all) $i \in I$. Since $\overline{\phi}$ is induced by ϕ we have $t_j = \phi(r_i) = r_i^n$ for all $i \in I$. Then the polynomial $x^n - \overline{w}_j = (x^m - \overline{w}_j)^{p^k} \in \overline{k}[x]$, induced by the factor $x^n - t_j$ of $f(x)$, should have a root of order $|I| > p^k$. This would imply $x^m - \overline{w}_j$ inseparable, a contradiction as $p \nmid m$.

The rest of the lemma follows. □

Theorem 3.23 *For all clusters $\mathfrak{s} \in \Sigma_f$ denote by $\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r)$, and let b be the denominator of $\lambda_{\mathfrak{s}}$. Then f is* `NP`*-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}}$.*

More precisely, let $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ but $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$, and let $r \in \mathfrak{s}$ with $v(r) = \lambda_{\mathfrak{s}}$. Then $\overline{f|_L}$ has a multiple root $\overline{u} = \left(\frac{r}{\pi^{\lambda_{\mathfrak{s}}}}\right) \in \overline{k}$, where L is the (unique) edge of `NP`*(f) of slope $-\lambda_{\mathfrak{s}}$. The multiplicity of $\overline{u}$ equals $|\mathfrak{s}^0|/p^{v_p(b)}$, where $\mathfrak{s}^0 = \left\{r \in \mathfrak{R} \mid \overline{u} = \left(\frac{r}{\pi^{\lambda_{\mathfrak{s}}}}\right)\right\}$. Furthermore, for every edge L of* `NP`*(f), the multiple roots of $\overline{f|_L}$ come from a proper cluster $\mathfrak{s}$ as described above.*

Proof. Let $f = g_1 \ldots g_t$ be a factorisation of Theorem 2.1 and let $-\rho_i$ be the slope of `NP`(g_i). Denote by $\mathfrak{R}$ the set of roots of f and by $\mathfrak{R}_i$ the set of roots of g_i. Note that the $\mathfrak{R}_i$'s are pairwise disjoint. For every edge L of `NP`(f) there exists i such that $\overline{f|_L} = \overline{g_i|_L}$. Hence, by Lemma 2.9 and Lemma 3.22, we need to prove that there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ if and only if for some $i = 1, \ldots, t$ there exists a proper cluster $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\lambda_{\mathfrak{s}_i}|_p = |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \lambda_{\mathfrak{s}_i} = \rho_i$. We will show that one can choose $\mathfrak{s} = \mathfrak{s}_i$.

First of all, note that if $\mathfrak{s} \in \Sigma_f$ contains roots of different valuations, that is $\mathfrak{s} \not\subseteq \mathfrak{R}_i$ for all i, then

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r') = \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}} = \min\{\rho_i \mid \mathfrak{R}_i \cap \mathfrak{s} \neq \varnothing\}.$$

Now suppose there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$. For the observation above, the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s} \subseteq \mathfrak{R}_i$ for some $i = 1, \ldots, t$. Let $\mathcal{D}$ be the v-adic disc such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$. Since $\mathfrak{s} \subseteq \mathfrak{R}_i$, one has $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}_i$ which means that $\mathfrak{s} \in \Sigma_{g_i}$, as required.

Finally suppose that for some i there exists $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \rho_i$. Let $r_i \in \mathfrak{s}_i$. Then

$$\mathfrak{s}_i = \{x \in \overline{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}_i.$$

Consider the cluster $\mathfrak{s} := \{x \in \overline{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}$ of f. Clearly $\mathfrak{s}_i \subseteq \mathfrak{s}$. Therefore

$$\lambda_{\mathfrak{s}_i} = \min_{r \in \mathfrak{s}_i} v(r) \geq \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}},$$

which implies

$$d_{\mathfrak{s}} = d_{\mathfrak{s}_i} > \rho_i = \lambda_{\mathfrak{s}_i} \geq \lambda_{\mathfrak{s}},$$

where $d_{\mathfrak{s}} = d_{\mathfrak{s}_i}$ by construction. Again from the observation above the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s}$ is contained in $\mathfrak{R}_j$ for some j. As $\mathfrak{s} \cap \mathfrak{R}_i \supset \mathfrak{s}_i \cap \mathfrak{R}_i = \mathfrak{s}_i$, we must have $\mathfrak{s} \subseteq \mathfrak{R}_i$. Thus $\mathfrak{s} = \mathfrak{s}_i$, that concludes the proof. □

Corollary 3.24 *Let $f \in K[x]$ be a separable polynomial. Recall the definition of radius $\rho_{\mathfrak{s}}$ of a proper cluster $\mathfrak{s} \in \Sigma_f$. Let $w \in K$. Then $f(x+w)$ is* NP*-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ have rational centre w and those with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$.*

Proof. If $f(x+w)$ is NP-regular, then, from the previous theorem, all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}}$, where $\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w)$. First let $\mathfrak{s} \in \Sigma_f$ and assume $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$. Then

$$d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w) \leq \max_{z \in K} \min_{r \in \mathfrak{s}} v(r-z) = \rho_{\mathfrak{s}} \leq d_{\mathfrak{s}},$$

so $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \rho_{\mathfrak{s}}$, and w is a rational centre of $\mathfrak{s}$. Now assume $|\mathfrak{s}| \leq |\lambda_{\mathfrak{s}}|_p$. In particular $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$, and so

$$\min_{r \in \mathfrak{s}} v(r-w) = \lambda_{\mathfrak{s}} \neq v(w - w_{\mathfrak{s}}),$$

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$. Then

$$\rho_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r - w + w - w_{\mathfrak{s}}) = \min\{\lambda_{\mathfrak{s}}, v(w - w_{\mathfrak{s}})\} \leq \lambda_{\mathfrak{s}}.$$

Clearly

$$\rho_{\mathfrak{s}} = \max_{z \in K} \min_{r \in \mathfrak{s}} v(r-z) \geq \min_{r \in \mathfrak{s}} v(r-w) = \lambda_{\mathfrak{s}},$$

that implies $\rho_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w)$. Hence w is a rational centre of $\mathfrak{s}$.

On the other hand, if all proper clusters $\mathfrak{s} \in \Sigma_f$ have rational centre $w \in K$ then $\rho_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w)$. Thus $f(x+w)$ is NP-regular again by Theorem 3.23. □

Definition 3.25 We say that Σ_f is *almost rational* if all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$.

Corollary 3.26 *Suppose that $K(\mathfrak{R})/K$ is a tame extension. Then Σ_f is almost rational if and only if every proper cluster $\mathfrak{s} \in \Sigma_f$ is G_K-invariant.*

Proof. Since $K(\mathfrak{R})/K$ is tame, every cluster $\mathfrak{s} \in \Sigma_f$ has $|\rho_{\mathfrak{s}}|_p \leq 1$. Therefore the corollary follows from Remark 3.12. □

Corollary 3.27 *Suppose that $K(\mathfrak{R})/K$ is a tame extension. Then $f(x+w)$ is* NP*-regular for some $w \in K$ if and only if Σ_f is nested.*

Proof. First note that every cluster $\mathfrak{s} \in \Sigma_f$ has $|\rho_{\mathfrak{s}}|_p \leq 1$, as $K(\mathfrak{R})/K$ is tame. Therefore from Corollary 3.24, we need to prove that Σ_f is nested if and only if all clusters $\mathfrak{s} \in \Sigma_f$ have $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$ and rational centre w, for some $w \in K$. But this follows from Remark 3.12. □

We conclude this section by showing that the cluster picture (centred at 0) completely determines the Newton polygon of f.

Definition 3.28 Let $z \in \overline{K}$. A *cluster centred at z* is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \overline{K} \mid v(x-z) \geq d\}$ for some $d \in \mathbb{Q}$.

Definition 3.29 Let $z \in \overline{K}$. The *cluster picture centred at z* of f is the set Σ_f^z of all clusters centred at z. Write $\mathring{\Sigma}_f^z$ for the set $\Sigma_f^z \smallsetminus \{\{z\}\}$.

The cluster picture centred at z is *nested*, i.e., every cluster $\mathfrak{s} \in \Sigma_f^z$ has at most one child in Σ_f^z.

Definition 3.30 Let $z \in \overline{K}$, and let $\mathfrak{s} \in \mathring{\Sigma}_f^z$ be a cluster centred at z. The *radius of $\mathfrak{s}$ with respect to the centre z* is

$$\rho_{\mathfrak{s}}^z = \min_{r \in \mathfrak{s}} v(r-z).$$

Finally set

$$\epsilon_{\mathfrak{s}}^{z} := v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}}^{z}.$$

Remark 3.31. From the definitions above, if $\mathfrak{s}$ is a cluster centred at $z \in K^{\mathrm{s}}$, then $\mathfrak{s} = \mathfrak{R} \cap \{x \in \overline{K} \mid v(x-z) \geq \rho_{\mathfrak{s}}^{z}\}$. But this does not mean z is a centre for $\mathfrak{s}$, that is false in general. For example, $\mathfrak{R}$ is clearly a cluster centred at any element of K^{s}, but any element of valuation lower than the valuation of a root $r \in \mathfrak{R}$ can not be a centre of $\mathfrak{R}$.

Remark 3.32. Let $\mathfrak{s} \in \Sigma_f$ be a proper cluster with centre z and rational centre w. Then $\mathfrak{s} \in \Sigma_f^z$, $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}^{z}$, $\nu_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^{z}$, $\rho_{\mathfrak{s}} = \rho_{\mathfrak{s}}^{w}$, and $\epsilon_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^{w}$. Furthermore, $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ if and only if $\mathfrak{s} \in \Sigma_f^w$.

Lemma 3.33 *Let $w \in K$. Then there is a* 1*-to-*1 *correspondence between the clusters in $\mathring{\Sigma}_f^w$ and the edges of* $\mathtt{NP}(f(x+w))$. *More explicitly, if $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ are the clusters in $\mathring{\Sigma}_f^w$, then* $\mathtt{NP}(f(x+w))$ *has vertices Q_i, $i = 0, \ldots, n$, where*

- $Q_n = (\deg f, v(c_f))$,
- $Q_i = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_i}^{w} - |\mathfrak{s}_i|\rho_{\mathfrak{s}_i}^{w}) = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_{i+1}}^{w} - |\mathfrak{s}_i|\rho_{\mathfrak{s}_{i+1}}^{w})$, *for* $i = 1, \ldots, n-1$,
- $Q_0 = \begin{cases} (0, \epsilon_{\mathfrak{s}_1}^{w}) & \text{if } f(w) \neq 0, \\ (1, \epsilon_{\mathfrak{s}_1}^{w} - \rho_{\mathfrak{s}_1}^{w}) & \text{if } f(w) = 0, \end{cases}$

and edges L_i of slope $-\rho_{\mathfrak{s}_i}^{w}$ linking Q_{i-1} and Q_i.

Proof. Without loss of generality we can assume $w = 0$. First note that the coordinates of Q_n are trivial. Now consider a factorisation $f = c_f \cdot g_0 \cdot g_1 \cdots g_s$ of Theorem 2.1 where the polynomials g_j are monic and

$$g_0 = \begin{cases} 1 & \text{if } x \nmid f \\ x & \text{if } x \mid f \end{cases}.$$

Let $\mathfrak{R}_j$ be the set of roots of g_j. It follows from the definition of cluster centred at 0 that

$$n = s, \quad \text{and} \quad \mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j.$$

Let $i = 1, \ldots, n-1$. Then the x-coordinate of Q_i follows as

$$|\mathfrak{s}_i| = \sum_{j=0}^{i} |\mathfrak{R}_j| = \sum_{j=0}^{i} \deg g_j = \deg \prod_{j=0}^{i} g_j.$$

The y-coordinate of Q_i equals the sum of $v(c_f)$ and the valuation of the constant term of $\prod_{j=i+1}^{n} g_j$, so

$$Q_i = \left(|\mathfrak{s}_i|, v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j) \right),$$

where r_j is any root in $\mathfrak{R}_j$. But since $\mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j$, we have $v(r_j) = \rho_{\mathfrak{s}_j}^{0}$. Therefore

$$v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=i+1}^{n} (|\mathfrak{s}_j| - |\mathfrak{s}_{j-1}|)\rho_{\mathfrak{s}_j}^{0} = \epsilon_{\mathfrak{s}_i}^{0} - |\mathfrak{s}_i|\rho_{\mathfrak{s}_i}^{0}.$$

Moreover,

$$\epsilon_{\mathfrak{s}_i}^{0} - |\mathfrak{s}_i|\rho_{\mathfrak{s}_i}^{0} = \epsilon_{\mathfrak{s}_{i+1}}^{0} - |\mathfrak{s}_i|\rho_{\mathfrak{s}_{i+1}}^{0}$$

from the easy computation $\epsilon^0_{\mathfrak{s}_i} - \epsilon^0_{\mathfrak{s}_{i+1}} = |\mathfrak{s}_i|\left(\rho^0_{\mathfrak{s}_i} - \rho^0_{\mathfrak{s}_{i+1}}\right)$. Finally the x-coordinate of Q_0 is trivial, while its y-coordinate equals

$$v(c_f) + \sum_{j=1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=1}^{n} |\mathfrak{R}_j| \rho^0_{\mathfrak{s}_j} + |\mathfrak{R}_0|\rho^0_{\mathfrak{s}_1} - |\mathfrak{R}_0|\rho^0_{\mathfrak{s}_1} = \epsilon^0_{\mathfrak{s}_1} - |\mathfrak{R}_0|\rho^0_{\mathfrak{s}_1},$$

that concludes the proof as $|\mathfrak{R}_0| = \deg g_0$. □

Notation 3.34 Following the notation of Lemma 3.33, let $i \in \{1, \ldots, n\}$ be such that $\mathfrak{s} = \mathfrak{s}_i$. We will write $L^w_{\mathfrak{s}}$ for the edge L_i.

4. Description of a regular model

For the following sections we will use the main definitions, notations and results of [Dok, §3].

Throughout this section, let C/K be a hyperelliptic curve of genus $g \geq 2$, given by the equation $y^2 = f(x)$. Recall from [M²D²] that the *cluster picture* Σ_C is Σ_f. Moreover, all definitions and notations attached to Σ_f (e.g. Σ_f^{rat}, Σ_f^z) given in §3 are given for Σ_C in the same way (e.g. Σ_C^{rat}, Σ_C^z).

Thanks to Lemma 3.33 we can explicitly relate the Newton polytope Δ^w_v of $g(x, y) = y^2 - f(x + w)$ and the cluster picture centred at w of f.

Lemma 4.1 *Let $w \in K$. Then there is a 1-to-1 correspondence between the clusters in $\mathring{\Sigma}^w_C$ and the faces of the Newton polytope Δ^w_v associated with $g_w(x, y) = y^2 - f(x + w)$. More explicitly, if $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ are the clusters in $\mathring{\Sigma}^w_C$ then Δ^w_v has vertices T,Q_i, $i = 0, \ldots, n$, where*

- $T = (0, 2, 0)$,
- $Q_n = (|\mathfrak{R}|, 0, v(c_f))$,
- $Q_i = (|\mathfrak{s}_i|, 0, \epsilon^w_{\mathfrak{s}_{i+1}} - |\mathfrak{s}_i|\rho^w_{\mathfrak{s}_{i+1}})$ *for* $i = 1, \ldots, n-1$,
- $Q_0 = \begin{cases} (0, 0, \epsilon^w_{\mathfrak{s}_1}) & \text{if } f(w) \neq 0, \\ (1, 0, \epsilon^w_{\mathfrak{s}_1} - \rho^w_{\mathfrak{s}_1}) & \text{if } f(w) = 0, \end{cases}$

and edges L_i ($i = 1, \ldots, n$), linking Q_{i-1} and Q_i, and V_j ($j = 0, \ldots, n$), linking Q_j and T.

Furthermore, (possible choices for) the slopes of the edges of Δ^w_v are:

- $$s_1^{V_n} = \delta_{V_n} \frac{-v(c_f) + (|\mathfrak{R}| - 2g)\rho^w_{\mathfrak{R}}}{2} \quad \text{and} \quad s_2^{V_n} = \lfloor s_1^{V_n} - 1 \rfloor;$$
- $$s_1^{V_i} = \delta_{V_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_i}\right), \quad s_2^{V_i} = \delta_{V_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_{i+1}}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_{i+1}}\right) \quad \text{for all } i = 1, \ldots, n-1;$$
- $$s_1^{V_0} = \delta_{V_0}\left(\frac{\epsilon^w_{\mathfrak{s}_1}}{2} - \rho^w_{\mathfrak{s}_1}\right) \quad \text{and} \quad s_2^{V_0} = \lfloor s_1^{V_0} - 1 \rfloor;$$
- $$s_1^{L_i} = \delta_{L_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_i}\right) \quad \text{and} \quad s_2^{L_i} = \lfloor s_1^{L_i} - 1 \rfloor,$$

  for all $i = 1, \ldots, n$. In particular, as δ_{L_i} is the denominator of $\rho^w_{\mathfrak{s}_i}$,

$$r_{L_i} = \begin{cases} 1 & \text{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \text{ is odd}, \\ 0 & \text{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \text{ is even}. \end{cases}$$

Proof. The first part of the lemma follows from Lemma 3.33. For the second part, we only need to individuate, for all the edges, the two points P_0 and P_1 of [Dok, Definition 3.12]. It is easy to see that the followings are admissible choices.

- For V_i and L_i ($i = 1, \dots, n$), choose $P_0 = (|\mathfrak{s}_i|, 0)$ and $P_1 = \left(\left\lfloor \frac{|\mathfrak{s}_i|-1}{2} \right\rfloor, 1\right)$.
- For V_0, choose $P_0 = (0, 2)$ and $P_1 = (1, 1)$;

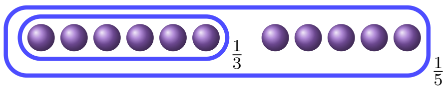
The second part of the lemma then follows from the first one. □

Notation 4.2 Let C be as above and let $w \in K$. For every cluster $\mathfrak{s} \in \Sigma_C^w$ denote by $F_\mathfrak{s}^w$ the face of the Newton polytope Δ_v^w of $g_w(x, y) = y^2 - f(x + w)$ that corresponds to $\mathfrak{s}$.

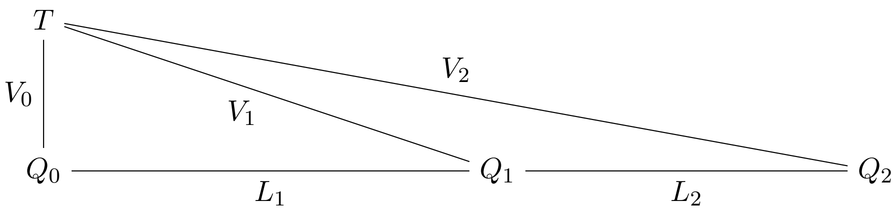
Following the notation of Lemma 4.1, let $i \in \{1, \dots, n\}$ be such that $\mathfrak{s} = \mathfrak{s}_i$. We will write $L_\mathfrak{s}^w$, $V_\mathfrak{s}^w$, V_0^w for the edges L_i, V_i, V_0, respectively.

Example 4.3. Let C be the hyperelliptic curve over $\mathbb{Q}_3$ given by the equation $y^2 = f(x)$ where $f(x) = x^{11} - 3x^6 + 9x^5 - 27$ is the polynomial of Example 3.20.

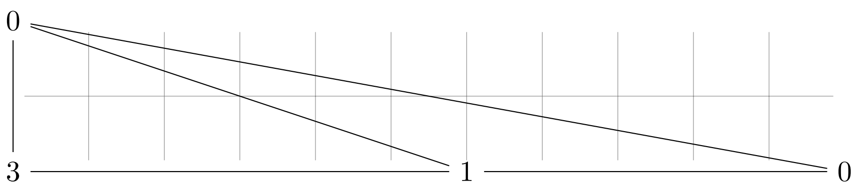
Its cluster picture centred at 0 is



where the subscripts represent the radii with respect to 0. As we can see, Σ_f^0 consists of two clusters: $\mathfrak{s}_1$ of size 6, radius $\frac{1}{3}$ and $\epsilon_{\mathfrak{s}_1}^0 = 3$, and $\mathfrak{s}_2 = \mathfrak{R}$ of size 11, radius $\frac{1}{5}$ and $\epsilon_{\mathfrak{s}_2}^0 = \frac{11}{5}$. Therefore the picture of Δ broken into v-faces will be



where $T = (0, 2)$, $Q_0 = (0, 0)$, $Q_1 = (6, 0)$, and $Q_2 = (11, 0)$. Denoting the values of v on vertices, the picture becomes



Before stating the theorems which describe the proper flat model $\mathcal{C}$ of C, constructed in §5, we need some definitions.

Definition 4.4 Let F/K be an unramified extension and let $\Sigma_F = \Sigma_{C_F}^{\mathrm{rat}}$ (i.e., set of clusters cut out by disk with centre in F). We define the following quantities:

$\mathfrak{s}\in\Sigma_F$, proper	
radius	$\rho_{\mathfrak{s}} = \max_{w\in F}\min_{r\in\mathfrak{s}} v(r-w)$
	$b_{\mathfrak{s}} =$ denominator of $\rho_{\mathfrak{s}}$
	$\epsilon_{\mathfrak{s}} = v(c_f)+\sum_{r\in\mathfrak{R}}\rho_{r\wedge\mathfrak{s}}$
	$D_{\mathfrak{s}} =$ 1 if $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ odd, 2 if $b_{\mathfrak{s}}\epsilon_{\mathfrak{s}}$ even
multiplicity	$m_{\mathfrak{s}} = (3-D_{\mathfrak{s}})b_{\mathfrak{s}}$
parity	$p_{\mathfrak{s}} =$ 1 if $\lvert\mathfrak{s}\rvert$ is odd, 2 if $\lvert\mathfrak{s}\rvert$ is even
slope	$s_{\mathfrak{s}} = \frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}}+p_{\mathfrak{s}}\rho_{\mathfrak{s}}-\epsilon_{\mathfrak{s}})$
	$\gamma_{\mathfrak{s}} =$ 2 if $\mathfrak{s}$ is even and $\epsilon_{\mathfrak{s}}-\lvert\mathfrak{s}\rvert\rho_{\mathfrak{s}}$ is odd, 1 otherwise
	$p^0_{\mathfrak{s}} =$ 1 if $\mathfrak{s}$ is minimal and $\mathfrak{s}\cap F\neq\varnothing$, 2 otherwise
	$s^0_{\mathfrak{s}} = -\epsilon_{\mathfrak{s}}/2+\rho_{\mathfrak{s}}$
	$\gamma^0_{\mathfrak{s}} =$ 2 if $p^0_{\mathfrak{s}}=2$ and $\epsilon_{\mathfrak{s}}$ is odd, 1 otherwise

Definition 4.5 We say that C is *y-regular* if either $p\neq 2$ or $\gamma_{\mathfrak{s}}=p_{\mathfrak{s}}$ for every proper $\mathfrak{s}\in\Sigma_C^{\text{rat}}$, and $\gamma^0_{\mathfrak{s}}=p^0_{\mathfrak{s}}$ when $\mathfrak{s}$ minimal.

Lemma 4.6 *The hyperelliptic curve C is Δ_v-regular if and only if C is y-regular and f is* NP*-regular.*

Proof. The proof follows from the structure of Δ_v. Indeed, if C is y-regular and f is NP-regular, then C is Δ_v-regular by Lemma 4.1. On the other hand, the converse also holds since if f is NP-regular, then all clusters have rational centre 0 by Theorem 3.23. □

Definition 4.7 Let $\mathfrak{s}\in\Sigma_F$ be a proper cluster and fix $c\in\mathbb{Z}$ such that $c\rho_{\mathfrak{s}}-\frac{1}{b_{\mathfrak{s}}}\in\mathbb{Z}$. Define

$$\tilde{\mathfrak{s}} = \{\mathfrak{s}'\in\Sigma_F\cup\{\varnothing\}\mid \mathfrak{s}'<\mathfrak{s} \text{ and } \tfrac{\lvert\mathfrak{s}'\rvert}{b_{\mathfrak{s}}}-c\epsilon_{\mathfrak{s}}\notin 2\mathbb{Z}\},$$

where $\varnothing<\mathfrak{s}$ if $p^0_{\mathfrak{s}}=2$.

The *genus* $g(\mathfrak{s})$ of a rational cluster $\mathfrak{s}\in\Sigma_F$ is defined as follows:

- If $D_{\mathfrak{s}}=1$, then $g(\mathfrak{s})=0$.
- If $D_{\mathfrak{s}}=2$, then $2g(\mathfrak{s})+1$ or $2g(\mathfrak{s})+2$ equals

$$\frac{\lvert\mathfrak{s}\rvert-\sum_{\mathfrak{s}'\in\Sigma_F,\mathfrak{s}'<\mathfrak{s}}\lvert\mathfrak{s}'\rvert}{b_{\mathfrak{s}}}+\lvert\tilde{\mathfrak{s}}\rvert.$$

Definition 4.8 Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma\subseteq\Sigma_C^{\min}$. For each cluster $\mathfrak{s}\in\Sigma$, fix a rational centre $w_{\mathfrak{s}}$; if possible, choose $w_{\mathfrak{s}}\in\mathfrak{s}$. Let W be the set of these rational centres and define $\Sigma^W=\bigcup_{w\in W}\Sigma_C^w$. For any proper cluster $\mathfrak{s}\in\Sigma^W$ fix a rational centre $w_{\mathfrak{s}}\in W$. Denote $r_{\mathfrak{s}}=\frac{w_{\mathfrak{s}}-r}{\pi^{\rho_{\mathfrak{s}}}}$ for $r\in\mathfrak{R}$ and define reductions $\overline{f_{\mathfrak{s}}}(x)\in k[x]$, $\overline{g_{\mathfrak{s}}}\in k[y]$, and for $\mathfrak{s}\in\Sigma$ also $\overline{g^0_{\mathfrak{s}}}\in k[y]$ by

$$x^{2-p^0_{\mathfrak{s}}}\overline{f^W_{\mathfrak{s}}}(x^{b_{\mathfrak{s}}}) = \tfrac{u}{\pi^{v(u)}}\prod_{r\in\mathfrak{s}\setminus\bigcup_{\mathfrak{s}'<\mathfrak{s}}\mathfrak{s}'}(x+r_{\mathfrak{s}}) \mod \pi, \qquad u=c_f\prod_{r\in\mathfrak{R}\setminus\mathfrak{s}} r_{\mathfrak{s}},$$

$$\overline{g_{\mathfrak{s}}}(y) = y^{p_{\mathfrak{s}}/\gamma_{\mathfrak{s}}}-\tfrac{u}{\pi^{v(u)}} \mod \pi, \qquad u=c_f\prod_{r\in\mathfrak{R}\setminus\mathfrak{s}} r_{\mathfrak{s}},$$

$$\overline{g^0_{\mathfrak{s}}}(y) = y^{p^0_{\mathfrak{s}}/\gamma^0_{\mathfrak{s}}}-\tfrac{u}{\pi^{v(u)}} \mod \pi, \qquad u=c_f\prod_{r\in\mathfrak{R}\setminus\{w_{\mathfrak{s}}\}} r_{\mathfrak{s}}.$$

where the union runs through all $\mathfrak{s}'\in\Sigma^W$, $\mathfrak{s}'<\mathfrak{s}$. Finally define the k-schemes

(1) $X_{\mathfrak{s}}:\{\overline{g_{\mathfrak{s}}}=0\}\subset\mathbb{G}_{m,k}$;
(2) $X^W_{\mathfrak{s}}:\{\overline{f^W_{\mathfrak{s}}}=0\}\subset\mathbb{G}_{m,k}$;
(3) $X^0_{\mathfrak{s}}:\{\overline{g^0_{\mathfrak{s}}}=0\}\subset\mathbb{G}_{m,k}$ if $\mathfrak{s}\in\Sigma$.

Notation 4.9 Given a scheme $\mathcal{X}/O_K$ we will denote by $\mathcal{X}_\eta$ its generic fibre $\mathcal{X}\times_{\text{Spec } O_K}\text{Spec } K$, and by $\mathcal{X}_s$ its special fibre $\mathcal{X}\times_{\text{Spec } O_K}\text{Spec } k$.

Notation 4.10 If $C = C_1 \cup \ldots C_r$ is a chain of $\mathbb{P}^1_k$s of length r and multiplicities $m_i \in \mathbb{Z}$ (meeting transversely), then $\infty \in C_i$ is identified with $0 \in C_{i+1}$, and $0, \infty \in C$ are respectively $0 \in C_1$ and $\infty \in C_r$. Finally, if $r = 0$, then $C = \operatorname{Spec} k$ and $0 = \infty$.

Notation 4.11 Let $\alpha, a, b \in \mathbb{Z}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1_k$s of length r and multiplicities αd_i. Furthermore, we denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor / \alpha)$.

Theorem 4.12 and Theorem 4.16 will be proved in §5.

Theorem 4.12 *Let C/K be a hyperelliptic curve given by a Weierstrass equation $y^2 = f(x)$, and let Σ, W and Σ^W as in Definition 4.8. Then there exists a proper flat model $\mathcal{C}/O_K$ of C with normal crossings such that its special fibre $\mathcal{C}_s/k$ consists of 1-dimensional schemes glued along 0-dimensional intersections as follows:*

1. *Every proper cluster $\mathfrak{s} \in \Sigma^W$ gives a 1-dimensional closed subscheme $\Gamma_\mathfrak{s}$ of multiplicity $m_\mathfrak{s}$. If $\Gamma_\mathfrak{s}$ is reducible then $D_\mathfrak{s} = 2$, $\Gamma_\mathfrak{s} = \Gamma_\mathfrak{s}^+ \cup \Gamma_\mathfrak{s}^-$, with $\Gamma_\mathfrak{s}^\pm = \mathbb{P}^1_k$, and there is a (birational) morphism $X_\mathfrak{s} \times \mathbb{P}^1_k \to \Gamma_\mathfrak{s}$.*
2. *Every proper cluster $\mathfrak{s} \in \Sigma^W$ with $D_\mathfrak{s} = 1$ gives the closed subscheme $X_\mathfrak{s}^W \times \mathbb{P}^1_k$, of multiplicity $b_\mathfrak{s}$, where $X_\mathfrak{s}^W \times \{0\} \subset \Gamma_\mathfrak{s}$.*
3. *Every proper cluster $\mathfrak{s} \in \Sigma^W$ such that $\mathfrak{s} \neq \mathfrak{R}$, gives the closed subscheme $X_\mathfrak{s} \times \mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s}, s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{\rho_\mathfrak{s} - \rho_{P(s)}}{2})$ where $X_\mathfrak{s} \times \{0\} \subset \Gamma_\mathfrak{s}$ and $X_\mathfrak{s} \times \{\infty\} \subset \Gamma_{P(\mathfrak{s})}$.*
4. *Every cluster $\mathfrak{s} \in \Sigma$ gives the closed subscheme $X_\mathfrak{s}^0 \times \mathbb{P}^1(\gamma_\mathfrak{s}^0, -s_\mathfrak{s}^0)$ where $X_\mathfrak{s}^0 \times \{0\} \subset \Gamma_\mathfrak{s}$ (the chain is open-ended).*
5. *Finally, the cluster $\mathfrak{R}$ gives the closed subscheme $X_\mathfrak{R} \times \mathbb{P}^1(\gamma_\mathfrak{R}, s_\mathfrak{R})$ where $X_\mathfrak{R} \times \{0\} \subset \Gamma_\mathfrak{s}$ (the chain is open-ended).*

Furthermore, if Σ_C is almost rational and C is y-regular, then, by choosing $\Sigma = \Sigma_C^{\min}$, the model $\mathcal{C}$ is regular. In that case, if $\mathfrak{s}$ is übereven and $\epsilon_\mathfrak{s}$ is even, then $\Gamma_\mathfrak{s} \simeq X_\mathfrak{s} \times \mathbb{P}^1_k$, otherwise $\Gamma_\mathfrak{s}$ is irreducible of genus $g(\mathfrak{s})$.

Definition 4.13 Let $\mathfrak{s} \in \Sigma_{K^{nr}}$. We say that

- $\mathfrak{s}$ is *removable* if either $|\mathfrak{s}| = 1$ or $\mathfrak{s} = \mathfrak{R}$ and it has a (rational) child of size $2g + 1$.
- $\mathfrak{s}$ is *contractible* if
  1. $|\mathfrak{s}| = 2$ and $\rho_\mathfrak{s} \notin \mathbb{Z}$, $\epsilon_\mathfrak{s}$ odd, $\rho_{P(\mathfrak{s})} \leq \rho_\mathfrak{s} - \frac{1}{2}$; or
  2. $\mathfrak{s} = \mathfrak{R}$ of size $2g + 2$ with a child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g$, and $\rho_\mathfrak{s} \notin \mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}'} \geq \rho_\mathfrak{s} + \frac{1}{2}$; or
  3. $\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, union of its 2 odd proper children $\mathfrak{s}_1, \mathfrak{s}_2 \in \Sigma_{K^{nr}}$, with $v(c_f)$ odd, $\rho_{\mathfrak{s}_i} \geq \rho_\mathfrak{s} + 1$ for $i = 1, 2$.

Notation 4.14 Write $\mathring{\Sigma} \subset \Sigma_{K^{nr}}$ for the set of non-removable clusters.

Definition 4.15 Choose rational centres $w_\mathfrak{s}$ for every $\mathfrak{s} \in \mathring{\Sigma}$, in such a way that $w_\mathfrak{s} \in \mathfrak{s}$ when $p_\mathfrak{s}^0 = 1$, and $\sigma(w_\mathfrak{s}) = w_{\sigma(\mathfrak{s})}$ for all $\sigma \in \operatorname{Gal}(K^{nr}/K)$. Denote $r_\mathfrak{s} = \frac{w_\mathfrak{s} - r}{\pi^{\rho_\mathfrak{s}}}$ for $r \in \mathfrak{R}$ and define $\overline{g_\mathfrak{s}}, \overline{g_\mathfrak{s}^0} \in k^{\mathrm{s}}[y]$ as in Definition 4.8, and $\overline{f_\mathfrak{s}}(x) \in k^{\mathrm{s}}[x]$, by

$$x^{2-p_\mathfrak{s}^0} \overline{f_\mathfrak{s}}(x^{b_\mathfrak{s}}) = \tfrac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s} \setminus \bigcup_{\mathfrak{s}' < \mathfrak{s}} \mathfrak{s}'} (x + r_\mathfrak{s}) \mod \pi, \quad u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_\mathfrak{s},$$

where the union runs through all $\mathfrak{s}' \in \mathring{\Sigma}$, $\mathfrak{s}' < \mathfrak{s}$. Let $G_\mathfrak{s} = \operatorname{Stab}_{G_K}(\mathfrak{s})$, $K_\mathfrak{s} = (K^{\mathrm{s}})^{G_\mathfrak{s}}$, and let $k_\mathfrak{s}$ be the residue field of $K_\mathfrak{s}$. Then $\overline{f_\mathfrak{s}} \in k_\mathfrak{s}[x]$, $\overline{g_\mathfrak{s}} \in k_\mathfrak{s}[y]$, and for

$\mathfrak{s}$ minimal $\overline{g^0_{\mathfrak{s}}} \in k_{\mathfrak{s}}[y]$. Finally define $\tilde{f}_{\mathfrak{s}} \in k_{\mathfrak{s}}[x]$ by

$$\tilde{f}_{\mathfrak{s}}(x) = \prod_{\mathfrak{s}' \in \tilde{\mathfrak{s}}} \left(x - \overline{u_{\mathfrak{s}',\mathfrak{s}}}\right) \cdot \overline{f_{\mathfrak{s}}}(x),$$

where $\overline{u_{\mathfrak{s}',\mathfrak{s}}} = \frac{w_{\mathfrak{s}'} - w_{\mathfrak{s}}}{\pi^{\rho_{\mathfrak{s}}}} \mod \pi$.

In the next theorem we describe the minimal regular model of C with normal crossings.

Theorem 4.16 (Minimal regular NC model) *Let C/K be a hyperelliptic curve. Suppose $C_{K^{nr}}$ is y-regular and $\Sigma_{C_{K^{nr}}}$ is almost rational. Then the minimal regular model with normal crossings $\mathcal{C}^{\min}/O_{K^{nr}}$ of C has special fibre $\mathcal{C}^{\min}_s/k^{\mathrm{s}}$ described as follows:*

1. *Every $\mathfrak{s} \in \mathring{\Sigma}$ gives a 1-dimensional subscheme $\Gamma_{\mathfrak{s}}$ of multiplicity $m_{\mathfrak{s}}$. If $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}}$ is even, then $\Gamma_{\mathfrak{s}}$ is the disjoint union of $\Gamma^{r_-}_{\mathfrak{s}} \simeq \mathbb{P}^1$ and $\Gamma^{r_+}_{\mathfrak{s}} \simeq \mathbb{P}^1$, otherwise $\Gamma_{\mathfrak{s}}$ is irreducible of genus $g(\mathfrak{s})$ (write $\Gamma^{r_-}_{\mathfrak{s}} = \Gamma^{r_+}_{\mathfrak{s}} = \Gamma_{\mathfrak{s}}$ in this case). The indices r_- and r_+ are the roots of $\overline{g_{\mathfrak{s}}}$.*
2. *Every $\mathfrak{s} \in \mathring{\Sigma}$ with $D_{\mathfrak{s}} = 1$ gives open-ended $\mathbb{P}^1$s of multiplicity $b_{\mathfrak{s}}$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{f_{\mathfrak{s}}}$.*
3. *Every non-maximal element $\mathfrak{s} \in \mathring{\Sigma}$ gives open-ended chains $\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(s)}}{2})$ from $\Gamma_{\mathfrak{s}}$ to $\Gamma_{P(\mathfrak{s})}$ indexed by roots of $\overline{g_{\mathfrak{s}}}$.*
4. *Every minimal element $\mathfrak{s} \in \mathring{\Sigma}$ gives open-ended chains $\mathbb{P}^1(\gamma^0_{\mathfrak{s}}, -s^0_{\mathfrak{s}})$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{g^0_{\mathfrak{s}}}$.*
5. *The maximal element $\mathfrak{s} \in \mathring{\Sigma}$ gives open-ended chains $\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}})$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{g_{\mathfrak{s}}}$.*
6. *Finally, blow down all $\Gamma_{\mathfrak{s}}$ where $\mathfrak{s}$ is a contractible cluster.*

In (3) and (5), a chain indexed by r goes from $\Gamma^r_{\mathfrak{s}}$ (to $\Gamma^r_{P(\mathfrak{s})}$ in (3)).

The Galois group G_k acts naturally, i.e., for every $\sigma \in G_k$, $\sigma(\Gamma^r_{\mathfrak{s}}) = \Gamma^{\sigma(r)}_{\sigma(\mathfrak{s})}$, and similarly, on the chains.

If $\Gamma_{\mathfrak{s}}$ is irreducible, then its function field is isomorphic to $k^{\mathrm{s}}(x)[y]$ with the relation $y^{D_{\mathfrak{s}}} = \tilde{f}_{\mathfrak{s}}(x)$.

Remark 4.17. Note that if $\Gamma_{\mathfrak{s}}$ or $\Gamma_{P(\mathfrak{s})}$ is reducible then $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = 2$.

As an application of Theorem 4.16 we suppose $p \neq 2$ and C to be semistable. In this setting [M^{2}D^2, Theorem 8.5] describes the minimal regular model of C in terms of its cluster picture Σ_C. We compare that result with the one obtained from Theorem 4.16 (Corollary 4.19).

From [M^{2}D^2, Definition 1.7], if C is semistable then

1. the extension $K(\mathfrak{R})/K$ is tamely ramified;
2. every proper cluster is $\mathrm{Gal}(K^{\mathrm{s}}/K^{nr})$-invariant;
3. every principal cluster has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.

It follows from Corollary 3.26 that $\Sigma_{C_{K^{nr}}}$ is almost rational.

In fact, (1) and (2) imply $\rho_{\mathfrak{s}} = d_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}} = \nu_{\mathfrak{s}}$ for any proper cluster $\mathfrak{s}$ (Remark 3.12). In particular, $\Sigma_{K^{nr}} = \Sigma_C$. Finally, note that $\tilde{\mathfrak{s}}$ is the set of odd children of $\mathfrak{s} \in \Sigma_C$.

Lemma 4.18 *Suppose $p \neq 2$. Assume C is semistable and let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Then $d_{\mathfrak{s}} \in \frac{1}{2}\mathbb{Z}$. Moreover, $d_{\mathfrak{s}} \notin \mathbb{Z}$ if and only if $\nu_{\mathfrak{s}}$ is odd and $p_{\mathfrak{s}} = 2$. In particular, $\mathfrak{s}$ is contractible if and only if*

1. *$|\mathfrak{s}| = 2$ and $d_{\mathfrak{s}} \notin \mathbb{Z}$; or*

(2) $\mathfrak{s} = \mathfrak{R}$ *of size* $2g+2$ *with only* 1 *proper child* $\mathfrak{s}' \in \Sigma_C$ *of size* $2g$*, and* $d_{\mathfrak{s}} \notin \mathbb{Z}$*; or*

(3) $\mathfrak{s} = \mathfrak{R}$ *of size* $2g+2$*, with exactly* 2 *odd children, and* $v(c_f)$ *odd.*

Proof. Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Clearly we need to study the cases when $\mathfrak{s}$ is not principal. First note that $d_{\mathfrak{s}} \in \frac{1}{2}\mathbb{Z}$ by Lemma3.11.

Suppose $\mathfrak{s} = \mathfrak{R}$, even, with exactly 2 children. By Lemma 3.18, $d_{\mathfrak{R}} \in \mathbb{Z}$. Since $\mathfrak{s}$ is non-removable, one of its children, say $\mathfrak{s}'$, is principal. Then $d_{\mathfrak{s}'} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}'} \in 2\mathbb{Z}$. Therefore

$$\nu_{\mathfrak{s}} = \nu_{\mathfrak{s}'} - |\mathfrak{s}'|(d_{\mathfrak{s}'} - d_{\mathfrak{s}}) \in 2\mathbb{Z}.$$

Assume $\mathfrak{s}$ is a cotwin. Let $\mathfrak{s}'$ denote its proper child. Then $\nu_{\mathfrak{s}} = \nu_{\mathfrak{s}'} - 2g(d_{\mathfrak{s}'} - d_{\mathfrak{s}})$. Since $\mathfrak{s}'$ is principal, we have $d_{\mathfrak{s}'} \in \mathbb{Z}$, $\nu_{\mathfrak{s}'} \in 2\mathbb{Z}$. Therefore $\nu_{\mathfrak{s}}$ is odd if and only if $d_{\mathfrak{s}} \notin \mathbb{Z}$. Moreover, if that happens, then $\mathfrak{s} = \mathfrak{R}$, even.

Finally suppose $\mathfrak{s}$ is a twin. Then $\nu_{\mathfrak{s}} = \nu_{P(\mathfrak{s})} + 2(d_{\mathfrak{s}} - d_{P(\mathfrak{s})})$. We have $d_{P(\mathfrak{s})} \in \mathbb{Z}$ and $\nu_{P(\mathfrak{s})} \in 2\mathbb{Z}$ (even if $P(\mathfrak{s})$ is not principal, from the first part of this proof). Thus $\nu_{\mathfrak{s}}$ is odd if and only if $d_{\mathfrak{s}} \notin \mathbb{Z}$. □

Corollary 4.19 (Minimal regular model (semistable reduction)) *Let* C/K *be a semistable hyperelliptic curve. The minimal regular model* $\mathcal{C}^{\min}/O_{K^{nr}}$ *of* C *has special fibre* $\mathcal{C}_s^{\min}/k^{\mathrm{s}}$ *described as follows:*

(1) Every proper cluster $\mathfrak{s} \in \Sigma_C$ *gives a* 1*-dimensional subscheme* $\Gamma_{\mathfrak{s}}$ *of multiplicity* $m_{\mathfrak{s}}$*. If* $\mathfrak{s}$ *is übereven, then* $\Gamma_{\mathfrak{s}}$ *is the disjoint union of* $\Gamma_{\mathfrak{s}}^{r_-} \simeq \mathbb{P}^1$ *and* $\Gamma_{\mathfrak{s}}^{r_+} \simeq \mathbb{P}^1$*, otherwise* $\Gamma_{\mathfrak{s}}$ *is irreducible of genus* $g(\mathfrak{s})$ *(write* $\Gamma_{\mathfrak{s}}^{r_-} = \Gamma_{\mathfrak{s}}^{r_+} = \Gamma_{\mathfrak{s}}$ *in this case). The indices* r_- *and* r_+ *are the roots of* $\overline{g_{\mathfrak{s}}}$*.*

(2) Every odd proper cluster $\mathfrak{s} \in \Sigma_C$*,* $|\mathfrak{s}| \leq 2g$ *gives a chain of* $\mathbb{P}^1$*s of length* $\lfloor \frac{d_{\mathfrak{s}} - d_{P(\mathfrak{s})} - 1}{2} \rfloor$ *from* $\Gamma_{\mathfrak{s}}$ *to* $\Gamma_{P(\mathfrak{s})}$ *indexed by the root of* $\overline{g_{\mathfrak{s}}}$*.*

(3) Every even proper cluster $\mathfrak{s}$*,* $|\mathfrak{s}| \leq 2g$*, gives a chain of* $\mathbb{P}^1$*s of length* $\lfloor d_{\mathfrak{s}} - d_{P(\mathfrak{s})} - \frac{1}{2} \rfloor$ *from* $\Gamma_{\mathfrak{s}}^{r_-}$ *to* $\Gamma_{P(\mathfrak{s})}^{r_-}$ *indexed by* r_- *and a chain of* $\mathbb{P}^1$*s of same length from* $\Gamma_{\mathfrak{s}}^{r_+}$ *to* $\Gamma_{P(\mathfrak{s})}^{r_+}$ *indexed by* r_+*.*

(4) Finally, blow down all $\Gamma_{\mathfrak{s}}$ *where* $\mathfrak{s}$ *is a contractible cluster.*

All components have multiplicity 1*, and the absolute Galois group* G_k *acts naturally, as in Theorem 4.16.*

Proof. Let $\mathfrak{s} \in \Sigma_C$. From Lemma 4.18, we have $D_{\mathfrak{s}} = 2$, $\gamma_{\mathfrak{s}} s_{\mathfrak{s}} \in \mathbb{Z}$ and if $\mathfrak{s}$ is minimal, $\gamma_{\mathfrak{s}}^0 s_{\mathfrak{s}}^0 \in \mathbb{Z}$. Therefore (2), (4) and (5) of Theorem 4.16 do not give any components. Finally, as $\gamma_{\mathfrak{s}} = 1$ for any $\mathfrak{s} \neq \mathfrak{R}$ and $p_{\mathfrak{s}} \frac{d_{\mathfrak{s}} - d_{P(\mathfrak{s})}}{2} \in \frac{1}{2}\mathbb{Z}$, the length of $\mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{d_{\mathfrak{s}} - d_{P(s)}}{2})$ in (3) is

$$\left\lfloor \gamma_{\mathfrak{s}} s_{\mathfrak{s}} - \gamma_{\mathfrak{s}} \left(s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{d_{\mathfrak{s}} - d_{P(s)}}{2} \right) - \frac{1}{2} \right\rfloor = \left\lfloor p_{\mathfrak{s}} \cdot \frac{d_{\mathfrak{s}} - d_{P(s)}}{2} - \frac{1}{2} \right\rfloor.$$

The corollary then follows from Theorem 4.16. □

5. Construction of the model

We are going to construct a proper flat model $\mathcal{C}/O_K$ of C by glueing models defined in [Dok, §4]. For this reason we will assume the reader has familiarity with the definitions and the results presented in that paper. Let us start this section by describing the strategy we will follow.

Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma \subseteq \Sigma_C^{\min}$. For any cluster $\mathfrak{s} \in \Sigma$ fix a rational centre $w_{\mathfrak{s}}$. If possible, choose $w_{\mathfrak{s}} \in \mathfrak{s}$.[2] Let W be the set of all such rational centres and define $\Sigma^W := \bigcup_{w \in W} \Sigma_C^w$. For

[2]This assumption is not necessary for the construction, but it will simplify the following.

every proper cluster $\mathfrak{t} \in \Sigma^W \smallsetminus \Sigma$ inductively fix a rational centre $w_\mathfrak{t} = w_\mathfrak{s}$ for some cluster $\mathfrak{s} < \mathfrak{t}$ (Lemma 3.14). For every cluster $w \in W$, consider the curve $C^w : y^2 = f(x+w)$, isomorphic to C, and construct the (proper flat) model $\mathcal{C}^w_\Delta/O_K$ by [Dok, Theorem 3.14]. We will define an open subscheme $\mathring{\mathcal{C}}^w_\Delta$ of $\mathcal{C}^w_\Delta$ and we will show that glueing these schemes along common opens gives a proper flat model $\mathcal{C}/O_K$ of C. Furthermore, if $\Sigma = \Sigma_C^{\min}$, Σ_C is almost rational and C is y-regular, then $\mathring{\mathcal{C}}^w_\Delta$ is the open subscheme of regular points of $\mathcal{C}^w_\Delta$ and therefore $\mathcal{C}$ is also regular.

5.1. **Charts.** Let $\Sigma = \{\mathfrak{s}_1 \dots, \mathfrak{s}_m\} \subseteq \Sigma_C^{\min}$ be a set of rationally minimal clusters and let $W = \{w_1, \dots, w_m\}$ be a set of corresponding rational centres, where $w_h \in \mathfrak{s}_h$, if possible. Define $\Sigma^W := \bigcup_{h=1}^m \Sigma_C^{w_h}$. Note that for the chioce of the rational centres w_h, the subset of proper clusters of $\Sigma_C^{\mathrm{w_h}}$ coincides with $\mathring{\Sigma}_C^{w_h}$. For any $h, l = 1, \dots, m$, $h \neq l$, define $w_{hl} := w_h - w_l$, and write $w_{hl} = u_{hl}\pi^{\rho_{hl}}$, where $u_{hl} \in O_K^\times$ and $\rho_{hl} \in \mathbb{Z}$. Note that $\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} = \rho_{lh}$, by Lemma 3.18.

Definition 5.1 Let $h = 1, \dots, m$ and let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. We say that a matrix M *is associated to* $\mathfrak{t}$ if $M = M_{L_\mathfrak{t}^{w_h},i}$ or $M = M_{V_\mathfrak{t}^{w_h},j}$ (or $M = M_{V_0^{w_h},j}$ if $\mathfrak{t} = \mathfrak{s}_h$). For a matrix M associated to $\mathfrak{t}$ we denote by δ_M and X_M respectively

- the quantity $\delta_{L_\mathfrak{t}^{w_h}}$ and the space $X_{\sigma_{L_\mathfrak{t}^{w_h},i,i+1}}$ if $M = M_{L_\mathfrak{t}^{w_h},i}$,
- the quantity $\delta_{V_\mathfrak{t}^{w_h}}$ and the space $X_{\sigma_{V_\mathfrak{t}^{w_h},j,j+1}}$ if $M = M_{V_\mathfrak{t}^{w_h},j}$,
- the quantity $\delta_{V_0^{w_h}}$ and the space $X_{\sigma_{V_0^{w_h},j,j+1}}$ if $M = M_{V_0^{w_h},j}$.

Finally, denote by

$$X_\Delta^h = \bigcup_M X_M,$$

the toric scheme defined in [Dok, §4.2].

The following lemma describes all possible matrices associated to $\mathfrak{t}$.

Lemma 5.2 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Consider the face $F_\mathfrak{t}^{w_h}$. Let $P_0, P_1 \in \mathbb{Z}^2$ and $n_i, d_i, k_i \in \mathbb{Z}$ be as in* [Dok, §4] *and define*

$$\delta := \delta_M, \quad \gamma_i := \frac{n_0}{\delta d_0} - \frac{n_i}{\delta d_i} \quad \text{and} \quad T_i = \begin{pmatrix} \frac{1}{\delta} & -d_{i+1}k_i & d_i k_{i+1} \\ 0 & \delta d_{i+1} & 0 \\ 0 & 0 & \delta d_i \end{pmatrix},$$

for each matrix M associated to $\mathfrak{t}$.

- *For all $i = 0, \dots, r_{L_\mathfrak{t}^{w_h}}$, we have*

$$M_{L_\mathfrak{t}^{w_h},i} = \begin{pmatrix} \delta & -d_i\left(\left\lfloor \frac{|\mathfrak{t}|}{2} \right\rfloor + 1\right) + k_i & d_{i+1}\left(\left\lfloor \frac{|\mathfrak{t}|}{2} \right\rfloor + 1\right) - k_{i+1} \\ 0 & d_i & -d_{i+1} \\ -\delta\rho_\mathfrak{t} & \frac{n_i}{\delta} - k_i\rho_\mathfrak{t} & -\frac{n_{i+1}}{\delta} + k_{i+1}\rho_\mathfrak{t} \end{pmatrix}, \quad M^{-1}_{L_\mathfrak{t}^{w_h},i} = T_i \cdot \begin{pmatrix} 1 & \left\lfloor \frac{|\mathfrak{t}|}{2} \right\rfloor + 1 & 0 \\ \rho_\mathfrak{t} & \frac{\epsilon_\mathfrak{t}}{2} + \gamma_{i+1} & 1 \\ \rho_\mathfrak{t} & \frac{\epsilon_\mathfrak{t}}{2} + \gamma_i & 1 \end{pmatrix},$$

  where $P_0 = (|\mathfrak{t}|, 0)$, $P_1 = (\lfloor |\mathfrak{t}|-1/2 \rfloor, 1)$ and $\delta = \delta_{L_\mathfrak{t}^{w_h}} = b_\mathfrak{t}$.

- *If $\mathfrak{t}$ is odd, then for all $j = 0, \dots, r_{V_\mathfrak{t}^{w_h}}$, we have*

$$M_{V_\mathfrak{t}^{w_h},j} = \begin{pmatrix} -|\mathfrak{t}| & -\frac{|\mathfrak{t}|+1}{2}d_j & \frac{|\mathfrak{t}|+1}{2}d_{j+1} \\ 2 & d_j & -d_{j+1} \\ -\epsilon_\mathfrak{t} + |\mathfrak{t}|\rho_\mathfrak{t} & n_j & -n_{j+1} \end{pmatrix}, \quad M^{-1}_{V_\mathfrak{t}^{w_h},j} = T_j \cdot \begin{pmatrix} 1 & \frac{|\mathfrak{t}|+1}{2} & 0 \\ \rho_\mathfrak{t} - 2\cdot\gamma_{j+1} & \frac{\epsilon_\mathfrak{t}}{2} - |\mathfrak{t}|\cdot\gamma_{j+1} & 1 \\ \rho_\mathfrak{t} - 2\cdot\gamma_j & \frac{\epsilon_\mathfrak{t}}{2} - |\mathfrak{t}|\cdot\gamma_j & 1 \end{pmatrix},$$

  where $P_0 = (|\mathfrak{t}|, 0)$, $P_1 = (\lfloor |\mathfrak{t}|-1/2 \rfloor, 1)$, $\delta = \delta_{V_\mathfrak{t}^{w_h}} = 1$ and $k_j = k_{j+1} = 0$.

- *If $\mathfrak{t}$ is even, then for all $j = 0, \dots, r_{V_\mathfrak{t}^{w_h}}$, we have*

$$M_{V_\mathfrak{t}^{w_h},j} = \begin{pmatrix} -\delta\frac{|\mathfrak{t}|}{2} & -\left(\frac{|\mathfrak{t}|}{2}+1\right)d_j - k_j\frac{|\mathfrak{t}|}{2} & \left(\frac{|\mathfrak{t}|}{2}+1\right)d_{j+1} + k_{j+1}\frac{|\mathfrak{t}|}{2} \\ \delta & d_j + k_j & -d_{j+1} - k_{j+1} \\ -\delta\frac{\epsilon_\mathfrak{t} - |\mathfrak{t}|\rho_\mathfrak{t}}{2} & \frac{n_j}{\delta} - k_j\frac{\epsilon_\mathfrak{t} - |\mathfrak{t}|\rho_\mathfrak{t}}{2} & -\frac{n_{j+1}}{\delta} + k_{j+1}\frac{\epsilon_\mathfrak{t} - |\mathfrak{t}|\rho_\mathfrak{t}}{2} \end{pmatrix},$$

$$M^{-1}_{V_{\mathfrak{t}}^{w_h},j}=T_j\cdot\begin{pmatrix}1 & \frac{|\mathfrak{t}|}{2}+1 & 0\\ \rho_{\mathfrak{t}}-\gamma_{j+1} & \frac{\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_{j+1} & 1\\ \rho_{\mathfrak{t}}-\gamma_j & \frac{\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_j & 1\end{pmatrix},$$

where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor |\mathfrak{t}|-1/2\rfloor,1)$ *and* $\delta=\delta_{V_{\mathfrak{t}}^{w_h}}$.

- *If* $f(w_h)=0$, *then for all* $j=0,\dots,r_{V_0^{w_h}}$, *we have*

$$M_{V_0^{w_h},j}=\begin{pmatrix}1 & d_j & -d_{j+1}\\ -2 & -d_j & d_{j+1}\\ \epsilon_{\mathfrak{s}_h}-\rho_{\mathfrak{s}_h} & n_j & -n_{j+1}\end{pmatrix},\quad M^{-1}_{V_0^{w_h},j}=T_j\cdot\begin{pmatrix}-1 & -1 & 0\\ \rho_{\mathfrak{s}_h}+2\cdot\gamma_{j+1} & \frac{\epsilon_{\mathfrak{s}_h}}{2}+\gamma_{j+1} & 1\\ \rho_{\mathfrak{s}_h}+2\cdot\gamma_j & \frac{\epsilon_{\mathfrak{s}_h}}{2}+\gamma_j & 1\end{pmatrix},$$

where $P_0=(0,2)$, $P_1=(1,1)$, $\delta=\delta_{V_0^{w_h}}=1$ *and* $k_j=k_{j+1}=0$.

- *If* $f(w_h)\neq 0$, *then for all* $j=0,\dots,r_{V_0^{w_h}}$, *we have*

$$M_{V_0^{w},j}=\begin{pmatrix}0 & d_j & -d_{j+1}\\ -\delta & -d_j-k_j & d_{j+1}+k_{j+1}\\ \delta\frac{\epsilon_{\mathfrak{s}_h}}{2} & \frac{n_j}{\delta}+k_j\frac{\epsilon_{\mathfrak{s}_h}}{2} & -\frac{n_{j+1}}{\delta}-k_{j+1}\frac{\epsilon_{\mathfrak{s}_h}}{2}\end{pmatrix},\quad M^{-1}_{V_0^{w_h},j}=T_j\cdot\begin{pmatrix}-1 & -1 & 0\\ \rho_{\mathfrak{s}_h}+\gamma_{j+1} & \frac{\epsilon_{\mathfrak{s}_h}}{2} & 1\\ \rho_{\mathfrak{s}_h}+\gamma_j & \frac{\epsilon_{\mathfrak{s}_h}}{2} & 1\end{pmatrix},$$

where $P_0=(0,2)$, $P_1=(1,1)$ *and* $\delta=\delta_{V_0^{w_h}}$.

Proof. We follow the notation of [Dok, §4]. Choose the points P_0 and P_1 as in the proof of Lemma 4.1.

First consider the edge $L_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. From Lemma 4.1 we have

$$\nu=(1,0,-\rho_t)\quad\text{and}\quad(w_x,w_y)=\left(-\left\lfloor\frac{|\mathfrak{t}|}{2}\right\rfloor-1,1\right).$$

Then $M_{L_{\mathfrak{t}}^{w_h},i}$ and $M^{-1}_{L_{\mathfrak{t}}^{w_h},i}$ follow from [Dok, §4.3] as

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{L_{\mathfrak{t}}^{w_h}}=v_{F_{\mathfrak{t}}^{w_h}}(P_1)-v_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+\left(\left\lfloor\frac{|\mathfrak{t}|}{2}\right\rfloor+1\right)\rho_{\mathfrak{t}}.$$

Now assume $\mathfrak{t}$ even and consider the edge $V_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. Since $\mathfrak{t}$ is even,

$$V_{\mathfrak{t}}^{w_h}(\mathbb{Z})=\left\{(|\mathfrak{t}|,0),\left(\frac{|\mathfrak{t}|}{2},1\right),(0,2)\right\},\quad\nu=\left(-\frac{|\mathfrak{t}|}{2},1,-\frac{\epsilon_t}{2}+\frac{|\mathfrak{t}|}{2}\rho_t\right)$$

and $(w_x,w_y)=\left(-\frac{|\mathfrak{t}|}{2}-1,1\right)$ as above. Then $M_{V_{\mathfrak{t}}^{w_h},j}$ and $M^{-1}_{V_{\mathfrak{t}}^{w_h},j}$ follow again from [Dok, (4.3)] as

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{V_{\mathfrak{t}}^{w_h}}=v_{F_{\mathfrak{t}}^{w_h}}(P_1)-v_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+\left(\frac{|\mathfrak{t}|}{2}+1\right)\rho_{\mathfrak{t}}.$$

Similar arguments and computations yield the remaining matrices. □

5.2. **Open subschemes.** Let $h=1,\dots,m$ and let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let M be a matrix associated to $\mathfrak{t}$. Write

$$M=\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{pmatrix}\quad\text{and}\quad M^{-1}=\begin{pmatrix}\tilde m_{11} & \tilde m_{12} & \tilde m_{13}\\ \tilde m_{21} & \tilde m_{22} & \tilde m_{23}\\ \tilde m_{31} & \tilde m_{32} & \tilde m_{33}\end{pmatrix}$$

Recall that $X_M=\operatorname{Spec}R$, where

$$R=\frac{O_K[X^{\pm1},Y,Z]}{(\pi-X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}})}\hookrightarrow\frac{O_K[X^{\pm1},Y^{\pm1},Z^{\pm1}]}{(\pi-X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}})}\overset{M}{\simeq}K\left[x^{\pm1},y^{\pm1}\right].$$

Let $l\neq h$. Set

$$T_M^{hl}(X,Y,Z):=\begin{cases}1+u_{hl}X^{\rho_{hl}\tilde m_{13}-\tilde m_{11}}Y^{\rho_{hl}\tilde m_{23}-\tilde m_{21}}Z^{\rho_{hl}\tilde m_{33}-\tilde m_{31}} & \text{if }\mathfrak{t}\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l,\\ u_{hl}^{-1}X^{\tilde m_{11}-\rho_{hl}\tilde m_{13}}Y^{\tilde m_{21}-\rho_{hl}\tilde m_{23}}Z^{\tilde m_{31}-\rho_{hl}\tilde m_{33}}+1 & \text{if }\mathfrak{t}\not\supseteq\mathfrak{s}_h\wedge\mathfrak{s}_l,\end{cases}$$

a polynomial in $R[Y^{-1}, Z^{-1}]$. Note that

$$\text{if} \quad \mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l \quad \text{then} \quad T_M^{hl}(X,Y,Z) \overset{M}{\longmapsto} \frac{x + w_{hl}}{x},$$
$$\text{if} \quad \mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l \quad \text{then} \quad T_M^{hl}(X,Y,Z) \overset{M}{\longmapsto} \frac{x + w_{hl}}{w_{hl}}.$$

The following two lemmas prove that $T_M^{hl}(X,Y,Z) \in R$.

Lemma 5.3 *Let $h, l = 1, \dots, m$, with $h \neq l$, let $t \in \Sigma_C^{w_h}$ be such that $\mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$ and let M be a matrix associated to $\mathfrak{t}$. Then*

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} \geq \rho_{\mathfrak{t}}\tilde{m}_{23} - \tilde{m}_{21} \geq 0 \quad \text{and} \quad \rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} \geq \rho_{\mathfrak{t}}\tilde{m}_{33} - \tilde{m}_{31} \geq 0.$$

Furthermore if $M = M_{L_{\mathfrak{t}}^{w_h}, i}$ then

- $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = 0$ *if and only if* $i = r_{L_{\mathfrak{t}}^{w_h}}$ *or* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$,
- $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ *if and only if* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$;

if $M = M_{V_{\mathfrak{t}}^{w_h}, j}$ then

- $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$,
- $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ *if and only if* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ *and* $j = 0$.

Proof. This result follows from Lemma 5.2, which gives us a complete description of M and M^{-1}. We show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h}, j}$, and leave the other cases for the reader. First of all recall that $\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}$ by Lemma 3.18. Then

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = \delta d_{j+1}\left(\rho_{hl} - \rho_t + \left(\frac{n_0}{\delta d_0} - \frac{n_{j+1}}{\delta d_{j+1}}\right)\right) > \delta d_{j+1}\left(\rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0,$$

where $\delta = \delta_M$. Note that if $\mathfrak{t} = \mathfrak{R}$ and $j = r_{V_{\mathfrak{R}}^{w_h}}$ then $d_{j+1} = 0$ and $n_{j+1} = -1$. But the inequality stays true since

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = \rho_{\mathfrak{t}}\tilde{m}_{23} - \tilde{m}_{21} = -n_{j+1} = 1 > 0.$$

Similarly,

$$\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = \delta d_j\left(\rho_{hl} - \rho_t + \left(\frac{n_0}{\delta d_0} - \frac{n_j}{\delta d_j}\right)\right) \geq \delta d_j\left(\rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0.$$

In particular $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ if and only if $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = 0$. □

Lemma 5.4 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster such that $\mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$, and let M be a matrix associated to $\mathfrak{t}$. Then*

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} \geq 0 \quad \text{and} \quad \tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0.$$

Furthermore, $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ if and only if

- $M = M_{L_{\mathfrak{t}}^{w_h}, i}$ *and* $i = r_{L_{\mathfrak{t}}^{w_h}}$, *or*
- $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$, $M = M_{V_{\mathfrak{t}}^{w_h}, j}$, *and* $j = r_{V_{\mathfrak{t}}^{w_h}}$.

Proof. This result follows again from Lemma 5.2. As in the previous lemma, we show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h}, j}$, and leave the other cases for the reader.

Let $r = r_{V_{\mathfrak{t}}^{w_h}}$. Note that $t \neq \mathfrak{R}$. Then

$$\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} = \delta d_j\left(\rho_t - \rho_{hl} - \left(\frac{n_0}{\delta d_0} - \frac{n_j}{\delta d_j}\right)\right) > \delta d_j\left(\rho_{P(\mathfrak{t})} - \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}\right) \geq 0.$$

since $\frac{n_0}{\delta d_0} - \frac{n_{r+1}}{\delta d_{r+1}} = \rho_{\mathfrak{t}} - \rho_{P(\mathfrak{t})}$. Similarly,

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = \delta d_{j+1}\left(\rho_t - \rho_{hl} - \left(\frac{n_0}{\delta d_0} - \frac{n_{j+1}}{\delta d_{j+1}}\right)\right) \geq \delta d_{j+1}\left(\rho_{P(\mathfrak{t})} - \rho_{hl}\right) \geq 0,$$

In particular $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ if and only if $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = r$. □

Let
$$T^h_M(X,Y,Z) := \prod_{l\neq h} T^{hl}_M(X,Y,Z),$$
and define
$$V^h_M := \mathrm{Spec}\ R[T^h_M(X,Y,Z)^{-1}] \subset X_M, \quad \text{and} \quad \mathring{X}^h_\Delta := \bigcup_M V^h_M \subseteq X^h_\Delta,$$
where M runs through all matrices associated to some proper cluster $\mathfrak{t}\in\Sigma^{w_h}_C$. We can then define the subscheme
$$\mathring{\mathcal{C}}^{w_h}_\Delta := \mathcal{C}^{w_h}_\Delta \cap \mathring{X}^h_\Delta \subset X^h_\Delta,$$
where $\mathcal{C}^{w_h}_\Delta/O_K$ is the regular model of the hyperelliptic curve $C^{w_h}: y^2 = f(x+w_h)$ constructed in [Dok, Theorem 3.14] (see [Dok, §4] for the explicit construction). More concretely, let $g_h(x,y) := y^2 - f(x+w_h)$ and write
$$y^2 - f(x+w_h) = Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}^h_M(X,Y,Z),$$
as in [Dok, 4.4], and consider the subscheme
$$U^h_M := \mathrm{Spec}\ \frac{R\left[T^h_M(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}^h_M(X,Y,Z)\right)} \subset V^h_M.$$
Then
$$\mathring{\mathcal{C}}^{w_h}_\Delta = \bigcup_M U^h_M \subset \mathring{X}^h_\Delta,$$
where M runs through all matrices associated to some proper cluster $\mathfrak{t}\in\Sigma^{w_h}_C$.

5.3. **Glueing.** Let $h,l = 1,\dots,m$, with $h\neq l$. Consider the isomorphism

(2) $$\phi : K\left[x^{\pm1}, y^{\pm1}, \prod_{o\neq l}(x+w_{lo})^{-1}\right] \xrightarrow{\simeq} K\left[x^{\pm1}, y^{\pm1}, \prod_{o\neq h}(x+w_{ho})^{-1}\right]$$

sending $x\mapsto x+w_{hl}$, $y\mapsto y$. If $\mathfrak{t}\supseteq \mathfrak{s}_h\wedge\mathfrak{s}_l$ and M is a matrix associated to $\mathfrak{t}$, then ϕ gives
$$R[Y^{-1}, Z^{-1}, T^l_M(X,Y,Z)^{-1}] \xrightarrow{M^{-1}\circ\phi\circ M} R[Y^{-1}, Z^{-1}, T^h_M(X,Y,Z)^{-1}],$$
which sends
$$F(X,Y,Z) \mapsto F(X\cdot T^{hl}_M(X,Y,Z)^{m_{11}}, Y\cdot T^{hl}_M(X,Y,Z)^{m_{12}}, Z\cdot T^{hl}_M(X,Y,Z)^{m_{13}}).$$
Hence it induces the isomorphisms

(3) $$R[T^l_M(X,Y,Z)^{-1}] \xrightarrow{\simeq} R[T^h_M(X,Y,Z)^{-1}], \qquad V^h_M \xrightarrow{\simeq} V^l_M.$$

Via these maps we see that $g_h(x,y) = Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}^h_M(X,Y,Z)$ also equals
$$Y^{n_{Y,l}}\cdot Z^{n_{Z,l}}\cdot (T^{hl}_M)^{n_{Y,l}m_{12}+n_{Z,l}m_{13}} \mathcal{F}^l_M\left(X\cdot (T^{hl}_M)^{m_{11}}, Y\cdot (T^{hl}_M)^{m_{12}}, Z\cdot (T^{hl}_M)^{m_{13}}\right),$$
where $T^{hl}_M = T^{hl}_M(X,Y,Z)$. Since neither Y nor Z divide $T^{hl}_M(X,Y,Z)$, we have $n_{Y,h} = n_{Y,l}$, $n_{Z,h} = n_{Z,l}$ and
$$\mathcal{F}^h_M(X,Y,Z) = (T^{hl}_M)^{n_{Y,l}m_{12}+n_{Z,l}m_{13}} \mathcal{F}^l_M\left(X\,(T^{hl}_M)^{m_{11}}, Y\,(T^{hl}_M)^{m_{12}}, Z\,(T^{hl}_M)^{m_{13}}\right).$$
Hence (3) induces the isomorphisms

(4) $$\frac{R\left[T^l_M(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}^l_M(X,Y,Z)\right)} \xrightarrow{\simeq} \frac{R\left[T^h_M(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}^h_M(X,Y,Z)\right)}, \qquad U^h_M \xrightarrow{\simeq} U^l_M.$$

Define the subschemes

$$V^{hl} := \bigcup_{M_l} V^h_{M_l} \subseteq \mathring{X}^h_\Delta, \qquad U^{hl} := V^{hl} \cap \mathcal{C}^{w_h}_\Delta \subseteq \mathring{\mathcal{C}}^{w_h}_\Delta,$$

where the union runs over all matrices M_l associated to some proper cluster $\mathfrak{t} \in \Sigma_C^{w_h} \cap \Sigma_C^{w_l}$ (i.e., $\mathfrak{t} \in \Sigma^W$, $\mathfrak{s}_h \wedge \mathfrak{s}_l \subseteq \mathfrak{t}$). From (2), (3) and (4) we have isomorphisms of schemes

$$V^{hl} \xrightarrow{\simeq} V^{lh}, \qquad U^{hl} \xrightarrow{\simeq} U^{lh}. \tag{5}$$

Now, $U^{hl} \subset V^{hl}$ are open subschemes respectively of $\mathring{\mathcal{C}}^{w_h}_\Delta \subset \mathring{X}^h_\Delta$ for any $l \neq h$. Glueing the schemes $\mathring{\mathcal{C}}^{w_h}_\Delta \subset \mathring{X}^h_\Delta$ respectively along the opens $U^{hl} \subset V^{hl}$ via (5) gives the schemes $\mathcal{C} \subset \mathcal{X}$. We will show that $\mathcal{C}/O_K$ is a proper flat[3] model of C.

5.4. **Generic fibre.** We start studying the generic fibre $\mathcal{C}_\eta$ of $\mathcal{C}$. Since it is the glueing of all $\mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}$ through the glueing maps

$$U^{hl}_\eta \longrightarrow U^{lh}_\eta$$

induced by (5), we start focusing on $\mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}$ for $h = 1, \dots, m$. In particular, as $\mathring{\mathcal{C}}^{w_h}_\Delta$ is an open subscheme of $\mathcal{C}^{w_h}_\Delta$, we study

$$\mathcal{C}^{w_h}_{\Delta,\eta} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta} = C^{w_h} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}.$$

For every choice of a proper cluster $\mathfrak{t} \in \Sigma_C^{w_h}$, and M associated to $\mathfrak{t}$, let

$$P_M := \left(C^{w_h} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta}\right) \cap X_M = \operatorname{Spec} \frac{R \otimes_{O_K} K}{(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z))}.$$

To study P_M we are going to use Lemma 5.2 and the definition of $T^h_M(X,Y,Z)$.

Suppose first $\mathfrak{t} \neq \mathfrak{R}$ and $M = M_{V_\mathfrak{t}^{w_h},j}$. Then $\tilde{m}_{23}, \tilde{m}_{33} > 0$, so

$$P_M = \operatorname{Spec} \frac{R[Y^{-1}, Z^{-1}]}{(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z))} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y^{\pm 1}]}{(g_h(x,y), \prod_o (x + w_{ho}))}, \tag{6}$$

where the product runs over all $o \neq h$. Now let $\mathfrak{t} = \mathfrak{R}$ and $M = M_{V_\mathfrak{t}^{w_h},j}$. If $j \neq r_{V_\mathfrak{R}^{w_h}}$, then P_M is as in the previous case (since $\tilde{m}_{23}, \tilde{m}_{33} > 0$). If $j = r_{V_\mathfrak{R}^{w_h}}$, then $\tilde{m}_{33} > 0$, $\tilde{m}_{23} = 0$, but $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$ by Lemma 5.3. So from the definition of $T^{hl}_M(X,Y,Z)$ we have once more the equality (6). Similarly, if $\mathfrak{t} = \mathfrak{s}_h$ and $M = M_{V_0^{w_h},j}$, then $\tilde{m}_{33} > 0$, and $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} > 0$ by Lemma 5.4. Hence we have (6) again.

It remains to study P_M when $M = M_{L_\mathfrak{t}^{w_h},i}$. If $i \neq r_{L_\mathfrak{t}^{w_h}}$, then $\tilde{m}_{23}, \tilde{m}_{33} > 0$ and so P_M is as in (6). Let $i = r_{L_\mathfrak{t}^{w_h}}$. Then $\tilde{m}_{33} > 0$ but both $\tilde{m}_{23}$ and $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21}$ equal 0. Hence $\tilde{m}_{23} = \tilde{m}_{21} = 0$, which also implies $m_{21} = m_{23} = 0$. Therefore M defines an isomorphism $R[Z^{-1}] \simeq K[x^{\pm 1}, y]$, which induces

$$P_M = \operatorname{Spec} \frac{R[Z^{-1}]}{(\mathcal{F}^h_M(X,Y,Z), T^h_M(X,Y,Z))} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y]}{\left(g_h(x,y), \prod_{o \neq h} (x + w_{ho})\right)}.$$

Therefore

$$C^{w_h} \smallsetminus \mathring{\mathcal{C}}^{w_h}_{\Delta,\eta} = \operatorname{Spec} \frac{K[x,y]}{\left(g_h(x,y), \prod_{o \neq h} (x + w_{ho})\right)}.$$

[3]Note that the flatness of $\mathcal{C}$ is trivial since it is a local property.

Regarding $\mathcal{C}_{\Delta}^{w_h}$ as a model over C via the natural isomorphism $C \xrightarrow{\sim} C^{w_h}$, we have

$$C \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h} = \operatorname{Spec} \frac{K[x,y]}{\left(y^2 - f(x), \prod_{o \neq h}(x - w_o)\right)}.$$

Thus the generic fibre of $\mathcal{C}$ is isomorphic to C.

5.5. **Special fibre.** We now study the structure of the special fibre $\mathcal{C}_s$ of $\mathcal{C}$. As for the generic fibre, we start considering

$$\mathcal{C}_{\Delta,s}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,s}^{w_h},$$

for any $h = 1, \dots, m$. So for every choice of a proper cluster $\mathfrak{t} \in \Sigma_C^{w_h}$, and M associated to $\mathfrak{t}$, let

$$S_M := \left(\mathcal{C}_{\Delta,s}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,s}^{w_h}\right) \cap X_M = \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}_M^h(X,Y,Z), T_M^h(X,Y,Z), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi\right)}.$$

Firstly suppose $M = M_{L_{\mathfrak{t}}^{w_h}, i}$. Fix $l \neq h$. If $\mathfrak{t} \not\supseteq \mathfrak{s}_l \wedge \mathfrak{s}_h$, then by Lemma 5.4, we have $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} \geq 0$ and $\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0$. Moreover, if $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$, then $i = r_{L_{\mathfrak{t}}^{w_h}}$, which implies $\tilde{m}_{23} = 0$ by Lemma 5.2. Therefore

$$\{T_M^{hl}(X,Y,Z) = Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}} = 0\} = \varnothing, \tag{7}$$

from the definition of $T_M^{hl}(X,Y,Z)$. On the other hand, if $\mathfrak{t} \supsetneq \mathfrak{s}_l \wedge \mathfrak{s}_h$, then by Lemma 5.3, we have $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} \geq 0$ and $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} > 0$. Moreover, if $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = 0$, then $i = r_{L_{\mathfrak{t}}^{w_h}}$, which implies $\tilde{m}_{23} = 0$ by Lemma 5.2 as above. Therefore we have (7) again. Now assume instead $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$. Since $\rho_{\mathfrak{t}} = \rho_{hl} \in \mathbb{Z}$, then $\rho_{hl}\tilde{m}_{13} - \tilde{m}_{11} = -1$, as $\delta_M = 1$ and we can choose $k_i = k_{i+1} = 0$ in the description of M of Lemma 5.2. Hence

$$T_M^{hl}(X,Y,Z) = 1 + u_{hl}X^{-1} = X^{-1}\left(X + u_{hl}\right),$$

by Lemma 5.3. Thus

$$S_M = \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}_M^h(X,Y,Z), \prod_l \left(X + u_{hl}\right), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi\right)} \subset \mathcal{C}_{\Delta}^{w_h},$$

where the product runs over all $l \neq h$ such that $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$.

We want to show that S_M consists of singular points of $\mathcal{C}_{\Delta}^{w_h}$.

Lemma 5.5 *Consider the model $\mathcal{C}_{\Delta}^{w_h}/O_K$ and let $f_h(x) = f(x + w_h)$. If*

$$P \in \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}_M^h(X,Y,Z), X + u_{hl}, Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi\right)} \subset \mathcal{C}_{\Delta}^{w_h},$$

for some $l \neq h$, where $(X,Y,Z) = (x, y, \pi) \bullet M_{L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}, i}$ for some $i = 0, \dots, r_{L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}}$, then P is a singular point of $\mathcal{C}_{\Delta}^{w_h}/O_K$.

If $\Sigma = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, Σ_C is almost rational and C is y-regular, then the converse also holds.

Furthermore, if $\mathfrak{t}_l \in \Sigma_C^{w_l}$, $\mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$, then u_{lh} is a multiple root of $\overline{f_h|_L}$ of order $|\mathfrak{t}_l|$, where $L = L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}$.

Proof. Singular points of $\mathcal{C}_{\Delta}^{w_h}$ may come from horizontal edges of $\Delta_v^{w_h}$, corresponding to multiple roots of $\overline{f_h|_L}$. More precisely, let F be the splitting field of f, of ring of integers O_F and uniformiser π_F. Then by [Dok, §4.5], $P \in \mathcal{C}_{\Delta}^{w_h}(O_F)$ is a singular point if

$$P \in \operatorname{Spec} \frac{O_F[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}_{M_{L,i}}^h(X,Y,Z), X - \alpha, Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi_F\right)},$$

for some horizontal edge L of $\Delta_v^{w_h}$, and some multiple root $\overline{\alpha}$ of $\overline{f_h|_L}$. If C is y-regular, then the structure of the Newton polytope $\Delta_v^{w_h}$ only allows singular points as above. By Theorem 3.23, $\overline{\alpha}$ is a multiple root of $\overline{f_h|_L}$ if and only if $\overline{\alpha} = \left(\overline{\frac{r}{\pi^{\lambda_{\mathfrak{s}}}}}\right)$ where $\mathfrak{s} \in \Sigma_{f_h}$ proper with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$, $r \in \mathfrak{s}$ such that $v(r) = \lambda_{\mathfrak{s}}$, and L is the unique edge of $\mathtt{NP}(f_h)$ of slope $-\lambda_{\mathfrak{s}}$. Let $\mathfrak{R}_h$ be the set of roots of f_h. Note that we have a bijection

$$\psi : \Sigma_{f_h} \longrightarrow \Sigma_f,$$

such that

$$\mathfrak{s} = \mathfrak{R}_h \cap \{x \in \overline{K} \mid v(x-z) > d\} \quad \Rightarrow \quad \psi(\mathfrak{s}) = \mathfrak{R} \cap \{x \in \overline{K} \mid v(x - w_h - z) > d\}.$$

If $r \in \mathfrak{s}$ then $r + w_h \in \psi(\mathfrak{s})$, so for any $\mathfrak{s}, \mathfrak{s}' \in \Sigma_{f_h}$, $|\mathfrak{s}| = |\psi(\mathfrak{s})|$ and if $\mathfrak{s}' < \mathfrak{s}$ then $\psi(\mathfrak{s}') < \psi(\mathfrak{s})$. In particular $d_{\mathfrak{s}} = d_{\psi(\mathfrak{s})}$ and

$$\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r) = \min_{r \in \psi(\mathfrak{s})} v(r - w_h).$$

Hence w_h is a rational centre of $\psi(\mathfrak{s})$ if and only if $\lambda_{\mathfrak{s}} = \rho_{\psi(\mathfrak{s})}$.

Let $\overline{\alpha}$ be a multiple root of $\overline{f_h|_L}$ and let $\mathfrak{s} \in \Sigma_{f_h}$ associated to $\overline{\alpha}$ as above. We want to prove that if Σ_C is almost rational and $\Sigma = \Sigma_C^{\min}$, then there exists $l \neq h$ so that $\overline{\alpha} = \overline{u_{lh}}$. Note first w_h is not a rational centre of $\psi(\mathfrak{s})$. Indeed, if w_h is a rational centre of $\psi(\mathfrak{s})$, then

$$|\psi(\mathfrak{s})| = |\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p = |\rho_{\psi(\mathfrak{s})}|_p,$$
$$d_{\phi(\mathfrak{s})} = d_{\mathfrak{s}} > \lambda_{\mathfrak{s}} = \rho_{\phi(\mathfrak{s})},$$

which contradicts our assumptions on Σ_C. As $\{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, we must have that w_l is a rational centre of $\psi(\mathfrak{s})$, for some $l \neq h$. Then $w_l - w_h = w_{lh}$ is a rational centre for $\mathfrak{s}$. Let $r \in \mathfrak{s}$ with $v(r) = \lambda_{\mathfrak{s}}$, and let $r' = r + w_h \in \psi(\mathfrak{s})$. Since $\lambda_{\mathfrak{s}} < \rho_{\psi(\mathfrak{s})} \leq v(r' - w_l)$, we have

$$\left(\overline{\frac{r'}{\pi^{\lambda_{\mathfrak{s}}}}}\right) = \left(\overline{\frac{w_l}{\pi^{\lambda_{\mathfrak{s}}}}}\right) \quad \text{in } \mathfrak{f}.$$

Therefore

$$\overline{\alpha} = \left(\overline{\frac{r}{\pi^{\lambda_{\mathfrak{s}}}}}\right) = \left(\overline{\frac{r' - w_h}{\pi^{\lambda_{\mathfrak{s}}}}}\right) = \left(\overline{\frac{w_{lh}}{\pi^{\lambda_{\mathfrak{s}}}}}\right) = \left(\overline{\frac{u_{lh}}{\pi^{\lambda_{\mathfrak{s}} - \rho_{hl}}}}\right).$$

We want to show $\lambda_{\mathfrak{s}} = \rho_{hl}$. Since $\mathfrak{s}_l \subseteq \psi(\mathfrak{s})$ but $\mathfrak{s}_h \not\subseteq \psi(\mathfrak{s})$, we have $\psi(\mathfrak{s}) \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$ and so

$$\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} < \rho_{\psi(\mathfrak{s})},$$

by Lemma 3.14. Moreover, for every root $r \in \psi(\mathfrak{s})$, one has

$$v(r - w_h) = v(r - w_l + w_l - w_h) = \min\{v(r - w_l), \rho_{hl}\} = \rho_{hl},$$

as $v(r - w_l) \geq \rho_{\psi(\mathfrak{s})} > \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l}$. In particular $\lambda_{\mathfrak{s}} = \rho_{hl}$.

Now, for any $l \neq h$ consider $\mathfrak{s}_l$, and its corresponding cluster $\psi^{-1}(\mathfrak{s}_l)$ of f_h. As above $v(r - w_h) = \rho_{hl} \in \mathbb{Z}$ for every root $r \in \psi^{-1}(\mathfrak{s}_l)$, so in particular, $|\lambda_{\psi^{-1}(\mathfrak{s}_l)}|_p \leq 1$. Therefore $|\psi^{-1}(\mathfrak{s}_l)^{w_h}| = |\psi^{-1}(\mathfrak{s}_l)| > |\lambda_{\psi^{-1}(\mathfrak{s}_l)}|_p$, and

$$d_{\psi^{-1}(\mathfrak{s}_l)} = d_{\mathfrak{s}_l} \geq \rho_{\mathfrak{s}_l} > \rho_{hl} = \lambda_{\psi^{-1}(\mathfrak{s}_l)},$$

which implies that $\overline{\alpha} = \overline{u_{lh}}$ is a multiple root of $\overline{f_h|_L}$ where L is the unique edge of $\mathtt{NP}(f_h)$ of slope ρ_{hl}. Since the edge of $\mathtt{NP}(f_h)$ of slope ρ_{hl} corresponds to $L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}$, the first part of the lemma follows.

Let $\mathfrak{t}_l \in \Sigma_C^{w_l}$, $\mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$. Then

$$\mathfrak{t}_l = \left\{ r \in \mathfrak{R} \mid \left(\overline{\frac{r - w_h}{\pi^{\rho_{hl}}}}\right) = \overline{u_{lh}} \right\},$$

as $v(r-w_h)>\rho_{hl}$ if and only if $\overline{\frac{r-w_h}{\pi^{\rho_{hl}}}}=\overline{u_{lh}}$. Thus the order of $\overline{u_{hl}}$ is $|\mathfrak{t}_l|$ by Theorem 3.23. □

In particular, if $\Sigma=\Sigma_C^{\min}$, Σ_C is almost rational and C is y-regular then $\mathring{\mathcal{C}}_\Delta^{w_h}$ is regular for every h. Thus $\mathcal{C}$ is a regular model of C.

It remains to compute S_M when $M=M_{V_{\mathfrak{t}}^{w_h},j}$. Fix $l\neq h$. Assume that if $\mathfrak{t}=\mathfrak{s}_l\wedge\mathfrak{s}_h$ then $j\neq 0$ and that if $\mathfrak{t}<\mathfrak{s}_l\wedge\mathfrak{s}_h$ then $j\neq r_{V_{\mathfrak{t}}^{w_h}}$. By Lemma 5.3 and Lemma 5.4, we have $\rho_{hl}\tilde{m}_{23}-\tilde{m}_{21}\neq 0$ and $\rho_{hl}\tilde{m}_{33}-\tilde{m}_{31}\neq 0$, that implies (7) as before. If $\mathfrak{t}=\mathfrak{s}_l\wedge\mathfrak{s}_h$ and $j=0$, then $\rho_{hl}\tilde{m}_{33}-\tilde{m}_{31}=0$ but $\rho_{hl}\tilde{m}_{23}-\tilde{m}_{21}>0$. So

$$\{T_M^{hl}(X,Y,Z)=Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}}=0\}=\{T_M^{hl}(X,Y,Z)=Z^{\tilde{m}_{33}}=0\}\subset \operatorname{Spec} R[Y^{-1}].$$

Similarly, if $\mathfrak{t}<\mathfrak{s}_l\wedge\mathfrak{s}_h$ and $j=r_{V_{\mathfrak{t}}^{w_h}}=:r$, then $\tilde{m}_{21}-\rho_{hl}\tilde{m}_{23}=0$, however $\tilde{m}_{31}-\rho_{hl}\tilde{m}_{33}>0$. Then

$$\{T_M^{hl}(X,Y,Z)=Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}}=0\}=\{T_M^{hl}(X,Y,Z)=Y^{\tilde{m}_{23}}=0\}\subset \operatorname{Spec} R[Z^{-1}].$$

In both cases $S_M\subseteq X_F$, where $F=F^{w_h}_{\mathfrak{s}_l\wedge\mathfrak{s}_h}$ ([Dok, Definition 3.7]). Let $L=L^{w_h}_{\mathfrak{s}_l\wedge\mathfrak{s}_h}$, and let $f_h(x)=f(x+w_h)$ and $g_h(x,y)=y^2-f_h(x)$. By Lemma 5.5, one has

$$S_M\subseteq X_{F^{w_h}_{\mathfrak{s}_l\wedge\mathfrak{s}_h}}\cap S_{M_{L,0}}=\varnothing,$$

as $\overline{g_h|_F}(X,Y)=Y^2-X^a f_h|_L(X)$ or $Y-X^a f_h|_L(X)$, for some $a\in\mathbb{Z}$ (see Lemma 5.7 for more details). Thus if $M=M_{V_{\mathfrak{t}}^{w_h},j}$, then $S_M=\varnothing$.

5.6. **Components.** Now that we have described the special fibre of $\mathcal{C}$, let us introduce some notation for closed subschemes that compose it. Let $\mathfrak{t}\in\Sigma^W$. For any $h=1,\dots,m$ such that $\mathfrak{s}_h\subseteq\mathfrak{t}$ recall the definition of $\overline{X}_{F_{\mathfrak{t}}^{w_h}}$, 1-dimensional closed subscheme of $\mathcal{C}^{w_h}_{\Delta,s}$. Let

$$\mathring{X}_{F_{\mathfrak{t}}^{w_h}}:=\overline{X}_{F_{\mathfrak{t}}^{w_h}}\cap\mathring{\mathcal{C}}_\Delta^{w_h}.$$

Denote by $\Gamma_{\mathfrak{t}}$ the 1-dimensional closed subscheme of $\mathcal{C}_s$, result of the glueing of the subchemes $\mathring{X}_{F_{\mathfrak{t}}^{w_h}}$ of $\mathring{\mathcal{C}}^{w_h}_{\Delta,s}$ for all h such that $\mathfrak{t}\in\Sigma_C^{w_h}$.

Lemma 5.6 *Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster and let $F=F_{\mathfrak{t}}^{w_h}$. Then $m_{\mathfrak{t}}=\delta_F$.*

Proof. Let $L=L_{\mathfrak{t}}^{w_h}$ and $M=M_{L,0}$. Then $\delta_F=\delta_L d_0$. The lemma follows as $\delta_L=b_{\mathfrak{t}}$ and $d_0=(3-D_{\mathfrak{t}})$ by Lemma 5.2. □

From the lemma above the multiplicity of $\Gamma_{\mathfrak{t}}$ is $m_{\mathfrak{t}}$. The lemma below gives the defining equation of $\Gamma_{\mathfrak{t}}$ on the chart X_M, for a certain matrix M associated to $\mathfrak{t}$.
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Lemma 5.7 *Let $f_h(x)=f(x+w_h)$ and let $L=L_{\mathfrak{t}}^{w_h}$, $F=F_{\mathfrak{t}}^{w_h}$ and $M=M_{L,0}$. Let $c,d\in\mathbb{Z}$ such that $\rho_{\mathfrak{t}}\cdot c+d=1/b_{\mathfrak{t}}$. Then the 1-dimensional subscheme $\overline{X}_F$ (or $\mathring{X}_F$) on the affine chart X_M is given by*

$$Y^{D_{\mathfrak{t}}}-X^{\frac{|\mathfrak{s}|}{b_{\mathfrak{t}}}-c\epsilon_{\mathfrak{t}}}\overline{f_h|_L}(X)=0,$$

where $\mathfrak{s}\in\Sigma_C^{w_h}\cup\{\varnothing\}$, $\mathfrak{s}<\mathfrak{t}$ (where $\varnothing<\mathfrak{t}$ only if $\mathfrak{t}=\mathfrak{s}_h$ and $p^0_{\mathfrak{s}_h}=2$).

In particular the points in S_M belong to all the irreducible components of $\overline{X}_F$.

Proof. Let $\Delta_v^{w_h}$ be the Newton polytope of $C^{w_h}: y^2-f(x+w_h)$. For its structure, $\overline{X}_F$ is given on X_M by $Y^b-X^a\overline{f_h|_L}(X)$, for some $b=1,2$ and $a\in\mathbb{Z}$. Moreover, if $b=2$ then $2-D_{\mathfrak{t}}=r_L=0$. We prove the converse also holds. Suppose $D_{\mathfrak{t}}=2$, that is $b_{\mathfrak{t}}\epsilon_{\mathfrak{t}} \in 2\mathbb{Z}$. We want to show that there exists a point $P \in \mathbb{Z} \times \{1\}$ with $v_F(P) \in \mathbb{Z}$. Let $c, d \in \mathbb{Z}$ as in the statement and set $P = (cb_{\mathfrak{t}}\epsilon_{\mathfrak{t}}/2, 1)$. Then

$$v_F(P) = \frac{v_F((cb_{\mathfrak{t}}\epsilon_{\mathfrak{t}}, 0))}{2} = \frac{\epsilon_{\mathfrak{t}} - (cb_{\mathfrak{t}}\epsilon_{\mathfrak{t}})\rho_{\mathfrak{t}}}{2} = \frac{db_{\mathfrak{t}}\epsilon_{\mathfrak{t}}}{2} \in \mathbb{Z},$$

by Lemma 4.1. Then a follows from the choice of P and Definition 2.5 (choose $k_0 = \lfloor |\mathfrak{t}|/2 \rfloor + 1 - c\epsilon_{\mathfrak{t}}b_{\mathfrak{t}}/2$ and $k_1 = -c$ for the description of M in Lemma 5.2).

Finally, the last part of the lemma follows from Lemma 5.5. □

Denote $X_{\mathfrak{t}}^W := X_{L_{\mathfrak{t}}^{w_h}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_h}$, $X_{\mathfrak{t}} := X_{V_{\mathfrak{t}}^{w_h}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_h}$ and $X_{\mathfrak{s}_h}^0 := X_{V_{\mathfrak{s}_h}^0} \cap \mathring{\mathcal{C}}_{\Delta}^{w_h}$. This notation agrees with Definition 4.8. Indeed, let $g_h(x, y) = y^2 - f_h(x)$ and $r_{\mathfrak{t}} = \frac{w_h - r}{\pi^{\rho_t}}$ for any $r \in \mathfrak{t}$. Then $\overline{g_h|_{V_{\mathfrak{t}}^{w_h}}} = \overline{g_{\mathfrak{t}}}$, $\overline{g_h|_{V_{\mathfrak{s}_h}^0}} = \overline{g_{\mathfrak{s}_h}^0}$ and $\overline{g_h|_{L_{\mathfrak{s}_h}^{w_h}}} = \overline{f_{\mathfrak{s}_h}}$. Finally, by Lemma 5.5, if $\mathfrak{t} \neq \mathfrak{s}_h$ the polynomial defining $X_{L_{\mathfrak{t}}^{w_h}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_h}$ is

$$\frac{\overline{g_h|_{L_{\mathfrak{t}}^{w_h}}}(x)}{\prod_{\mathfrak{t}_h \neq \mathfrak{t}_l < \mathfrak{t}} (x + \overline{u_{hl}})} = \left(\frac{\frac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{t} \setminus \mathfrak{t}_h} (x + r_{\mathfrak{t}})}{\prod_{r \in \mathfrak{t}_l \neq \mathfrak{t}_h} (x + r_{\mathfrak{t}})} \mod \pi \right) = \overline{f_{\mathfrak{t}}^W}(x),$$

where $u = c_f \prod_{r \in \mathfrak{R} \notin \mathfrak{s}} r_{\mathfrak{t}}$ and $\mathfrak{t}_l \in \Sigma_C^{w_l}$, $\mathfrak{t}_l < \mathfrak{t}$ for any l. Indeed

$$\overline{g_h|_{L_{\mathfrak{t}}^{w_h}}}(x) = \overline{f_h|_{L_{\mathfrak{t}}^{w_h}}}(x) = \tfrac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{t} \setminus \mathfrak{t}_h} (x + r_{\mathfrak{t}}) \mod \pi$$

from Lemma 3.33 and $u_{hl} = r_{\mathfrak{t}}$ for every $r \in \mathfrak{t}_l$ as $v(w_l - r) \geq \rho_{\mathfrak{t}_l} > \rho_{\mathfrak{t}} = \rho_{hl}$.

Proposition 5.8 *Let* $\mathfrak{t} \in \Sigma_C^{w_h}$ *and let* $w_{\mathfrak{t}} = w_h$. *On the affine open* U_M^h, *the* 1*-dimensional scheme* $\Gamma_{\mathfrak{t}}$ *is given by*

$$Y^{D_{\mathfrak{t}}} = \prod_{\mathfrak{s} \in \tilde{\mathfrak{t}} \cap \Sigma^W} (X - \overline{u_{lh}}) \overline{f_{\mathfrak{t}}^W}(X)$$

where w_l *is a rational centre of* $\mathfrak{s}$ *and* $\overline{u_{hh}} := 0$.

Proof. Let $c \in \mathbb{Z}$ be as in Definition 4.7. If $\mathfrak{t}$ has two or more children in Σ_C^{rat}, then $b_{\mathfrak{t}} = 1$, and so $c = 0$. Hence $\tilde{\mathfrak{t}}$ is the set of odd rational children and so the proposition follows from Lemma 5.7 and Lemma 5.5. □

5.7. **Separatedness.** It remains to prove that $\mathcal{C}$ is a proper scheme. We first show it is separated. Clearly it suffices to prove that $\mathcal{X}/O_K$ is separated. Since the schemes X_{Δ}^h are separated, then the open subschemes $\mathring{X}_{\Delta}^h$ are separated as well by [LiA, Proposition 3.3.9]. Consider the open cover $\{V_M^h\}_{h,M}$ of $\mathcal{X}$. Let $h, l = 1, \ldots, m$ and let M_h and M_l be matrices associated to proper clusters $\mathfrak{t}_h \in \Sigma_C^{w_h}$ and $\mathfrak{t}_l \in \Sigma_C^{w_l}$ respectively. By [LiA, Proposition 3.3.6] we want to show

(i) $V_{M_h}^h \cap V_{M_l}^l$ is affine,

(ii) The canonical homomorphism

$$O_{\mathcal{X}}(V_{M_h}^h) \otimes_{\mathbb{Z}} O_{\mathcal{X}}(V_{M_l}^l) \longrightarrow O_{\mathcal{X}}(V_{M_h}^h \cap V_{M_l}^l)$$

is surjective.

The definition of the glueing map 5 implies (i). If $h = l$, or $\mathfrak{s}_l \subset \mathfrak{t}_h$, or $\mathfrak{s}_h \subset \mathfrak{t}_l$, then ((ii)) follows from the separatedness of $\mathring{X}_{\Delta}^h$ and $\mathring{X}_{\Delta}^l$. So assume $l \neq h$, and $\mathfrak{t}_h, \mathfrak{t}_l \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$. Consider the Moebius transformation

$$\psi_l : \quad x \mapsto \frac{x}{xw_{hl}^{-1} + 1}, \quad y \mapsto \frac{y}{(xw_{hl}^{-1} + 1)^{g+1}}.$$

It sends the curve C^{w_l} to the isomorphic hyperelliptic curve

$$C_l^h : y^2 = (xw_{hl}^{-1} + 1)^{2g+2} f\left(x(xw_{hl}^{-1} + 1)^{-1} + w_l\right).$$

As
$$\begin{aligned} f_l^h(x) :&= (xw_{hl}^{-1}+1)^{2g+2} f\left(x(xw_{hl}^{-1}+1)^{-1}+w_l\right) \\ &= c_f w_{hl}^{|\mathfrak{R}|}(xw_{hl}^{-1}+1)^{2g+2-|\mathfrak{R}|} \prod_{r\in\mathfrak{R}\smallsetminus\{w_h\}} \frac{r-w_l}{w_{lh}}\left(xw_{hl}^{-1}+\frac{r-w_l}{r-w_h}\right), \end{aligned}$$
every cluster $\mathfrak{s}\in\Sigma_C^{w_l}$ such that $\mathfrak{s}\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$, corresponds to a unique cluster $\mathfrak{s}^h\in\Sigma^0_{C_l^h}$ of same size, radius and rational centre 0. Moreover,
$$\epsilon_{\mathfrak{s}^h} = v(c_{f_l^h}) + \sum_{r'\in\mathfrak{s}^h}\rho_{\mathfrak{s}^h} + \sum_{r'\notin\mathfrak{s}^h} v(r') = \epsilon_{\mathfrak{s}}.$$
Call $\mathfrak{t}_l^h$ the cluster in $\Sigma^0_{C_l^h}$ corresponding to $\mathfrak{t}_h$. Let Δ_v^{lh} be the Newton polytope of $y^2-f_l^h(x)$ and let X_Δ^{lh} be its attached toric scheme (defined in [Dok]). Since $\mathfrak{t}_l\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$, the faces $F_{\mathfrak{t}_l}$ of $\Delta_v^{w_l}$ and $F_{\mathfrak{t}_l^h}$ of Δ_v^{lh} are identical by Lemma 4.1 and so the matrix $M:=M_l$ is also associated to $\mathfrak{t}_l^h$. For every $o=1,\dots,m$, with $o\neq h$, define
$$w_{hlo} = \begin{cases} \frac{w_{lo}w_{hl}}{w_{ho}} & \text{if } o\neq l,\\ w_{hl} & \text{if } o=l,\end{cases}$$
and write $w_{hlo}=u_{hlo}\pi^{\rho_{hlo}}$, where $u_{hlo}\in O_K^\times$ and $\rho_{hlo}\in\mathbb{Z}$, i.e.
$$u_{hlo} = \begin{cases} \frac{u_{lo}u_{hl}}{u_{ho}} & \text{if } o\neq l,\\ u_{hl} & \text{if } o=l,\end{cases} \quad\text{and}\quad \rho_{hlo} = \begin{cases} \rho_{hl}+\rho_{lo}-\rho_{ho} & \text{if } o\neq l,\\ \rho_{hl} & \text{if } o=l.\end{cases}$$
Define
$$\tilde{T}_M^{hlo}(X,Y,Z) := \begin{cases} 1+u_{hlo}X^{\rho_{hlo}\tilde{m}_{13}-\tilde{m}_{11}}Y^{\rho_{hlo}\tilde{m}_{23}-\tilde{m}_{21}}Z^{\rho_{hlo}\tilde{m}_{33}-\tilde{m}_{31}} & \text{if } \mathfrak{t}_l\supseteq\mathfrak{s}_o,\\ u_{hlo}^{-1}X^{\tilde{m}_{11}-\rho_{hlo}\tilde{m}_{13}}Y^{\tilde{m}_{21}-\rho_{hlo}\tilde{m}_{23}}Z^{\tilde{m}_{31}-\rho_{hlo}\tilde{m}_{33}}+1 & \text{if } \mathfrak{t}_l\not\supseteq\mathfrak{s}_o.\end{cases}$$
We want to show $\tilde{T}_M^{hlo}(X,Y,Z)\in R$. If $o=l$ then
$$\tilde{T}_M^{hlo}(X,Y,Z) = T_M^{hl}(X,Y,Z)\in R.$$
So assume $o\neq l$. If $\mathfrak{s}_o\subseteq\mathfrak{t}_l$, then it follows from Lemma 5.3 as $\mathfrak{s}_l\wedge\mathfrak{s}_o\subsetneq\mathfrak{s}_l\wedge\mathfrak{s}_h$ and so $\rho_{hlo}=\rho_{lo}$. On the other hand, if $\mathfrak{s}_o\not\subseteq\mathfrak{t}_l$, then it follows from Lemma 5.4 as $\rho_{hlo}\leq\max\{\rho_{hl},\rho_{lo}\}$. Let
$$\tilde{T}_M^{hl}(X,Y,Z) := \prod_{o\neq h}\tilde{T}_M^{hlo}(X,Y,Z).$$
The Moebius transformation
$$K[x^{\pm1},y^{\pm1},\prod_{o\neq l}(x+w_{lo})^{-1}] \xrightarrow{\psi_l} K[x^{\pm1},y^{\pm1},\prod_{o\neq h}(x+w_{hlo})^{-1}]$$
considered above induces an isomorphism
$$R[T_M^l(X,Y,Z)^{-1}] \xrightarrow{M^{-1}\circ\psi_l\circ M} R[\tilde{T}_M^{hl}(X,Y,Z)^{-1}],$$
sending
$$\begin{aligned} X &\mapsto X\cdot T_M^{hl}(X,Y,Z)^{-m_{11}-(g+1)m_{21}},\\ Y &\mapsto Y\cdot T_M^{hl}(X,Y,Z)^{-m_{12}-(g+1)m_{22}},\\ Z &\mapsto Z\cdot T_M^{hl}(X,Y,Z)^{-m_{13}-(g+1)m_{23}}.\end{aligned}$$
Then
$$\tilde{V}_M^{lh} := \operatorname{Spec} R[\tilde{T}_M^{hl}(X,Y,Z)^{-1}]$$

is an open subscheme of X_Δ^{lh}, isomorphic to V_M^l. We can clearly carry out similar constructions for t_h, M_h.

By comparing the Newton polytopes Δ_v^{lh} and Δ_v^{hl}, we see that the Moebius transformation

$$\psi : K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{hlo})^{-1}] \longrightarrow K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{lho})^{-1}]$$
$$x \longmapsto -w_{hl}^2/x$$
$$y \longmapsto -y/(w_{hl}^{-1}x)^{g+1}$$

induces a birational map $X_\Delta^{hl} \dashrightarrow X_\Delta^{lh}$, defined on the open set $\tilde{V}_{M_h}^{hl}$ of X_Δ^{hl}. In particular, there exists an open set $\tilde{V}_{M_h}^{lh}$ of X_Δ^{lh}, isomorphic to $V_{M_h}^h$ via $\psi_h^{-1}\circ\psi$.

Recall the definition of ϕ in (2), which induces the glueing map between $V_{M_l}^l$ and $V_{M_h}^h$. Since the following diagram

$$\begin{array}{ccc} K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{lo})^{-1}] & \xrightarrow{\phi} & K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{ho})^{-1}] \\ \downarrow{\psi_l} & & \downarrow{\psi_h} \\ K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq h}(x+w_{hlo})^{-1}] & \xrightarrow{\psi} & K[x^{\pm 1}, y^{\pm 1}, \prod_{o\neq l}(x+w_{lho})^{-1}] \end{array}$$

is commutative, then the surjectivity of

$$O_{\mathcal{X}}(V_{M_h}^h)\otimes_{\mathbb{Z}} O_{\mathcal{X}}(V_{M_l}^l) \longrightarrow O_{\mathcal{X}}(V_{M_h}^h\cap V_{M_l}^l)$$

follows from the separatedness of X_Δ^{lh}.

5.8. **Properness.** By [EGA, IV.15.7.10], it remains to show that $\mathcal{C}_s$ is proper. From [LiA, Exercise 3.3.11], we only need to prove that the 1-dimensional subscheme $\Gamma_{\mathfrak{t}}$ is proper for every $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$. Indeed every other component is entirely contained in a model $\mathcal{C}_\Delta^{w_h}$, which is proper (see §5.5). Let $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$ for some $h,l=1,\dots,m$, with $h\neq l$. For any $o=1,\dots,m$ such that $\mathfrak{s}_o\subset\mathfrak{t}$, let $\mathfrak{t}_o$ be the unique child of $\mathfrak{t}$ with $\mathfrak{s}_o\subseteq\mathfrak{t}_o<\mathfrak{t}$. Then $\Gamma_{\mathfrak{t}}$ is equal to the glueing of the schemes

$$\mathrm{Spec}\,\frac{R[U_o(X,Y,Z)^{-1}]}{\big(\mathcal{F}_M^o(X,Y,Z),Z,\pi\big)},\qquad M=M_{L_{\mathfrak{t}}^{w_o},0}, M_{V_{\mathfrak{t}}^{w_o},0},$$

and

$$\mathrm{Spec}\,\frac{R[U_o(X,Y,Z)^{-1}]}{\big(\mathcal{F}_M^o(X,Y,Z),Y,\pi\big)},\qquad M=M_{V_{\mathfrak{t}_o}^{w_o},r_{V_{\mathfrak{t}_o}^{w_o}}},$$

for all o such that $\mathfrak{s}_o\subset\mathfrak{t}$, through the isomorphism (5) and the glueing maps in the definition of $\mathcal{C}_\Delta^{w_o}$. In particular, for any o as above there exists a natural birational map $s_o:\Gamma_{\mathfrak{t}}\dashrightarrow\overline{X}_{F_{\mathfrak{t}}^{w_o}}$ which is defined as the identity morphism on the dense open $\Gamma_{\mathfrak{t}}\cap\mathring{\mathcal{C}}_\Delta^{w_o}$.

Let Y/k be a non-singular curve, let $P\in Y$ and let $Y\smallsetminus\{P\}\xrightarrow{g}\Gamma_{\mathfrak{t}}$ be a non-constant morphism of curves. We want to show that g extends to Y. For every o as above, $\overline{X}_{F_{\mathfrak{t}}^{w_o}}$ is proper, so the birational map

$$g_o := s_o\circ g : Y\smallsetminus\{P\}\dashrightarrow\overline{X}_{F_{\mathfrak{t}}^{w_o}}$$

extends to a morphism $\overline{g}_o : Y\longrightarrow\overline{X}_{F_{\mathfrak{t}}^{w_o}}$.

If

$$P_o := \overline{g}_o(P)\in\left(\overline{X}_{F_{\mathfrak{t}}^{w_o}}\cap\mathring{\mathcal{C}}_\Delta^{w_o}\right) = s_o\left(\Gamma_{\mathfrak{t}}\cap\mathring{\mathcal{C}}_\Delta^{w_o}\right)$$

for some o such that $\mathfrak{s}_o \subset \mathfrak{t}$ (we will later show this is always the case), then there exists an open neighbourhood U of P_o such that $U \subseteq \left(\overline{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_o}\right)$ and so $s_o|^{U}_{s_o^{-1}(U)}$ is an isomorphism. Since $P \in \overline{g}_o^{-1}(U)$, the map

$$\overline{g}_o^{-1}(U) \xrightarrow{\overline{g}_o|^{U}_{\overline{g}_o^{-1}(U)}} U \xrightarrow{\left(s_o|^{U}_{s_o^{-1}(U)}\right)^{-1}} s_o^{-1}(U) \hookrightarrow \Gamma_{\mathfrak{t}},$$

induces an extension $Y \longrightarrow \Gamma_{\mathfrak{t}}$ of g.

Suppose that $P_o \notin \overline{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_o}$ for any o such that $\mathfrak{s}_o \subset \mathfrak{t}$. From §5.5 we have

$$P_o \in S_M = \operatorname{Spec} \frac{R}{\left(\mathcal{F}_M^o(X,Y,Z), \prod_l (X + u_{ol}), Z, \pi\right)}, \tag{8}$$

where $M = M_{L_{\mathfrak{t}}^{w_o}}$, and the product runs over all $l \neq o$ such that $\mathfrak{t} = \mathfrak{s}_o \wedge \mathfrak{s}_l$. In particular P_o is a point of each irreducible component of $\overline{X}_{F_{\mathfrak{t}}^{w_o}}$ by Lemma 5.7. Let $h \neq o$ such that $X + u_{oh}$ vanishes at P_o. Let ξ be the generic point of Y and let $\xi_o = g_o(\xi)$, $\xi_h = g_h(\xi)$ be generic points of $\overline{X}_{F_{\mathfrak{t}}^{w_o}}$ and $\overline{X}_{F_{\mathfrak{t}}^{w_h}}$ respectively. Then the birational maps s_o and s_h give

$$\begin{array}{ccccccc} & & \overline{X}_{F_{\mathfrak{t}}^{w_o}} & & & & k\left(\overline{\xi_o}\right) \\ & & \overset{s_o}{\nearrow} & & & \overset{\phi_{g_o}}{\swarrow} & \\ Y \smallsetminus \{P\} \xrightarrow{g} \Gamma_{\mathfrak{t}} & & & \Longrightarrow & k(Y) & & \downarrow \simeq \\ & & \underset{s_h}{\searrow} & & & \underset{\phi_{g_h}}{\nwarrow} & \\ & & \overline{X}_{F_{\mathfrak{t}}^{w_h}} & & & & k\left(\overline{\xi_h}\right) \end{array}$$

where we denote by ϕ_{g_o} and ϕ_{g_h} the homomorphisms between function fields induced by g_o and g_h. The vertical isomorphism is induced by the map

$$\frac{R[T_M^o(X,Y,Z)^{-1}]}{\left(\mathcal{F}_M^o(X,Y,Z), Z\right)} \longrightarrow \frac{R[T_M^h(X,Y,Z)^{-1}]}{\left(\mathcal{F}_M^h(X,Y,Z), Z\right)}$$

which sends (see §5.3)

$$X + u_{oh} \mapsto X \cdot T_M^{ho}(X,Y,Z)^{m_{11}} + u_{oh} = X\left(1 + u_{ho}X^{-1}\right) + u_{oh} = X.$$

But the rational function $X + u_{oh}$ vanishes at P_o, while X does not vanish at P_h by (8). This gives a contradiction, as $\overline{g}_o(P) = P_o$ and $\overline{g}_h(P) = P_h$.

5.9. **Genus.** Suppose $\Sigma = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, Σ_C is almost rational and C is y-regular. Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster.

Lemma 5.9 *If C is y-regular, then $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ is übereven if and only if it equals the union of its even rational children. Furthermore, if $p \neq 2$, every even cluster $\mathfrak{s} \in \Sigma_C$ is rational.*

Proof. Let $\mathfrak{s} \in \Sigma_C$ such that $\mathfrak{s} \neq \mathfrak{s}^{\mathrm{rat}} = \mathfrak{t}$, for some $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$. In particular $\mathfrak{s}^{\mathrm{rat}} = \mathfrak{t}$, and $d_{\mathfrak{s}} \neq \rho_{\mathfrak{s}} = \rho_{\mathfrak{t}}$. Since Σ_C is almost rational, we must have $|\mathfrak{s}| \leq |\rho_{\mathfrak{s}}|_p = |\rho_{\mathfrak{t}}|_p$. Then $v_p(b_{\mathfrak{t}}) > 1$. Let σ be an element of the wild inertia subgroup of G_K. For any $r \in \mathfrak{s}$, we have $\sigma(r) \in \mathfrak{s}$, since $v(r - \sigma(r)) > \rho_{\mathfrak{t}}$ and $\mathfrak{s} < \mathfrak{t}$. Therefore $p^{v_p(b_{\mathfrak{t}})} \leq |\mathfrak{s}|$. Thus we have showed $|\mathfrak{s}| = p^{v_p(b_{\mathfrak{t}})}$, that implies $\mathfrak{s}$ is odd if $p \neq 2$.

It remains to show the first part of the lemma when $p = 2$. It suffices to show that every child of a übereven rational cluster $\mathfrak{t}$ is rational. Suppose not. Then $2 \mid b_{\mathfrak{t}}$ from above, and then $\mathfrak{t}$ has at most one child in Σ_C^{rat} (Lemma 3.18). Let $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}} \cup \{\varnothing\}$, $\mathfrak{s} < \mathfrak{t}$. If $\mathfrak{s} \neq \varnothing$ then by Lemma 3.15 it is also a (proper) child of

$\mathfrak{t}$. Let a be the number of non-rational children of $\mathfrak{t}$. Then $|\mathfrak{t}| - |\mathfrak{s}| = a \cdot 2^{v_p(b_\mathfrak{t})}$. Therefore $(|\mathfrak{t}| - |\mathfrak{s}|)\rho_\mathfrak{t}$ is odd by Lemma 3.11. Thus one between $p_\mathfrak{t}/\gamma_t$ and $p_\mathfrak{s}/\gamma_\mathfrak{s}$ (or $p_\mathfrak{t}^0/\gamma_\mathfrak{t}^0$ if $\mathfrak{s} = \varnothing$) equals 2. But this is impossible as C is y-regular. □

Proposition 5.10 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ with $D_\mathfrak{t} = 2$ and let $w_\mathfrak{t} = w_h$. On the affine open U_M^h the 1-dimensional scheme $\Gamma_\mathfrak{t}$ is given by*

$$Y^{D_\mathfrak{t}} = \prod_{\mathfrak{s}\in\tilde{\mathfrak{t}}} (X - \overline{u_{lh}})\overline{f_\mathfrak{t}}(X)$$

where w_l is a rational centre of $\mathfrak{s}$ and $\overline{u_{hh}} := 0$.

In particular,

1. *if $D_\mathfrak{t} = 1$, then $\Gamma_\mathfrak{t} \simeq \mathbb{P}^1_k$;*
2. *if $D_\mathfrak{t} = 2$ and $\mathfrak{t}$ is übereven, then $\Gamma_\mathfrak{t}$ over k^{s} is the disjoint union of two $\mathbb{P}^1s$;*
3. *in all other cases, $\Gamma_\mathfrak{t}$ is a hyperelliptic curve of genus $g(\mathfrak{t})$.*

Proof. The first part of the proposition follows from Proposition 5.8.

Since $g(X) = \prod_{\mathfrak{s}\in\tilde{\mathfrak{t}}}(X - \overline{u_{lh}})\overline{f_\mathfrak{t}}(X)$ is a separable polynomial, by Lemma 5.9 it only remains to prove that $\mathfrak{t}$ equals the union of its even children if and only if $g(X) \in k$.

Suppose $\mathfrak{t}$ equals the union of its own even rational children. In particular $b_\mathfrak{t} = 1$ by Lemma 3.18 and so $\tilde{\mathfrak{t}} = \varnothing$ since it is the set of odd rational children. Therefore $\mathfrak{t} \setminus \bigcup_{\mathfrak{s}<\mathfrak{t}} \mathfrak{s} = \varnothing$, and so $\overline{f_\mathfrak{t}} \in k$. Thus $g \in k$.

Now suppose $g \in k$. Then $\tilde{\mathfrak{t}} = \varnothing$ and $\mathfrak{t} = \bigcup_{\mathfrak{s}<\mathfrak{t}} \mathfrak{s}$, $\mathfrak{s}$ rational. In particular $\mathfrak{t}$ has two or more children in Σ_C^{rat}, and so $b_\mathfrak{t} = 1$, again by Lemma 3.18. But then $\tilde{\mathfrak{t}}$ is the set of odd children of $\mathfrak{t}$, and so all rational children of $\mathfrak{t}$ are even. □

5.10. **Minimal regular SNC model.** Suppose the base extended curve $C_{K^{nr}}$ satisfies the condition of §5.9, and consider the model $\mathcal{C}/O_{K^{nr}}$ constructed before. we want to see what components of $\mathcal{C}_s$ can be blown down to obtain the minimal regular model with normal crossings. Recall [Dok, §5]. Let $\Sigma_{K^{nr}} = \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$ and ix a proper cluster $\mathfrak{s} \in \Sigma_{K^{nr}}$ of rational centre w_h.

Suppose $\mathfrak{s} \neq \mathfrak{s}_h \wedge \mathfrak{s}_l$ for all $h, l = 1, \ldots, m$ with $l \neq h$. Then $\Gamma_\mathfrak{s} = \overline{X}_{F_\mathfrak{s}^{w_h}}$. In particular, if $\Gamma_\mathfrak{s}$ can be blown down then $F_\mathfrak{s}^{w_h}$ is a removable or contractible face. By Lemma 4.1, we find

- $F_\mathfrak{s}^{w_h}$ is removable if and only if $\mathfrak{s} = \mathfrak{R}$ even with a (rational) child of size $2g + 1$.
- $F_\mathfrak{s}^{w_h}$ is contractible if and only if either $|\mathfrak{s}| = 2$ and $\frac{\epsilon_\mathfrak{s}}{2} - \rho_\mathfrak{s} \in \mathbb{Z}$ or $|\mathfrak{s}| > 2g$ with a unique (rational) child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g$ and $\frac{\epsilon_\mathfrak{s}}{2} - g\rho_\mathfrak{s} \in \mathbb{Z}$.

First of all note that $F_\mathfrak{s}^{w_h}$ is removable if and only if $\mathfrak{s}$ is removable. In this case $F_\mathfrak{s}^{w_h}$ can be ignored for the construction of $\mathcal{C}_\Delta^{w_h}$ (for any h since $\mathfrak{s} = \mathfrak{R}$), and so $\mathfrak{s}$ can be ignored for the construction of $\mathcal{C}$.

Assume now $F_\mathfrak{s}^{w_h}$ contractible. We want to understand when $\Gamma_\mathfrak{s}$ can be blown down. First consider the case $|\mathfrak{s}| = 2$ and $\frac{\epsilon_\mathfrak{s}}{2} - \rho_\mathfrak{s} \in \mathbb{Z}$. Then $\Gamma_\mathfrak{s}$ intersects other components of $\mathcal{C}_s$ in 2 points (as $V_\mathfrak{s}^{w_h}$ gives two chains of $\mathbb{P}^1$s and the edges $V_0^{w_h}$ and $L_\mathfrak{s}^{w_h}$ give no component in $\mathcal{C}_{\Delta,s}^{w_h}$). To have self-intersection -1, $\Gamma_\mathfrak{s}$ has to have multiplicity > 1. It follows from Lemma 5.6 that $\rho_\mathfrak{s} \notin \mathbb{Z}$, as $\frac{\epsilon_\mathfrak{s}}{2} - \rho_\mathfrak{s} \in \mathbb{Z}$. Moreover, by Lemma 3.11, one has $\rho_\mathfrak{s} \in \frac{1}{2}\mathbb{Z}$. Therefore $\epsilon_\mathfrak{s}$ is odd and the multiplicity of $\Gamma_\mathfrak{s}$ is 2. Let $r := r_{V_\mathfrak{s}^{w_h}}$ and consider the chain

$$\frac{n_0}{d_0} > \frac{n_1}{d_1} > \cdots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}}$$

given by $V_{\mathfrak{s}}^{w_h}$. If $\Gamma_{\mathfrak{s}}$ can be blown down then $d_1=1$. Since $\frac{n_0}{d_0}=-\frac{\epsilon_{\mathfrak{s}}}{2}+2\rho_{\mathfrak{s}}$, we have $d_0=2$. In particular $d_1=1$ if and only if $\rho_{\mathfrak{s}}-\rho_{P(\mathfrak{s})}=\frac{n_0}{d_0}-\frac{n_{r+1}}{d_{r+1}}\geq\frac{1}{2}$. Thus if $|\mathfrak{s}|=2$, then $\Gamma_{\mathfrak{s}}$ can be blown down if and only if $\rho_{\mathfrak{s}}\notin\mathbb{Z}$, $\epsilon_{\mathfrak{s}}$ odd, $\rho_{P(\mathfrak{s})}\leq\rho_{\mathfrak{s}}-\frac{1}{2}$. Note that this is case (1) of Definition 4.13.

Second consider the case $|\mathfrak{s}|=2g+2$ with a unique proper rational child $\mathfrak{s}'$ of size $2g$. The argument is very similar to the previous one. If $\Gamma_{\mathfrak{s}}$ can be blown down then it must have multiplicity >1 and this implies $\rho_{\mathfrak{s}}\notin\mathbb{Z}$ again by Lemma 5.6. From Lemma 3.11 it follows that $(|\mathfrak{s}|-|\mathfrak{s}'|)\rho_{\mathfrak{s}}\in\mathbb{Z}$, so $\rho_{\mathfrak{s}}\in\frac{1}{2}\mathbb{Z}$. Then $m_{\mathfrak{s}}=2$ and

$$\frac{v(c_f)}{2}=\frac{\epsilon_{\mathfrak{s}}}{2}-(g+1)\rho_{\mathfrak{s}}\notin\mathbb{Z},$$

so $v(c_f)$ odd. Let $r:=r_{V_{\mathfrak{s}'}^{w_h}}$ and consider the chain

$$\frac{n_0}{d_0}>\frac{n_1}{d_1}>\cdots>\frac{n_r}{d_r}>\frac{n_{r+1}}{d_{r+1}}$$

given by $V_{\mathfrak{s}'}^{w_h}$. If $\Gamma_{\mathfrak{s}}$ can be blown down then $d_r=1$. Since $\frac{n_{r+1}}{d_{r+1}}=-\frac{\epsilon_{\mathfrak{s}}}{2}+(g+1)\rho_{\mathfrak{s}}$, we have $d_0=2$. In particular $d_r=1$ if and only if $\rho_{\mathfrak{s}'}-\rho_{\mathfrak{s}}=\frac{n_0}{d_0}-\frac{n_{r+1}}{d_{r+1}}\geq\frac{1}{2}$. Thus if $\mathfrak{s}$ has size $2g+2$ and it has a unique proper child $\mathfrak{s}'\in\Sigma_{K^{nr}}$ of size $2g$, then $\Gamma_{\mathfrak{s}}$ can be blown down if and only if $\rho_{\mathfrak{s}}\notin\mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}'}\geq\rho_{\mathfrak{s}}+\frac{1}{2}$. This is case (2) of Definition 4.13.

Finally, if $|\mathfrak{s}|=2g+1$, $\mathfrak{s}$ has a proper child $\mathfrak{s}'\in\Sigma_{K^{nr}}$ of size $2g$ and $\frac{\epsilon_{\mathfrak{s}}}{2}-g\rho_{\mathfrak{s}}\in\mathbb{Z}$, then $\rho_{\mathfrak{s}}\in\mathbb{Z}$, as $(|\mathfrak{s}|-|\mathfrak{s}'|)\rho_{\mathfrak{s}}\in\mathbb{Z}$. It follows that $m_{\mathfrak{s}}=1$, but then the self-intersection of $\Gamma_{\mathfrak{s}}$ is not -1, since it intersects the rest of $\mathcal{C}_{\mathfrak{s}}$ in at least two points as before. Hence in this case $\Gamma_{\mathfrak{s}}$ can never be blown down.

Now assume there exists $l\neq h$ such that $\mathfrak{s}=\mathfrak{s}_h\wedge\mathfrak{s}_l$. Then $\mathfrak{s}$ is not minimal. Let $\mathfrak{s}_h',\mathfrak{s}_l'\in\Sigma_{K^{nr}}$ be such that $\mathfrak{s}_h\subseteq\mathfrak{s}_h'<\mathfrak{s}$ and $\mathfrak{s}_l\subseteq\mathfrak{s}_l'<\mathfrak{s}$. Suppose $\Gamma_{\mathfrak{s}}$ irreducible. If $|\mathfrak{s}|\leq 2g$ (or, equivalently, $\mathfrak{s}$ is not the largest non-removable cluster), then $\Gamma_{\mathfrak{s}}$ intersects at least other 3 components of $\mathcal{C}_s$ (given by $\mathfrak{s}_h',\mathfrak{s}_l'$, and $P(\mathfrak{s})$). So it cannot be contracted to obtain a model with normal crossings. The same argument holds if there exists $o\neq l$ such that $\mathfrak{s}_o\wedge\mathfrak{s}_h=\mathfrak{s}$. Assume then $|\mathfrak{s}|>2g$ and $\mathfrak{s}_o\wedge\mathfrak{s}_h\neq\mathfrak{s}$ for all $o\neq l$. Then $\Gamma_{\mathfrak{s}}$ intersects at least other 2 components of $\mathcal{C}_s$ given by $V_{\mathfrak{s}_h'}^{w_h}$ and $V_{\mathfrak{s}_l'}^{w_l}$. Firstly, if $\Gamma_{\mathfrak{s}}$ can be blown down, then $m_{\mathfrak{s}}>1$. But $\rho_{\mathfrak{s}}=\rho_{hl}\in\mathbb{Z}$. Then

$$\frac{\epsilon_{\mathfrak{s}}}{2}-\left\lfloor\frac{|\mathfrak{s}|-1}{2}\right\rfloor\rho_{\mathfrak{s}}\in\frac{1}{2}\mathbb{Z}\smallsetminus\mathbb{Z},$$

so $\epsilon_{\mathfrak{s}}$ odd as well as $v(c_f)$. Hence $D_{\mathfrak{s}}=1$ and $\Gamma_{\mathfrak{s}}\simeq\mathbb{P}^1$ by Proposition 5.10. However, if $\mathfrak{s}$ is odd then this implies that $V_{\mathfrak{s}}^{w_h}$ gives a $\mathbb{P}^1$ intersecting $\Gamma_{\mathfrak{s}}$. Since that would be a third component intersecting $\Gamma_{\mathfrak{s}}$, the cluster $\mathfrak{s}$ has to be even. It follows that $\mathfrak{s}=\mathfrak{R}$ and $|\mathfrak{s}|=2g+2$. Now, $L_{\mathfrak{s}}^{w_h}$ gives some $\mathbb{P}^1$s intersecting $\overline{X}_{F_{\mathfrak{s}}^{w_h}}\subset\mathcal{C}_{\Delta,s}^{w_h}$. All these $\mathbb{P}^1$s are not in $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ (and so in $\mathcal{C}_s$) if and only if $\mathfrak{s}_h'\cup\mathfrak{s}_l'=\mathfrak{s}$. In particular, $\mathfrak{s}_h'$ and $\mathfrak{s}_l'$ are either both even or both odd. If $\mathfrak{s}_h'$ is even, then $\delta_{V_{\mathfrak{s}_h'}^{w_h}}=2$, and so the component given by $V_{\mathfrak{s}_h'}^{w_h}$ has multiplicity at least 2. Once again, the self-intersection of $\Gamma_{\mathfrak{s}}$ could not be -1 in this case. Assume $\mathfrak{s}_h'$ is odd. Let $r:=r_{V_{\mathfrak{s}}^{w_h}}$ and consider the chain

$$\frac{n_0}{d_0}>\frac{n_1}{d_1}>\cdots>\frac{n_r}{d_r}>\frac{n_{r+1}}{d_{r+1}}$$

given by $V_{\mathfrak{s}_h'}^{w_h}$. We want $d_r=1$. Since

$$\frac{n_{r+1}}{d_{r+1}}=-\frac{\epsilon_{\mathfrak{s}}}{2}+\frac{|\mathfrak{s}_h'|-1}{2}\rho_{\mathfrak{s}}\in\frac{1}{2}\mathbb{Z}\smallsetminus\mathbb{Z},$$

we have $d_{r+1} = 2$. As before $d_r = 1$ if and only if $\frac{\rho_{\mathfrak{s}'_h} - \rho_{\mathfrak{s}}}{2} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$ and similarly for $\mathfrak{s}'_l$. Thus if $\mathfrak{s}$ has two or more rational children and $\Gamma_{\mathfrak{s}}$ is irreducible then it can be blown down if and only if $v(c_f)$ is odd and $\mathfrak{s} = \mathfrak{R}$ is union of its 2 rational children $\mathfrak{s}'_h$ and $\mathfrak{s}'_l$, satisfying $\rho_{\mathfrak{s}'_h} \geq \rho_{\mathfrak{s}} + 1$, $\rho_{\mathfrak{s}'_l} \geq \rho_{\mathfrak{s}} + 1$. This is case (3) of Definition 4.13.

Suppose now $\Gamma_{\mathfrak{s}}$ reducible. By Proposition 5.10 $\mathfrak{s}$ is übereven, $\epsilon_{\mathfrak{s}}$ is even and $\Gamma_{\mathfrak{s}}$ is the disjoint union of $\Gamma_{\mathfrak{s}}^- \simeq \mathbb{P}^1$ and $\Gamma_{\mathfrak{s}}^+ \simeq \mathbb{P}^1$. As before, both $\Gamma_{\mathfrak{s}}^-$ and $\Gamma_{\mathfrak{s}}^+$ intersect at least other two components (given by the proper children of $\mathfrak{s}$). But then neither $\Gamma_{\mathfrak{s}}^-$ nor $\Gamma_{\mathfrak{s}}^+$ has self-intersection -1, as $m_{\mathfrak{s}} = 1$.

It remains to show that after blowing down all components $\Gamma_{\mathfrak{s}}$ where $\mathfrak{s}$ is a contractible cluster, no other component can be blown down. First note that if $\mathfrak{s}$ is a contractible cluster, then $b_{\mathfrak{s}} = 2$ and $\Gamma_{\mathfrak{s}}$ intersects one or two other components of multiplicity 1 in two points. If it intersects only one component, then after the blowing down, the latter will have a node and will not be isomorphic to $\mathbb{P}^1$. If $\Gamma_{\mathfrak{s}}$ intersects two components and those intersect something else in $\mathcal{C}_s$, then they will not have self-intersection -1 also when $\Gamma_{\mathfrak{s}}$ is blown down. Therefore suppose that one of those two does not intersect any other component of $\mathcal{C}_s$. If we are in case (1) or case (2), it is easily to see that this never happens. Then without loss of generality assume to be in case (3) and that $\Gamma_{\mathfrak{s}'_h}$ is the component that can be blown down once $\Gamma_{\mathfrak{s}}$ has been contracted. This implies $\mathfrak{s}_h = \mathfrak{s}'_h$ and $\rho_{\mathfrak{s}_h} = \rho_{\mathfrak{s}} + 1$. But then $D_{\mathfrak{s}_h} = 2$ and $|\mathfrak{s}_h|/2 \geq 1$. Hence $g_{\mathfrak{s}_h} \geq 2$ and so $\Gamma_{\mathfrak{s}_h}$ cannot be blown down.

5.11. **Galois action.** Consider the base extended hyperelliptic curve $C_{K^{nr}}/K^{nr}$. The rational clusters and their corresponding rational centres of $C_{K^{nr}}$ are then over K^{nr}. Assume $\Sigma_{C_{K^{nr}}}$ is almost rational and let $\Sigma_{C_{K^{nr}}}^{\min} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\}$ be the set of rationally minimal clusters of $C_{K^{nr}}$. Fix a set $W = \{w_1, \dots, w_m\}$ of corresponding rational centres $w_h \in K^{nr}$. By Lemma A.1, we can assume this choice to be G_K-equivariant, i.e., for any $\sigma \in G_K$, one has $\sigma(w_l) = w_h$ if and only if $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$.

For any $h = 1, \dots, m$, define $f_h(x) = f(x + w_h) \in K^{nr}[x]$ and let C^{w_h}/K^{nr} : $y^2 = f_h(x)$. Fix $\sigma \in G_K$. If $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$, then $\sigma(f_l) = f_h$. Now, let $\mathfrak{t} \in \Sigma_{C_{K^{nr}}}^{w_l}$ be a proper cluster. Then $\sigma(\mathfrak{t}) \in \Sigma_{C_{K^{nr}}}^{w_h}$ and $\rho_{\mathfrak{t}} = \rho_{\sigma(\mathfrak{t})}$. In particular, if M is a matrix associated to $\mathfrak{t}$ then M is associated to $\sigma(\mathfrak{t})$ as well. So $\sigma(\mathcal{F}_M^l) = \mathcal{F}_M^h$. Finally, as $\sigma(\prod_{o \neq l}(x + w_{lo})^{-1}) = \prod_{o \neq h}(x + w_{ho})^{-1}$ we also have $\sigma(T_M^l) = T_M^h$.

Hence the natural K^{nr}-isomorphism $C^{w_h} \xrightarrow{\sigma} C^{w_l}$ induces $O_{K^{nr}}$-isomorphisms of schemes

$$\mathcal{C}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathcal{C}_{\Delta}^{w_l}, \qquad \mathring{\mathcal{C}}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathring{\mathcal{C}}_{\Delta}^{w_l}, \qquad U_M^h \xrightarrow{\sigma} U_M^l.$$

These maps describe the action of σ on $\mathcal{C}$ (see §5.3). In particular, if $\mathfrak{s}_h \wedge \mathfrak{s}_l \subseteq \mathfrak{t}$, then the glueing map (5) and σ are equal.

Let $\mathfrak{s} \in \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$ be a proper cluster, $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$ and $k_{\mathfrak{s}}$ be the residue field of $K_{\mathfrak{s}}$. Let $w_{\mathfrak{s}} \in K_{\mathfrak{s}}$ be a rational centre of $\mathfrak{s}$. Let $\Gamma_{\mathfrak{s}}$ be the regular 1-dimensional scheme defined in §5.6 and let $\tilde{\Gamma}_{\mathfrak{s}}/k_{\mathfrak{s}}$ be the regular 1-dimensional scheme given by

$$Y^{D_{\mathfrak{t}}} = \prod_{\mathfrak{s}' \in \tilde{\mathfrak{s}}} (X - \overline{u_{\mathfrak{s}'\mathfrak{s}}}) \overline{f_{\mathfrak{s}}}(X),$$

where $\overline{u}_{\mathfrak{s}'\mathfrak{s}} = \frac{w_{\mathfrak{s}'} - w_{\mathfrak{s}}}{\pi^{\rho_{\mathfrak{s}}}}$, and $\overline{f_{\mathfrak{s}}}$ as in Definition 4.15. Then there is a natural k^{s}-isomorphism $\Gamma_{\mathfrak{s}} \to \tilde{\Gamma}_{\mathfrak{s}}$ by Proposition 5.10. Furthermore, it is actually a $k_{\mathfrak{s}}$-isomorphism, since the action of $\sigma \in \mathrm{Gal}(k^{\mathrm{s}}/k_{\mathfrak{s}})$ corresponds to the glueing maps in $\mathcal{C}$ and is trivial on $\Gamma_{\mathfrak{s}}$.

A similar argument shows that $X_{\mathfrak{s}} : \{\overline{g_{\mathfrak{s}}} = 0\}$, where $w_{\mathfrak{s}}$ is the rational centre chosen in the definition of $\overline{g_{\mathfrak{s}}}$.

6. Integral differentials

Let C be a hyperelliptic curve of genus $g \geq 2$ defined over K by a Weierstrass equation $y^2 = f(x)$. Assume that C is y-regular. It is well-known that the K-vector space of global sections of the sheaf of differentials of C, namely $H^0(C, \Omega^1_{C/K})$, is spanned by the basis

$$\underline{\omega} = \left\{ \frac{dx}{2y}, x\frac{dx}{2y}, \dots, x^{g-1}\frac{dx}{2y} \right\}.$$

Let $\mathcal{C}$ be a regular model of C over O_K and consider its canonical sheaf $\omega_{\mathcal{C}/O_K}$. The free O_K-module of its global sections $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ can be viewed as an O_K-lattice of $H^0(C, \Omega^1_{C/K})$ by [LiA, Corollary 9.2.25(a)]. The aim of this section is to present a basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ as an O_K-linear combination of the elements in $\underline{\omega}$. Note that by [LiA, Corollary 9.2.25(b)] the problem is independent of the choice of model but it does depend on the choice of the equation $y^2 = f(x)$ since the basis $\underline{\omega}$ does. Throughout this section let C and $\mathcal{C}/O_K$ be as above.

If C is Δ_v-regular, [Dok, Theorem 8.12] gives an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$, as required. We rephrase it in terms of cluster invariants, by using results of §3.

Theorem 6.1 *For any cluster $\mathfrak{s} \in \Sigma_C$, set $\lambda_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r)$. Suppose that all proper clusters $\mathfrak{s} \in \Sigma_C$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}}$. Let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the proper clusters in Σ^0_C. For every $j = 0, \dots, g-1$, define*

$$i_j := \min\{i \in \{1, \dots, n\} \mid 2(j+1) \leq |\mathfrak{s}_i|\}$$

and

$$e_j := \frac{1}{2}\epsilon^0_{\mathfrak{s}_{i_j}} - (j+1)\rho^0_{\mathfrak{s}_{i_j}}.$$

Then the differentials

$$\mu_j = \pi^{\lfloor e_j \rfloor} x^j \frac{dx}{2y} \qquad j = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. The theorem is a consequence of [Dok, Theorem 8.12], Theorem 3.23, Lemma 4.6 and Lemma 4.1. □

Corollary 6.2 *Let $w \in K$. Suppose that all proper clusters $\mathfrak{s} \in \Sigma_C$ have rational centre w and those with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$. Let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the proper clusters in Σ^w_C. For every $j = 0, \dots, g-1$, define*

$$i_j := \min\{i \in \{1, \dots, n\} \mid 2(j+1) \leq |\mathfrak{s}_i|\}$$

and

$$e_j := \frac{1}{2}\epsilon_{\mathfrak{s}_{i_j}} - (j+1)\rho_{\mathfrak{s}_{i_j}}.$$

Then the differentials

$$\mu_j = \pi^{\lfloor e_j \rfloor} (x-w)^j \frac{dx}{2y} \qquad j = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Let $C^w : y^2 = f(x+w)$ be the hyperelliptic curve isomorphic to C through the change of variable $y \mapsto y$, $x \mapsto x+w$. By Corollary 3.24 and Lemma 4.6, the curve C^w is Δ_v-regular and so satisfies the hypothesis of Theorem 6.1 (alternatively they can be checked directly). Therefore

$$\mu_j = \pi^{\lfloor e_j \rfloor} x^j \frac{dx}{2y} \qquad j = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ as a subset of $H^0(C^w, \Omega^1_{C^w/K})$ (that is if $\mathcal{C}$ is regarded as a model of C^w). Changing variables concludes the proof. □

From now on, suppose that Σ_C is almost rational. Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters and let $W = \{w_{\mathfrak{s}} \mid \mathfrak{s} \in \Sigma_C^{\min}\}$ be a corresponding set of rational centres, where $w_{\mathfrak{s}} \in \mathfrak{s}$ if possible. For every proper cluster $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$, choose a minimal cluster $\mathfrak{s} \subseteq \mathfrak{t}$ and set $w_{\mathfrak{t}} := w_{\mathfrak{s}}$. Consider the regular model $\mathcal{C}/O_K$ of C of Theorem 4.12 constructed in §5 that the model $\mathcal{C}$ by glueing the open subschemes $\mathring{\mathcal{C}}_\Delta^w$ of $\mathcal{C}_\Delta^w$ for $w \in W$. We want to describe the canonical morphism $C \to \mathcal{C}$. Let $C^w : y^2 = f(x+w)$ and

$$y^2 - f(x+w) = Y^{n_Y} Z^{n_Z} \mathcal{F}_M^w(X,Y,Z),$$

as in [Dok, 4.4]. Let $\mathfrak{t} \in \Sigma_C^w$ be a proper cluster and let M be a matrix associated to $\mathfrak{t}$. Then, on the affine chart X_M the projection $C \to \mathcal{C}_\Delta^w$ is induced by

$$\frac{R}{(\mathcal{F}_M^w(X,Y,Z))} \xrightarrow{M} \frac{K[(x')^{\pm1},(y')^{\pm1}]}{((y')^2 - f(x'+w))} \xrightarrow{\simeq} \frac{K[x^{\pm1},y^{\pm1}]}{(y^2 - f(x))},$$

where $(X,Y,Z) = (x',y',\pi) \bullet M$ and $(x',y') = (x-w,y)$. In particular it follows that $(X,Y,Z) = (x-w,y,z) \bullet M$ and so

$$\begin{pmatrix} x-w \\ y \\ \pi \end{pmatrix} = \begin{pmatrix} X^{\tilde m_{11}} Y^{\tilde m_{21}} Z^{\tilde m_{31}} \\ X^{\tilde m_{12}} Y^{\tilde m_{22}} Z^{\tilde m_{32}} \\ X^{\tilde m_{13}} Y^{\tilde m_{23}} Z^{\tilde m_{33}} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \bullet M^{-1}.$$

For a proper cluster $t \in \Sigma_C^{\mathrm{rat}}$ recall the definitions of $\Gamma_{\mathfrak{t}}$ and $m_{\mathfrak{t}}$.

Proposition 6.3 *Let $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ be a proper cluster. Then*[4]

$$\begin{aligned} \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}}) &= m_{\mathfrak{t}} \rho_{\mathfrak{t}}, \\ \mathrm{ord}_{\Gamma_{\mathfrak{t}}} \tfrac{dx}{y} &= -m_{\mathfrak{t}} \left(\tfrac{1}{2}\epsilon_{\mathfrak{t}} - \rho_{\mathfrak{t}} - 1\right) - 1. \end{aligned}$$

for every proper cluster $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$, $\mathfrak{s} \subseteq \mathfrak{t}$.

Proof. Assume first $\Gamma_{\mathfrak{t}}$ irreducible. Let $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ proper, $\mathfrak{s} \subseteq \mathfrak{t}$. Then from the proof of [Dok, Proposition 8.1] and Lemma 4.1 it follows that

$$\begin{aligned} \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}}) &= \delta_{F_{\mathfrak{t}}^{w_{\mathfrak{s}}}} \rho_{\mathfrak{t}}^{w_{\mathfrak{s}}}, \\ \mathrm{ord}_{\Gamma_{\mathfrak{t}}} \frac{dx}{2y} &= -\delta_{F_{\mathfrak{t}}^{w_s}} \left(\tfrac{1}{2}\epsilon_{\mathfrak{t}}^{w_{\mathfrak{s}}} - \rho_{\mathfrak{t}}^{w_{\mathfrak{s}}} - 1\right) - 1. \end{aligned}$$

by [LiA, Lemma 9.2.17(a)]. Since $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{t}$ for Lemma 3.14, Remark 3.32 implies $\rho_{\mathfrak{t}}^{w_{\mathfrak{s}}} = \rho_{\mathfrak{t}}$ and $\epsilon_{\mathfrak{t}}^{w_{\mathfrak{s}}} = \epsilon_{\mathfrak{t}}$. Then the proposition follows since $\delta_{F_{\mathfrak{t}}^{w_s}} = m_{\mathfrak{t}}$.

Suppose now that $\Gamma_{\mathfrak{t}}$ is reducible. Let $L = L_{\mathfrak{t}}^{w_{\mathfrak{s}}}, 0$, let $M = M_L$ and let X, Y, Z be the transformed variables $(X,Y,Z) = (x - w_{\mathfrak{s}}, y, \pi) \bullet M$. Consider the open set $U : \{Z = 0\}$ of $\Gamma_{\mathfrak{t}} \subset \mathcal{C}_s$. Since Z vanishes of order 1 on U, it follows from Lemma 5.2 that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}}) = \tilde m_{31} = m_{\mathfrak{t}} \rho_{\mathfrak{t}} \quad \text{and} \quad \mathrm{ord}_{\Gamma_{\mathfrak{t}}} y = \tilde m_{32} = m_{\mathfrak{t}} \tfrac{\epsilon_{\mathfrak{t}}}{2},$$

[4] If $\Gamma_{\mathfrak{t}}$ is reducible, say $\Gamma_{\mathfrak{t}} = \Gamma_{\mathfrak{t}}^- \cup \Gamma_{\mathfrak{t}}^+$, then $\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(\cdot)$ means $\min\{\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}(\cdot), \mathrm{ord}_{\Gamma_{\mathfrak{t}}^+}(\cdot)\}$

as $d_0\delta_M = m_{\mathfrak{t}}$.

By Proposition 5.10 the defining equation of the $\Gamma_{\mathfrak{t}}$ on U is of the form

$$Y^2 - h(X)^2 = X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}} = 0,$$

where $h(X) \in O_K[X]$ such that $\overline{h}(X)^2 = \overline{f|_L}(X) \in k[X]$. Let

$$\Gamma_{\mathfrak{t}}^- := \{Y - h(X) = X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}} = 0\},$$
$$\Gamma_{\mathfrak{t}}^+ := \{Y + h(X) = X^{\tilde{m}_{13}}Y^{\tilde{m}_{23}}Z^{\tilde{m}_{33}} = 0\},$$

be the two components of $\Gamma_{\mathfrak{t}}$ on U. Consider $\Gamma_{\mathfrak{t}}^-$. By [LiA, Corollary 6.4.14], the sheaf $\omega_{\mathcal{C}/O_K}$ is generated on $\Gamma_{\mathfrak{t}}^-$ by

$$\begin{vmatrix} -h'(X) & 1 \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}^{-1} dZ$$

if this determinant is non-zero. As $(\tilde{m}_{13}, \tilde{m}_{23}) \neq (0,0)$, the determinant above is non-zero. Recall the polynomials G_0, H in the proof of [Dok, Proposition 8.1]. Then $G_0(X,Y,Z) = Y^2 - G(X)^2 + Z \cdot E(X,Y,Z)$, for some polynomial $E \in K[X,Y,Z]$. Hence

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} \frac{dx}{\pi(x-w_{\mathfrak{s}})y^2} = -m_{\mathfrak{t}}\epsilon_{\mathfrak{t}} - 1 + \mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}\left(\begin{vmatrix} -h'(X)h(X)+ZE'_X & 2Y+ZE'_Y \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}^{-1} dZ \right)$$

from the proof of [Dok, Proposition 8.1]. This implies that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} \frac{dx}{2y} = -m_{\mathfrak{t}}\left(\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - 1\right) - 1 - \mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} \frac{\begin{vmatrix} -h'(X)h(X)+ZE'_X & 2Y+ZE'_Y \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}}{\begin{vmatrix} -h'(X) & 1 \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}}.$$

Then we want to show

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} a(X,Y,Z) = 0, \quad a(X,Y,Z) := \frac{\begin{vmatrix} -h'(X)h(X)+ZE'_X & 2Y+ZE'_Y \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}}{\begin{vmatrix} -h'(X) & 1 \\ \tilde{m}_{13}\pi X^{-1} & \tilde{m}_{23}\pi Y^{-1} \end{vmatrix}}.$$

We have

$$\begin{aligned} a(X,Y,Z) &= \frac{2\big(\tilde{m}_{23}h'(X)h(X)X + \tilde{m}_{13}Y^2\big) + Z(\tilde{m}_{13}YE'_Y - \tilde{m}_{23}XE'_X)}{\tilde{m}_{23}Xh'(X) + \tilde{m}_{13}Y} \\ &= 2Y - \frac{2\frac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)(Y-h(X)) - Z\left(YE'_Y - \frac{\tilde{m}_{23}}{\tilde{m}_{13}}XE'_X\right)}{Y + \frac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)}. \end{aligned}$$

As $\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}(Y - h(X)) = \mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} Z = 1$, we have

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}\left(2\tfrac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)\left(Y - h(X)\right) - Z\left(YE'_Y - \tfrac{\tilde{m}_{23}}{\tilde{m}_{13}}XE'_X\right)\right) \geq 1.$$

Therefore if $\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}\left(Y + \frac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)\right) = 0$ then $\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-} a(X,Y,Z) = 0$, as Y is a unit on U. Suppose by contradiction that $\mathrm{ord}_{\Gamma_{\mathfrak{t}}^-}\left(Y + \frac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)\right) \geq 1$. Since $Z \nmid \left(Y + \frac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)\right)$, we must have

$$(Y - h(X)) \mid \left(Y + \tfrac{\tilde{m}_{23}}{\tilde{m}_{13}}Xh'(X)\right),$$

that trivially implies $h(X) = X^n$, where $n = -\frac{\tilde{m}_{13}}{\tilde{m}_{23}} \in \mathbb{Z}_{\geq 0}$. Thus $\overline{f|_L}(X) = X^{2n}$, but this is impossible since $\mathtt{NP}(\overline{f|_L}(X)) \neq \mathtt{NP}(X^{2n})$. ∎

Theorem 6.4 *Let C/K be a hyperelliptic curve of genus $g \geq 2$ defined by the Weierstrass equation $y^2 = f(x)$ and let $\mathcal{C}/O_K$ be a regular model of C. Suppose C is y-regular and Σ_C is almost rational. For $i = 0, \dots, g-1$ choose inductively proper clusters $\mathfrak{s}_i \in \Sigma_C^{\mathrm{rat}}$ so that*

$$e_i := \frac{\epsilon_{\mathfrak{s}_i}}{2} - \sum_{j=0}^{i} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_i} = \max_{\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}} \left\{ \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\},$$

where if $\mathfrak{s}$ and $\mathfrak{s}'$ are two possible choices for $\mathfrak{s}_i$ satisfying $\mathfrak{s}' \subset \mathfrak{s}$, then choose $\mathfrak{s}_i = \mathfrak{s}$. Then the differentials

$$\mu_i = \pi^{\lfloor e_i \rfloor} \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j}) \frac{dx}{2y}, \qquad i = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Since $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ is independent of the choice of regular model, we consider $\mathcal{C}$ to be the model described in Theorem 4.12 (and constructed in §5).

We first show that the differentials μ_i are global sections of $\omega_{\mathcal{C}/O_K}$. It suffices to prove they are regular along all components $\Gamma_{\mathfrak{t}}$, where $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ proper. Indeed for the construction of $\mathcal{C}$ and the definition of the e_i's, the differentials μ_i are regular along all other components of $\mathcal{C}_s$ by Corollary 6.2. Fix $i = 1, \dots, g-1$ and let $j = 0, \dots, i-1$. Let $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ be a proper cluster. If $\mathfrak{s}_j \subseteq \mathfrak{t}$ then

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) = m_{\mathfrak{t}} \rho_{\mathfrak{t}} = m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}},$$

by Proposition 6.3. If $\mathfrak{t} \subsetneq \mathfrak{s}_j$ then $w_{\mathfrak{t}}$ is a rational centre of $\mathfrak{s}_j$. Hence

$$v(w_{\mathfrak{t}} - w_{\mathfrak{s}_j}) \geq \min_{r \in \mathfrak{t}} \min\{v(r - w_{\mathfrak{t}}), v(r - w_{\mathfrak{s}_j})\} \geq \min\{\rho_{\mathfrak{t}}, \rho_{\mathfrak{s}_j}\} = \rho_{\mathfrak{s}_j} = \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}.$$

Therefore Lemma 3.18 implies

$$\begin{aligned} \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) &\geq \min\{\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{t}}), \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(w_{\mathfrak{t}} - w_{\mathfrak{s}_j})\} \\ &\geq \min\{m_{\mathfrak{t}} \rho_{\mathfrak{t}}, m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}\} = m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}. \end{aligned}$$

If $\mathfrak{s}_j \not\subseteq \mathfrak{t}$ and $\mathfrak{t} \not\subseteq \mathfrak{s}_j$ then

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) = \min\{m_{\mathfrak{t}} \rho_{\mathfrak{t}}, m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}\} = m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}.$$

as $\rho_{\mathfrak{t}} > \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}$. Thus we have proved that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) \geq m_{\mathfrak{t}} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}, \qquad \text{where the equality holds if } \mathfrak{t} \not\subset \mathfrak{s}_j. \tag{9}$$

Therefore it follows from Proposition 6.3 that

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}} \mu_i \geq m_{\mathfrak{t}} \left(\lfloor e_i \rfloor + \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} - \frac{\epsilon_{\mathfrak{t}}}{2} + \rho_t + 1 \right) - 1.$$

But

$$\lfloor e_i \rfloor \geq \left\lfloor \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\rfloor > \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} - 1,$$

then $\mathrm{ord}_{\Gamma_{\mathfrak{t}}} \mu_i > -1$, that implies $\mathrm{ord}_{\Gamma_{\mathfrak{t}}} \mu_i \geq 0$, as required.

Now we need to show that the differentials μ_i span $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$, i.e., the lattice they span is saturated in the global sections of $\omega_{\mathcal{C}/O_K}$. Suppose not. Then there exist $I \subseteq \{0, \dots, g-1\}$ and $u_i \in O_K^\times$ for $i \in I$ such that the differential $\frac{1}{\pi} \sum_{i \in I} u_i \mu_i$ is regular along $\Gamma_{\mathfrak{t}}$, for every proper cluster $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$. Let $I_1 \subseteq I$ be the set of indices i such that $\gamma_i := e_i - \lfloor e_i \rfloor$ is maximal. Let $I_2 \subseteq I$ be the set of indices $i \in I_1$ such that $\mathfrak{s}_i$ is maximal with respect to the inclusion. Define $i_0 := \min I_2$ and denote by Γ_0 the closed subscheme $\Gamma_{\mathfrak{s}_{i_0}}$. First we want to show that $\mathfrak{s}_{i_0} \not\subset \mathfrak{s}_j$ for all $j = 0, \dots, i_0$. Suppose by contradiction that there exists $j_0 < i_0$ such that $\mathfrak{s}_{i_0} \subsetneq \mathfrak{s}_{j_0}$. Then by the definitions of $\mathfrak{s}_{i_0}$ and $\mathfrak{s}_{j_0}$ one has

$$\begin{aligned}\frac{\epsilon_{\mathfrak{s}_{j_0}}}{2} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{j_0}} = e_{j_0} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=j_0+1}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{j_0}} &\geq e_{j_0} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=j_0+1}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} \\ &\geq \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{j_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=j_0+1}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} \\ &= \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \sum_{j=0}^{i_0} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} = e_{i_0} \geq \frac{\epsilon_{\mathfrak{s}_{j_0}}}{2} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{j_0}}.\end{aligned}$$

Therefore

$$\max_{\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}} \left\{ \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\} = e_{i_0} = \frac{\epsilon_{\mathfrak{s}_{j_0}}}{2} - \rho_{\mathfrak{s}_{j_0}} - \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{j_0}},$$

and this means that $\mathfrak{s}_{j_0}$ is a possible choice for the i_0-th cluster $\mathfrak{s}_{i_0}$. But $\mathfrak{s}_{i_0} \subsetneq \mathfrak{s}_{j_0}$, so the i_0-th cluster should have been $\mathfrak{s}_{j_0}$, a contradiction. Then $\mathfrak{s}_{i_0} \not\subset \mathfrak{s}_j$ for all $j = 0, \dots, i_0$. From this fact and (9) we have

$$\begin{aligned} m := \mathrm{ord}_{\Gamma_0} \frac{1}{\pi} \mu_{i_0} &= -m_{\mathfrak{s}_{i_0}} \gamma_{i_0} + m_{\mathfrak{s}_{i_0}} \left(e_{i_0} - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i_0-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} \right) - 1 \\ &= -m_{\mathfrak{s}_{i_0}} \gamma_{i_0} - 1 < 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{ord}_{\Gamma_0} \frac{1}{\pi} \mu_i &\geq -m_{\mathfrak{s}_{i_0}} \gamma_i + m_{\mathfrak{s}_{i_0}} \left(e_i - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} \right) - 1 \\ &\geq -m_{\mathfrak{s}_{i_0}} \gamma_i - 1 \geq -m_{\mathfrak{s}_{i_0}} \gamma_{i_0} - 1 = m, \end{aligned}$$

for all $i \in I$. Let $J := \{ i \in I \mid \mathrm{ord}_{\Gamma_0} \frac{1}{\pi} \mu_i = m \}$. Then $J \neq \varnothing$ since $i_0 \in J$ and the differential $\frac{1}{\pi} \sum_{i \in J} u_i \mu_i$ must cancel along Γ_0. Let $i \in I$. Then $i \in J$ if and only if $\mathrm{ord}_{\Gamma_0} \frac{1}{\pi} \mu_i = m$ which is equivalent to

$$\gamma_i = \gamma_{i_0} \quad \text{and} \quad \frac{\epsilon_{\mathfrak{s}_i}}{2} - \sum_{j=0}^{i} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_i} = e_i = \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}$$

by the computations above. We want to show that $J = I_2$. We have already noted that $i_0 \in J$. Assume $i \in J$, $i \neq i_0$. Then the equality $e_i = \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}$ implies $\mathfrak{s}_{i_0} \subseteq \mathfrak{s}_i$, while it follows from $\gamma_i = \gamma_{i_0}$ that $i \in I_1$. Hence $\mathfrak{s}_i = \mathfrak{s}_{i_0}$, as $i_0 \in I_2$, and so $i \in I_2$. On the other hand, if $i \in I_2$ then $\mathfrak{s}_i = \mathfrak{s}_{i_0}$ by definition and this trivially implies $e_i = \frac{1}{2} \epsilon_{\mathfrak{s}_{i_0}} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}$. Moreover, $\gamma_i = \gamma_{i_0}$ as $I_2 \subseteq I_1$. Therefore $J = I_2$. For any $i \in I_2$ we have

$$\lfloor e_i \rfloor - \lfloor e_{i_0} \rfloor = e_i - \gamma_i - e_{i_0} + \gamma_{i_0} = e_i - e_{i_0} = - \sum_{j=i_0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}.$$

Hence

$$\frac{1}{\pi} \sum_{i \in I_2} u_i \mu_i = \frac{1}{\pi} \mu_{i_0} \left(\sum_{i \in I_2} \frac{u_i}{\pi^{\sum_{j=i_0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}}} \prod_{j=i_0}^{i-1} (x - w_{\mathfrak{s}_j}) \right),$$

and since $\mathrm{ord}_{\Gamma_0} \frac{1}{\pi} \mu_{i_0} = m < 0$ we must have

$$\mathrm{ord}_{\Gamma_0} \left(\sum_{i \in I_2} \frac{u_i}{\pi^{\sum_{j=i_0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}}} \prod_{j=i_0}^{i-1} (x - w_{\mathfrak{s}_j}) \right) > 0. \tag{10}$$

Let $i_1 := \max I_2$. We have already proved that $\mathfrak{s}_j \not\subset \mathfrak{s}_{i_0}$ and $\mathfrak{s}_{i_0} \not\subset \mathfrak{s}_j$ for any $j \in I$ with $i_0 \leq j \leq i_1$, as $\mathfrak{s}_{i_1} = \mathfrak{s}_{i_0}$. Therefore for any $j \in I$ with $i_0 \leq j \leq i_1$ such that $\mathfrak{s}_j \neq \mathfrak{s}_{i_0}$, we have $\rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} \in \mathbb{Z}$ and

$$\begin{aligned}\mathrm{ord}_{\Gamma_0}(x - w_{\mathfrak{s}_j}) &= \min\{\mathrm{ord}_{\Gamma_0}(x - w_{\mathfrak{s}_{i_0}}), \mathrm{ord}_{\Gamma_0}(w_{\mathfrak{s}_{i_0}} - w_{\mathfrak{s}_j})\} \\ &= m_{\mathfrak{s}_{i_0}} \min\{\rho_{\mathfrak{s}_{i_0}}, \rho_{\mathfrak{s}_{i_0} \wedge \mathfrak{s}_i}\} = m_{\mathfrak{s}_{i_0}} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}} = \mathrm{ord}_{\Gamma_0}(w_{\mathfrak{s}_j} - w_{\mathfrak{s}_{i_0}}),\end{aligned}$$

by Lemma 3.18. Since $\frac{w_{\mathfrak{s}_j} - w_{\mathfrak{s}_{i_0}}}{\pi^{\rho_{\mathfrak{s}_j \wedge \mathfrak{s}_{i_0}}}} \in O_K^\times$, it follows from (10) that

$$\mathrm{ord}_{\Gamma_0}\left(\sum_{i \in I_2} v_i \frac{(x - w_{\mathfrak{s}_{i_0}})^{\beta_i}}{\pi^{\beta_i \rho_{\mathfrak{s}_{i_0}}}}\right) > 0,$$

for some $v_i \in O_K^\times$, where $\beta_i = \#\{j \in I \mid i_0 \leq j < i \text{ and } \mathfrak{s}_j = \mathfrak{s}_{i_0}\}$.

To find a contradiction, we will use the explicit description of an open affine subset of Γ_0. Let $w = w_{\mathfrak{s}_{i_0}}$, $L = L^w_{\mathfrak{s}_{i_0}}$, $M = M_{L,0}$, and consider the affine open subset

$$U_M^w = \mathrm{Spec}\, \frac{R[T_M^w(X, Y, Z)^{-1}]}{(\mathcal{F}_M^w(X, Y, Z), Z)} \subset \Gamma_{\mathfrak{t}}.$$

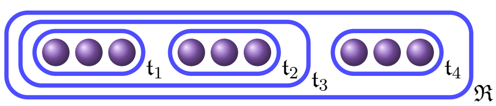
From Lemma 5.2,

$$\sum_{i \in I_2} v_i \frac{(x - w)^{\beta_i}}{\pi^{\beta_i \rho_{\mathfrak{s}_{i_0}}}} = \sum_{i \in I_2} v_i X^{\beta_i / b_{\mathfrak{s}}},$$

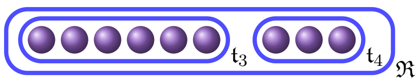
which is a unit since, for the structure of Δ_v^w, the polynomial $\mathcal{F}_M^w(X, Y, Z)$ in $\{Z = 0\}$ is of the form $Y^2 - G(X)$ or $Y - G(X)$ for some non-constant $G(X) \in K[X]$. This gives a contradiction and concludes the proof. $\square$

We conclude this section with an application of Theorem 6.4.

Example 6.5. Let C be a hyperelliptic curve over $\mathbb{Q}_3$ of genus 4 described by the equation $y^2 = f(x)$, where $f(x) = (x^3 - 3^4)(x^3 + 3^4)((x-3)^3 - 3^{11})$. The cluster picture of C is



where $d_{\mathfrak{t}_1} = d_{\mathfrak{t}_2} = \frac{11}{6}$, $d_{\mathfrak{t}_3} = \rho_{\mathfrak{t}_1} = \rho_{\mathfrak{t}_2} = \rho_{\mathfrak{t}_3} = \frac{4}{3}$, $d_{\mathfrak{t}_4} = \frac{25}{6}$, $\rho_{\mathfrak{t}_4} = \frac{11}{3}$ and $d_{\mathfrak{R}} = \rho_{\mathfrak{R}} = 1$. Then C satisfies the hypothesis of Theorem 6.4. Its rational cluster picture is



where the set of minimal clusters is $\Sigma_C^{\min} = \{\mathfrak{t}_3, \mathfrak{t}_4\}$. We choose rational centres for $\mathfrak{t}_3$ and $\mathfrak{t}_4$: $w_{\mathfrak{t}_3} = 0$ and $w_{\mathfrak{t}_4} = 3$. Since $\mathfrak{R} = \mathfrak{t}_3 \wedge \mathfrak{t}_4$, we can choose either $w_{\mathfrak{R}} = w_{\mathfrak{t}_3}$ or $w_{\mathfrak{R}} = w_{\mathfrak{t}_4}$. Let us fix $w_{\mathfrak{R}} = w_{\mathfrak{t}_3} = 0$. Then to choose $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ as in Theorem 6.4 we draw the following table:

	$\rho_{\mathfrak{t}}$	$\epsilon_{\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\rho_{\mathfrak{s}_0\wedge\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{2}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$
$\mathfrak{t}_3$	$\frac{4}{3}$	11	$\frac{25}{6}$	$\frac{19}{6}$	$\frac{11}{6}$	$\frac{1}{2}$
$\mathfrak{t}_4$	$\frac{11}{3}$	17	$\frac{29}{6}$	$\frac{7}{6}$	$\frac{1}{6}$	$-\frac{5}{6}$
$\mathfrak{R}$	1	9	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$

The red numbers indicate that $\mathfrak{s}_0 = \mathfrak{t}_4$, $\mathfrak{s}_1 = \mathfrak{s}_2 = \mathfrak{t}_3$ and $\mathfrak{s}_3 = \mathfrak{R}$. Thus the differentials

$$\mu_0 = 3^4 \cdot \frac{dx}{2y}, \quad \mu_1 = 3^3 \cdot (x-3)\frac{dx}{2y}, \quad \mu_2 = 3 \cdot (x-3)x\frac{dx}{2y}, \quad \mu_3 = (x-3)x^2\frac{dx}{2y}$$

form a $\mathbb{Z}_3$-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/\mathbb{Z}_3})$, for any regular model $\mathcal{C}/\mathbb{Z}_3$ of C.

Appendix A. Rational centres over tame extensions

Let C/K be a hyperelliptic curve given by $y^2 = f(x)$.

Lemma A.1 *Let L/K be a field extension. Consider the base extended curve C_L/L and its associated cluster picture Σ_{C_L}. Let $\mathfrak{s} \in \Sigma_{C_L}$ be a proper cluster $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, and $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$. If $LK_{\mathfrak{s}}/K_{\mathfrak{s}}$ is either the maximal tamely ramified extension or the maximal unramified extension, then $\mathfrak{s}$ has a rational centre $w_{\mathfrak{s}} \in L \cap K_{\mathfrak{s}}$.*

Proof. This proof follows the spirit of [M²D², Lemma B.1]. Write $K_{\mathfrak{s}}^{nr}$ for the maximal unramified extension of $K_{\mathfrak{s}}$. Let $r \in \mathfrak{s}$. Then $r \in K_{\mathfrak{s}}^{nr}(\sqrt[b]{\pi_{\mathfrak{s}}})$ for b large enough and some uniformiser $\pi_{\mathfrak{s}}$ of $K_{\mathfrak{s}}$ (we fix here a choice of $\sqrt[b]{\pi_{\mathfrak{s}}}$). Write the p-adic expansion of r as

$$r = u_t \sqrt[b]{\pi_{\mathfrak{s}}}^t + u_{t+1}\sqrt[b]{\pi_{\mathfrak{s}}}^{t+1} + \dots$$

for a suitable $\mathfrak{t} \in \mathbb{Z}$ and $u_t \in K_{\mathfrak{s}}^{nr}$ roots of unity of order prime to p. Let $w_{\mathfrak{s}} \in L$ be a rational centre of $\mathfrak{s}$. For $\sigma \in G_{\mathfrak{s}}$ we have $\sigma(r) \equiv w_{\mathfrak{s}} \bmod \pi_K^{\rho_{\mathfrak{s}}}$, where

$$\rho_{\mathfrak{s}} = \max_{w \in L} \min_{r \in \mathfrak{s}} v(r-w).$$

Hence the terms in the p-adic expansions of $\sigma(r)$ and $w_{\mathfrak{s}}$ agree up to $\sqrt[b]{\pi_{\mathfrak{s}}}^{e_{K_{\mathfrak{s}}/K} b\rho_{\mathfrak{s}}}$. Define

$$w = \sum_{l < e_{K_{\mathfrak{s}}/K} b\rho_{\mathfrak{s}}} u_l \sqrt[b]{\pi_{\mathfrak{s}}}^l.$$

We want to show that $w \in L$. It trivially follows if $w = 0$. Suppose $0 \neq w \notin L$, and that $u_{l_0}\sqrt[b]{\pi_{\mathfrak{s}}}^{l_0}$ is the lowest valuation term of the expansion which is not in L. Without loss of generality we can assume $t = l_0$ (consider $w' = w - \sum_{l<l_0} u_l \sqrt[b]{\pi_{\mathfrak{s}}}^l$). As $v(w - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}$, we have $v(w_{\mathfrak{s}}) = v(w) = t/b$. If $LK_{\mathfrak{s}} = K_{\mathfrak{s}}^{nr}$, then $b \nmid t$, which gives a contradiction, as $v(w_{\mathfrak{s}})$ has to be an integer. On the other hand, if $LK_{\mathfrak{s}}$ is the maximal tamely ramified extension of $K_{\mathfrak{s}}$, then $p \mid b$, that again contradicts $v(w_{\mathfrak{s}}) = t/b$.

Therefore w is a rational centre of $\mathfrak{s}$, as $v(w - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}$. We only need to show it is $G_{\mathfrak{s}}$-invariant. Suppose not, and that $u_l\sqrt[b]{\pi_{\mathfrak{s}}}^l$ is the lowest valuation term of the expansion which is not $G_{\mathfrak{s}}$-invariant. Note that the denominator of l/b is coprime with p since $w \in L$ and L tame. If $b \nmid l$, then there is some element σ of tame inertia of $K_{\mathfrak{s}}$ which fixes $u_l \in K_{\mathfrak{s}}^{nr}$ and maps $\sqrt[b]{\pi_{\mathfrak{s}}}^l$ to $\zeta\sqrt[b]{\pi_{\mathfrak{s}}}^l$, where

$\zeta \neq 1$ is a root of unity; this contradicts the fact that $\sigma(r) \equiv r \bmod \pi_K^{\rho_{\mathfrak{s}}}$. If $b \mid l$, then $\sigma(r) \equiv w_{\mathfrak{s}} \bmod \sqrt[b]{\pi_{\mathfrak{s}}}^{e_{K_{\mathfrak{s}}/K} b\rho_{\mathfrak{s}}}$, so we must have $u_l \notin K_{\mathfrak{s}}$. Then there exists some element $\sigma \in \mathrm{Gal}(K_{\mathfrak{s}}^{nr}/K_{\mathfrak{s}})$ so that $\sigma(u_l) \neq u_t$; this contradicts the fact that $\sigma(r) \equiv r \bmod \sqrt[b]{\pi_{\mathfrak{s}}}^{e_{K_{\mathfrak{s}}/K} b\rho_{\mathfrak{s}}}$. □

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