Simplicial complexes: higher-order spectral dimension and dynamics

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Abstract. Simplicial complexes constitute the underlying topology of interacting complex systems including among the others brain and social interaction networks. They are generalized network structures that allow to go beyond the framework of pairwise interactions and to capture the many-body interactions between two or more nodes strongly affecting dynamical processes. In fact, the simplicial complexes topology allows to assign a dynamical variable not only to the nodes of the interacting complex systems but also to links, triangles, and so on. Here we show evidence that the dynamics defined on simplices of different dimensions can be significantly different even if we compare dynamics of simplices belonging to the same simplicial complex. By investigating the spectral properties of the simplicial complex model called "Network Geometry with Flavor" (NGF) we provide evidence that the up and down higher-order Laplacians can have a finite spectral dimension whose value depends on the order of the Laplacian. Finally we discuss the implications of this result for higher-order diffusion defined on simplicial complexes showing that the n-order diffusion dynamics have a return type distribution that can depends on n as it is observed in NGFs.

1. Introduction

Simplicial complexes are generalized network structures that allow to capture the many body interactions existing between the constituents of complex systems [1–3]. They are becoming increasingly popular to represent brain data [3–6], social interacting systems [7–10], financial networks [11, 12] and complex materials [13, 14], beyond the framework of pairwise interactions. A simplicial complex is formed by a set of simplices such as nodes, links, triangles, tetrahedra and so on glued to each other along their faces. Being built by geometrical building blocks, simplicial complexes represent an ideal setting to investigate the properties of emergent network geometry and topology in complex systems [1, 15–17]. Moreover they reveal the rich interplay between network geometry and dynamics [18–20, 26–29].

The recently proposed non-equilibrium growing simplicial complex model called "Network Geometry with Flavor" (NGF) [16] is able to display emergent hyperbolic network geometry [17] together with the major universal properties of complex networks including scale-free degree distribution, small-word distance property, high clustering coefficient and significant modular structure. Interestingly, the simplicial complexes generated by the NGF model display also a finite spectral dimension [18, 19, 30, 31]. The coexistence of a finite spectral dimension and the small-world properties might be strongly related to the hyperbolicity of the simplicial complexes. Indeed the presence of a finite spactral dimension is not an exclusive property of NGFs but it has also been observed in a recent modelling framework that uses hyperbolic and small-world simplicial complex to model nano-networks [32].

The spectral dimension [34–37] characterises the spectrum of the graph Laplacian of network geometries and is well known to affect the return-time probability of classical [34] critical phenomena [38, 39] and quantum diffusion [40]. Additionally, the spectral dimension strongly affects the synchronization properties of the Kuramoto model which display a thermodynamically stable synchronized phase only if the spectral dimension d_S is greater than four [18, 19]. Finally the spectral dimension is also used in quantum gravity to probe the geometry of different model of quantum space-time [41–46]

Recent works [20, 47–49] have emphasised that simplicial complexes can sustain dynamical processes whose variables can be located not only on their nodes but also on their higher dimensional simplices such as links, triangles and so on. These signal includes dynamical fluxes of relevance in biological transport [21–24] but can also indicate other types of signals [48] such as brain signals [25] and signals travelling in social networks such as collaboration networks or face-to-face networks [7, 8].

Despite the field is only at its infancy few works have started to investigate dynamical processes defined on higher dimensional simplices. In Ref. [20] the higher order Kuramoto model has been introduced showing that it can display either a continuous and a discontinuous synchronization phase transition. Additionally, Ref. [49] defines a higher-order diffusion dynamics. To give an intuition of the relevance of higher-order diffusion on a simplicial complex we consider the case of social simplicial complexes. In a scientific collaboration network a simplex is formed by a the set of nodes indicating all the co-authors of at least a paper. In this case it is very plausible to define a diffusion going from a collaborative team (working at a given paper) to another team (working at another paper) if the two corresponding simplices have an overlap, i.e. they share a subset of the authors. Similarly, in face-to-face interactions it is plausible that some social behavior can spread from one small gathering to another small gathering but in order to propagate might require that the groups share more than one person in common leading to a higher-order diffusion instead of a simple diffusion.

The higher-order diffusion dynamics and the higher-order Kuramoto model depend on the higher-order boundary maps of the simplices and the higher-order Laplacian matrix. The higher-order Laplacian matrix [48–51] of order $n > 0$ describes a diffusion dynamics taking place between simplices of order n and can be decomposed in the sum between the up-Laplacian and the down-Laplacian. The higher-order discrete Laplacian has been studied by several mathematicians [50, 51], however as far as we know, there is no previous result showing that the high-order Laplacian can display a finite spectral dimension.

In this work we investigate the spectral properties of the higher-order Laplacian on NGF. We show that the n-order up and down-Laplacians have a finite spectral dimension that depends on the order n. By investigating the properties of higher-order diffusion on NGF we find that the higher-order spectral dimension has a significant effect on the return-time probability of the process. Therefore, we provide evidence that the diffusion, occurring on the same simplicial complex but taking place on simplices of different order n, can induce significantly different dynamical behavior.

2. Methods

In this section we provide the mathematical background needed to appreciate our results. In particular, we define the simplicial complexes, we introduce the boundary map needed to define the higher order Laplacian and finally we present the main properties of higher-order Laplacians. A final paragraph is devoted to summarize main characteristics of simplicial complexes having a finite spectral dimension of the graph Laplacian.

2.1. Simplicial complexes

Simplicial complexes are able to capture higher-order interactions between two or more nodes described by simplices. A n-dimensional *simplex* μ is formed by $n+1$ nodes

$$\mu = [i_0, i_1, \dots, i_n]. \tag{1}$$

Therefore, a 0-dimensional simplex is a node, a 1-dimensional simplex is a link, and so on. A face of n-dimensional simplex is a n'-dimensional simplex formed by a proper subset of $n'+1$ nodes of the original simplex. Consequently, we necessarily have $n' < n$. A simplicial complex $\mathcal{K}$ is a set of simplices that is closed under the inclusion of the faces of the simplices. We will indicate with d the dimension of the simplicial complex determining the maximum dimension of its simplices. The "skeleton" of a simplicial complex indicates the networks derived by the simplicial complex under consideration retaining only its nodes and its links. Moreover, we will indicate with $N_{[n]}$ the number of n-dimensional simplices of the simplicial complex $\mathcal{K}$. Therefore, in the following $N_{[0]}$ indicates the number of nodes, $N_{[1]}$ the number of links, $N_{[2]}$ the number of triangles and so on. Simplicial complexes can sustain a diffusion dynamics occurring on its n-dimensional faces. This higher-order diffusion dynamics is determined by the properties of the higher-order Laplacians. In order to introduce here the higher-order Laplacian we will devote the next paragraph to some fundamental quantities in network topology.

2.2. Oriented simplices and boundary map

In topology each n-dimensional simplex μ

$$\mu = [i_0, i_1, \dots, i_n]. \tag{2}$$

has an orientation given by the sign of the permutation of the label of the nodes. Therefore, we have

$$[i_0, i_1, \dots, i_n] = (-1)^{\sigma(\pi)}[i_{\pi(0)}, i_{\pi(1)}, \dots, i_{\pi(n)}] \tag{3}$$

where $\sigma(\pi)$ indicates the parity of the permutation π.

The boundary map ∂_n is a linear operator acting on linear combinations of n-dimensional simplices and defined by its action on each of the simplices $\mu = [i_0, i_2 \dots, i_n]$ of the simplicial complex as

$$\partial_n[i_0, i_1 \dots, i_n] = \sum_{p=0}^{n}(-1)^p[i_0, i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_n]. \tag{4}$$

Therefore the boundary map of a link $[i, j]$ is given by

$$\partial_1[i, j] = [j] - [i]. \tag{5}$$

Similarly the boundary map of a triangle $[i, j, k]$ is given by

$$\partial_2[i, j, k] = [j, k] - [i, k] + [i, j]. \tag{6}$$

From this definition it follows directly that

$$\partial_{n-1}\partial_n = \mathbf{0}, \tag{7}$$

relation that is often referred to by saying that the "boundary of the boundary is null". For instance, the reader can easily check using the above definitions that $\partial_1\partial_2[i, j, k] = 0$. The boundary map ∂_n can also be represented by the incidence matrix $\mathbf{B}_{[n]}$ of dimension $N_{[n-1]} \times N_{[n]}$. Then, Eq.(7) can be expressed using the incidence matrices as

$$\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}. \tag{8}$$

In Figure 1 we show a small 2-dimensional simplicial complex formed by the set of nodes $\{[1], [2], [3], [4], [5]\}$, the set of links $\{[12], [13], [23], [34], [24], [25], [45]\}$ and triangles $\{[123], [234], [245]\}$. Its boundary maps are given by

$$\mathbf{B}_{[1]} = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{9}$$

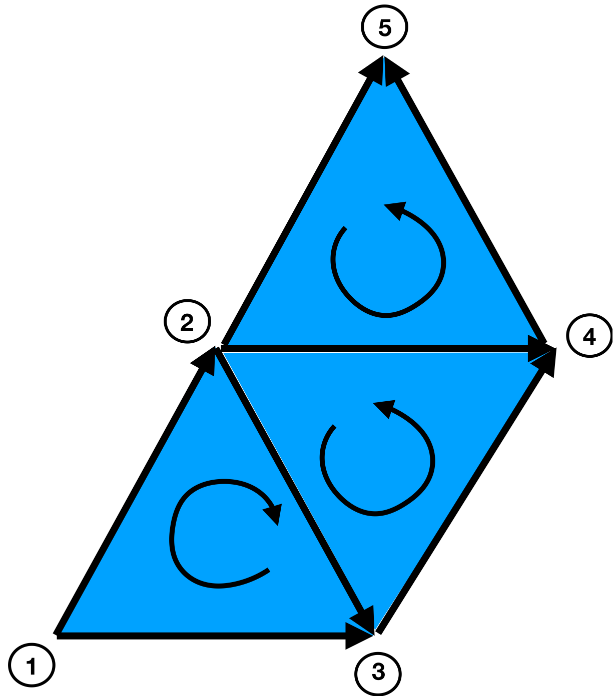

Figure 1. We show a small 2-dimensional simplicial complex of $N_{[0]} = 5$ nodes, $N_{[1]} = 7$ links and $N_{[3]} = 3$ triangles whose incidence matrices $\mathbf{B}_{[1]}$ and $\mathbf{B}_{[2]}$ are given in Eqs. (9) and (10).

$$\mathbf{B}_{[2]} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

2.3. Higher-order Laplacians

The zero-order Laplacian $\mathbf{L}_{[0]}$ is the usual graph Laplacian defined on networks and is a $N_{[0]} \times N_{[0]}$ matrix of elements

$$\left(L_{[0]}\right)_{ij} = \delta_{ij} - a_{ij}, \tag{11}$$

where here and in the following δ_{ij} indicates the Kronecker delta, and a_{ij} indicates the element (i, j) of the adjacency matrix. The graph Laplacian $\mathbf{L}_{[0]}$ can be also expressed in terms of the incidence matrix $\mathbf{B}_{[1]}$ as

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}. \tag{12}$$

This definition can be extended to higher-order Laplacians using higher-order incidence matrices $\mathbf{B}_{[n]}$. In particular the higher-order Laplacian $\mathbf{L}_{[n]}$ (with $n > 0$) [48–50] is the $N_{[n]} \times N_{[n]}$ matrix defined as

$$\mathbf{L}_{[n]} = \mathbf{L}_{[n]}^{down} + \mathbf{L}_{[n]}^{up}, \tag{13}$$

with

$$\begin{aligned} \mathbf{L}_{[n]}^{down} &= \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]}, \\ \mathbf{L}_{[n]}^{up} &= \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}. \end{aligned} \tag{14}$$

The higher-order Laplacians are independent on the orientation of the simplices as long as the orientation of the simplices is induced by the label of the nodes. The degeneracy of the zero eigenvalue of the n Laplacian $\mathbf{L}_{[n]}$ is equal to the Betti number β_n. The eigenvectors associated to the zero eigenvalue of the n-Laplacian are localized on the corresponding n-dimensional cavities of the simplicial complex. Therefore, the higher-order Laplacians with $n > 0$ are not guaranteed to have a zero eigenvalue as simplicial complexes with $\beta_n = 0$ for some $n > 0$ exist. In particular, if the topology of the simplicial complex is trivial, i.e. $\beta_0 = 1$ and $\beta_n = 0$ for all $n > 0$, the Laplacians of order $n > 0$ do not admit a zero eigenvalue.

Another important property of the n-Laplacian is that each non-zero eigenvalue is either a non-zero eigenvalue of the n-order up-Laplacian or is a non-zero eigenvalue of the n-order down-Laplacian. Consider an eigenvector $\mathbf{v}$ of the up-Laplacian with eigenvalue $\lambda \neq 0$. Then, we have

$$\mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}\mathbf{v} = \lambda\mathbf{v}, \tag{15}$$

or equivalently

$$\mathbf{v} = \frac{1}{\lambda}\mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}\mathbf{v}. \tag{16}$$

Let us apply the down-Laplacian to the eigenvector $\mathbf{v}$. Thus we obtain

$$\mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]}\mathbf{v} = \frac{1}{\lambda}\mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]}\mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}\mathbf{v} = \mathbf{0}, \tag{17}$$

where we have used Eq. (8). It follows that if $\mathbf{v}$ is an eigenvector associated to a non-zero eigenvalue λ of the n-order up-Laplacian then it is an eigenvector of the n-order down-Laplacian with zero eigenvalue. Therefore, in this case $\mathbf{v}$ is an eigenvector of the n-order Laplacian with the eigenvalue λ. Similarly, it can be easily shown that if $\mathbf{v}$ is an eigenvector associated to a non-zero eigenvalue λ of the n-order down-Laplacian then it is also an eigenvector of the n-order Laplacian with the same eigenvalue. Consequently the spectrum of the n-order Laplacian includes all the eigenvalues of the n-order up-Laplacian and the n-order down-Laplacian.

Another important property of the high-order up and down Laplacians is that the spectrum of the n-order up-Laplacian coincides with the spectrum of the $(n+1)$-order down-Laplacian as the two are related by

$$\mathbf{L}_{[n]}^{up} = \left(\mathbf{L}_{[n+1]}^{down}\right)^{\top}. \tag{18}$$

The n-Laplacian is positive semi-definite and, therefore, it has $N_{[n]}$ non negative eigenvalues that we indicate as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_r \leq \ldots \leq \lambda_{N_{[n]}}. \tag{19}$$

Moreover, in the following we will indicate by $\mathbf{v}^{(r)}$ the eigenvector corresponding to eigenvalue λ_r of the n-Laplacian.

2.4. Spectral dimension of the graph Laplacian

The spectral dimension is defined for networks (1-dimensional simplicial complexes) with distinct geometrical properties, and determines the dimension of the network as "experienced" by a diffusion process taking place on it [18,19,34–36,40]. These networks might well be the skeleton of simplicial complexes of dimension $d > 1$. In particular the spectral dimension is traditionally defined starting from the density of eigenvalues $\rho(\lambda)$ of the 0-Laplacian. We say that a network has *spectral dimension* $d_S^{[0]}$ if the density of eigenvalues $\rho(\lambda)$ of the 0-Laplacian follows the scaling relation

$$\rho(\lambda) \simeq \tilde{C}_{[0]} \lambda^{d_S^{[0]}/2-1} \tag{20}$$

for $\lambda \ll 1$. In d-dimensional Euclidean lattices $d_S^{[0]} = d$. Additionally, $d_S^{[0]}$ is related to the Hausdorff dimension d_H of the network by the inequalities [45, 46]

$$d_H \geq d_S^{[0]} \geq 2\frac{d_H}{d_H+1}. \tag{21}$$

Therefore, for small-world networks, which have infinite Hausdorff dimension $d_H = \infty$, it is only possible to have finite spectral dimension $d_S^{[0]} \geq 2$. If the density of eigenvalues $\rho(\lambda)$ follows Eq.(20) it results that the cumulative distribution $\rho_c(\lambda)$ evaluating the density of eigenvalues $\lambda' \leq \lambda$ satisfies

$$\rho_c(\lambda) \simeq C_{[0]} \lambda^{d_S^{[0]}/2}, \tag{22}$$

for $\lambda \ll 1$. In presence of a finite spectral dimension the Fiedler eigevalue λ_2 satisifies

$$\lambda_2 \propto N_{[0]}^{-2/d_S^{[0]}}. \tag{23}$$

Therefore, the Fidler eigenvalue $\lambda_2 \to 0$ as $N_{[0]} \to \infty$ and we say that in the large network limit the spectral gap closes. This is another distinct property of networks with a geometrical character, i.e. significantly different from random graphs and expanders. In particular in the framework of diffusion dynamics as a finite Fiedler eigenvalue is known to be related to a finite relaxation time, the fact that the Fiedler eigenvalue is vanishing on networks with a finite spectral dimension implies very slow, power-law relaxation dynamics to the steady state distribution [34].
The spectral dimension has also been proven to be essential to determine the stability of the synchronized state of the Kuramoto model which can be thermodynamically stable only if $d_S^{[0]} > 4$.

In the next section we will show that the notion of spectral dimension can be generalized to up-Laplacians of order $n > 0$ with important consequences for higher-order simplicial complex dynamics.

3. Results

In this section we will investigate the spectral properties of a recently proposed model of simplicial complexes called "Network Geometry with Flavor". We will show that the higher-order up-Laplacians of these simplicial complexes display a finite spectral dimension depending on the order n of the up-Laplacian considered, the dimension of the simplicial complex d and a parameter of the model called flavor s. Therefore given a single instance of a NGF we can define different spectral dimensions $d_S^{[n]}$ for $0 < n < d-1$. Here we will show that this implies that the dynamics defined on simplices of different dimension n of the same simplicial complex can be significantly different.

3.1. Network Geometry with Flavor

The model "Network Geometry with Flavor" (NGF) [16, 17] generates d-dimensional simplicial complexes. Each simplex is obtained by performing a non-equilibrium process consisting in the continuous addition of new d-simplices attached to the rest of the simplicial complex along a single $(d-1)$-face. To every $(d-1)$-face μ of the simplicial complex, (i.e. a link for $d = 2$, or a triangular face for $d = 3$) we associate an *incidence number* n_μ given by the number of d-dimensional simplices incident to it minus one. The evolution of NGF depends on a parameter $s \in \{-1, 0, 1\}$ called *flavor*. Starting from a single d-dimensional simplex, with $d \geq 2$ at each time we add a d-dimensional simplex to a $(d-1)$-face μ. The face μ is chosen randomly with probability Π_μ given by

$$\Pi_\mu = \frac{1 + sn_\mu}{\sum_\nu 1 + sn_\nu}. \tag{24}$$

According to a classical result in combinatorics, under this dynamics we obtain a discrete manifold only if n_μ can take exclusively the values $n_\mu = 0, 1$. This occurs only for $s = -1$. In fact for $n_\mu = 0$ we obtain $\Pi_\mu = 1/(\sum_\nu 1 + sn_\nu) > 0$ but for $n_\mu = 1$ we obtain $\Pi_\mu = 0$. Therefore, the resulting simplicial complex is a discrete manifold, with each $(d-1)$-face incident at most to two d-dimensional simplices, i.e. $n_\mu = 0, 1$. For $s = 0$ the attachment probability Π_μ is uniform while for $s = 1$ the attachment probability Π_μ increases linearly with the number of simplices already incident to the face μ implementing a generalized preferential attachment. Therefore for $s = 0$ as for $s = 1$ the incidence number n_μ can take arbitrary large values $n_\mu = 0, 1, 2, 3 \ldots$.

This model generates emergent hyperbolic geometry, and the underlying network is small-word (has infinite Hausdorff dimension, i.e. $d_H = \infty$), has high clustering coefficient and high modularity. Interestingly, this model reduces to well known models in specific cases: for $d = 1$ and $s = 1$ it reduces to the tree Barabasi-Albert model [52], for $d = 2$ and $s = 0$ it reduces to the model first studied in Ref. [53] and finally for $d = 3$ and $s = -1$ it reduces to the random Apollonian model [54, 55].

Given a d-dimensional NGF simplicial complex with $N_{[0]}$ nodes, the number of n-dimensional simplices is given by

$$N_{[n]} = \binom{d+1}{n+1} + [N_{[0]} - (d+1)] \binom{d}{n}, \tag{25}$$

independently on the flavor s. In fact $\binom{d+1}{n+1}$ is the number of n-dimensional simplices of the initial d simplex, and for any addition of a new d-simplex (and therefore a new node) we get $\binom{d}{n}$ new n-dimensional simplices that contain the new node.

3.2. Spectral properties of NGF

The graph Laplacian of NGFs has been recently show to display a finite spectral dimension $d_S^{[0]}$ and localized eigenvectors with important consequences on dynamics [18,19,30]. Interestingly, here we show that also the higher-order up-Laplacians $\mathbf{L}_{[n]}^{up}$ and the higher-order down-Laplacians $\mathbf{L}_{[n]}^{down}$ of NGFs display a finite spectral dimension.

Since the up-Laplacian is defined as $\mathbf{L}_{[n]}^{up} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}$ the eigenvalues λ of the n-order up-Laplacian are the square of the singular values of the incidence matrix $\mathbf{B}_{[n+1]}$. The incidence matrix $\mathbf{B}_{[n+1]}$ is a rectangular $N_{[n]} \times N_{[n+1]}$ matrix, therefore the non-zero singular values cannot be more than $\min(N_{[n]}, N_{[n+1]})$. For NGFs, that have a trivial topology, the Hodge decomposition [48] guarantees that the number $\mathcal{N}_{[n]}$ of non-zero eigenvalues of the n-order up-Laplacian with $n > 0$ achieves this limit and consequently we have

$$\mathcal{N}_{[n]} = \begin{cases} N_{[0]} - 1 & \text{if} \quad n = 0, \\ \min(N_{[n]}, N_{[n+1]}) & \text{if} \quad 0 < n < d. \end{cases} \tag{26}$$

In Figure 2 we plot the cumulative density of eigenvalues $\rho_c(\lambda)$ of the n-order Laplacian and the cumulative density of non-zero eigenvalues $\rho_c^{up}(\lambda)$ of the n-order up-Laplacians of NGF with $d = 3$ and flavor $s = -1, 0, 1$. The n-order up-Laplacians display a finite spectral dimension, i.e. their cumulative density of eigenvalues obeys the scaling

$$\rho_c^{up}(\lambda) \simeq C_{[n]}\lambda^{d_S^{[n]}/2}, \tag{27}$$

for $\lambda \ll 1$. Therefore by plotting $\rho_c^{up}(\lambda)$ versus λ for small values of λ on a log-log plot, by performing a linear fit (see Figure 3) we can extract the values of the spectral dimensions $d_S^{[n]}$. The fitted values of these higher-order spectral dimensions are reported in Table 1 for different values of the order n and the flavor s of the 3-dimensional NGF. From this table it can be clearly shown that the values of the higher-order spectral dimension $d_S^{[n]}$ increase with n i.e.

$$d_S^{[n+1]} > d_S^{[n]} \tag{28}$$

for any value of the flavor s and have values greater than two. We note that our numerical results (not shown) clearly show that while the higher-order spectral dimension of up-Laplacian remains finite also for $d \neq 3$, the values of the spectral dimension might not be monotonic, so they might not satisfy Eq. (28).

Since the n-order up-Laplacian is the transpose matrix of the $(n+1)$-order down-Laplacian (as given in Eq. (18)) the two matrices have the same spectrum. It follows directly that the $(n+1)$-order down-Laplacian has spectral dimension $d_S^{[n]}$.

From these results on the higher-order up-Laplacian we can easily determine the scaling of the density of eigenvalues for the higher-order Laplacian of NGFs. In particular for $0 < n < d$ we have

$$\rho_c(\lambda) = \frac{\mathcal{N}_{[n-1]}}{N_{[n]}}C_{[n-1]}\lambda^{d_S^{[n-1]}/2} + \frac{\mathcal{N}_{[n]}}{N_{[n]}}C_{[n]}\lambda^{d_S^{[n]}/2}, \tag{29}$$

for $n = 0$ we have instead

$$\rho_c(\lambda) = C_{[0]}\lambda^{d_S^{[0]}/2}, \tag{30}$$

Table 1. Fitted value of the spectral dimension $d_S^{[n]}$ of the n-order up-Laplacian of the NGF for different values of n and s and for $d = 3$. The fitted values have been estimated from a single realization of the NGF with $N_{[0]} = 2000$ nodes. The error over the fitted spectral dimension is the 0.01 confidence interval of the corresponding linear regression model.

d/s	$n=0$	$n=1$	$n=2$
$s=-1$	3.03 ± 0.03	5.36 ± 0.04	14.3 ± 0.5
$s=0$	3.98 ± 0.05	5.48 ± 0.02	11.3 ± 0.03
$s=1$	4.82 ± 0.08	6.04 ± 0.05	7.8 ± 0.3

and for $n = d$ we have

$$\rho_c(\lambda) = C_{[d-1]}\lambda^{d_S^{[d-1]}/2}. \tag{31}$$

Therefore the density of eigenvalues $\rho(\lambda)$ of the n-order Laplacian reads

$$\rho(\lambda) = \begin{cases} \tilde{C}_{[0]}\lambda^{d_S^{[0]}/2-1} & \text{for} \quad n=0, \\ \tilde{C}_{[n-1]}\lambda^{d_S^{[n-1]}/2-1} + \tilde{C}_{[n]}\lambda^{d_S^{[n]}/2-1} & \text{for} \quad 0<n<d, \\ \tilde{C}_{[d-1]}\lambda^{d_S^{[d-1]}/2-1} & \text{for} \quad n=d, \end{cases} \tag{32}$$

where $\tilde{C}_{[n]}$ are constants. Therefore, the cumulative density of the eigenvalues of the higher-order Laplacian will asymptotically scale as a power-law dictated by the minimum between $d_S^{[n-1]}$ and $d_S^{[n]}$.

3.3. Diffusion using higher-order Laplacian

Higher-order Laplacians $\mathbf{L}_{[n]}$ can be used to define a diffusion process defined over n-dimensional simplices. For instance, one can consider a classical quantity x_μ defined on the n-dimensional simplices μ of the simplicial complex and use the n-Laplacian $\mathbf{L}_{[n]}$ to study its diffusion using the dynamical equation

$$\frac{dx_\mu}{dt} = -\sum_{\nu \in S_n} \left(\mathbf{L}_{[n]}\right)_{\mu,\nu} x_\nu. \tag{33}$$

where with S_n we indicate the set of all simplices of dimension n (of cardinality $|S_n| = N_{[n]}$). For $n = 0$ there is always a stationary state as β_1 indicates at the same time the number of connected components of the simplicial complex (therefore we have $\beta_1 \geq 1$) and the degeneracy of the zero eigenvalue of the Laplacian matrix $\mathbf{L}_{[0]}$. Additionally, for a connected network the stationary state is uniform over all the nodes of the network. However, Eq. (33) for $n > 0$ will have a stationary state only if the Betti number $\beta_n > 0$, i. e. if there is at least a n-dimensional cavity in the simplicial complex. Note, however, that also if this stationary state exists the stationary state will be non-uniform over the network but localized on the n-dimensional cavities. In order

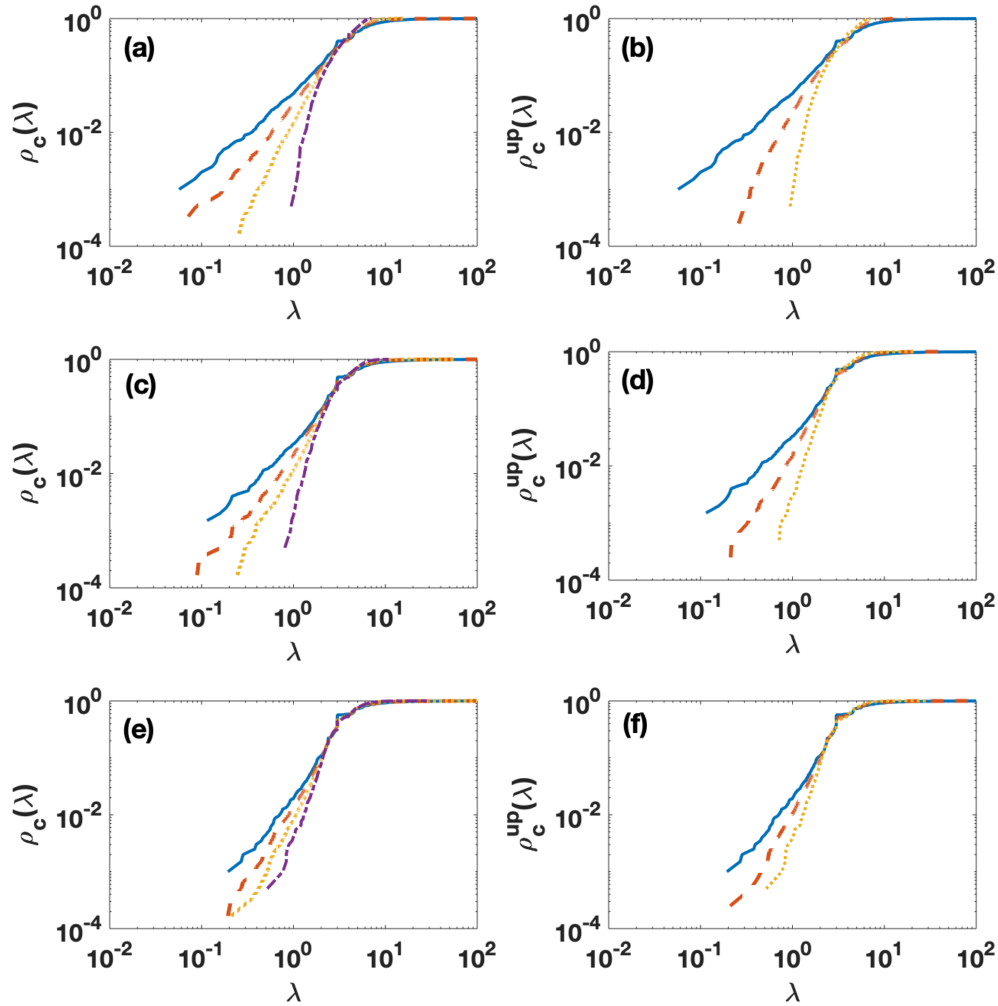

Figure 2. The cumulative density of eigenvalues $\rho_c(\lambda)$ of the n-order Laplacian and the cumulative density of non-zero eigenvalues $\rho_c^{up}(\lambda)$ of the n-order up-Laplacians are shown for the NGF of dimension $d = 3$ and flavor $s = -1$ (panels a and b), $s = 0$ (panels c and d) and $s = 1$ (panels e and f). Here the blue solid lines indicate $n = 0$, the red dashed lines indicate $n = 1$, the yellow dotted line indicates $n = 2$ and the purple dot-dashed lines indicates $n = 3$. The NGF under consideration are single instance of NGFs with $N_{[0]} = 2000$ nodes $N_{[1]} = 5994$ links $N_{[2]} = 5992$ triangles and $N_{[3]} = 1997$ tetrahedra.

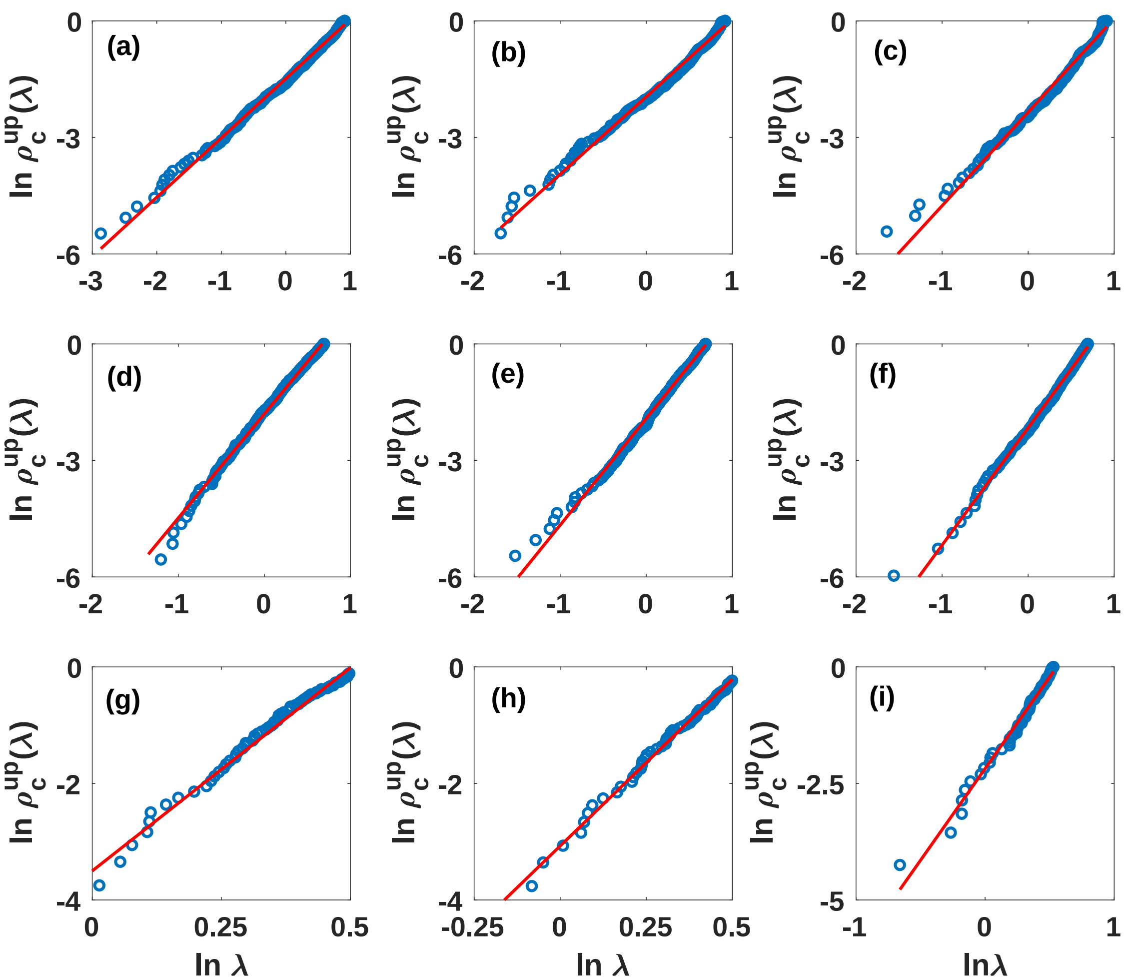

Figure 3. The tail of the cumulative density of non-zero eigenvalues $\rho_c^{up}(\lambda)$ of the n-order up-Laplacians is shown on a log-log plot (data points) for the NGF of dimension $d = 3$ and flavor $s = -1$ (panels a, d, g for $n = 0, 1, 2$ respectively), $s = 0$ (panels b, e and g for $n = 0, 1, 2$) and $s = 1$ (panels c, f and i for $n = 0, 1, 2$ respectively) together with its fit. The NGF under consideration are single instances of NGFs with $N_{[0]} = 2000$ nodes $N_{[1]} = 5994$ links $N_{[2]} = 5992$ triangles and $N_{[3]} = 1997$ tetrahedra. In Table 1 we report the corresponding values for the spectral dimension d_S.

to describe a diffusion equation that has a non trivial stationary state also when $\beta_n = 0$ we can modify the diffusion equation and consider instead the dynamics

$$\frac{dx_\mu}{dt} = -\sum_{\nu \in S_n} \left(\mathbf{L}_{[n]}\right)_{\mu,\nu} x_\nu - \lambda_1 v_\mu^{(1)} \sum_{\nu \in S_n} v_\nu^{(1)} x_\nu. \tag{34}$$

This equation reduces to Eq. (33) if the smallest eigenvalue of the n-Laplacian is zero (i. e. $\lambda_1 = 0$) and admits always a non-trivial stationary state localized along the eigenvector $\mathbf{v}^{(1)}$ corresponding to the smallest eigenvalue. The NGFs have Betti numbers $\beta_0 = 1$ and $\beta_n = 0$ for every $n > 0$. In this case, when $n > 0$ the dynamics defined by

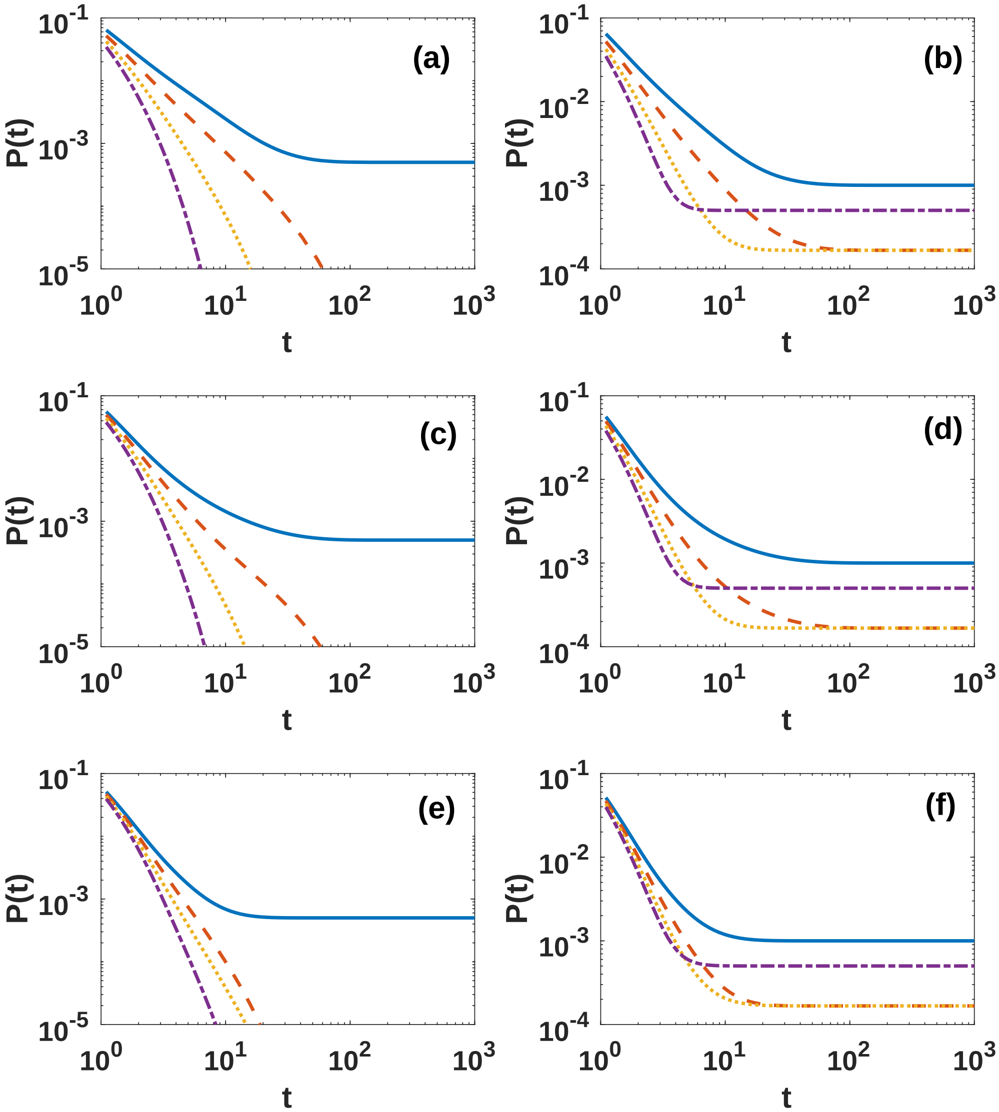

Figure 4. The return-time probability $P(t)$ of the higher-order diffusion dynamics determined by Eq. (33) (panels a, c, e) and Eq. (34) (panels b, d, f) is shown for NGF with $N_{[0]} = 2000$ nodes, $d = 3$ and flavor $s = -1$ (panels a and b), $s = 0$ (panels c and d) and $s = 1$ (panels e and f). Here the blue solid lines indicate $n = 0$, the red dashed lines indicate $n = 1$, the yellow dotted line indicates $n = 2$ and the purple dot-dashed line indicates $n = 3$. Here the return-time probability $P(t)$ is obtained using Eq. (36) and the numerically evaluated higher-order spectrum of single instances of NGFs.

Eq. (33) gives a transient to a vanishing solution $x_\nu = 0$ for every n-dimensional face ν. On the contrary, the dynamics defined by Eq. (34) gives a transient to a non-vanishing steady state solution. The solution for the two dynamical equations (33) and (34) can be written as

$$x_\mu(t) = \sum_{r=1}^{N_{[n]}} e^{-\lambda_r(1-c\delta_{r,1})t} v_\mu^{(r)} \sum_{\nu \in S_n} v_\nu^{(r)} x_\nu(0) \tag{35}$$

where for the dynamics defined in Eq. (33) we put $c = 0$ while for the dynamics defined in Eq. (34) we put $c = 1$.

For both dynamics, we investigate the return-time probability $P(t)$ as a function of time. The return-time probability $P(t)$ is defined as the probability that the diffusion process starting from a localized configuration on a given simplex μ returns back to the simplex μ at time t, averaged over all simplices $\mu \in S_n$ of the simplicial complex. Therefore $P(t)$ is given by

$$P(t) = \sum_{\mu \in S_n} \sum_{r=1}^{N_{[n]}} e^{-\lambda_r(1-c\delta_{r,1})t} v_\mu^{(r)} v_\mu^{(r)} = \sum_{r=1}^{N_{[n]}} e^{-\lambda_r(1-c\delta_{r,1})t} \tag{36}$$

where in the last expression we have used the normalization of the eigenvectors $\mathbf{v}^{(r)}$, i. e.

$$\sum_{\mu \in S_n} v_\mu^{(r)} v_\mu^{(r)} = 1. \tag{37}$$

Note that Eq. (36) is very general and can be applied to any simplicial complex of which we know the spectrum of the higher-order Laplacians. In particular, here use Eq.(36) together with numerical diagonalization of the higher-order Laplacians of single instance of the NGFs to predict the return time distribution of higher-order diffusion dynamics. Interestingly, for large NGF the return-time probability $P(t)$ decays in time at different rates depending on the dimension n over which the diffusion dynamics is defined. In particular, for a large simplicial complex when $N_{[n]} \to \infty$ we can approximate the return-time probability $P(t)$ as

$$P(t) = \int_{\lambda_1}^{\lambda_{N_{[n]}}} d\lambda \; \rho(\lambda) e^{-\lambda(1-c\delta(\lambda,\lambda_1))t}. \tag{38}$$

By inserting the scaling of the density of states given by Eq. (32), we easily obtain

$$P(t) \begin{cases} A_{[0]} t^{-d_S^{[0]}/2} & \text{for} \quad n = 0, \\ A_{[n-1]} t^{-d_S^{[n-1]}/2} + A_{[n]} t^{-d_S^{[n]}/2} & \text{for} \quad 0 < n < d, \\ A_{[d-1]} t^{-d_S^{[d-1]}/2} & \text{for} \quad n = d. \end{cases} \tag{39}$$

where $A_{[n]}$ are constants. In Figure 4 we provide evidence of the different power-law scaling of the return-time probability $P(t)$ for diffusion processes occurring on the simplices of different dimension n of the NGF. This result shows that the diffusion dynamics defined on nodes, links or triangles of the same instance of simplicial complex generated by the model NGF, can display significantly different dynamical properties. This effect is due to the fact that the process is affected by the value of a higher-order spectral dimension that increases with n.

4. Discussion

Simplicial complexes can sustain dynamics defined not only on nodes but also on higher-order simplices. Linear and non-linear processes such as diffusion and synchronization can be extended to higher-order thanks to the higher-order Laplacian. Therefore, the investigation of the spectral properties of the higher-order Laplacian is rather crucial to reveal the properties of higher-order dynamical processes on simplicial complexes. In this work we reveal that the higher-order up and down-Laplacian can display a finite spectral dimension by providing a concrete example where this phenomenon is displayed, the simplicial complex model called "Network Geometry with Flavor". In particular, we numerically show that the up-Laplacians have a spectral dimension that depends on their order n and the other parameters of the model, i. e. the flavor s and the dimension d of the simplicial complex. Finally, we show how this spectral property of the higher-order up-Laplacian affects the diffusion dynamics defined on the simplices. Notably, we show that different spectral dimensions can cause significant effects in the return-time probability of the diffusion process. These results provide evidence that the same simplicial complex can sustain diffusion processes with rather distinct dynamical signatures depending on the dimension n of the simplices over which the diffusion dynamics is defined.

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Data availability statement

Data sharing is not applicable to this article as no new data were created or analysed in this study. The codes will be available upon request to the authors.

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