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Reduced Normal Forms Are Not Extensive Forms

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Abstract Fundamental results in the theory of extensive form games have singled out the reduced normal form as the key representation of a game in terms of strategic equivalence. In a precise sense, the reduced normal form contains all strategically relevant information. This note shows that a difficulty with the concept has been overlooked so far: given a reduced normal form alone, it may be impossible to reconstruct the game's extensive form representation.

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1 Introduction

Most game-theoretic solution concepts are defined in the normal form, that is, as (sets of) strategy profiles, e.g., iteratively undominated strategies, rationalizable strategies (Bernheim, 1984; Pearce, 1984), curb sets (Basu and Weibull, 1991), tenable blocks (Myerson and Weibull, 2015), and of course Nash equilibrium (Nash, 1950, 1951). A number of refinements of Nash equilibrium, on the other hand, rely on the extensive form representation, e.g., subgame

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perfection (Selten, 1965), quasi-perfect equilibrium (van Damme, 1984), or sequential equilibrium (Kreps and Wilson, 1982). Some refinements are defined in the *agent* normal form and then translated to the extensive form game (Selten, 1975).

All known refinement concepts that are defined in the extensive form suffer from a fragility, though. An extensive form game can often be written down in different ways, reflecting the inessential transformations introduced by Thompson (1952) and Elmes and Reny (1994) (see also Battigalli et al., 2020). And yet, these transformation do affect refinement criteria that are based on the extensive form. That is, a subgame perfect or sequential equilibrium may cease to be so after an inessential (Thompson) transformation; or a mere Nash equilibrium may become subgame perfect or sequential after such a transformation. Since these transformations of the extensive form only generate different "framings" of the same problem, this fragility appears undesirable, at least from a rationalistic viewpoint.[1]

In the normal form two pure strategies of a player are *strategically equivalent* if they give the same payoffs to all players for all strategy profiles among the opponents. The (pure-strategy) *reduced* normal form is obtained by collapsing all strategically equivalent strategies to single representatives. Two different extensive form representations can be obtained from each other through inessential transformations (Thompson, 1952; Elmes and Reny, 1994) if and only if their reduced normal forms coincide (see, e.g., Thompson, 1952; Perea, 2001; Ritzberger, 2002). That is, the reduced normal form is unaffected by those transformations. This fact, and the fragility of equilibrium concepts based on the extensive form, have led many researchers to insist on the priority of the reduced normal form, or even the mixed-strategy reduced normal form, as in the strategic stability debate (e.g., Kohlberg and Mertens, 1986; Mertens, 1989, 1991; Hillas, 1990; Vermeulen and Jansen, 1998). Yet, the gist of extensive form refinements is to capture backwards induction, which by its very nature is an extensive form notion, and hence normal forms remain unsatisfactory. A major advance, and a vehicle to bring backwards induction to bear on normal form concepts, was the seminal result by van Damme (1984): A proper equilibrium (Myerson, 1978) induces a quasi-perfect (hence, sequential) equilibrium in every extensive form game with the given normal form. So,

[1] This is especially important for research programs adopting the position that implicit or self-referential equilibrium concepts should be formalized in a framework which clearly specifies the details, order, and dependence of the possible actions and interactions of all involved agents. This is because the latter necessarily results in an extensive form game, which may in principle be susceptible to inessential transformations. An example of such a research program is given by Glycopantis et al. (2001, 2003, 2005, 2009). This also extends to the Nash program (Nash, 1953) about giving non-cooperative foundations to cooperative games, because those often lead to extensive form games. A case in point are bargaining games where the extensive form explicitly specifies the bargaining protocol (e.g., Rubinstein, 1982; Britz et al., 2010; Glycopantis, 2020).

proper equilibrium appears to be a sufficient condition to integrate backwards induction into normal form analysis.[2]

Two extensive form games that have the same normal form also share the same (pure-strategy) reduced normal form. Hence, to find an equilibrium that satisfies backwards induction and is robust to inessential transformations of the extensive form it is enough to determine a proper equilibrium in the reduced normal form—or apply one of the stronger set-valued solution concepts. This is comforting and provides support to the position that the normal form is sufficient.

Yet, one aspect is easily overlooked. This position is preference-dependent. It depends on payoffs because strategic equivalence depends on payoffs, hence on the players' preferences. In the reduced normal form, the payoffs of two different strategies against all profiles of strategies of the other players may coincide because the strategies lead to the same "outcome" (for example, because the strategies differ only on unreached parts of the tree), or because they lead to different outcomes which happen to give the same payoffs. That is, the (payoff-based) reduced normal form cannot distinguish between coincidental payoff ties (preference-based) and fundamental ties arising from the extensive form. It is well-known, however, that an analogous notion applies to a pure representation of the rules of the game without payoffs: Two strategies of a player are *strategically equivalent* if they induce the *same plays* (or "outcomes") for all strategy profiles among the opponents.[3] If such a pure representation of the rules of the game without payoff assignments is adopted,[4] there is still a "normal form" where now strategies map into plays rather than payoff vectors, and there is a corresponding reduced normal form.

From this perspective, we show that a new problem emerges: The reduced normal form may not be the normal form of *any* extensive form, if both representations are preference-free (alternatively, the players' preference orderings over plays are robust to sufficiently small perturbations). Put differently, with an equilibrium in a given reduced normal form, in order to go back and reconstruct the extensive form and "what happens in equilibrium," one would still need information about the reduction steps that led to the reduced normal form.

To emphasize that again, the example below, by which we show that a reduced normal form may not be the normal form of any extensive form, is formulated as a "game form," that is, without payoffs. Instead, strategy profiles

[2] Many of the stronger set-valued refinement concepts always contain a proper equilibrium, e.g., M-stable sets (Mertens, 1989, 1991) or equilibrium components with non-zero index (Ritzberger, 1994). As far as these sets are contained in connected components of Nash equilibria, they therefore guarantee that the probability distribution on plays associated with the solution set generically satisfies backwards induction, since for generic extensive form games the probability distributions on plays are constant across every connected component of Nash equilibria (see Kreps and Wilson, 1982).

[3] This concept is slightly weaker than its preference-dependent analogue, precisely because for the latter coincidental payoff ties can render two strategies equivalent.

[4] Such a preference-free extensive form representation amounts to insisting on robustness of the players' preference ordering over plays against (sufficiently small) payoff perturbations.

map into plays; plays are the domain of preferences and hence may be mapped into payoff profiles in the full "game." This is crucial, as the formulation as a *game form* guarantees that ties only arise from extensive form structures and not by coincidence. Hence, this is equivalent to a generic payoff assignment where no two terminal nodes (or plays) yield the same payoff profile.

It is important to note that the difficulty we point out here will be completely obscured if one ignores the role of plays, which is highlighted by the focus on game forms. If one focuses on payoffs instead, any normal form game can be cast into a trivial (but non-equivalent) extensive form game where all players choose at the root between their strategies and all the remaining nodes are terminal. This extensive form game will have as many plays (or outcomes) as *cells* in the normal form game, which will typically exceed the number of plays in the original extensive form game that the normal form game was derived from. This is inappropriate, since, first, by artificially changing the space of plays/outcomes, it also changes the strategic elements of the problem,[5] and, second, backwards induction makes little sense in such a trivial extensive form representation. If the goal is to integrate backwards induction into a normal form solution concept, then the structure of the normal form must supply information about the extensive form—and this is achieved by the game form, that rules out coincidental, knife-edge ties by its very construction.

2 The Example

To begin with, a few clarifications are in order. An *extensive form* with $n \geq 1$ players is specified by a game tree and by choices and/or information sets for each player. (There are no payoff assignments.) A *play* is a full path in the tree from its root to a terminal node.[6] A pure strategy for a player is a function that maps this player's information sets into choices that are available there. The *normal form* is given by the function that maps strategy profiles into plays. The *reduced* normal form is obtained by collapsing, for each player, into single representatives all strategies that induce the same plays for all strategy profiles among the opponents.

Now consider the extensive form depicted in Figure 1, which is a short "centipede." Player 1 starts and can stop (choice S_1), which ends the game with play ω_1, or continue (choice C_1). In the latter case, player 2 decides between stopping (choice S_2), which ends the game with play ω_2, or continue (choice C_2). If the latter is chosen, player 1 decides again, choosing between S_3 and C_3, which ends the game at plays ω_3 or ω_4, respectively.

[5] For instance, a generic cell-based *payoff* perturbation of the reduced normal form game can only originate from a trivial extensive form game were all players play simultaneously, while a perturbation of the preferences defined on plays leaves the link between the original extensive form game and its reduced normal form unaffected.

[6] This at least applies if all plays end after finitely many moves. An infinitely repeated game cast in extensive form provides an example of a game tree without terminal nodes. Plays, on the other hand, are still well defined objects. See Alós-Ferrer and Ritzberger (2016).

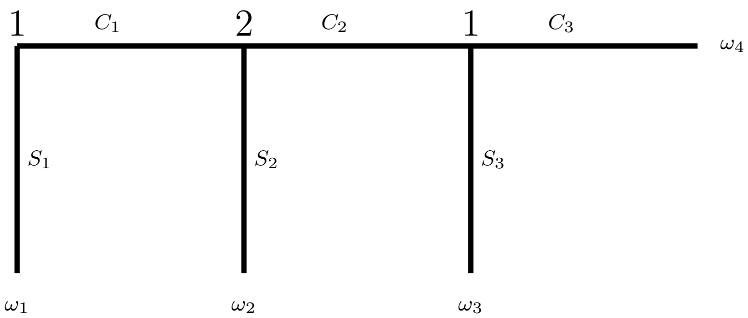


Fig. 1 Graphical representation of the example.

$1 \setminus 2$	S_2	C_2
S_1S_3	ω_1	ω_1
S_1C_3	ω_1	ω_1
C_1S_3	ω_2	ω_3
C_1C_3	ω_2	ω_4

$1 \setminus 2$	S_2	C_2
S	ω_1	ω_1
C_1S_3	ω_2	ω_3
C_1C_3	ω_2	ω_4

Fig. 2 The normal form (left) and reduced normal form (right) of the example.

The normal form of the game is given on the left-hand side of Figure 2. Since player 2 just decides once, among two choices, she has two pure strategies that correspond to the choices. Player 1 decides at two different nodes, each with two available choices. Hence he has four pure strategies, corresponding to the combinations of the choices, which are called S_1S_3, S_1C_3, C_1S_3, and C_1C_3. As the table shows, strategies S_1S_3 and S_1C_3 are strategically equivalent, since they always yield the same play for any given strategy of player 2 (actually, they always simply yield the play ω_1). Hence the reduced normal form, depicted on the right-hand side of Figure 2, combines them into a single strategy S.

We now ask the question of whether an extensive form can be defined whose normal form is the one shown on the right-hand side of Figure 2, i.e. it coincides with the reduced normal form of our example. That is, we seek an extensive form having exactly four plays, such that the allocation of decision nodes results in two pure strategies for player 2 and three pure strategies for player 1, such that the six combinations induce the plays as summarized by the right-had normal form in Figure 2. We keep the discussion free of any particular formalism, but it can be adapted to any formalization of extensive form games one might wish to employ (see Alós-Ferrer and Ritzberger, 2016).

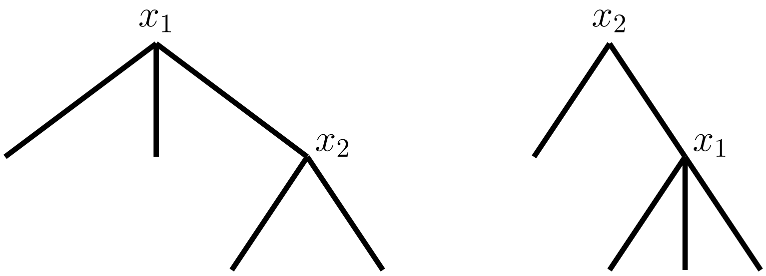


Fig. 3 Possible game trees to represent the reduced normal form.

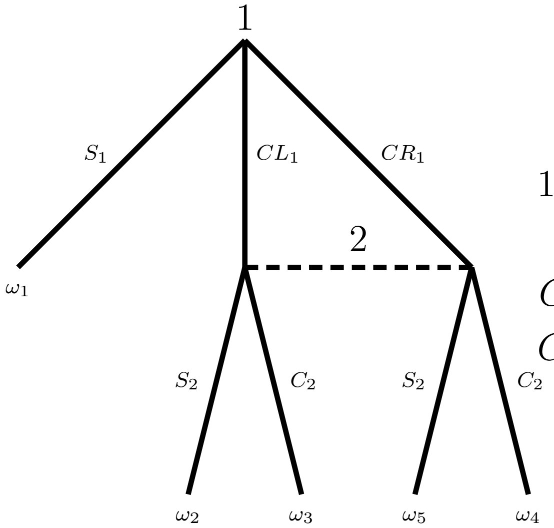
Suppose such an extensive form existed. Since player 2 has exactly two strategies, she can only decide in one node of this form, since if she decided in more than one node, the smallest number of strategies would be four.[7] Analogously, player 1, who has exactly three strategies, can only be active at a single node. Thus, the tree of the hypothesized extensive form can only have two moves (nodes that are not terminal), x_1 (where player 1 decides), and x_2 (where player 2 decides). There are then exactly two possibilities, whose trees are depicted in Figure 3: either x_1 follows x_2, or vice versa.

Both trees have four plays, as should be the case. However, if x_1 precedes x_2, player 1 must have two choices (hence pure strategies) that lead to fixed plays (i.e. plays that do not change with the choice of player 2), in contradiction with the reduced normal form (Figure 2, right). If x_2 precedes x_1, player 2 must have one choice leading to a fixed play independently of the choice of player 1, again in contradiction with the reduced normal form. Therefore, there exists no extensive form whose normal form corresponds to the sought one. Hence, we have proven that, in general, reduced normal forms derived from extensive forms can *not* be represented as extensive forms.

Remark 1 It is important to note that the result does not depend on whether or not imperfect information is allowed. The nonexistence of an extensive form corresponding to the reduced normal form described above follows simply from the fact that it is not possible to construct an extensive form with only four plays that can be reached in a manner consistent with the strategic decisions recorded in the reduced normal form. To clarify this point, consider the following attempt to build an approximation of the strategic problem by relying on imperfect information.[8] Player 1 decides either to stop (S_1), to continue left (CL_1), or to continue right (CR_1). After S_1, the game ends with the play ω_1. Both choices CL_1 and CR_1 lead to the same information set of player 2, that is, the second player cannot distinguish whether player 1 has continued left or right, but knows that she has not stopped. Then player 2 has two choices, S_2

[7] We assume here non-trivial decisions, where a player active at a node has at least two choices. Some formalisms allow for trivial decision where one player has one and only one choice. Allowing for those does not change the argument.

[8] We are grateful to a referee for inspiring this clarification.



$1 \setminus 2$	S_2	C_2
S_1	ω_1	ω_1
CL_1	ω_2	ω_3
CR_1	ω_5	ω_4

Fig. 4 An imperfect information game failing to generate the reduced normal form.

and C_2. Figure 4 depicts the game and the resulting normal form. Superficially, one may impose that the two possible continuations of player 1 are analogous to the two continuation possibilities of player 1 in the game depicted in Figure 1, namely C_1S_3 and C_1C_3. But this would be a statement about the players' preferences: Neither player cares about whether player 1 continued "right" or "left." In the game in Figure 1, the strategy combinations (C_1S_3, S_2) and (C_1C_3, S_2) induce the same play ω_2, but in the game in Figure 4, the strategy combinations (CL_1, S_2) and (CR_1, S_2) induce different plays, ω_2 and ω_5.

Specifically, suppose that one adds payoffs to the representation, and imposes that both players are exactly indifferent between ω_2 and ω_5. Then, the payoff-based representations of the normal form games in Figures 1 and 4 would indeed coincide. But the reason is just a coincidental payoff tie, unrelated to the structure of the extensive form game. The reduced normal forms, expressed in terms of plays, can never coincide.

3 Conclusion

It is well known from the work by Mailath et al. (1993) that payoff ties in the reduced normal form of a finite game reflect extensive form structures, like information sets or subgames. Yet, when players are assigned specific payoff functions these ties may get confounded with coincidental ones. If one insists that the only relevant payoff ties are those that are induced by the extensive form, then the observation in this note casts doubt on the sufficiency of the reduced normal form: A given reduced normal form, where all equivalence classes with respect to strategic equivalence are singletons, may not be representable as an extensive form. To reconstruct an extensive form representation of a given reduced normal form therefore takes information about the reduction steps, which is not included in the normal form.

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