ISSN 2347-3487
Volume 12 Number 4
Journal of Advances in Physics

Conserved Quantities for the General Linear Heat Equation

Adil Jhangeer, Fahad Al-Mufadi

Deanship of Educational Services, Preparatory Year Unit,
Qassim University, P.O. Box 6595, Al-Qassim,Buraidah: 51452,
Kingdom of Saudi Arabia.
adil.jhangeer@gmail.com

College of Engineering, Department of Mechanical Engineering,
Qassim University, P.O. Box 6595, Al-Qassim,Buraidah: 51452,
Kingdom of Saudi Arabia.
almufadi@qec.edu.sa

ABSTRACT

In this paper, conserved quantities are computed for a class of evolution equation by using the partial Noether approach [2]. The partial Lagrangian approach is applied to the considered equation, infinite many conservation laws are obtained depending on the coefficients of equation for each *n*. These results give potential systems for the family of considered equation, which are further helpful to compute the exact solutions.

.**Keywords:** Partial Noether approach, Partial Lagrangian, Conserved quantity.

Subject Classification: 76M60

1 Introduction

The (1+1)-dimensional equation of the form ([3] and reference therein):

$$u_t = a(t,x_1)u_{x_1x_1} + f(t,x_1) \tag{1}$$

represents one dimensional thermal processes in quiescent media. It also describes the solids with constant thermal diffusivity in the presence of a volume thermal source dependent on space coordinate and time.

The equation of the form ([3] and reference therein):

$$u_t = a(t,x_1)u_{x_1x_1} + b(t,x_1)u_{x_1} + c(t,x_1)u + f(t,x_1) \tag{2}$$

plays a vital role in physics. The homogenous form with b=0 discusses mass transfer in a quiescent medium with a first-order volume chemical reaction. It also explains the thermal processes and heat transfer in one dimensional rod whose lateral surface exchanges heat with ambient medium having constant temperature. For c=0, Eq. (2) describes the non-stationary problems of convective mass transfer in a continues medium that moves with a constant velocity where f represents absorption or release of substance.

The following equation ([3] and reference therein):

$$u_t = a(t,r)(u_{rr} + \frac{1}{r}u_r) + f(t,r) \tag{3}$$

explains the heat transfer in a plane, where f is proportional to amount of heat released per unit time in the volume. Eq. (3) also represents thermal processes having axial symmetry while unsteady thermal processes with central symmetry is described by:

$$u_t = a(t,r)(u_{rr} + \frac{2}{r}u_r) + f(t,r). \tag{4}$$

The models (1) to (4) are included in the equation of the form:

$$u_{x_0}=\sum_{j=1}^{n}\sum_{i=1}^{n}(a_{ii}u_{x_ix_i}+a_{ij}u_{x_ix_j}+a_iu_{x_i})+cu+f,\ \ a_{ii}\neq 0,\ \ i<j, \tag{5}$$

where a_{ii}, a_{ij}, a_i, c and f are the smooth functions of independent variables $(x_0,x_1,x_2,\cdots x_n)$ and u is considered as dependent variable. It is important to note here that conservation laws of Eqs. (1)-(4) cannot be computed by using the results given in [1] because of the smooth function f. Hence it is meaningful to discuss the Eq. (5) by means of partial Noether approach. In the next section, we will present the main results.

2 Main Result

In this section, we will present the conserved quantities of Eq. (5). It should be noted that Eq. (5) belongs to the non-variational problem and thus does not admit a standard Lagrangian but it does have a partial Lagrangian viz.

$$L=\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\left(\frac{a_{ii}u_{x_i}^2}{2}+\frac{a_{ij}u_{x_i}u_{x_j}}{2}\right)\right],\ \ i<j \tag{6}$$

and associated partial Euler-Lagrange equation is

$$\frac{\delta L}{\delta u}=\sum_{j=1}^{n}\left[\sum_{i=1}^{n}\left(a_i-(a_{ii})_{x_i}-\frac{(a_{ij})_{x_j}}{2}\right)u_{x_i}\right]-u_{x_0}+cu+f, \tag{7}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-\sum_{i=0}^{n}D_{x_i}\frac{\partial}{\partial u_{x_i}}\cdots, \tag{8}$$

while

$$D_{x_i}=\frac{\partial}{\partial x_i}+u_{x_i}\frac{\partial}{\partial u}+u_{x_ix_i}\frac{\partial}{\partial u_{x_i}}+u_{x_ix_j}\frac{\partial}{\partial u_{x_j}}+\cdots. \tag{9}$$

The partial Noether equation is:

$$X^{[1]}L+L\left(\sum_{i=0}^{n}D_{x_i}\xi_{x_i}\right)=\left(\phi-\sum_{i=0}^{n}D_{x_i}\xi_{x_i}\right)\frac{\delta L}{\delta u}+\sum_{i=0}^{n}D_{x_i}B^i, \tag{10}$$

where B^i's are known as the gauge terms, while $X^{[1]}$ is the first prolongation of the generator X and defined as

$$X^{[1]}=\sum_{i=0}^{n}\xi^i\frac{\partial}{\partial x_i}+\phi\frac{\partial}{\partial u}+\sum_{i=0}^{n}\phi^{x_i}\frac{\partial}{\partial u_{x_i}}+\ldots \tag{11}$$

Using Eq. (6) and Eq. (7) in Eq.(10) one can get an over-determined system of linear PDEs which gives

$$\xi^i=0,\ \ i=1,2,\cdots,n\ \ and\ \ \phi=\phi(x_0,x_1,\ldots,x_n)$$

The conserved vectors [2] for Eq. (5) are

$$T^0 = \phi u + \alpha^0,$$

$$T^i = \sum_{j=1}^{n} [\{(a_{ii})_{x_i} - a_i + \frac{(a_{ij})_{x_j}}{2}\}\phi u + a_{ii}(\phi_{x_i} u - \phi u_{x_i}) + + \frac{a_{ij}}{2}(\phi_{x_j} u - \phi u_{x_j})] + \alpha^i, \qquad i < j, \quad i = 1,2\ldots,n,$$

where ϕ satisfies the following system of PDEs

$$\phi_{x_0} + \sum_{j=1}^{n} \sum_{i=1}^{n} [(a_{ii}\phi_{x_i x_i} + a_{ij}\phi_{x_i x_j}) + (2(a_{ii})_{x_i} - a_i + (a_{ij})_{x_j})\phi_{x_j} + ((a_{ii})_{x_i x_i} - (a_i)_{x_i} + (a_{ij})_{x_i x_j} + c)\phi] = 0, \quad i<j,$$

while

$$\sum_{i=0}^{n} \alpha^i_{x_i} + f\phi = 0.$$

For Eq. (5), an infinite number of conserved vectors are obtained.

3 Conclusions

A special type of (1+*n*)-dimensional linear evolution equation was considered. Conserved quantities using the partial Lagrangian approach was derived in terms of the coefficients of the discussed equation. An infinite number of conserved quantities were computed for the considered equation.

Acknowledgement

Authors acknowledge Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number 2457 during the academic year 1436 AH/ 2014 AD.

REFERENCES

1. A. Jhangeer and I. Naeem, Conserved quantities for a class of (1+*n*)-dimensional linear evolution equation, Commun. Nonlinear Sci. Numer. Simul., 17(7), (2012) 2804-2814.
2. A. H. Kara and F. M. Mahomed, Noether-type symmetries and conservation laws via partial Lagrangian, Nonlinear Dynam., 45, (2006) 367-383.
3. A. D. Polyanin, Handbook of Linear Partial Differential Equations for Engineers and Scientists, Chapman and Hall/CRC Press, Boca Raton–London, 2002.