On Spectral-Homotopy Perturbation Method Solution of Nonlinear Differential Equations in Bounded Domains

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Abstract

In this study, a combination of the hybrid Chebyshev spectral technique and the homotopy perturbation method is used to construct an iteration algorithm for solving nonlinear boundary value problems. Test problems are solved in order to demonstrate the efficiency, accuracy and reliability of the new technique and comparisons are made between the obtained results and exact solutions. The results demonstrate that the new spectral homotopy perturbation method is more efficient and converges faster than the standard homotopy analysis method. The methodology presented in the work is useful for solving the BVPs consisting of more than one differential equation in bounded domains.

Keywords: Chebyshev spectral method; Homotopy perturbation method; nonlinear boundary value problems.

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1 Introduction

Many problems in the fields of physics, engineering and biology are modeled by coupled systems of boundary value problems of ordinary differential equations. The existence and approximations of the solutions of these systems have been investigated by many authors and some of them are solved using numerical solutions and some are solved using the analytic solutions. One of these analytic solutions is the homotopy perturbation method (HPM). This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields. It was proposed first by the Chinese researcher J. Huan He in 1998 [4,6]. The method has been applied successfully to solve different types of linear and nonlinear differential equations such as Lighthill equation[4], Duffing equation [5] and Blasius equation [10], wave equations [6], boundary value problems [11,12]. HPM method has been recently intensively studied by scientists and they used it for solving nonlinear problems and some modifications of this method have published [13,14] to facilitate and accurate the calculations and accelerate the rapid convergence of the series solution and reduce the size of work. The application of the HPM in linear and non-linear problems has been developed by many scientists and engineers [7,8,9], because this method continuously deforms some difficult problems into a simple problems which are easy to solve. The limited selection of suitable initial approximations and linear operators and are some of the main limitations of the HPM. Complicated linear operators and initial approximations may result in higher order differential equations that are difficult or impossible to integrate using the standard HPM.

The purpose of the present paper to introduce a new alternative and improved of the HPM called Spectral Homotopy Perturbation method (SHPM) in order to address some of the perceived limitations of the HPM uses the Chebyshev pseudospectral method to solve the higher order differential equations. This study proposes a standard way of choosing the linear operators and initial approximations for the SHPM. The obtained results suggest that this newly improvement technique introduces a powerful for solving nonlinear problems. Numerical examples of nonlinear second order BVPs are used to show the efficiency of the SHPM in comparison with the HPM. The new modification demonstrates an accurate solution compared with the exact solution.

2 The Spectral-Homotopy Perturbation Method

For the convenience of the reader, we first present a brief review of the standard HPM. This is then followed by a description of the algorithm of the SHPM solving nonlinear ordinary differential equations.

To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{2}$$

where A is a general operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. The operator A can, in generally, be divided into two parts L and part N so that equation (1) can be written as

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

where L is a simple part which is easy to handle and N contains the remaining parts of A. By the homotopy technique [2, 3], we construct a homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \ r \in \Omega \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (1), wich satisfies the boundary conditions. Obviously, from equation (4) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{7}$$

The changing process of p from zero to unity is equivalent to the deformation of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$, $A(v) - f(r)$ are homotopic. We can assume that the solution of equation (4) can be written as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{8}$$

setting $p = 1$, results in the approximation to the solution of equation (1)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{9}$$

The coupling of the perturbation method and the homotopy method gives the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation methods.

To describe the basic ideas of the spectral-homotopy perturbation method, we consider the following second order boundary value problem

$$u''(x) + a(x)u'(x) + b(x)u(x) + N[u,x] = F(x) \tag{10}$$

subject to the boundary conditions:

$$u(-1) = u(1) = 0 \tag{11}$$

where $x \in [-1,1]$ is an independent variable, $a(x), b(x)$ and $F(x)$ are known functions defined on $[-1,1]$ and N is a nonlinear function. The differential equation (10) can be written in the following operator form:

$$L[u] + N[u] = F(x) \tag{12}$$

where

$$L = \frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x) \tag{13}$$

Here u_0 is taken to be an initial solution of the nonhomogeneous linear part of governing differential equation (10) given by:

$$L[u_0] = F(x) \tag{14}$$

subject to the boundary conditions:

$$u_0(-1) = u_0(1) = 0. \tag{15}$$

Equation (14) together with the boundary conditions (15) can easily be solved using any numerical methods methods such as finite differences, finite elements, Runge-Kutta or collocation methods. In this work we used the Chebyshev spectral collocation method. This method is based on approximating the unknown functions by the Chebyshev interpolating polynomials in such a way that the are collocated at the Gauss-Lobatto points (see [1,15] for details). The unknown function $u_0(x)$ is approximated as a truncated series of Chebyshev polynomials of the form

$$u_0(x) \approx u_0^N(x) = \sum_{k=0}^{N} u_k T_k(x_j), \quad j = 0,1,\cdots,N, \tag{16}$$

where T_k is the k th Chebyshev polynomial, u_k are coefficients and $x_0, x_1, \cdots, x_N$ are Gauss-Lobatto collocation points defined on the interval $[-1,1]$ by

$$x_j = \cos\frac{\pi}{N}, \quad j = 0, 1, \cdot, N \tag{17}$$

The derivatives of the function $u_0(x)$ at the collocation points are represented as

$$\frac{d^r u_0(x)}{dx^r} = \sum_{k=0}^{N} \mathrm{D}_{kj}^r u_0(x_j) \tag{18}$$

where r is the order of differentiation and D being the Chebyshev spectral differentiation matrix whose entries are defined as (see for example,[1,15]);

$$
\left.
\begin{aligned}
\mathrm{D}_{jk} &= \frac{c_j}{c_k}\frac{(-1)^{j+k}}{x_j - x_k} && j \neq k;\ j,k = 0,1,\ldots,N, \\
\mathrm{D}_{kk} &= -\frac{x_k}{2(1-x_k^2)} && k = 1,2,\ldots,N-1, \\
\mathrm{D}_{00} &= \frac{2N^2+1}{6} = -\mathrm{D}_{NN}.
\end{aligned}
\right\} \tag{19}
$$

Substituting Equations (16)-(18) in (14) yields

$$Au_0(x) = F(x), \tag{20}$$

where

$$A = \mathrm{D}^2 + a(x)\mathrm{D} + b(x)I, \tag{21}$$

where I is a diagonal matrix of size $N \times N$. The matrix A has dimensions $N \times N$ while matrix $F(x)$ has dimensions $N \times 1$. To incorporate the boundary conditions (15) to the system (20) we delete the first and the last rows and columns of A and delete the first and last elements of u_0 and $F(x)$, this showing as follows

$$
\begin{pmatrix}
A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} & A_{0,N} \\
A_{1,0} & & & & A_{1,N} \\
\vdots & & A & & \vdots \\
A_{N-2,0} & & & & A_{N-2,N} \\
A_{N,0} & & \cdots & & A_{N,N} \\
A_{N,0} & A_{N,1} & \cdots & A_{N,N-1} & A_{N,N}
\end{pmatrix}
\begin{pmatrix}
u(x_0) \\
\vdots \\
u(x_N)
\end{pmatrix}
=
\begin{pmatrix}
F(x_0) \\
\vdots \\
F(x_N)
\end{pmatrix}
\tag{22}
$$

Thus, the solution u_0 is determined from the equation

$$\tilde{u}_0 = \tilde{A}^{-1}\tilde{F}(x), \tag{23}$$

where $\tilde{u}_0, \tilde{A}$ and $\tilde{F}(x)$ are the modified matrices of u_0, A and $F(x)$, respectively. The solution (23) provide us with the initial approximation of the Equation (10). The higher approximations are obtained by construct a homotopy for the government Equation (10) as follows

$$\mathrm{H}(U,p) = \mathrm{L}[U] - \mathrm{L}[u_0] + p\mathrm{L}[u_0] + p\big(N[U] - F(x)\big) = 0, \tag{24}$$

where $p \in [0,1]$ is an embedding parameter and U is assumed a solution of equation (10) given as power series in p as follows

$$U = u_0(x) + pu_1(x) + p^2 u_2(x) + ... = \sum_{n=0}^{\infty} p^n u_n(x) \tag{25}$$

Substitute (25) into (24) and compare between the coefficients of p^i of both sides of resulting equation, we have

$$\mathrm{L}u_i - \Phi_i = 0, \ i = 1, 2, ..., n \tag{26}$$

where

$$\Phi_i = (\chi - 1)\left(\mathrm{L}u_0 + N[u_0] - F(x) \right) - \frac{\chi}{n!}\left[\frac{d^n}{d\lambda^n} N\left[\sum_{i=0}^{n} \lambda^i u_i \right] \right]_{\lambda=0} = 0, \ i = 1, 2, ... \tag{27}$$

where

$$\chi = \begin{cases} 0, & i = 1 \\ 1, & i > 1 \end{cases} \tag{28}$$

From the set of equations (26), the i th order approximation for $i = 1, 2, 3, ..$ are given by the following system of matrices

$$Au_i = \Phi_i, \tag{29}$$

subject to the boundary conditions

$$u_i(-1) = u_i(1) = 0, \tag{30}$$

To incorporate the boundary conditions (30) to the system (29) we delete the first and the last rows and columns of A and delete the first and last elements of u_i and Φ_i. This reduces the dimension of A to $(N-2) \times (N-2)$ and those of u_i and Φ_i to $(N-2) \times 1$. Finally, the solution f_i is determined from the equation

$$\tilde{u}_i = \tilde{A}^{-1}\tilde{\Phi}_i, \tag{31}$$

where $\tilde{u}_i, \tilde{A}$ and $\tilde{\Phi}_i$ are the modified matrices of u_i, A and Φ_i, respectively. The solutions for (29) provides us with the highes order approximations of the governing equation (10). The series u_i is convergent for most cases. However, the convergence rate depends on the nonlinear operator of (10). The following opinions are suggested and proved by He [16,17]

1. The second derivative of $N(u)$ with respect to u must be small because the parameter p may be relatively large, i.e $p \rightarrow 1$.

2. The norm of $L^{-1}\frac{\partial N}{\partial u}$ must be smaller than one so that the series converges.

This is the same strategy that is used in the SHPM approach. We observe that the main difference between the HPM and the SHPM is that the solutions are obtained by solving a system of higher order ordinary differential equations in the HPM while for the SHPM solutions are obtained by solving a system of linear algebraic equations that are easier to solve.

3 Solution of Test Problems

In this section, we illustrate the use of SHPM by solving systems of nonlinear boundary value problems whose exact solutions are known.

Problem 1:

Consider the nonlinear second order boundary value problem:

$$f'' + 2xf' + 2f + 4xff' + 2f^2 + \frac{1}{2} = 0, \tag{32}$$

subject to the boundary conditions

$$f(-1) = f(1) = 0. \tag{33}$$

The exact solution for (32) is

$$f(x) = \frac{1}{x^2 + 1} - \frac{1}{2}. \tag{34}$$

To apply the SHPM on this problem we may construct the homotopy:

$$\mathrm{H}(F, p) = \mathrm{L}[F] - \mathrm{L}[f_0] + p\mathrm{L}[f_0] + p\left[N(F) + \frac{1}{2}\right] = 0, \tag{35}$$

where F is an approximate series solution of (32) given by

$$F = f_0 + pf_1 + p^2 f_2 + \cdots. \tag{36}$$

and

$$\mathrm{L} = \frac{d^2}{dx^2} + 2x\frac{d}{dx} + 2. \tag{37}$$

The initial approximation for the solution of (32) is obtained from the solution of the linear equation

$$f_0'' + 2xf_0' + 2f_0 + \frac{1}{2} = 0. \tag{38}$$

subject to the boundary conditions

$$f_0(-1) = f_0(1) = 0. \tag{39}$$

The higher order approximations for (32) obtained by compared between the coefficients of $p^i, (i \geq 1)$ of both sides of (35) to get the following system of matrices

$$Af_i = \Phi_i \tag{40}$$

subject to the boundary conditions

$$f_i(-1) = f_i(1) = 0 \tag{41}$$

where

$$\Phi_i = \left[(\chi - 1)\left(L[f_0] + \frac{1}{2}\right) - 4x\sum_{j=0}^{i-1} f_j Df_{i-1-j} - 2\sum_{j=0}^{i-1} f_j f_{i-1-j}\right], \quad j = 1, 2, \ldots \tag{42}$$

Finally, the solution of (32) is given by substitute $f_i's$ in (36) after setting $p = 1$.

Table 1: Comparison of the values of the SHPM (shaded) and HPM (unshaded) approximate solutions for $f(x)$ with the exact solution for various values of x ..

x	4th order	5th order	6th order	7th order	Exact
-0.9	0.052486	0.052486	0.052486	0.052486	0.052486
	0.052441	0.052482	0.052486	0.052486	
-0.7	0.171141	0.171141	0.171141	0.171141	0.171141
	0.168303	0.170417	0.170956	0.171094	
-0.5	0.299999	0.300000	0.300000	0.300000	0.300000
	0.284180	0.294067	0.297775	0.299166	
-0.3	0.417420	0.417430	0.417431	0.417431	0.417431
	0.378111	0.399540	0.409291	0.413727	
0	0.499959	0.499994	0.499999	0.500000	0.500000
	0.437500	0.468750	0.484375	0.492188	
0.2	0.461516	0.461535	0.461538	0.461538	0.461538
	0.410496	0.437038	0.449778	0.455894	
0.4	0.362065	0.362069	0.362069	0.362069	0.362069
	0.335244	0.350802	0.357337	0.360082	
0.6	0.235294	0.235294	0.235294	0.235294	0.235294
	0.227584	0.232827	0.234505	0.235041	
0.8	0.109756	0.109756	0.109756	0.109756	0.109756
	0.109116	0.109641	0.109735	0.109752	

Table 2: Maximum absolute errors of the approximate solution of $f(x)$ for test problem 1 for different values of N

N	2nd order	4th order	6th order	8th order
30	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386776×10^{-8}
50	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386776×10^{-8}
60	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386773×10^{-8}
100	1.693407×10^{-3}	4.090457×10^{-5}	9.880794×10^{-7}	2.386771×10^{-8}
200	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386777×10^{-8}

Table 1 gives a comparison of SHPM and HPM results at different orders of approximation against the exact solution at selected values of x when $N = 40$ for SHPM. It can be seen from Table 1, the HPM results converge slowly to the exact solution while the SHPM results converge rapidly to the exact solution. The SHPM convergence is achieved up to 6 decimal places at the 6th order of approximation. It is clear that the results obtained by the present method are more convergence to the exact solution compared to the HPM. As with most approximation techniques, the accuracy further improves with an increase in the order of the SHPM approximations.

Table 2: Maximum absolute errors of the approximate solution of $f(x)$ for test problem 1 for different values of N

N	2nd order	4th order	6th order	8th order
30	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386776×10^{-8}
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60	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386773×10^{-8}
100	1.693407×10^{-3}	4.090457×10^{-5}	9.880794×10^{-7}	2.386771×10^{-8}
200	1.693407×10^{-3}	4.090457×10^{-5}	9.880795×10^{-7}	2.386777×10^{-8}

Table 2 shows the maximum absolute errors between the SHPM and the exact solution at different order of approximate for different values of N. The maximum absolute errors are generally very small and very further decrease with an increase in order of the SHPM approximation. However, increasing the number of nodes (or increasing N) does not result in a significant improvement in the accuracy of the SHPM approximations.

Figure 1 shows a comparison between the 3rd order of both SHPM and HPM approximate solutions against the exact solution for test problem 1. It can be seen that the accuracy is not achieved at the 3rd order for HPM approximation whereas there is very good agreement between the SHPM and exact results at the same order of approximations. This shows that the efficiency of the SHPM approach and it gives superior accuracy and convergence to the exact solution compared with HPM.

Problem 2:

We consider the following coupled system of nonlinear second order BVP:

$$f''(x) + \ln(2) f'(x) + \frac{1}{8}(x+3)g + [f'(x)]^2 = \phi_1(x) \tag{43}$$

$$g''(x) + 3f'(x) - \frac{3}{32}(x+9)^2 g - \frac{3}{8}(x+9)g^2 - \frac{1}{2}g^3 = \phi_2(x) \tag{44}$$
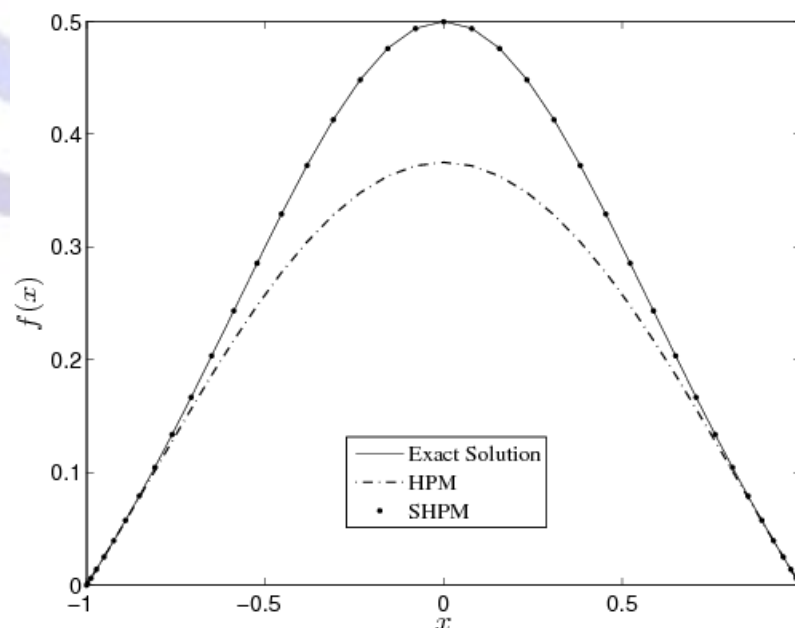
Fig 1: Comparison between the exact solution of $f(x)$, 3rd order HPM and 3rd order SHPM of problem 1.

subject to the boundary conditions

$$f(-1) = f(1) = g(-1) = g(1) = 0 \tag{45}$$

where

$$\phi_1(x) = \frac{1}{32}(x^2 - 1) - \frac{1}{4}\ln^2(2) \tag{46}$$

$$\phi_2(x) = \frac{1}{128}(225 - 69x - 45x^2 - 7x^3) - \frac{3}{2}\ln(2) \tag{47}$$

The exact solutions for $f(x)$ and $g(x)$ are

$$f(x) = \ln\left(\frac{x+3}{2}\right) - \frac{1}{2}(x+1)\ln(2) \tag{48}$$

$$g(x) = \frac{x-3}{4} + \frac{2}{x+3} \tag{49}$$

The initial approximations of (43) and (44) are solutions of the following system of equations

$$f_{0''}(x) + \ln(2)f_{0'}(x) + \frac{1}{8}(x+3)g_0 = \phi_1(x) \tag{50}$$

$$g_{0''}(x) + 3f_{0'}(x) - \frac{3}{32}(x+9)^2 g_0 = \phi_2(x) \tag{51}$$

subject to the boundary conditions

$$f_0(-1) = f_0(1) = g_0(-1) = g_0(1) = 0 \tag{52}$$

Applying the Chebyshev spectral collocation method we obtain the following system of matrices

$$[A][F_0 \quad G_0]^T = [P_0 \quad Q_0]^T \tag{53}$$

where

$$A = \begin{bmatrix} D^2 + \ln(2)D & \frac{1}{8}diag[(x+3)] \\ 3D & D^2 - \frac{3}{32}diag\left[(x+9)^2\right] \end{bmatrix} \tag{54}$$

$$F_0 = [f_0(x_0), f_0(x_1), ..., f_0(x_N)], \quad G_0 = [g_0(x_0), g_0(x_1), ..., g_0(x_N)] \tag{55}$$

and

$$P_0 = [\phi_0(x_0), \phi_0(x_1), ..., \phi_0(x_N)], \quad Q_0 = [\phi_1(x_0), \phi_1(x_1), ..., \phi_1(x_N)] \tag{56}$$

where diag[ ] is a diagonal matrix of size $N \times N$ and T is the transpose. The matrix **A** has dimensions $2N \times 2N$ while matrices $[F_0 \quad G_0]^T$ and $[P_0 \quad Q_0]^T$ have dimensions $2N \times 1$. To implement the boundary conditions (52) to the system (53) we delete the first, $N, N+1$ and the last rows of **A**, we also delete the first, N, $N+1$ and last elements of $[F_0 \quad G_0]^T$ and $[P_0 \quad Q_0]^T$. The first, $N, N+1$ and the last columns of **A** are also deleted. This reduce the dimensions

of **A** to $2(N-2)\times 2(N-2)$, and reduce the dimensions of $[F_0 \; G_0]^T$ and $[P_0 \; Q_0]^T$ to $2(N-2)\times 1$. The solution F_0 and G_0 for the system (53) gives the first approximations of the system equations (43) and (44) for $f(x)$ and $g(x)$, respectively. To compute higher order approximations we may construct the homotopy:

$$H_1(F,G,p) = L_{11}[F] + L_{12}[G] - L_{11}[f_0] - L_{12}[g_0] + pL_{11}[f_0]$$

$$+ pL_{12}[g_0] + p\left[[F']^2 - F_1\right] = 0 \tag{57}$$

$$H_2(F,G,p) = L_{21}[G] + L_{22}[F] - L_{21}[g_0] - L_{22}[f_0] + pL_{21}(g_0)$$

$$+ pL_{22}[f_0] + p[-\frac{3}{8}(x+9)G^2 - \frac{1}{2}G^3 - F_2] = 0 \tag{58}$$

where

$$F = \sum_{i=0}^{n} f_i \qquad and \qquad G = \sum_{i=0}^{n} g_i \tag{59}$$

are series solutions for (43) and (44), respectively, and we choose the linear operators as:

$$L_{11} = D^2 + \ln(2)D, \quad L_{12} = \frac{1}{8}(x+3), \quad L_{21} = D^2 - \frac{3}{32}(x+9)^2, \quad L_{22} = 3D \tag{60}$$

By substituting (59) into (57) and (58) and compare the powers of p, we have the following system of matrices:

$$[A][F_i \; G_i]^T = [P_i \; Q_i]^T, \; i = 1, 2, ... \tag{61}$$

subject to the boundary conditions

$$f_i(-1) = f_i(1) = g_i(-1) = g_i(1) = 0 \tag{62}$$

where

$$A = \begin{bmatrix} D^2 + \ln(2)D & \frac{1}{8}diag[(x+3)] \\ 3D & D^2 - \frac{3}{32} diag\left[(x+9)^2\right] \end{bmatrix} \tag{63}$$

$$F_i = [f_i(x_0), f_i(x_1), ..., f_i(x_N)], \quad G_i = [g_i(x_0), g_i(x_1), ..., g_i(x_N)] \tag{64}$$

$$\left. \begin{aligned} P_i &= (\chi - 1)\left(L_{11}[f_0] + \frac{1}{8}(x+3)g_0 - \phi_1\right) - \sum_{j=0}^{i-1} Df_j Df_{i-1-j} \\ Q_i &= (\chi - 1)\left(L_{21}[g_0] - L_{22}[f_0] - \phi_2\right) + \frac{3}{8}\sum_{j=0}^{i-1} g_j g_{i-1-j} + \frac{1}{2}\sum_{j=0}^{i-1} g_{i-1-j}\sum_{k=0}^{j} g_i g_{j-k} \end{aligned} \right\} \tag{65}$$

Starting from the initial approximations f_0 and g_0, higher order approximations f_i and g_i for F_i and F_i $(i = 1, 2, 3, ...)$, respectively, can be obtained through the iterative formula (61) together with the boundary conditions (62). Finally, the solution of (43) and (44) is obtained by substitute the series f_i and g_i in (59).

Table 3 gives a comparison between the SHPM and the exact solutions at selected nodes for Problem 2. In general, convergence of the SHPM is achieved at the 4th order of approximation. The results again point to the faster convergence of the SHPM.

Table 3: Comparison of the values of the SHPM approximate solutions for $f(x)$ **and** $g(x)$ **at different orders with the exact solution for various values of** x.

x	2nd order	3rd order	4th order	Exact	2nd order	3rd order	4th order	Exact
	$f(x)$				$g(x)$			
-0.9	0.014133	0.014133	0.014133	0.014133	0.022618	0.022619	0.022619	0.022619
-0.7	0.035790	0.035790	0.035790	0.035790	0.055432	0.055435	0.055435	0.055435
-0.5	0.049857	0.049857	0.049857	0.049857	0.074995	0.075000	0.075000	0.075000
-0.3	0.057504	0.057503	0.057503	0.057503	0.084253	0.084259	0.084259	0.084259
0	0.058892	0.058892	0.058892	0.058892	.083327	0.083333	0.083333	0.083333
0.2	0.054116	0.054115	0.054115	0.054115	0.074994	0.075000	0.075000	0.075000
0.4	0.045426	0.045425	0.045425	0.045425	0.061761	0.061764	0.061765	0.061765
0.6	0.033269	0.033269	0.033269	0.033269	0.044442	0.044444	0.044444	0.044444
0.8	0.018021	0.018021	0.018021	0.018021	0.023683	0.023684	0.023684	0.023684

Table 4 shows the maximum absolute errors $f(x)$ and $g(x)$ of the SHPM solution at different orders of approximation for different values of N. As pointed out earlier however, increasing the number of nodes (increasing N) does not result in a significant improvement in the accuracy of the SHPM approximation.

Table 4: Maximum absolute errors of the approximate solution of $f(x)$ **and** $g(x)$ **for test problem 2 for different values o**f N.

	2nd order	4th order	6th order	8th order
N	$\max\lvert f_{Exact} - f_{SHPM} \rvert$			
30	8.773578×10^{-7}	8.672349×10^{-9}	5.230253×10^{-11}	3.237272×10^{-13}
50	8.783291×10^{-7}	8.678809×10^{-9}	5.234172×10^{-11}	3.248096×10^{-13}
60	8.777888×10^{-7}	8.678587×10^{-9}	5.234038×10^{-11}	3.250109×10^{-13}
100	8.783291×10^{-7}	8.678813×10^{-9}	5.234540×10^{-11}	3.284872×10^{-13}
200	8.783587×10^{-7}	8.680182×10^{-9}	5.234731×10^{-11}	3.225614×10^{-13}
	$\max\lvert g_{Exact} - g_{SHPM} \rvert$			
30	6.931877×10^{-6}	3.4275×10^{-8}	2.042015×10^{-10}	1.344633×10^{-12}
50	6.941926×10^{-6}	3.436920×10^{-8}	2.048715×10^{-10}	1.349518×10^{-12}
60	6.935148×10^{-6}	3.439727×10^{-8}	2.051871×10^{-10}	1.352599×10^{-12}
100	6.941926×10^{-6}	3.439727×10^{-8}	2.051877×10^{-10}	1.353279×10^{-12}
200	6.941926×10^{-6}	3.4423×10^{-8}	2.051966×10^{-10}	1.362133×10^{-12}

(a)   (b)
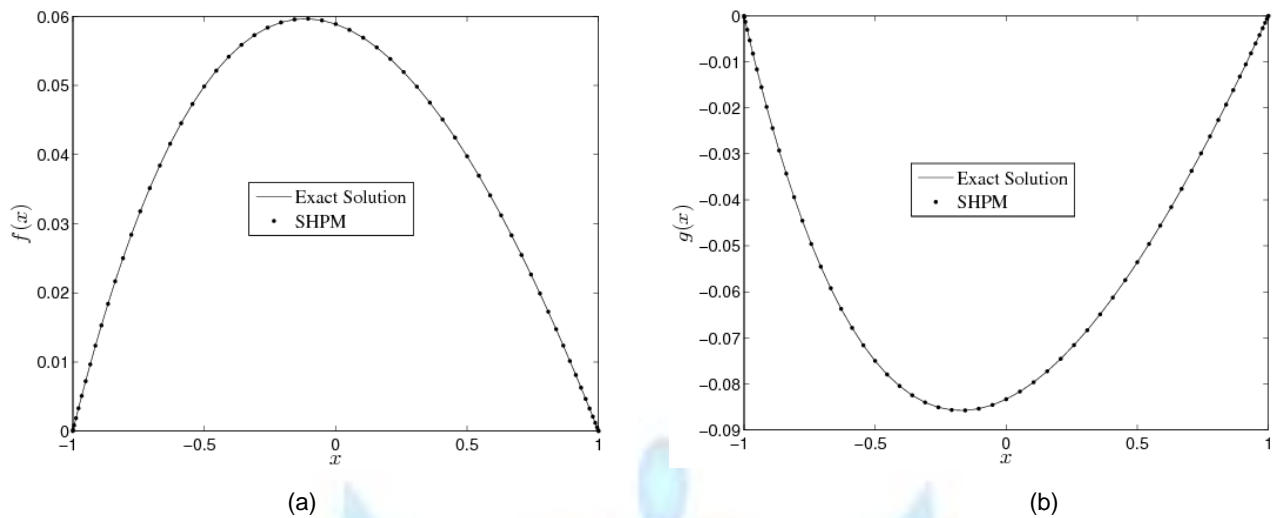
Fig 2: Comparison of the exact solution $f(x)$ **and** $g(x)$ **of test problem 2 with 2nd order SHAM solutions.**

Figure 2 shows a comparison between the exact solution for $f(x)$ and $g(x)$ against the 2nd order SHPM approximate solutions for Problem 2. Again, we note that there is good agreement between the exact solutions and the SHPM approximations even at very low orders of approximation.

4 Conclusion

In this paper, we have shown that the proposed SHPM can be used successfully for solving nonlinear boundary value problems in bounded domains. The merit of the SHPM is that it converges faster to the exact solution with a few terms necessary to obtain accurate solution, this was demonstrated through examples which proved the convergency of the SHPM, it was also found that is has best selection method to the initial approximation than HPM.

The main conclusions emerging from this study are follows:

1. SHPM proposes a standard way of choosing the linear operators and initial approximations by using any form of initial guess as long as it satisfies the boundary conditions while the initial guess in the HPM can be selected that will make the integration of the higher order deformation equations possible.
2. SHPM is simple and easy to use for solving the nonlinear problems and useful for finding an accurate approximation of the exact solution because the obtained governing equations are presented in form of algebraic equations.
3. SHPM is highly accurate, efficient and converges rapidly with a few iterations required to achieve the accuracy of the numerical results compared with the standard HPM, for example, in this study it was found that for a few iterations of SHPM was sufficient to give good agreement with the exact solution.

Finally, the spectral homotopy perturbation method described above has high accuracy and simple for nonlinear boundary value problems compared with the standard homotopy perturbation method. Because of its efficiency and easy of use. The extension to systems of nonlinear BVPs allows the method to be used as alternative to the traditional Runge-Kutta, finite difference, finite element and Keller-Box methods.

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